Discrete Mathematics
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Lecture -25
Tutorial 4: Part I

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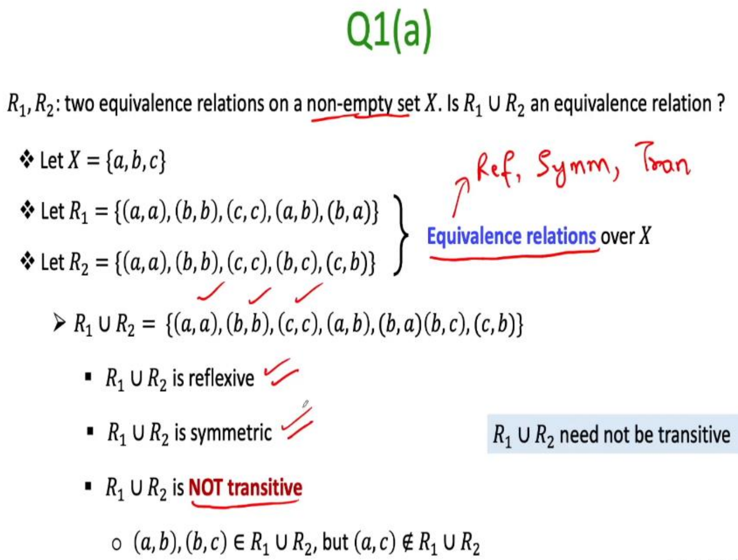

Hello everyone, welcome to the first part of tutorial 4. So, let us start with question number 1(a). This question you are given two equivalence relations on a non-empty set X. You are asked to prove or disprove whether $R_1 \cup R2$ is an equivalence relation or not? So, it turns out at a union of two equivalence relations need not be an equivalence relation.

And this is demonstrated by this counter example. So, I consider my set X={a,b,c}. And let me have equivalence relations R_1 and R_2. So, $R_1 = \{ (a,a),(b,b),(c,c),(a,b),(b,a)\}$ and $R_2 = \{ (a,a),(b,b),(c,c),(b,c),(c,b)\}$. It is easy to see that both of them are equivalence relations over the set X they are reflexive. Each of them is a reflexive relations symmetric relation and a transitive relation I am not going through that, it is easy to verify that.

Now, if you take the union of these two relations, $R_1 \cup R_2$, you will have these ordered pairs. And it is easy to verify that $R_1 \cup R_2$ is a reflexive relation over the set X, because you have $\{(a,a),(b,b),(c,c)\}$ present in the relation. The union is also symmetric because if you have any

(a, b) in the union present and ordered pair (b, a) is also present in the union.

But it is easy to see that union here is not a transitive relation, specifically you have (a, b) and an ordered pair (b, c) present in $R_1 \cup R_2$, but a, c is not present in the union and hence the transitivity properties violated. So, in general the union of two equivalence relations need not be transitive. We can prove that they will be always it will be always reflexive and symmetric. If I take the union it will be always reflexive and symmetric, but the union need not be a transitive relation.

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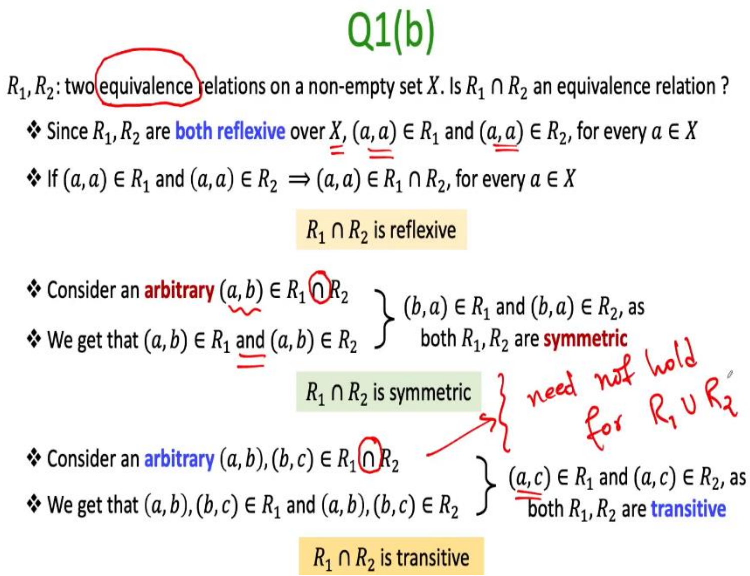

In part b of question 1, we are supposed to prove whether the obvious as to prove or disprove, whether the intersection is an equivalence relation or not. And it turns out that intersection of two equivalence relations over the same set X is always an equivalence relation. So, let us prove the three required properties. If I take R_1 and R_2 to be equivalence relations over the set X, then since R_1 and R_2 are individually reflexive relations. You will have ordered pair of the form (a, a) present in both R_1 as well as in R_2. And as a result, you will have ordered pairs of the form (a, a) present in the intersection of R_1 and R_2 as well. That shows that $R_1 \cap R_2$ is a reflexive relation. Well let us try to prove the symmetric property for $R_1 \cap R_2$. So, for that I consider an arbitrary ordered pair (a, b) to be present in $R_1 \cap R_2$.

Since it is present in the intersection as per the definition of intersection, it will be present in both R_1 as well as R_2. Now it is given that my relation R_1 and relation R_2 are equivalence relations, so individually both of them are symmetric relations. And as a result of that the ordered pair (b, a) will be present in R_1 as well as the ordered pair where (b, a) also will be present in R_2. So, since (b, a) is present in both R_1 as well as in R_2, it will be present in their intersection as well. And which shows that the intersection will be a symmetric relation.

Now consider the transitivity property for which I consider arbitrary ordered pairs (a, b) and (b, c) to be present in $R_1 \cap R_2$. So, since it is present in $R_1 \cap R_2$, there two ordered pairs will be individually present in both R_1 as well as in R_2 and since R_1 and R_2 are individually equivalence relations. Each of them satisfies the transitivity property, due to which I get that ordered pairs (a, c) is present in both R_1 as well as in R_2. And hence the ordered pair (a, c) will be present in their intersection as well.

So, now you can see that the argument that we have given here for $R_1 \cap R_2$ to be a transitive relation need not hold for the union. And this is precisely the reason due to which the union of two equivalence relations need not be transitive relation and hence any equivalence relation you can verify it.

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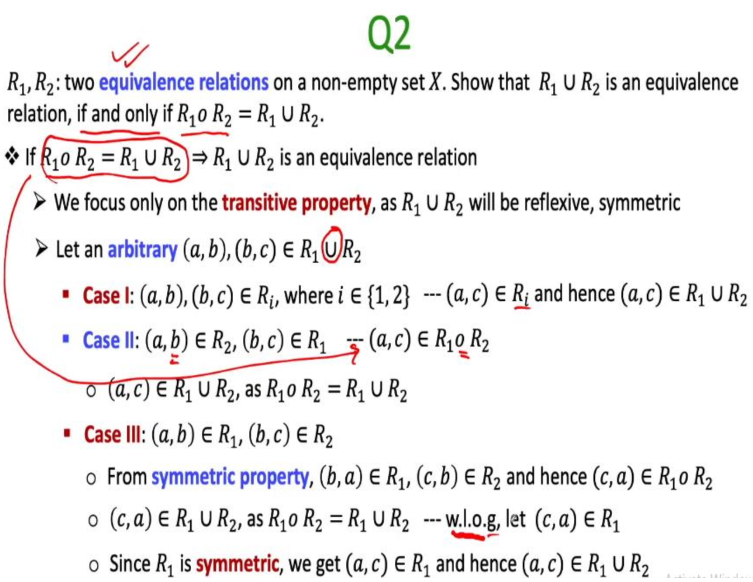

Now on question number 2 again, you are given two equivalence relations over a non- empty set and we want to prove that $R_1 \cup R_2$ will be an equivalence relation if and only if the composition $R_1 o\, R_2$ is equal to the $R_1 \cup R_2$. And this is, an if and only if statement. So, we have to give two proofs. We have to prove the implication in both the directions.

So, let us first prove the implication in the direction where I assume $R_1 o\ R_2 = R_1 \cup R_2$. Under that assumption I will be showing that $R_1 \cup R_2$is an equivalence relation. And I will be just focusing on proving that $R_1 \cup R_2$ is a transitive relation. Because we can always show that $R_1 \cup R_2$ will satisfy the reflexive property and symmetric property.

It is only the transitive property which is missing. And what we will show is that if this premise hold, then $R_1 \cup R_2$ will be satisfying the transitivity property. So, let us consider arbitrary ordered pairs (a, b) and (b, c) to be present in $R_1 \cup R_2$. Now because of the union there could be three possible cases depending upon where exactly the ordered pair (a, b) and ordered pair (b, c) belongs.

So, case 1 could be that the ordered pairs (a, b) as well as (b, c) are present in one of these two relations, at least in one of these two relations. Say in the relation R_i, there R_i could be either R_1 or R_2. If that is the case, then since that relation R_i where both (a, b) and (b, c) are present is also an equivalence relation mind it. We are given that individually, we are given that both R_1 as well as R_2, are equivalence relations.

So, if both (a, b) and (b, c) are present in the relation R_i and as for the transitivity property the ordered pair (a, c) will be present in R_i as well and hence it will be present in $R_1 \cup R_2$. Case 2 could be where the ordered pair $(a, b) \in\ R_2$ and ordered pair $(b, c) \in\ R_1$. In that case, you will have (a, b), (b, c) present in the union as well. Now that is the case, then as per the definition of composition of two relations, the ordered pair (a, c) will be present in $R_1 o\ R_2$because you have an intermediate b. So, (a, b) is in R_2 and (b, c) is in R_1. So, hence (a, c) will be in $R_1 o\ R_2$. But our premise says that $R_1 o\ R_2$ is exactly the same as the union. So, (a, c) is present in $R_1 o\ R_2$, it will present in the union as well and hence in this case also we proved a transitivity property.

So, tricky cases when you have the ordered pair $(a, b) \in\ R_1$ and ordered pair $(b, c) \in\ R_2$. In that case, I cannot apply the same argument as I applied in case 2, I have to do something extra here. So, what I do here is I apply the symmetric property for the relation R_1. Mind it R_1 is an equivalence relation and hence it has to symmetric property. So, since $(a, b) \in\ R_1$ we will have

$(b, a) \in R_2$.

Due to the same reason, since R_2 it also satisfies the symmetric property. We will have $(c, b) \in R_2$. And as a result I can say that the ordered pair (c, a) is present in the composition. But now that is not my goal. My goal is to show that (a, c) is present in the composition and then I can use the fact that $R_1 o\, R_2$ is same as $R_1 \cup R_2$ and conclude that (a, c) is in the union as well.

So, what I do here is since I know I have the premise says that $R_1 o\, R_2$ is same as their union, I can conclude that ordered pair (c, a) is present in the union as well. Now if it is present in $R_1 \cup R_2$. It will be present in at least one of these two relations either in R_1 or R_2. So, without loss of generality let it be present in R_1. So, this w.l.o.g here means, without loss of generality.

So, whatever argument I am going to give assuming that (c, a) is present in R_1, can be applied in also symmetrically for the case when (c, a) is present R_2. So, assuming $(c, a) \in R_1$, since R_1 is also symmetric we get (a, c) also present R_2 and hence (a, c) will be present in the union of R_1. We have proved the implication in this direction.

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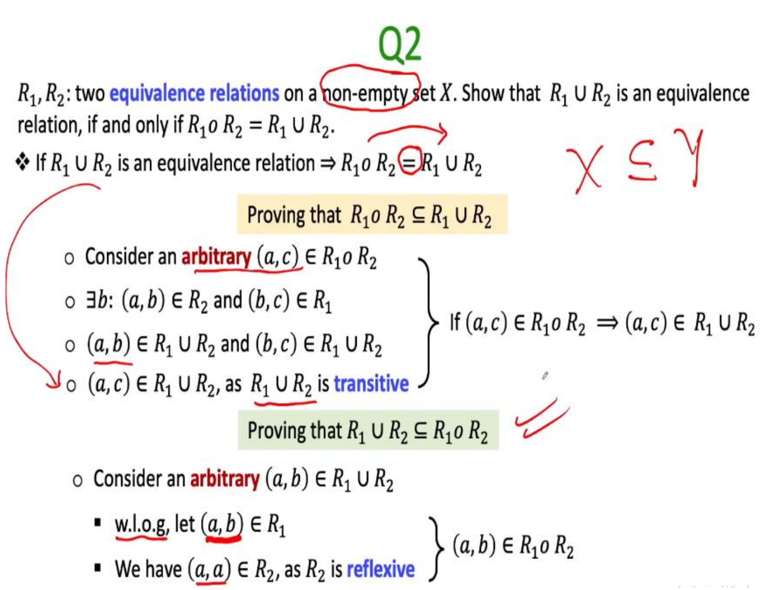

Let us prove the implication in the other direction. So, I am going to prove that if $R_1 \cup R_2$ is an equivalence relation, then there and $R_1 o\, R_2$ is equal to their union. So, I have to prove the equality of two sets. I have to prove that $R_1 o\, R_2 \subseteq R_1 \cup R_2$. And I have to prove that $R_1 \cup R_2 \subseteq R_1 o\, R_2$.

Then only I can conclude that these two sets are equal.

So, let us prove that $R_1 o\ R_2 \subseteq R_1 \cup\ R_2$. And how do I prove that a set X is a subset of set Y. I have to show that you take any element from the set X it will be present in the set Y as well. So, that is why I am taking an arbitrary (a, c) present in $R_1 o\ R_2$. So, since it is present in the $R_1 o\ R_2$ what I can say here is that, as for the definition of composition.

There should, exist some intermediate element b. Such that (a, b) will be present in R_2 and (b, c) will present in R_1. That is the definition of composition. And that means that ordered pair (a, b) is present in $R_1 \cup\ R_2$ and (b, c) is also present in $R_1 \cup\ R_2$. And since I am assuming here, that $R_1 \cup\ R_2$ is an equivalence relation that is the premise. It will be transitive as well.

And if (a, b) and (b, c) are present in the union and it is transitive that means (a, c) is present in the union as well. So, what we have shown here is now. You take any element any ordered pair (a, c) in $R_1 o\ R_2$, it will present in $R_1 \cup\ R_2$ as well. Let us prove, now Y is the subset of X. Namely we will prove that $R_1 \cup\ R_2 \subseteq R_1\ o\ R_2$.

So, again, let us take an arbitrary element namely ordered pair arbitrary ordered pair (a, b) present in $R_1 \cup\ R_2$, we will show it is present in $R_1 o\ R_2$. Again there could be two cases depending upon whether (a, b) is in R_1 or whether (a, b) is in R_2. So, again without loss of generality assume that it is present in the first relation. And I know that my relation R_2 is reflexive and I am assuming here that the relations are over a non-empty set.

That means my set X has at least one element a. So, since my relation R_2 is reflexive, I will also have element ordered pair (a, a) in my relation R_2 and now I can use the definition of $R_1\ o\ R_2$. I have (a, a) present in R_2 and I have (a, b) present in R_1. So I can say (a, b) is present in $R_1\ o\ R_2$. So, that shows Y is also a subset of X and hence the two sets are equal.

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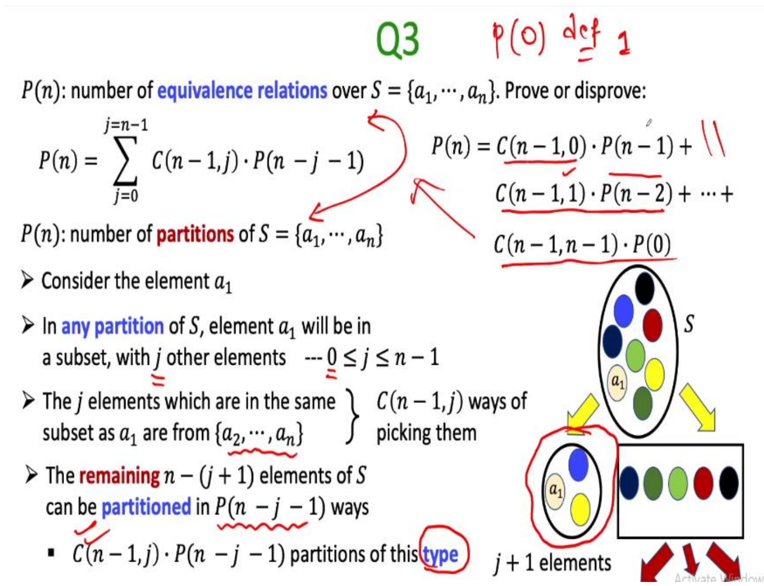

In question number 3, we are defining a function *P(n)* which denotes the number of equivalence relations over set S consisting of n elements. So, *P(1)* means, number of possible equivalence relations over the set consisting of one element. *P(2)* will give you the number of equivalence relations over set consisting of two elements and so on. Now the question ask you to either prove or disprove whether *P(n)* satisfies this condition.

So, here the C function is the combinatoric function, namely it denotes here. So, *C(n-1,j)* here the notation denotes the number of ways of selecting j objects, j distinct objects or j objects we say from a collection of n - 1 objects. That is a notation *C(n-1, j)*. Basically this is a recurrence equation, what exactly we mean by a recurrence equation here. We are trying to express the value of the function P on input n in terms of the value of function P on previous input namely on inputs of size less than n.

So, we are supposed, we are asked to either prove or disprove whether this condition holds or not. In fact, we will prove that this is true, this equation is true. The first thing to observe here is that the function *P(n)* also denotes the number of partitions of a set S consisting of n elements. Because remember we have proved that every equivalence relation gives a partition. And every partition corresponds to an equivalence relation.

So, the number of equivalence relations is nothing but the number of partitions over that is it. Now we will focus our argument based on the fact that we open denotes the number of partitions of a set S consisting of n elements. So, imagine your S set is has n elements, and what we are going to do is, we are going to discuss what are the various ways in which we could partition this set S.

So, for that I consider the first element a_1. Now in order to partition the set S into various subsets definitely the element a_1 will be present in one of the subsets in that partition. And along with the element a_1, there could be j other elements from the set S in the subset in which the element a_1 is present. Now the j ranges from 0 to n - 1, what does that mean? That is either the element a_1 might be the only element in the subset in its partition, in the partition that means, when you are partitioning the set S into various $\{a_1\}$ is the solitary element in $\{a_1\}$ is present. That is one case in which case my j will be 0 or my partition could be such that that, the a_1 is present along with all other elements of the set S in its subset. In which case j can take the value n -1. So, that is why the range of this j here is from 0 to n -1.

That means what I am saying here is that irrespective of the way you partition the set S, the subset in which a1 is present along with a1 you will have j other elements. So, in total that subset will have j + 1 element. And the j elements will be chosen from the remaining elements a_2 to a_n. So, how many ways you can pick those j elements from the remaining n- 1 elements, that is why the notation c of n- 1, j is coming into picture here.

It is not the case that all only the first j elements outside a1 will be present along with an. You can pick any j elements from the set a_2 to a_n. That is why this expression $C(n-1, j)$ will be picturing here. Now once we have decided which j elements are going to come together with a1 in its subset, the remaining elements which are now n - j + 1 in number, have to be partitioned. And there are these many numbers of ways of partitioning a smaller set consisting of n- j + 1 number of elements.

That means once you have decided that I am deciding, I am defining a partition where along with a1 these other j elements are going to come, once you have decided which j elements are going to take the position along with a_1 , now your remaining elements are n - j + 1. And now you have to worry about how you are going to partition that is smaller subset. And now as for the definition of my P function there are, *P(n)* - j - 1 number of base of dividing that subsets.

So, once you have decided which j elements to occupy or which j elements to put along with a1. These will be the total number of ways in which you can partition the set S. So, that gives you one type of partition. Now since j ranges from 0 to n - 1, you have n number of such possible types of partitions. That is why we get this overall formula. My first class of partition is there, the element a_1 is the only element in this subset.

Remaining n - 1 elements are now partitioned into various subsets. So, that is one category of partition there are these many number of partitions of that type. My second category of partitioning of set S is where along the a1, I also put one additional element in that subset. That additional element will be chosen in these many numbers of ways *P(n)* – 1, 1 and now the remaining n - 2 elements are partitioned into *P(n)* - 2 number of ways.

And continuing like that my last category of partitioning is the following, where I put n - 1 element along with the element a_1 in its subset. That means that whole set is the only partition in which case I have to now partition the remaining elements which are now 0 in number. So, that is why $P(0)$, I can define to be 1 that means if you are set as only if a set is an empty set and there is only one way of partitioning it, namely knowing.

So, I can define $P(0) = 1$ and now you can check that *P(n)* satisfies this equation and all these different types of partitioning are disjoint. There will not be any partitioning which will present in which can be considered as two different types of partitions, because the value of j is different for each category.

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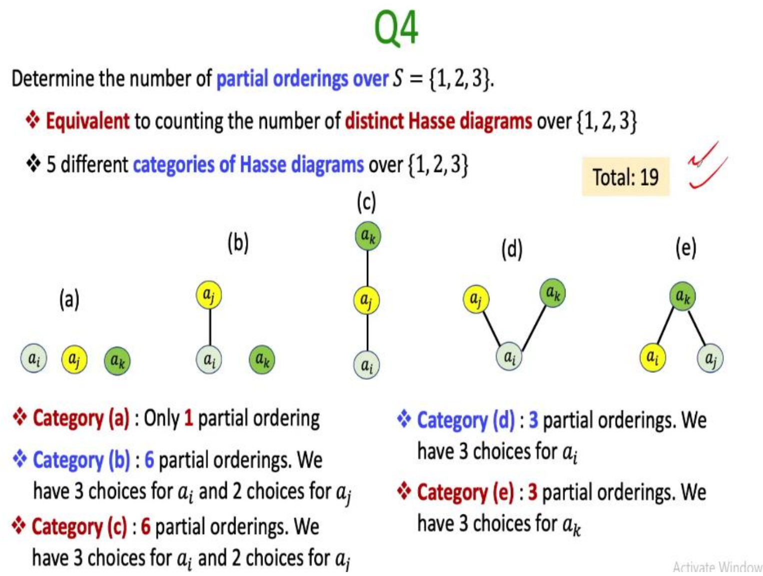

Now in question number 4, we are supposed to find out the number of partial orderings over a set S consisting of three elements. So, instead of enumerating all possible partial ordering, so what over the set consisting of three elements remember partial ordering means your relation is reflexive, antisymmetric and transitive. So, instead of enumerating all such relations what we will do is we will count the number of distinct Hasse diagrams, which we can draw using these three sets. Because remember, each partial ordering can be represented by Hasse diagram.

So, it turns out that we can draw five different categories of Hasse diagram over the set. And let us consider each of them and each category, we will count how many Hasse diagrams we can draw. The first category of Hasse diagram is where I have no edges among the nodes.

So, I have the nodes a_i, a_j, a_k, where a i can be any value in the set $\{1, 2, 3\}$, a_j can be any value in the set $\{1, 2, 3\}$ and a_k is any value in the set $\{1, 2, 3\}$. What exactly is the relation corresponding to this Hasse diagram? The relation here will be a_i is related to a_i, a_j is related to a_j and a_k is related to a_k. Remember in Hasse diagram, the directions are not there, self- loops are always implicit, transitively implied edges are also there and so on.

So, the relation corresponding to this Hasse diagram is this relation, which is a partial order. Now, the question is how many types of Hasse diagrams of this category I can draw? I can draw only one Hasse diagram like this, because it does not matter whether a_i is 1 or 2 or 3. The resultant partial ordering will be the same. So, I can have only one partial ordering whose Hasse diagram will be of category a.

My category b Hasse diagram will be like this, where I will have. So, this is a relation this corresponds to the relation where I have a_i is related to a_i, a_j is related to a_j and a_k is related to a_k, remember self-loops are always implicitly there. And we have the ordered pair a_i related to a_j, because the directions are always assumed to be from bottom to up. So, the question is how many partial ordering of this type we can have?

In terms of we can have six different partial ordering depending upon what is your value of a_i and what is your value of a_j. Because your, a_i could be either 1 or 2 or 3. If my, a_i is 1, then that is different from the case when my a_i is 2 and so on. But you have three choices for the element a_i. And once you have decided what is your a i you have now two choices for a_j because a_j has to be different from a_i.

And once you have decided a_i and a_j you do not have any other choice remaining for a_k the third element which is now left has to be a_k. That is why I can have only six possible partial orderings in this category. My category three could be where I have a total ordering among a_i, a_k, namely by a Hasse diagram is a chain. And it turns out that we can have six partial orderings of this category depending upon whatever my values of a_i and a_j.

So, I have three choices for a_i. By a_i could be either 1, 2, 3. Once I have fixed a_i, I have two choices for a_j and once I have fixed a_i and a_j the third element will be mine. So, that is why will have three different Hasse diagrams in this category. Fourth category is where you have a Hasse diagram where you have a least element and two maximal elements. In this category, we can have three partial orderings depending upon the choice of your least element.

Your least element a_i could be either 1 or 2 or 3. Once you have decided your, a_i, it does not matter whether what is your a_j and a_k you are going to be the remaining two elements. You can have only three partial orderings of this category. And your last category is when you have only the greatest element namely a_k and two minimal elements a_i, a_k. In this category, you can have three partial orderings depending upon what is your greatest element.

The choice of your greatest element, the greatest element would be either 1 or 2 or 3. Once you have decided what is your greatest element, it does not matter whether a_i is the what are your remaining elements a_i, they are going to be the remaining two elements. So, that is why if I now count all the different partial orderings and the various categories I get 19 different relations over the set $\{1, 2, 3\}$ which will be reflexive, anti- symmetric and transitive.

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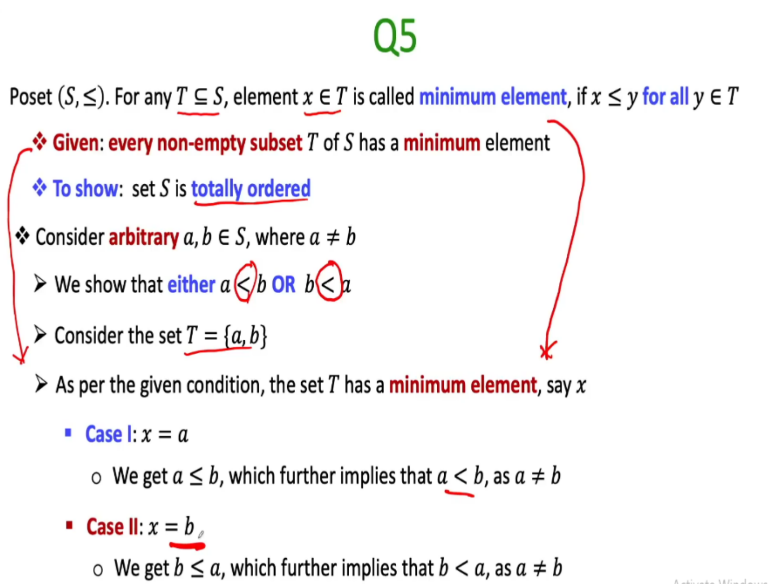

In question 5, you are given the following, you are given an arbitrary poset. And for any subset T of that set S an element x from that set T will be called as a minimum element, if that element x is related to all other elements y of depth subset T. So, here I am defining minimum element with respect to the subsets here. It is not a global minimum element it is defined with respect to a subset of the set S.

Now and the question you are given, the condition that is your poset is such that every non- empty subset T of S as a minimum element. That means it does not matter what the size of your subset T. You take any subset T of the set S is a minimum element as per this definition is bound to exist. So, your poset is like that. Under that condition you have to show that your poset is actually a total ordering. It is not a partial ordering but it is actually a total ordering.

So, remember the definition of total ordering is you take any pair of elements they will be comparable either the first element is related to the second or the second is related to the first. You will not have incomparable elements. So, that is what we are going to do here. We will take an arbitrary pair of elements a, b which are distinct and we will show they are comparable. That means either $a \leq b$ or $b \leq a$, remember this less than notation does not mean the numerical less than notation. It means that a is related to b or b is related to a as per your relation less than equal to, where less than equal to is not the numeric less than equal to relation, it is an arbitrary relation which is reflexive, anti- symmetric and transitive.

So, how I am going to show that $a \leq b$ or $b \leq a.$ I will take the subset T consisting of the elements a, b. And as per the given condition this subset T also will have a minimum element. Let us call that, denote that minimum element by x. Now the definition also says that a minimum element will be within that subset itself. And my subset here is the set $\{a, b\}$, that means that minimum element can be either a or b. If the minimum element x is a then we get that $a \leq b$as per the definition of the given minimum element.

And $a \neq b$, that means $a \leq b$ and hence a and b are comparable. Case 2, when my minimum element x is the element b. In this case, again since $a \neq b$ and as per the definition of the minimum element there is set element $b \leq a$ and hence a and b are comparable. So, with that we finish the first part of this tutorial. Thank you!