Discrete Mathematics
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Lecture -17
Operations on Relations

Hello everyone, welcome to this lecture on operations on relations.

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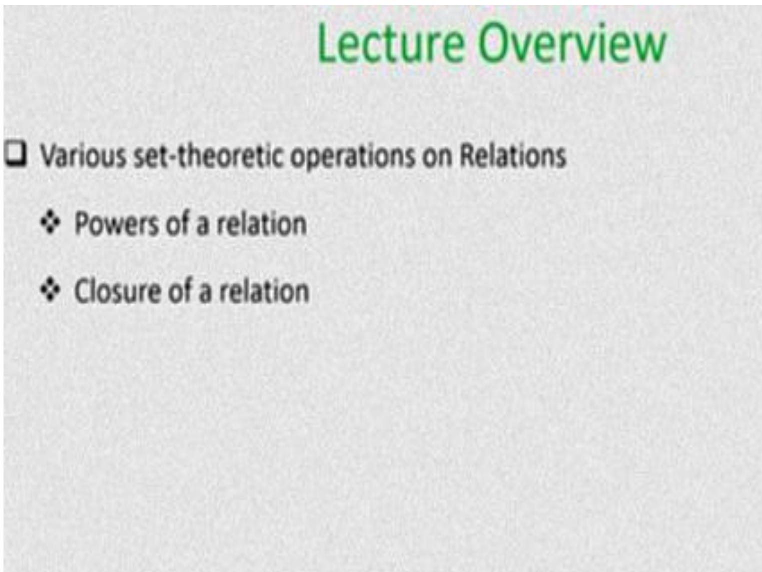

Just to recap in the last lecture we introduced the definition of relations and we saw various types of relations. The plan for this lecture is as follows. We will see various set theoretic operations which we can perform on relations. Specifically, we will be discussing on the powers of a relation and closure of a relation.

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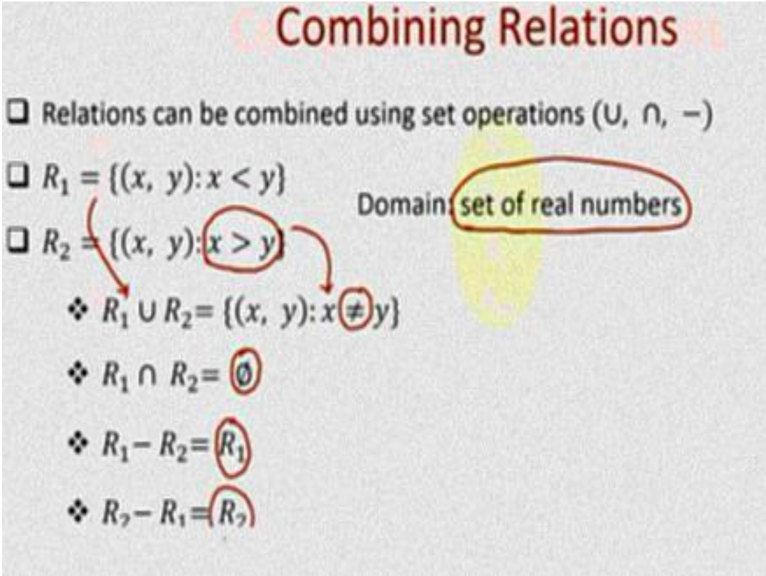

So, it turns out that since relations are nothing but a set so we can perform various set theoretic operations like union, intersection, set difference on relations as well. So, to demonstrate this, let us consider two relations R_1 and R_2, you have a relation R_1 which consists of all (x, y) pairs or all real numbers (x, y) where x < y. So, here my relation R_1 is defined over the set of real numbers, my domain is that of real numbers.

So, R_1 will have all tuples of the form (x, y) where real number x is less than real number y and similarly, relation R_2 is another relation defined over the set of real numbers consisting of all (x, y) pairs where the real number x is greater than the real number y. Now, if I take the union of these two relations and the union of these two relations is well defined because both R_1 and R_2 are sets and we can perform the union of two sets.

So, it turns out that the union of these two relations will have all pairs of the form (x, y) where the real number x is not equal to real number y because the union will have all the elements of R_1 and a union also will have all the elements of R_2. So, one way of describing the union of the two relations is that it have all (x, y) pairs where either x < y or x > y. But, if you want to represent the same if you want to state the same thing in a compact way, we can say that it has all (x, y) pairs where x is different from y.

Whereas, if I take the intersection of these two relations R_1 and R_2 and it turns out to be an empty set, because you cannot have real numbers x and y, where x is simultaneously less than y as well as x is simultaneously greater than y. So, you cannot have any (x, y) pairs satisfying simultaneously the conditions for the relation R_1 and R_2. In the same way it is easy to see that if I take the difference of the relation R_1 and R_2.

That means, if I subtract R_2 from R_1, then I will be getting the relation R_1 namely it will have only the elements of the form (x, y) we are x is less than y and similarly, difference of R_2 - R_1 will be the relation R_2.

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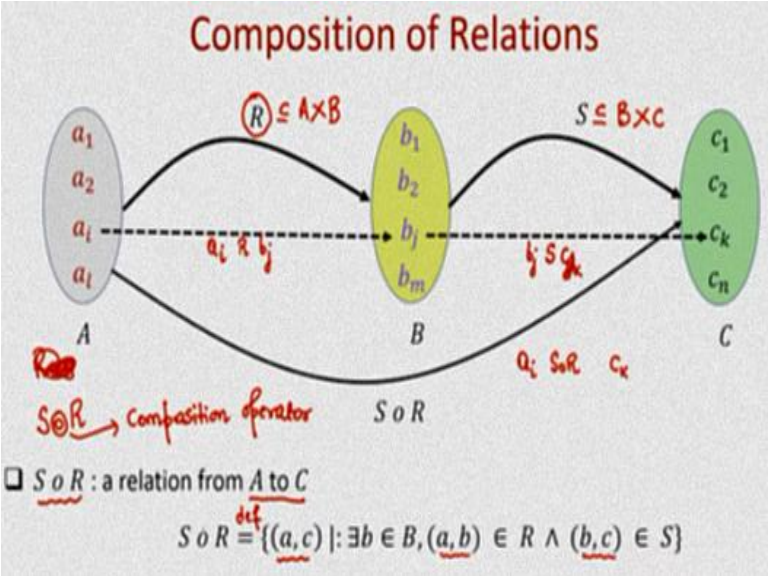

Now, we can perform another interesting operation on the relations which is called as composition of relations. So, imagine you are given two relations R and S, the relation R is from the set A to B. So, R is a subset of A x B. And your relation S is from the set B to C, so S is a subset of B x C, I am taking here three arbitrary sets A, B, C; A, B, C may be same, they may be different there is no relationship, they are just arbitrary sets here.

And I am numbering the naming the elements of this set A as a_1 to a_l. Similarly, the elements of B are named as b_1 to b_m and elements of C are named as c_1 to c_n. Now the composition of these two relations is defined as follows. So, first of all we use this notation S o R, and this means that

I am going to apply the relation R first and then the relation S. It is not a relation S applied first and then the relation R.

We are going to apply the relation R and on top of that we are going to apply the relation S. So, this will be a relation from the set A to C and that is why these ordering matters are lots. If I am saying S composed with R, so this operation o is called as the composition operator here and right now, I am composing the relation S with the relation R. If I write R o S, then that is a different relation.

That means, I am composing the relation R with the relation S that means here S will be applied first and on top of that relation R will be applied. So, the notation specifies clearly which relation is composed with which relation. This S composition R will be a relation from the set A to C. That means it will have ordered pairs of the form (a, c) and definition here is it will have all ordered pairs of the form (a, c) provided, you have some element b in the set B such that a is related to b as for the relation R and b is related to c as per the relation S. So, pictorially, if you have say a_i related to b_j as per the relation R and the same b_j is related to c_k as per the relation S. Then we will say that, a_i is related to c_k as per the relation S composition R. So, it is some kind of transitive property here that transitive pairs or transitive tuples of the form (a,c) which we will be including in the relation S composition R.

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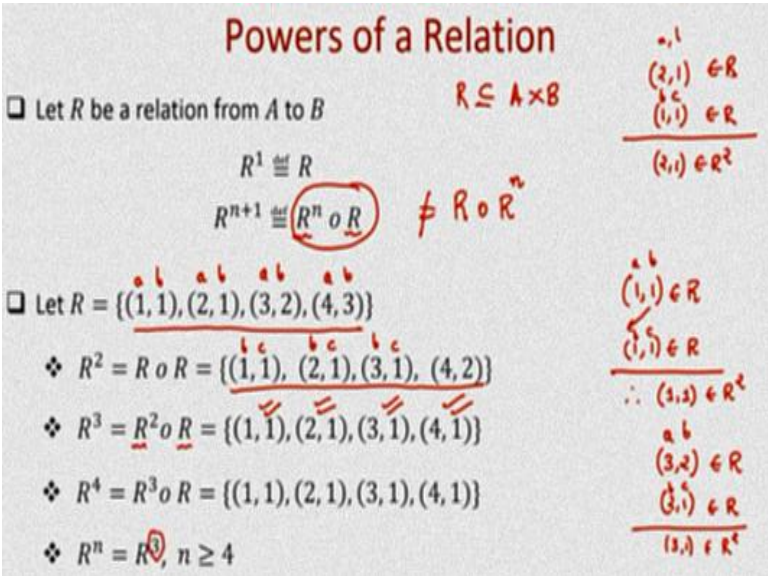

So, once the definition of compositions of relations is given, we can define what we call as powers of a relation and how it is defined. So, imagine R is a relation from A to B that means R is a subset of A x B. Then the definition of powers of a relation is as follows; R^1 is defined to be the relation R itself and then recursively I define the n + 1th power of R to be the composition of the relation which is n^{th} power of R with the original relation R.

So, again, I stress here the order matters here, the n + 1th power is defined to be the composition of R^n with R that means you have to apply the relation R first and then you have to apply the relation R^n. It need not be equal to the composition of R with the nth power of R that may or may not be the case because in general the composition of two relations need not be commutative.

Let me demonstrate the powers of a relation with an example. Here I have defined a relation R consisting of the pairs (1, 1), (2, 1), (3, 2) and (4, 3). So, R^1 as per my definition will be the relation R itself. Now, R^2 will be the composition of R with R and my claim is that R^2 will consist of these four pairs {(1,1), (2,1), (3,1), (4,2)}, why so? Let us start with (1, 1); So, you have (1,1) present in R and the same 1 is again related to 1 itself as per R, therefore I can say that transitively this is your a here this is your b here and this is your b here and this is your c here.

So, as per the definition (a, c) should be there in R^2. So, that is why (1, 1) will be present in R^2. Now take the tuple (2, 1). So, this is your (a, b) and you have the tuple (1, 1) also present in R, this is your (b, c). So, therefore (a, c) should be in R^2, that means (2, 1) will be in R^2. If you take (3, 2) present in R, this is your a, this is your b and you have (2, 1) present in R as well, this is your b this is your c therefore (a, c) namely (3, 1) will be in R^2.

And in the same way you have (a, b) here namely (4, 3) and (b, c) here can be (3, 2) that is why (4, 2) which is (a, c) will be present in R^2 and no other tuples will be there in R^2, that is how we take the powers of relation. If I want to compute R^3 then it will be the composition of R^2 with relation R. So, I will take this R^2 here first and compose it with the original relation R that means the relation R has to be applied first and then on top of that the relation R^2 has to be applied.

So, for instance this is your (a, b) here present in R and you have (b, c) as well present in R^2, therefore (a, c) which is (1,1) will be present in R^3. Similarly (2, 1) is present in R and you have again (1, 1) present in R^2. So, that is why (2, 1) is present in R^3 , you have (3, 2) present in R and (2, 1) is present in R^2. So, that is why you will have (a, c) which is (3, 1) here present in R^3, you have (4, 3) present in R and you have (3, 1) present in R^2.

So, that is why (4, 1) is present in R^3 and similarly, if you take the R^4 relation, then it turns out to be the same as R^3 and now you can check yourself that after the 4th power you take any power of the relation R it is going to give you the same relation as R^3, you will not get any new tuples added in the relation R, in the next powers of R.

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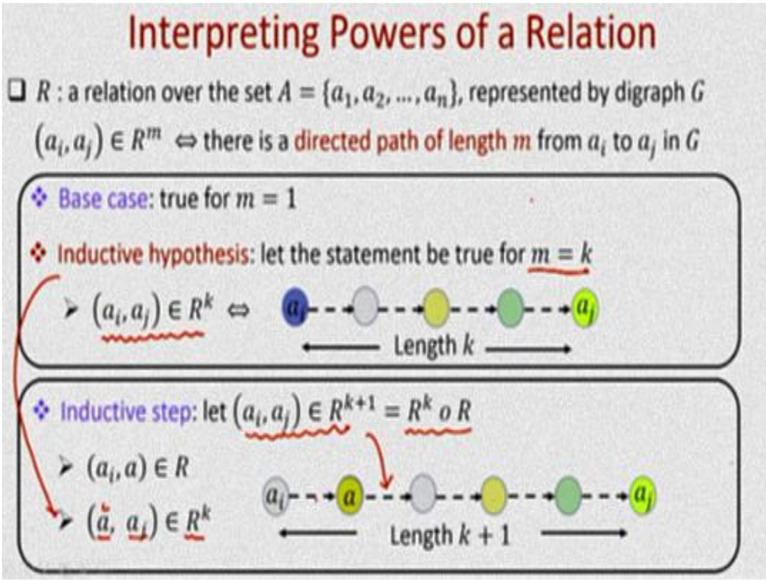

Now, next what we are going to discuss is a very nice interpretation of the powers of a relation in terms of a property in the corresponding graph. So, imagine you are given a relation over some set A consisting of n elements and suppose the relation is represented by a directed graph G. Now the claim here is the following the ordered pair (a_i, a_j) will be present in the mth power of the relation, only a_i is related to a_j in m^{th} power of the relation, if and only if, there is a directed path of length m from the node a_i to the node a_j in the digraph of your relation. So, what do I mean by directed path here? So, remember G is a graph where you have where you have vertices a_1, a_2, a_i, a_j, a_n and you have directed edges. So, if there is a direct edge between two node; from one node to another one other say from a_1 to a_2 then this will be considered as a directed path of length 1.

Whereas say, if you have a path of the form from a_2 you have an edge to a_1 and from a_1 you have edge to a_n, then I will say that a sequence of edges a_2 to a_1 and a_1 to a_n, this will be considered as a directed path of length 2 from a_2 to a_n. Like that you can imagine a directed path of length m from a_i to a_j, that means your source or the starting point will be a_i the end point will be a_j and in intermediate you will have m number of intermediate directed edges in that path.

That is the interpretation of a directed path, later on when we will discuss graph theory in detail we will formally define what exactly is a path, path length and so on. But right now we want to prove this nice theorem regarding the power of a relation and since this is a universally quantified statement because even though the universal quantification is not explicitly available here, but the statement is for all m >= 1.

So, that is why we will use a proof by induction, induction on the power m to prove this theorem or the property and the base case will be m equal to 1 and the statement is of course trivially true for the base case because if at all a_i is related to a_j in R that means that there will be a direct edge from the node a_i to the node a_j in my graph that comes from the definition of the digraph, so the base case is trivial to prove.

Now, assume that my inductive hypothesis is true, that means I assume that the statement is true for m = k by that I mean that if a_i is related to a_j in the kth power of the relation then there exists a directed path of length k starting at a_i and ending at a_j in the graph G. Now I go to the inductive step and I consider an ordered pair, an arbitrary ordered pair, in the k + 1^{th} power of the relation.

So, suppose the ordered pair (a_i, a_j) is present in the k + 1th power of the relation and the k + 1th power of the relation is basically the composition of R with the kth power of the relation as per the definition of powers of the relation. Now, when can it be possible that a_i is related to a_j in the k + 1th power of the relation that is possible only if the following two conditions hold. There should be some element a such that a_i is related to a and that element a should be related to a_j, then only I can say that a_i is related to a_j in the $k + 1^{th}$ power of the relation.

But a_i related to a means there is a structure of this form in your graph namely there is a direct edge from a_i to a. This should not be a dotted arrow this should be actually a straight arrow because indeed I have a direct edge from a_i to a and the interpretation of a being related to a_j in the k^{th} power means I have a directed path of length k starting at a ending at a_j this comes from my inductive hypothesis.

Now what can I say regarding a path from a_i to a_j? I can say that if I concatenate these two paths that means if take the edge from a_i to a first and then once I reach the node a from there I traverse through this path and come to the node a_j that will give me a path starting with a_i ending with a_j and now what can you say about the length of that path. The length of that path will be k + 1 that means I have shown here that if the ordered pair (a_i, a_j) is present in the $k + 1^{th}$ power of the relation, then in the directed graph of your relation you indeed have a path of length k + 1 starting at a_i ending at a_j.

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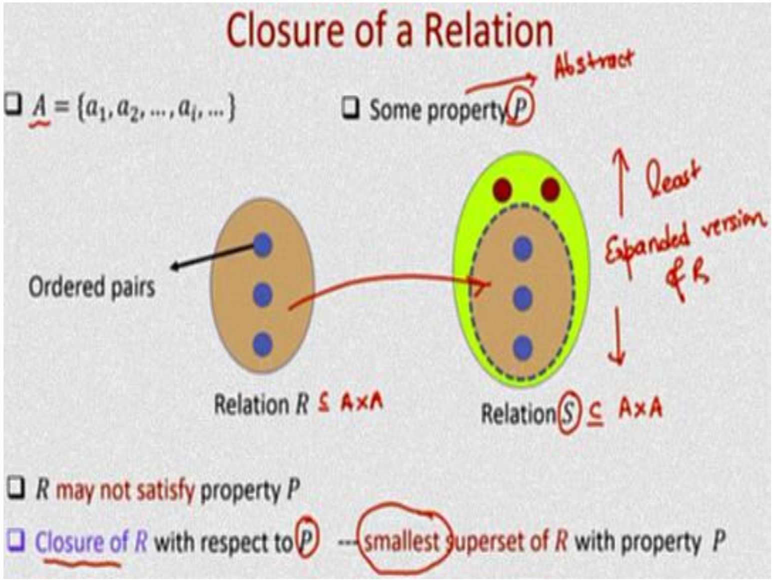

Next let us define what we call as closure of a relation. So, what do we mean by this? So, imagine you are given a set A that may be finite or infinite and you are given a relation R over the set A that means a relation R is a subset of A x A and I have some abstract property P, it is some abstract property and I am interested to check whether the relation R over the set A satisfies this property P or not? If it is satisfies the property well and good.

But, that may not be the case R that is given to you need not satisfy the property. So, the closure of the relation R is defined with respect to this property P. If you change the property P, the closure will change. So, what is that is the closure of the relation R with the property P. Well, it is the smallest superset of R which has the property P. Pictorially what I am trying to do here is if my relation R already satisfies the property P, I do not need to add anything to the relation R.

I do not need to actually add any extra element to the relation R to satisfy the property P. But if my relation R does not satisfy the property P then I will be interested to introduce new ordered pairs in the relation R and convert it into another relation S, so that the expanded relation S satisfies the property P that is what I am trying to do here, this S you can imagine as expanded version of R and this S is also going to be a relation over the set A itself.

I am including the original relation R that is carried as it is. On top of that I am adding or I may add few extra elements and try to ensure that expanded R which is S satisfies the property P, but I am not going to do the expansion arbitrarily; I am interested in the least possible expansion, least expanded version, what I mean by least? That means this is the minimal expansion which I need to do in order to ensure that the relation S satisfies the property P that is important.

Otherwise, what is a big deal in expanding the relation R? You keep on adding any arbitrary number of elements definitely you will more or less soon get an expanded version which will satisfy the property P. So, we are interested in the smallest possible expansion.

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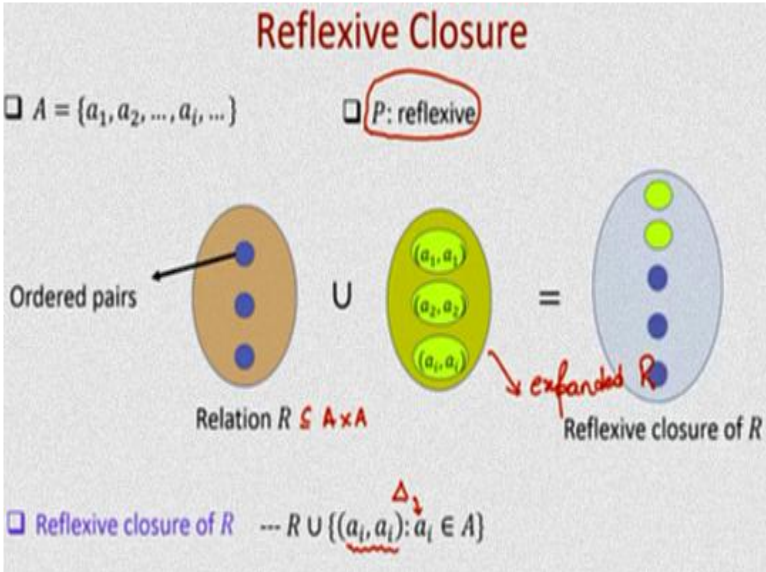

So, let us see some examples of this abstract property P and how the resulting closure looks like. So, my first abstract property is the reflexive property and this gives us what we call as reflexive closure of a relation. So, you are given a relation R over the set A and I am interested to see whether this relation R satisfies the reflexive property or not, reflexive over the set A, so that is what is the reflexive closure of the relation R.

So, how can you construct a reflexive closure of R? Well you just take the union of R with all ordered pairs of the form (a_i, a_i) where a_i is present in your set A. If your, (a_i, a_i) is already there in the relation R then as per the union definition, you will not be including it again. Remember, union means if (a_i, a_i) is present in R as well as in this new relation, so this new relation I am calling it as Δ relation.

So, this Δ relation you can imagine it is consisting of all ordered pairs of the form (a_i, a_i) such that a_i is present in A. So, if (a_i, a_i) is already there in R, it will not be included again but if (a_i, a_i) is not present in R then due to the union, due to taking union with this Δ, it will be now added to the relation R and now you can see that this is your expanded R that may be same as R itself, in case if your relation R is already reflexive then you are not going to add any extra elements.

So, this expanded R will have the original elements of the relation R plus this expanded R will satisfy the reflexive property.

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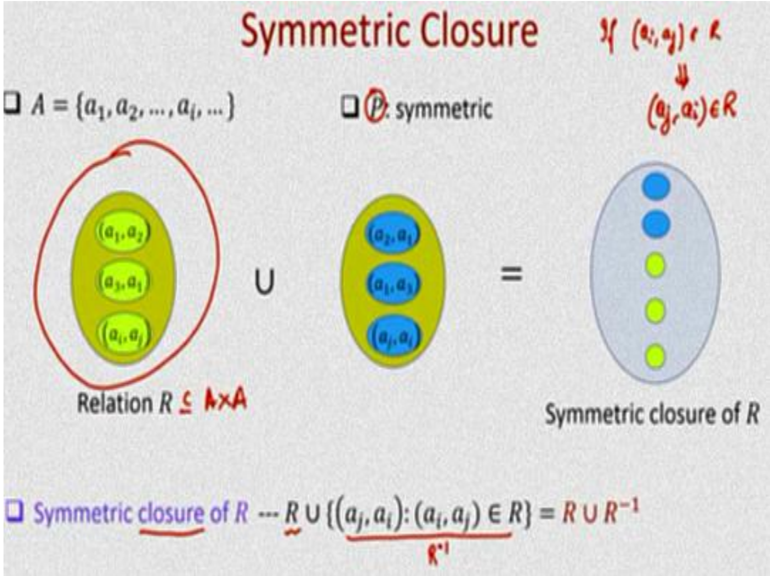

Now, let us take the case when my abstract property P is the symmetric property. So, again, I am given a relation R defined over the set A and I am interested to form a symmetric closure of R. Why I am calling it closure? Because I am trying to put a layer over, I am trying to enclose the original relation R and get a relation which has the original R as well as it satisfies the property P that is why it is called a closure.

So, how do I form the Symmetric Closure? So, if you recall the property of symmetric relation then the requirement here is that if (a, b) or if (a_i, a_j) is present in R, then I need the guarantee that (a_j, a_i) should also be present in R, this is the requirement from a symmetric relation and I need to include the original relation R, what I am going to do is, I am going to take the union of R with what I call as the inverse of the relation R.

So, this is the inverse relation and what is this inverse relation? It is defined to consist of all ordered pairs of the form (a_j, a_i) such that a_i is related to a_j in the original relation. So, if in your original relation a_i is related to a_j then in the R^{-1} relation a_j will be related to a_i and it is easy to see that if you take the union of R with its inverse then the resultant relation will be symmetric and it will have the original relation R and this will be the smallest possible expansion of your relation R which satisfies the symmetric property.

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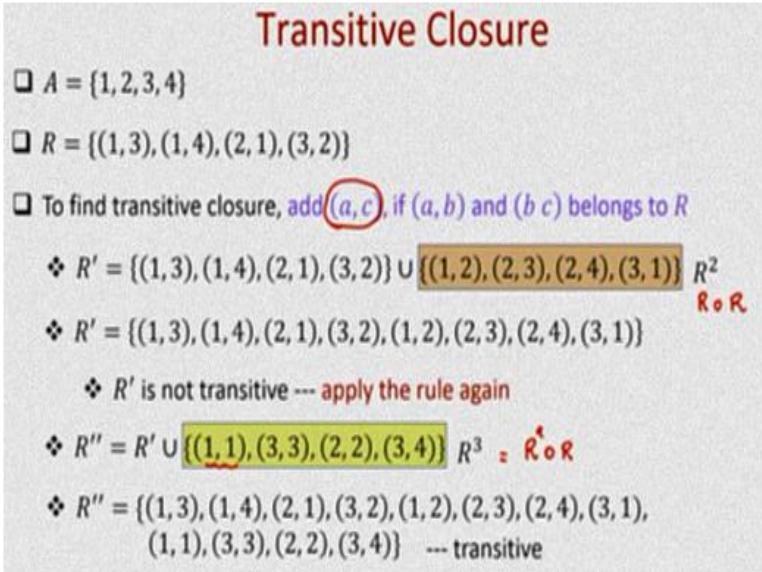

Now, what about the transitive closure? So, it turns out that finding transitive closure is not that simple, why so? Let me demonstrate that with an example. So, you might say that intuitively to find the transitive closure I add all ordered pairs of the form (a, c). Such that a is related to b as well as b is a related to c in the relation R because only when I add such order tuples of the form (a, c) in the relation R, it will ensure that the transitive property is satisfied.

So, let us try to do that, you are given the original relation R and, I am forming, I am taking the union of the original R and I am adding all the ordered pairs of the form (a, c) which are needed to ensure the transitivity property. For instance, I need to add (1, 2), if you are wondering why I need to add (1, 2)? Because I have (a, b) present here, I have (b, c) present here. So, I need (a, c) which is (1, 2) also to be present which is not there in R so I say let me add it.

In the same way I have (2, 1) present, which is your (a, b) and you have (1, 3) also present which is (b, c). So, you should add (2, 3) and so on. So, these are the new things which I add and my R’ will be this new relation. It has my original relation R, but now let us stop here and ask whether R’ is transitive or not. It turns out that R’ is not transitive. So, for instance, you have (2, 3) present here which is your (a, b) and you have (3, 2) present but you do not have (a, c) namely (2, 2) present in the relation R’.

So, even though you have expanded R and got R’, your R’ is not satisfying the transitivity property. So, what you can do is? You can again apply the rule and to apply the rule what I do now is, I take the union of R’ namely the expanded R and add all the extra ordered pairs of the form (a, c) which were missing in R’ to satisfy the transitivity property. So, see I am adding (1, 1) here, why I am adding (1, 1)? Because I have (a, b) here and (b, c) here but (a, c) namely (1, 1) is not present in R’. So, I am adding here and so on.

And now I call this expanded relation as R’’ and now you can check that is R’’ is indeed satisfying the transitivity property. So, if you see closely here what happened in this whole process is the following. First time when I expanded the relation R, I took the union of R with all the tuples, which will be present in R^2. You can verify that as per our definition of R^2 the tuples which I have highlighted here are the tuples in R^2.

So, there are tuples which I have highlighted here and I have added with the tuples which were present in R and nothing but a tuples of R^2. In the same way, when I expanded R’ to R’’ the tuples which I added actually are nothing but elements of R^3. And recall R^3 is nothing but R^2 composition R and so on. It turns out that when we want to find a transitive closure; we need not be a one step process.

I need to apply this rule namely keep on adding the ordered pairs of the form (a, c), such that (a, b) and (b, c) are there in your expanded relation and I may need to keep on applying this process several times and then only I can obtain the transitive closure. So, that brings me to the end of this lecture just to summarize; in this lecture we introduced some set theoretic operations on relations and we discussed various closure properties with respect to a given relation. Thank you.