Discrete Mathematics
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Lecture -13
Tutorial 2: Part 1

Hello everyone, welcome to the part 1 of tutorial 2. So let us start with question number 1.

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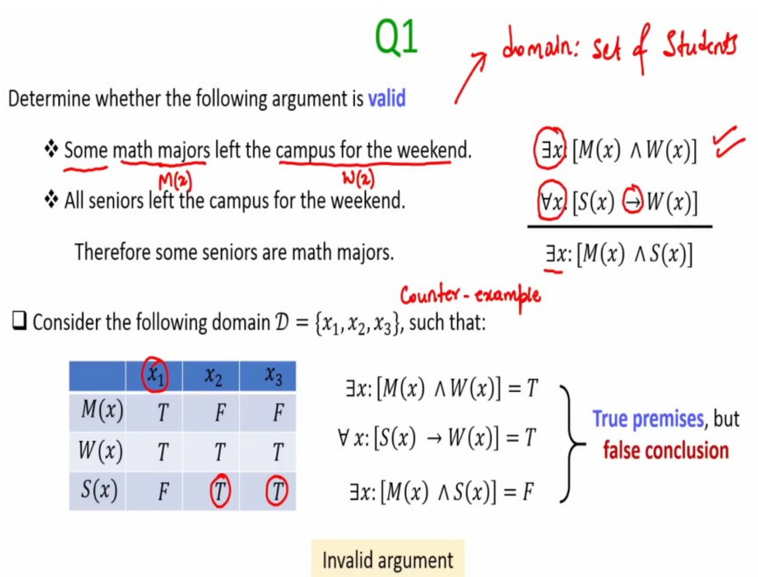

Here, you are supposed to find out whether the following argument is valid or not. So you are given some premises and conclusion. So the first thing that we have to do is we have to convert everything in terms of predicate functions. So we introduce appropriate predicates here. So of course, the domain is explicitly not given here. But domain, the implicit domain here is the set of students.

So the first statement here, the premise here is some math majors left the campus for the weekend. So it is easy to see that this is an existential quantified statement, it is not making an assertion about all the math majors. But let us first decide what are the predicates that we need here. So the assertion is about math majors. So let M(x) be the predicate which is true if the student x is a math major.

And we are saying something regarding whether he has left the campus for the weekend or not. So that is the second property for the subject x. So that is why I introduce a predicate the W(x) which is true, if the subject x or if the student x is left for the weekend. And I am making a statement that there is some student x for which both these conditions are true, so that is why this is an existentially quantified statement with conjunction inside.

The second statement here or the premise here is that all seniors left the campus for the weekend. So this is a universally quantified statement. And if you see clearly or closely here, the interpretation of this statement is that, if a student x is senior then he has left the campus. So there is an implicit implication here and that is why this premise can be represented as $\forall x, S(x) \rightarrow W(x)$.

The conclusion that I am making here is, some seniors that means existentially quantified statement, are math majors. That means at least one student is there for which the property that he is a math major as well as, he is a senior are true. Now we have to verify whether this is a valid argument and as per the definition it will be a valid argument if, based on the premises I can draw the conclusion for every possible domain.

However, it turns out that this is not a valid argument and we can give a counterexample. You can give multiple counter examples here. Even if you show one counter example that is sufficient to show that this argument form is not valid. So the domain that I consider is the following imagine you have a college where you have 3 students x_1, x_2, x_3. And say with respect to those 3 students the status of the 3 predicate functions are as follows.

For x_1, the property M is true, W is true, but S is false. Student x_2, the property M is false, property W is true and the property S is true and so on. Now you can verify that with respect to this domain and this assignment, the premises are true. Indeed there exists a student for which the property that he is a math major and he has left for the weekend, he has left the campus for the weekend are both true.

Namely x_1 is one such student. And similarly the second premise namely all seniors have left the campus is also true. So who are the seniors here? The seniors are x_2 and x_3. And indeed x_2 has left the campus and x_3 has also left the campus. So both your premises are true but what about the conclusion, is there any student who is a math major as well as senior? Well, it turns out the answer is no. That means my premises are true here, but my conclusion is false and that is why this is a invalid argument.

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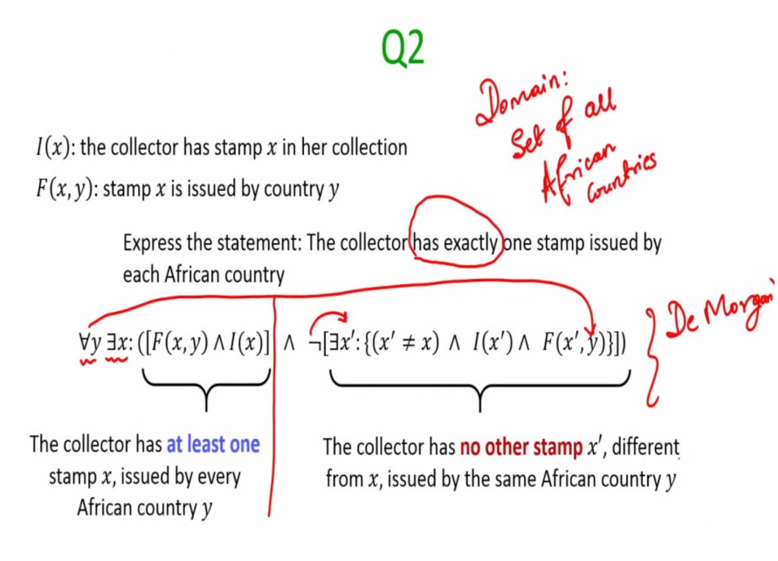

Let us see question number 2. In this question, you are given two defined or two predicates which are defined for you. I(x) denotes that a stamp collector has stamp x in her collection and F(x, y) denotes that stamp x is issued by country y and you have to express the statement that this collector has exactly one stamp issued by each African country. So I am making a statement about a specific collector and I want to state that, for each African country, she has exactly one stamp issued by that country in her collection. So, of course my domain here is set of all of African countries. So if you recall from the lecture whenever we face this scenario where we want to represent a property p is true for exactly one element of the domain then there are two things which we have to represent. The first thing; that the property is true for at least one element of the domain.

In this case, the property is that for every African country, there is one stamp at least issued by that country, which is there in the collection of the collector. That is the first part here, which is represented by this expression. So, this expression means that for every African country y, there is at least one stamp x, such that the stamp x was issued by that country y and the collector has that stamp x.

For the moment forget about what is there in the remaining part of the expression forget it. Just focus on this part of the expression. But this is not what we want to represent because I cannot stop with this expression because this expression also means that there might be multiple x values for the same y, for the same country y where those other x stamps are also issued by the same country y and the collector has those other x stamps in her collection.

That is not what we want to represent. We want to represent that exactly one value of x or one stamp x is there for each country y. So that is why we have to put this second part of the expression. For the moment forget about this negation. And whatever is there before the conjunction forget about that as well. The second part of the expression denotes that, there can be other stamps x' issued by the same country y and which is also there in the collection of the collector and you see I have very carefully put the parentheses here.

The scope of this y is still covered by this ∀ y, the scope of this universal quantification is carried over to this y as well. And the scope of this x', this x' is again within it is a nested quantification here, there exists x prime it is nested quantification falling within this ∀ y. So the second part of the color expression denotes that there might be other stamps x' issued by the same country y which can be there in the collection of the collector.

But I do not want that to happen that is not what I want to represent. So that is why I put a negation here and if I put a negation that means there cannot be any other stamp x' different from x, which is also there in the collection of the collector and x' was issued by the African country y. And that is why the conjunction of these two things represents the required statement.

Of course, you can simplify this, apply the De Morgan's law and take this negation inside convert everything, make everything in the form of an implication and so on that also you can do but even if you write this expression, that is correct.

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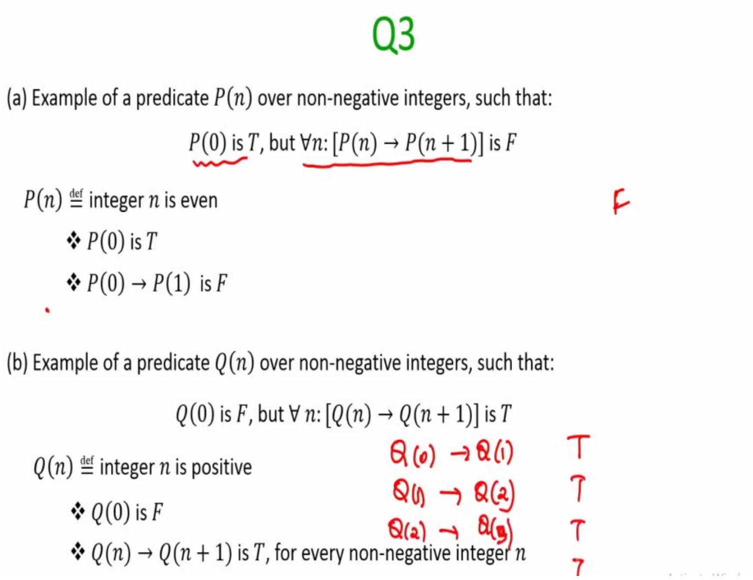

Part a of question 3, asks you to do the following. It asks you to give an example of a predicate P(n) over the domain of non-negative integers such that the proposition P(0) is true, but the universal quantification ∀ n P(n) → P(n+1) is false. So if you want to make P(n) → P(n+1) to be false, ∀ n that does not mean you have to make it false for every value of n in the domain.

Remember the meaning of this universal quantification is that it will be true for all the universal quantification will be true if it is true for every value of n in the domain. But even if it is false for one value of n in the domain that shows that this universal quantification is false. So here there can be multiple examples of such a predicate P. A very simple example is the following.

Say my property P is that integer n is even. When I substitute n = 0, the resultant proposition is that 0 is an even integer, which is a true proposition but what about the statement P(0) → P(1). It is false, because P(0) is, if 0 is even → 1 is odd. Which is clearly a false implication and that means, since P(0) → P(1) is false, ∀ n P(n) → P(n+1) is automatically false. It does not matter that P(1) → P(2), this is true.

Because P(1) is false, P(2) is true, false → true is true. Whereas P(2) → P(3) is false and so on. So I have a statement here which is, for which this universal implication is not coming out to be true for every value of n in the domain and that is why this is an example of such a predicate.

The part b of the question is an opposite of part a here. You are asked to give a predicate Q, such that Q(0) is false, but the universal implication Q(n) → Q(n+1) is true.

So now my example here is that property Q(n) is defined that integer n is positive. It turns out that Q(0) is false, because Q(0) is the proposition that 0 is positive and definitely 0 is not positive. So, this proposition is false, but it turns out that Q(0) → Q(1) is true. Because Q(0) is false the false implies anything is true and now any statement of the form Q(n) → Q(n+1) everything will be true, that means now I can say that this universal quantification is true.

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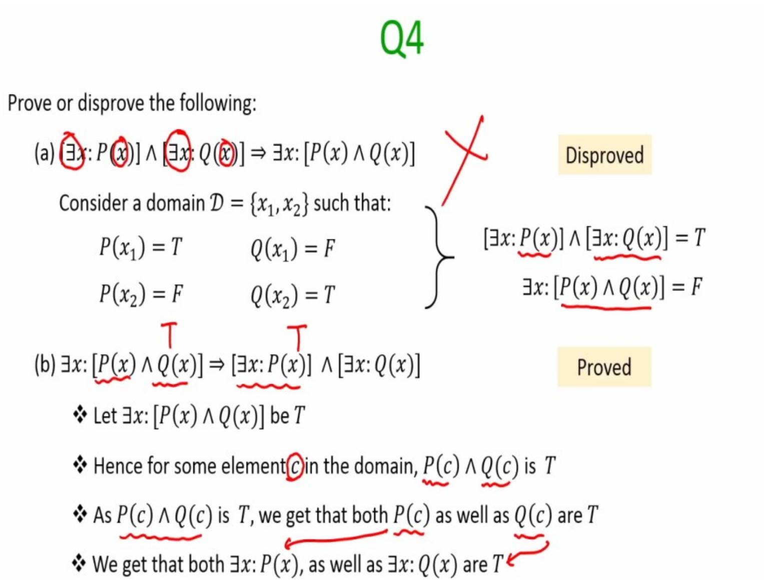

Now let us see question number 4. Here I have to show or I have to either prove or disprove that the left hand side expression implies the right hand side expression. So you see the left hand side expression, I have explicitly added the parenthesis here, so the x within the P and x within the Q are different here whereas in the right hand side the x both within P and Q are the same. Because both of them are covered by the same 'there exist'.

Whereas, in the left hand side, the first x is covered by the first 'there exist' (∃) and the second x is covered by the second 'there exist' (∃). The informal way to interpret the statement is if you are given that property P is true for some element in the domain and if you are given that property Q is true for some element of the domain, then can you conclude that both P and Q property are true for some element of the domain.

And this need not be true. I can give you a very simple counter example, imagine a domain where you have two values of x possible and say property P is true for x_1, but false for x_2 whereas, Q is false for x_1 and true for x_2. In this case, you can check that your left hand side is true, because indeed the property P is true for at least one value of the domain and indeed the property Q is true for at least one value of the domain.

But that does not mean that it is the same x for which both P and Q are true. Individually P might be true for some x and Q might be true for a different x. That does not mean ∃ an x for which both P and Q property are true and which is happening in this case. So this is not a correct statement. What about the part b is the implication in the reverse direction. It says that if you are given that ∃ some x value in the domain for which both property P and property Q are true.

Then you can conclude that individually the property P and Q are true for some value in the domain. So we can prove this and the way we prove this is as follows. So since you are given, so to prove that this implication is true, we have to show that if I assume left hand side is true, then I have to show that the right hand side is also true. Because for all other cases an implication always turns out to be true that means by false implies anything is true and so on.

So assume your left hand side is true, that means there exists some x value in the domain for which both property P and Q are true. I do not know the exact value of that x, because my domain could be very large. But I can say that that element x for which the left hand side is true can be represented by c. So this is your existential instantiation. So I know that proposition P(c) is true as well as the proposition Q(c) is true. I stress the value of c is not known here.

It is an arbitrary element, but it exists. Now since the conjunction of the two propositions P(c) and Q(c) is given to be true. This is possible only if the individual propositions P(c) and Q(c) are true. And if the proposition P(c) is true, that means I can say that existential quantification, ∃ P(x) is true. And in the same way since the proposition Q(c) is true, I can say that the existential quantification ∃ Q(x) is true.

Both of them are true, that means the right hand side is true. That means assuming left hand side to be true I can conclude the right hand side is true and hence this identity is a correct identity.

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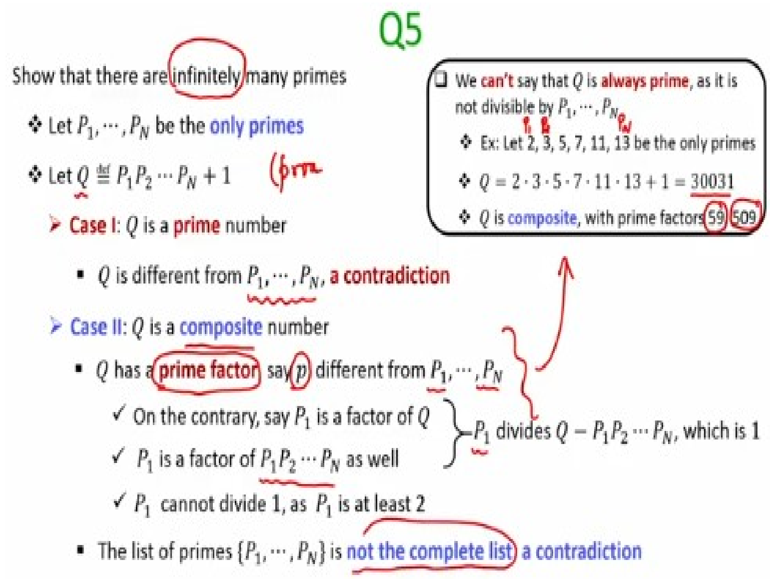

The 5 question is a very interesting question. It asks you to show that there are infinitely many prime numbers and there are several interesting proofs possible for this statement, let me show one of them. So I am trying here a proof by contradiction. So that the statement I want to prove is there are infinitely many primes but I assume a contradictory statement that there are only a finite number of primes.

Say n number of primes, n could be anything it could be 2, 3 or 4 anything. Now what I do is I define a new number Q which is the product of my finite number of primes, which I am assuming to exist plus 1. Now what can I say about the number Q. There can be 2 possible cases. Now, I apply the proof by cases here. My Q could be a prime number itself, my Q could be a composite number and there cannot be any third case possible with respect to Q.

It turns out that, if Q is a prime number, then definitely Q is different from all your numbers prime numbers P_1, P_2, P_n that are the only prime numbers you assumed to exist. That means now you have found a new prime number. That means your listing of P_1 to P_n is not an exhaustive listing of all the prime numbers that exist. So you got a contradiction. Whereas it might be possible that Q is a composite number.

If Q is a composite number I can show that none of these prime numbers P_1, P_2, P_n will be a factor of Q. None of them will divide Q. On contrary, assume say for instance P_1 divides your number Q. Now, if P_1 divides Q, that means P_1 is a factor of Q, and we know that P_1 is a factor of the product of P_1 to P_n, because that has P_1 in it. So that means you have now a number P_1, a prime number P_1 which divides both Q as well as the product of P_1 to P_n.

That means it will divide the difference of Q and the product of P_1 to P_n. But the difference of Q and the product of P_1 to P_n is 1. That means P_1 divides 1. But that is not possible because P_1 is at least 2, because you are assuming that P_1 to P_n are primes and a smallest prime that is possible is 2. And 2 cannot divide 1. That means we have shown here that P_1 cannot divide your number Q. In the same way we can show that P_2 does not divide Q.

In the same way we can show that P_i also does not divide Q. And in the same way I can show that P_n does not divide Q. But Q definitely has a prime factor because that comes from my fundamental theorem of arithmetic you take any number it can be expressed as product of prime powers. That means it has definitely one prime factor, say P. But at the same time I am showing that P cannot be P_1, it cannot be P_2, it cannot be P_n.

That again shows that I am missing a prime number P in my listing of prime number. That means my list of prime numbers P_1 to P_n which I assumed is not the complete list. So that is a contradiction I will get in case 2. It turns out that very often students just give the following argument which is an incorrect argument. They say that for surety since Q is not divisible by P_1, since Q is not divisible by P_1, Q is not divisible by P_2, Q is not divisible by P_n.

They end up with the conclusion that Q is definitely prime. That is not correct, let me demonstrate that. Imagine that these are the only prime numbers which you assume to exist. Now your Q in this case will be the product of all these prime numbers plus 1. And as per your argument Q should be always prime, because it is not divisible by 2, it is not divisible by 3, it is not divisible by 5, not divisible by 7, not divisible by 11 and 13.

But it turns out that Q is composite here, where the prime factors of this composite Q are 59 and 509. And these are the 2 primes which are not there because missing from your list of exhaustive prime numbers, which you are assuming to exist. And that is why in case 2 we cannot simply stop with the argument that Q is also a new prime number which I am finding because it is not divisible by P_1, P_2, P_n.

The correct argument is that we will show that Q will have at least 1 prime factor, which is not present in the list of prime numbers, which I am assuming to exist which is demonstrated by this example.

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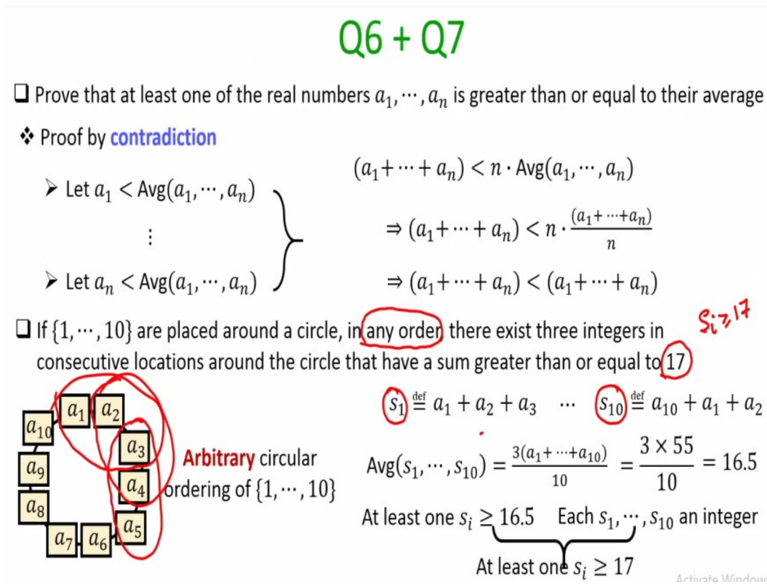

Now, let us see question number 6 and 7 together. We will first equation number 6 and the solution of question 6 will be used for question number 7. The question 6 says you have to prove that there exists at least one real number among a set of n real numbers which is greater than equal to their average. I stress here that a_1 to a_n are arbitrary here. You cannot show concrete values of a_1 to a_n and prove this statement for those concrete values and conclude that this statement is true.

This is a universally quantified statement. So how do we prove it? We have to take arbitrary values of a_1 to a_n and prove the statement with respect to those arbitrarily chosen values of a_1 to a_n. What we do here is we give proof by contradiction. So our goal is to prove that average of a_1

to a_n is less than equal to some a_i. But instead, I assume that each of the individual numbers among these n numbers is less than their average.

That means the first number is less than their average. The second number is less than the average of the n numbers and similarly the last number is less than the average of the n numbers. That is a contradiction. Now if I add this n equations I get this inequality. And if I substitute the value of the average by this formula I come to the conclusion that the summation of n numbers is less than the summation of n numbers which is not possible which is a contradiction.

That means assuming this contradiction leads to a false conclusion that means the statement is a true statement. That means you take any n real numbers, any n arbitrary real numbers, they could be positive, negative, they may be the same, different. At least one of them will be greater than or equal to their average. Based on this I want to solve question 7. In question 7, you are given the following.

You are given the numbers 1 to 10 which are placed around the circle in any arbitrary order. Maybe in ascending order, descending order, maybe the odd numbers first, next even numbers and so on. So the order is not given. It is an arbitrary order. And the question says that it does not matter in what order you arrange the numbers 1 to 10, there always exist 3 integers in that arrangement which will be in consecutive locations.

Such that the sum of those three numbers will be greater than or equal to 17. So I stress here this is with respect to any arbitrary arrangement of the numbers 1 to 10. So pictorially, you can imagine that you are given this arbitrary circular ordering of 1 to 10 where, a_1 can be any number from 1 to 10, a_2 could be any number from 1 to 10 and so on. I have to show that once I freeze this arbitrary ordering.

In this arbitrary ordering, there exist collections of 3 integers, in 3 locations such that their sum is greater than equal to 17. And I want to take the help of question number 6, whatever I have proved in question number 6. So what I do here is since the question involves sum of 3 numbers what I do here is once I freeze this circular arrangement of 1 to 10, I take the following sums, I take the sum of first 3 numbers namely a_1, a_2, a_3.

That is my S_1. In the same way, I take the sum of next 3 numbers namely a_2, a_3, a_4. I call it S_2. I take the sum of a_3, a_4, a_5 that I call it as S_3 and in the same way I take the sum of a_{10}, a_1, a_2, that will be my last sum namely S_{10}. And what is my goal? The question says that either S_1 is greater than equal to 17 or S_2 is greater than equal to 17 or sum S_i is greater than equal to 17. That is what I want to prove here.

Because I have taken the different possible sum of 3 consecutive numbers in this circular arrangement. Now what I can do here is I can interpret S_1, S_2 up to S_{10} as 10 possible values. That means let n = 10. Now, what can be what will be the average of these sums S_1, S_2, S_{10}. It does not matter what are the values in your circular arrangement. If you take the average of these 10 sum values, then in the denominator, you will have 10. Because n is 10 but when you take the sum of S_1 up to S_{10} each number in this arrangement will occur three times because a_1 will be occurring in S_1. a_1 will be occurring in S_{10} and a_1 will be also occurring in S_9.

In the same way, a_2 will be occurring in S_2, it will be occurring in S_3 and S_4, and so on. So each of this value a_1 to a_{10} will be occurring thrice when you take the average and when taking the average you will be adding S_1 to S_{10}. And if my claim here is if you add S_1 to S_{10} each of this value say a_1 to a_{10} will be occurring thrice. And possible values of a_1 to a_{10} each of them belong to 1 to 10 and only once they occur.

That means i know that if you add the value say a 1 to a 10 you are basically adding the numbers 1 to 10. And the summation of the numbers 1 to 10 is nothing but 55. That means I know that it does not matter in what order the numbers are arranged. If I define sums like this and take the average it will be 16.5. And from previous question I know that either S_1 is greater than equal to the average of S_1 to S_{10} or S_2 is greater than equal to the average of S_1 to S_{10} and so on.

That means at least one S_i is there which is greater than equal to 16.5. And each S_i is an integer because S_i, S_1 is the summation of three integers S_2 is the summation of three integers. In the same way, S_{10} is the summation of three integers, so each S_i is an integer. So what is the smallest integer, which is greater than equal to 16.5, well it is 17. So that shows that either S_1 is equal to 17 or S_2 is 17 and so on. So that solves your question number 7.