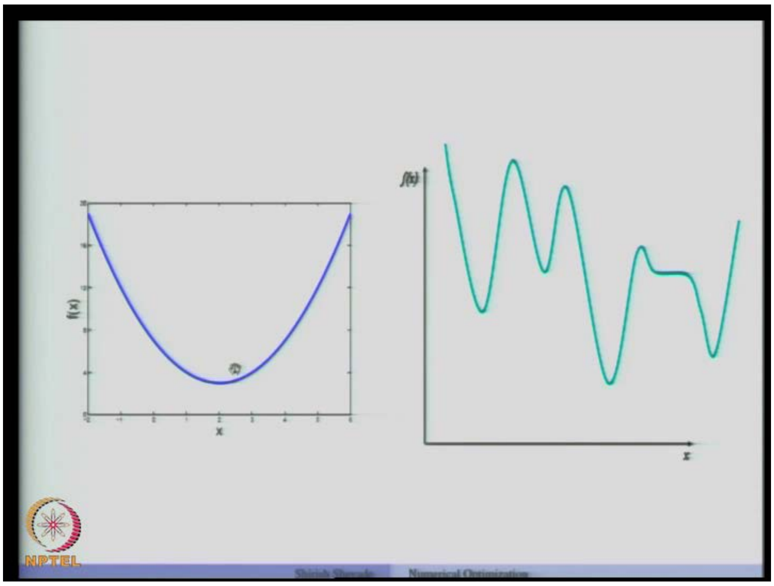
Numerical Optimization
Prof. Shirish K. Shevade
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 8
Convex Functions

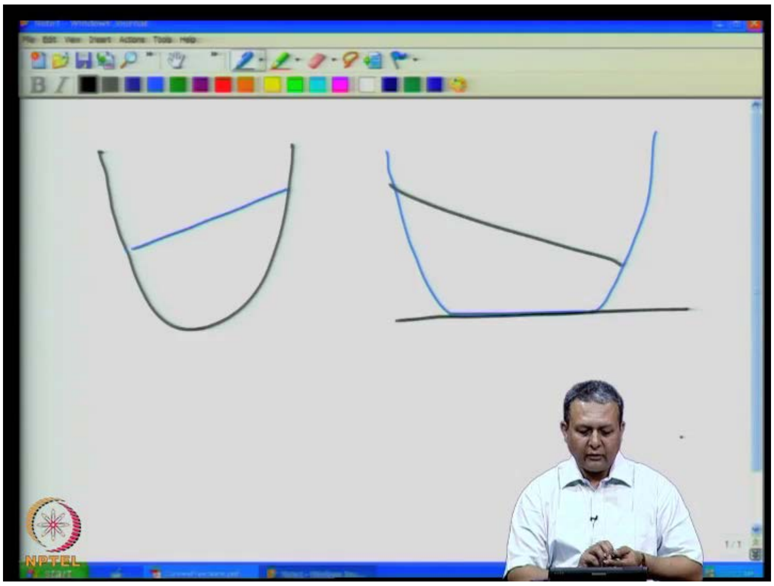
(Refer Slide Time: 00:58)



Hello. So, welcome back to this series of lectures on numerical optimization. In the last class, we looked at convex sets and properties. How do we characterize a convex set? In this class, we will discuss about convex functions and in general convex programming problems. So, we will first study about the convex functions and their properties, and what are the nice properties of convex programming problems, and what are the usefulness of convex functions. So, let us look at these two functions which are shown here. So, on the left side, you will see a function which is nice quadratic function and the function increases when x increases or x decreases.

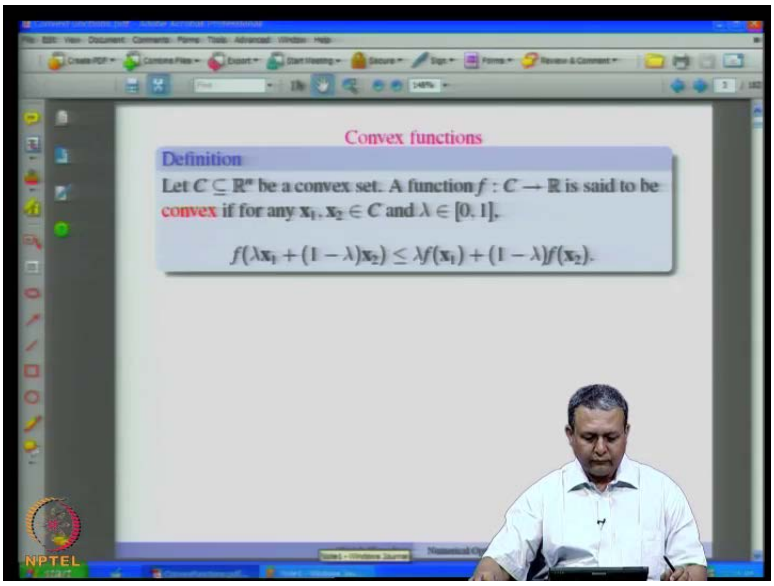
Now, on the right side you will see a function with lots of peaks and lots of valleys. On the left side, if you see the function, if you take this point which happens to be local minimum of the function because in the neighborhood, the function is increasing. Now, you will see that if x increases, the function value also increases or x decreases, the function value also increases, so this also turns out to be a global minimum. Now, that is not the case, in the case of this function. For example, these points as we saw earlier, these are all local minima. Now, among all this local minima, this is the point at which the function attends the least value. Assuming that the function increases as x increases, in this way or x decreases in the other way. So, you will see that this function is very nice function. It has a local minimum which is also a global minimum.

(Refer Slide Time: 02:24)



Now, what are the properties of such functions? So, let us quickly look at this function. So, the function which we are considering, they are of this type. Now, so what are some important properties of such functions? Now, if I take any two points on this graph of the function and draw a line segment joining that graph. Now, you will see that the value of the function always is less than the point on this chord. So, the function always lies below on or below this chord. So, suppose, if we take a function like this and then suppose we take a chord joining any two points. You will see that the function always lies below this chord. Now, it may so happen that if you take these two points and then take a chord joining them, then you will see that the value of the function is equal to the same value as obtained using the chord.

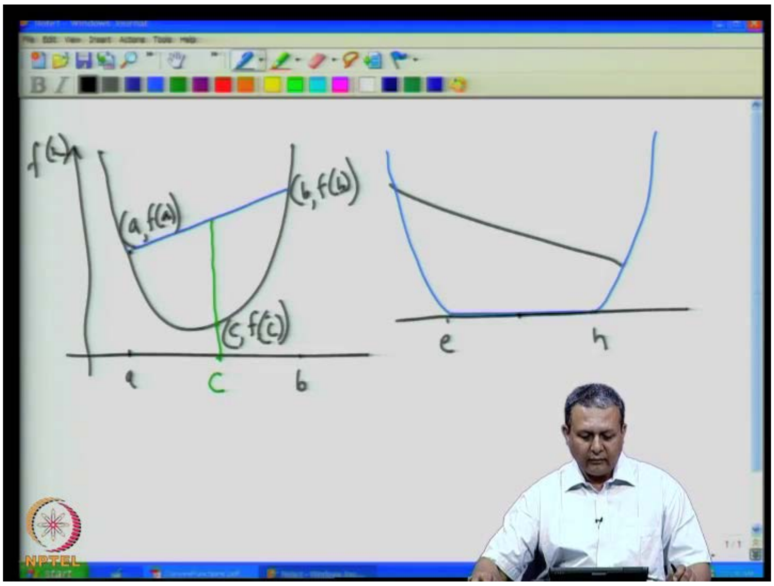
(Refer Slide Time: 04:23)



So, you will see that, in this case, the function always lies. So, if you take any point on the line segment joining these two, the value of the function will always be less than the value obtained here. Here also, it is similar and here, if you take any point, the value of the function will always be on the cord. Such functions are called the convex functions and good thing about such functions is that they have a local minimum, which is also a global minimum.

So, we will now formally see the definition of a convex function. So, let us consider a convex set c and a function f from c to r, a real valued function defined on the set c which is convex. The function is said to be convex if you take any two points x1 and x2 belonging to the set c and any lambda, which is a scalar in the closed interval 0 to 1. Then, the value of the function at any point on the line segment joining x1 and x2 in the set c will always be less than or equal to the value of the function interpolated between the two cords.

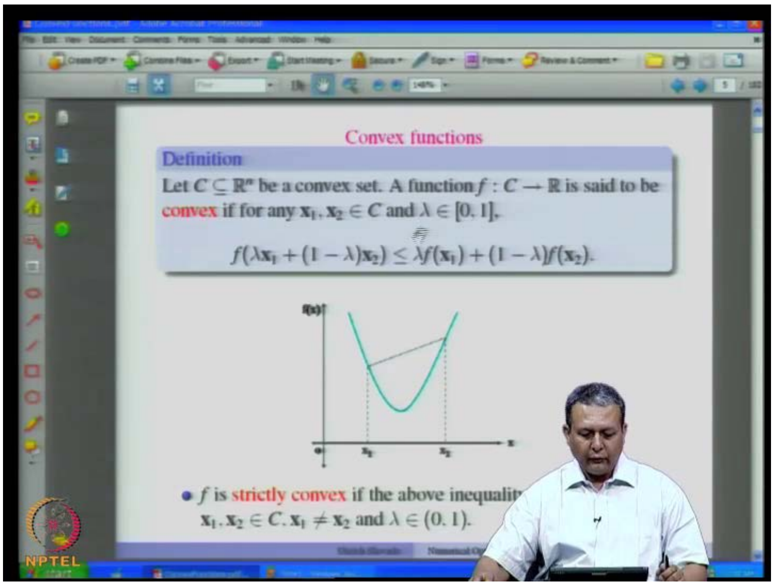
(Refer Slide Time: 05:17)



So, let us look at the example again. If we define this function to be on the close interval, a to b. Now, at this point, this is our function f of x. So, at this point, we have the coordinates a, f of a, and the coordinates are b, f of b. Now, if you take any point on this line segment joining a and b, so suppose let us take a point c. Now, if you take the value of the function at c, this is c, f c. So, you will see that if you now take a line segment joining a f a and b f b and interpolate the inter mediate points, then you will see that f c, the value of the function will always be less than or equal to the value represented by the corresponding point on the cord. We just have to interpolate the function values using this line segment.

Now, similar is the case here, and you will see that if we take these two points, let us call them as e and h. Now, if we take a point here, you will see that the value of the function is equal to the value obtained by using the interpolation of the points e f e and h f h. So, either the function value is less than the value attained by a point on the cord or it is in this case equal the value attained by a point on the cord. So, this is a convex function.

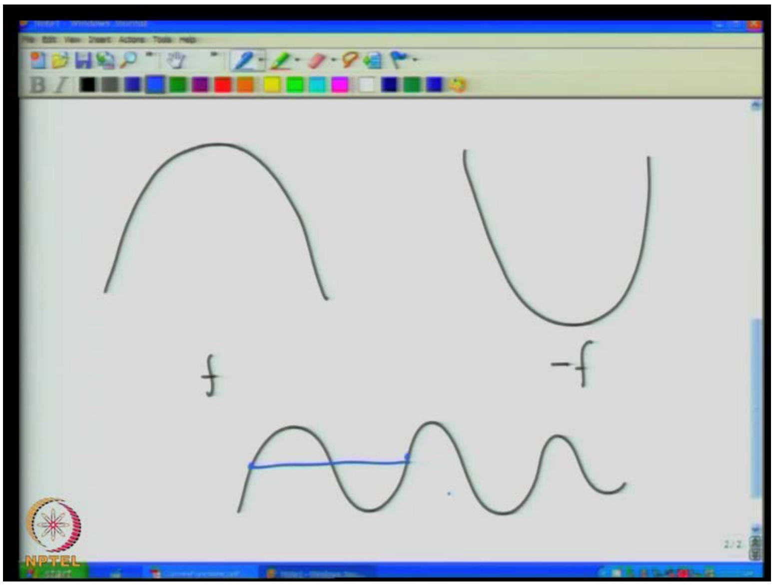
(Refer Slide Time: 07:29)



So, what does this definition says that, if you take any two points x1 and x2 in the set c and take a lambda in the close interval 0 to 1 and you will get point on the line segment joining x1 and x2, so lambda x1 plus 1 minus lambda x2. As we have seen earlier, if lambda is in the close interval 0 to 1, it represents a point on the line segment joining x1 and x2 and the convexity says is that the function is convex if f of lambda x 1 plus 1 minus lambda x 2 is less than or equal to lambda x lambda f of x1 plus 1 minus lambda f of x2.

So, here is the example of a convex function. So, we take two points, x1 and x2. Take a cord joining the two points. At any intermediate point on the line segment joining x1 and x2, the value of the function is less than the value obtained by interpolating the end points of the cord and finding the appropriate value on this cord of the function. Now, a function is strictly convex if this inequality is strict for any two distinct x1 and x2 in the set c and lambda is in the open interval 0 to 1. So, if this inequality is strict, then the function is said to be strictly convex.

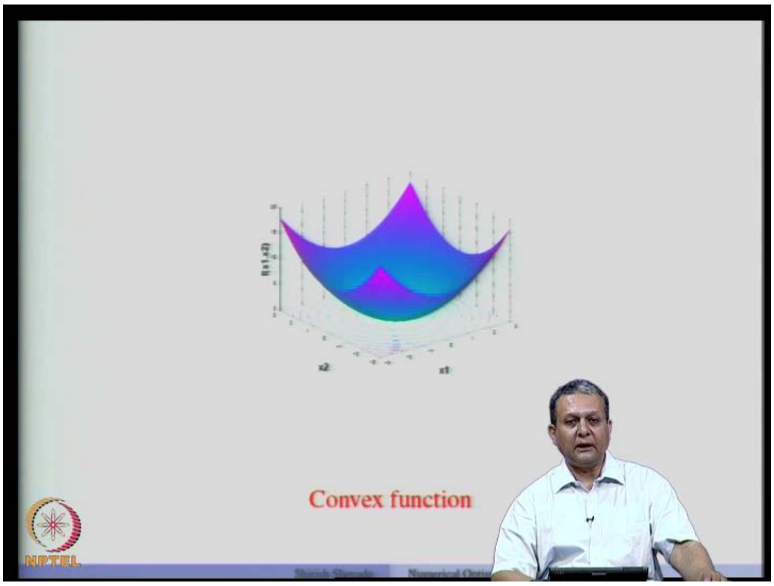
(Refer Slide Time: 09:52)



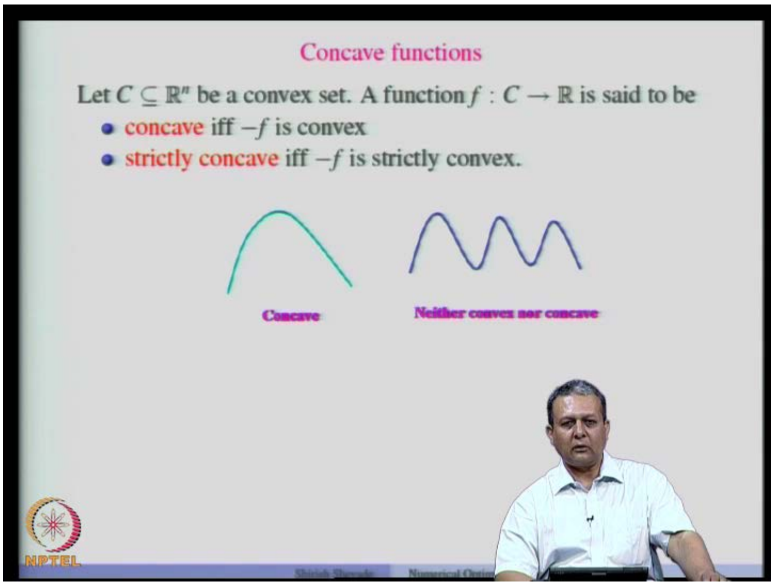
So, now if you look at our four figures, you will see that this function is strictly convex, while the other function is not strictly convex because there do exist some points, where the inequality does not hold strictly. So, the equality holds in that definition. Hence, that is why this function is not strictly convex, while this function is strictly convex. Now, if we consider a function which is of this type, so such functions are called concave functions or in other words, if we take the negative of this function, so if suppose this is f and if you take minus f, so this is our function f of x, and if you take a negative of the function which is minus f. Now, minus f is a convex function. So, if f is said to be concave, so minus f is convex, and so there would exist some functions which are neither convex nor concave.

For example, if we draw a function which is like this. Now, you will see that this is not a convex function because if you take any two points on the graph of the function and then take a cord joining them, you will see that in some regions, the function lies above this cord and in some regions, the function lies below this cord. So, even if you take a minus of this function that also will have the similar property. So, such functions are neither convex nor concave.

(Refer Slide Time: 11:21)

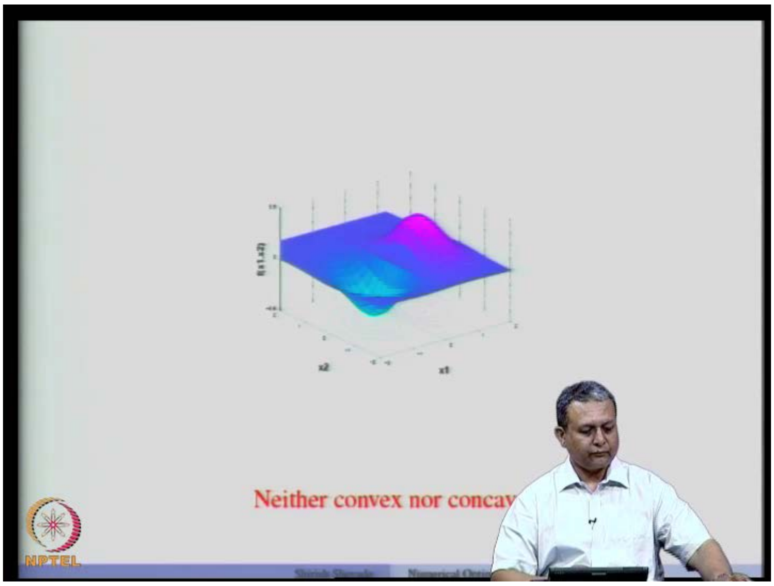


(Refer Slide Time: 11:56)

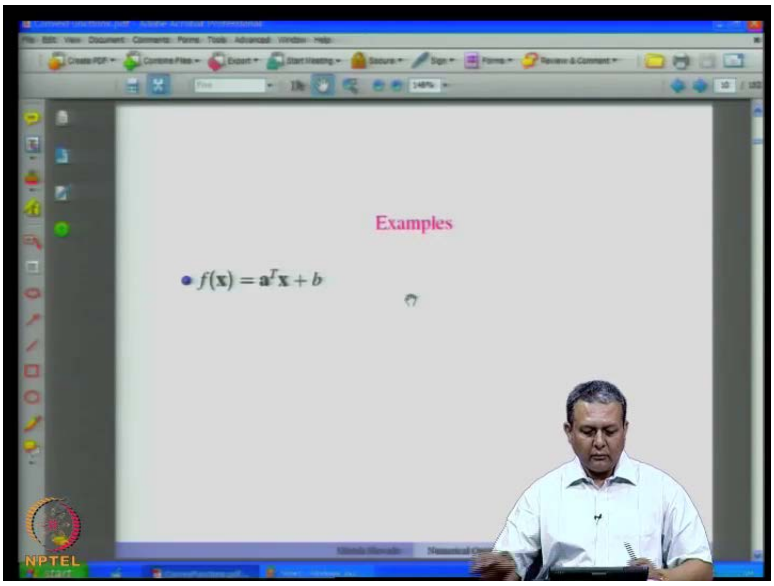


So, now let us look at some examples. So, this is the function in two variables. So, you will see that it has a nice surface, the cup shaped surface and such a function is a convex function because if you take any two points on the graph of this function and take a chord joining those two points, that chord always lies either on or above the function. In fact, in this case the function is strictly convex. Now, as I mentioned earlier that if we take a function f on a convex set c and that function is said to be concave if and only if the minus f is convex and is said to be strictly concave if and only if minus f is strictly convex. So, here is an example of a concave function and here is an example of a function which is neither convex nor concave. Now, you will see one more example shown which is neither convex nor concave.

(Refer Slide Time: 12:28)

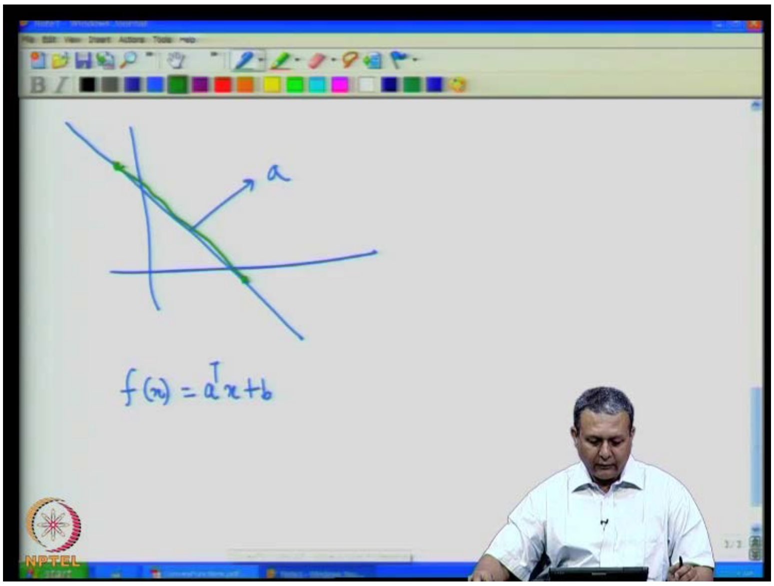


(Refer Slide Time: 12:58)



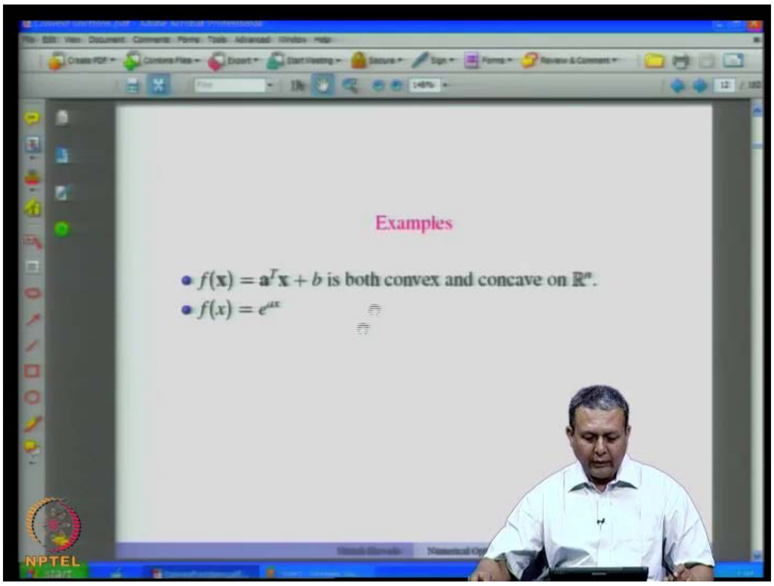
So, you will see that if the variable x 1 is greater than 0, then the function is concave. If the variable x 1 is less than 0, then the function is convex, but if we retain x 1 to be belonging to the set of real numbers, then this function is neither convex nor concave. Now, let us look at some example of functions and try to identify whether they are convex or not by drawing their graphs. So, here we have function f x equal to a to the power x a transpose x plus a b. Now, if we draw the plot of this function, so we have already seen that f x equal to a transpose x plus b will be a function like this, where a is normal to the hyperplane, so it is a hyperplane.

(Refer Slide Time: 13:35)

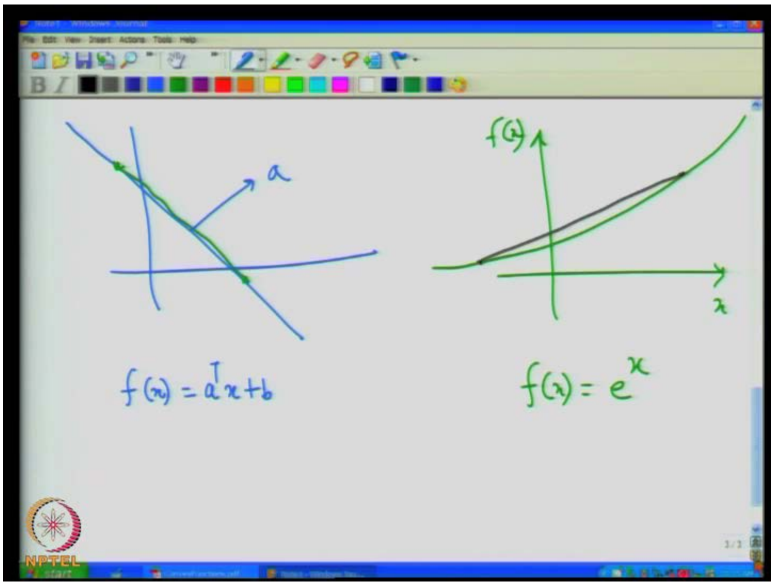


Now, if you take any two points on this hyperplane and take a line segment joining them, we will see that it always lies on the function. So, f x equal to a transpose x plus b is a affine function which is convex. Now, you will see that this also satisfies the definition of a concave function because minus f, if we take minus f, then that also is a convex function. So, in fact this is the only function, the affine function is the only function which is both convex and concave. Now, let us look at. So, this function is both convex and concave r n.

(Refer Slide Time: 14:30)

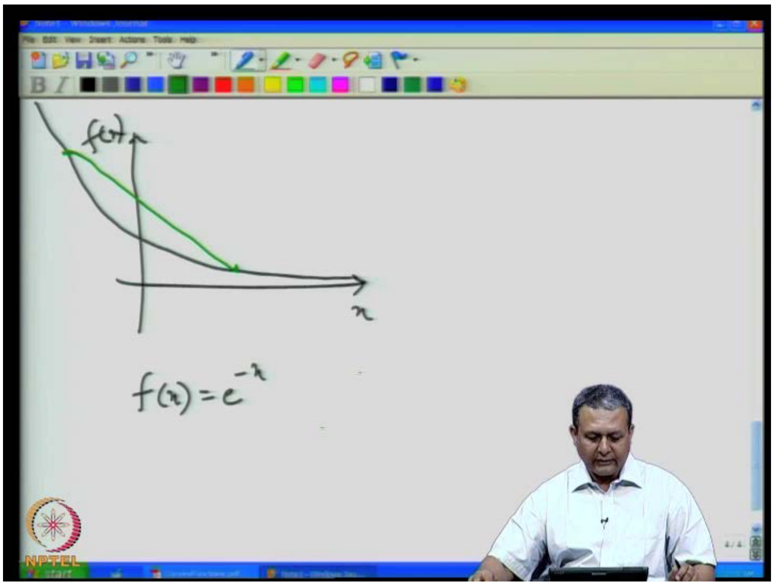


(Refer Slide Time: 14:58)

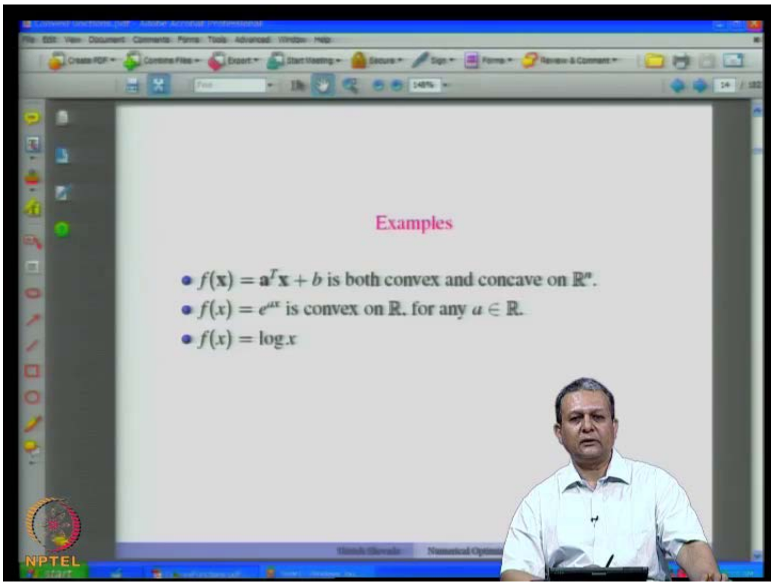


Now, let us look at another function x to the power a, where a is a real number. Now, if we look at the graph of the function, so we have x and f of x and we want to draw the graph of the function. So, e to the power x. So, this graph would look something like this. Now, if we take any two points on the chord of the function and joint them, you will see that the function is always below the chord. So, in fact this function is strictly convex.

(Refer Slide Time: 15:46)

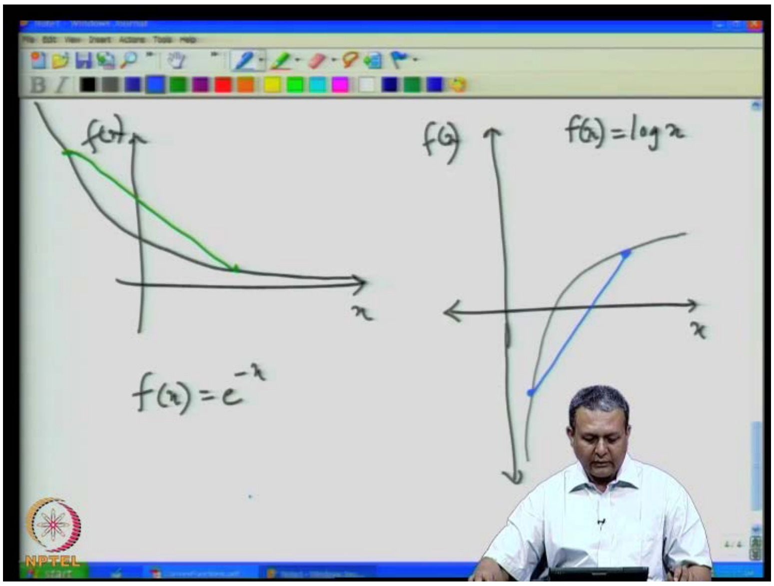


(Refer Slide Time: 16:30)



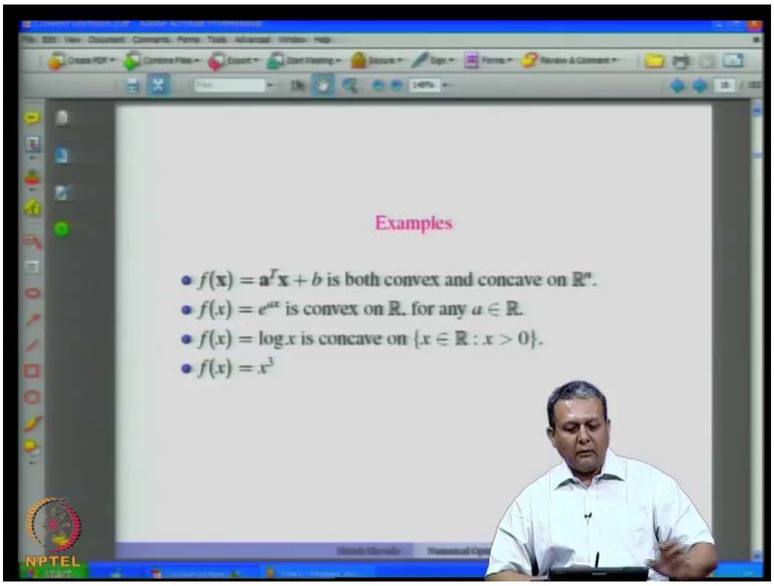
Now, if we take a e to the power minus x, so suppose if we take a function f x to f of x equal to e to the power minus x, so this function would look like this, and if we take any two points on the function on the plot of the function, you will see that the lines segment joining those two always lies above the function. So, this is a strictly convex function. Now, let us look at some more examples. So, e to the power x is convex on r for any a in from the set of real numbers. Now, the f x is equal to log x. So, this function now if you try to draw the graph of this function, so you have f of x. So, the function that we are going to plot f of x is equal to log x.

(Refer Slide Time: 16:41)

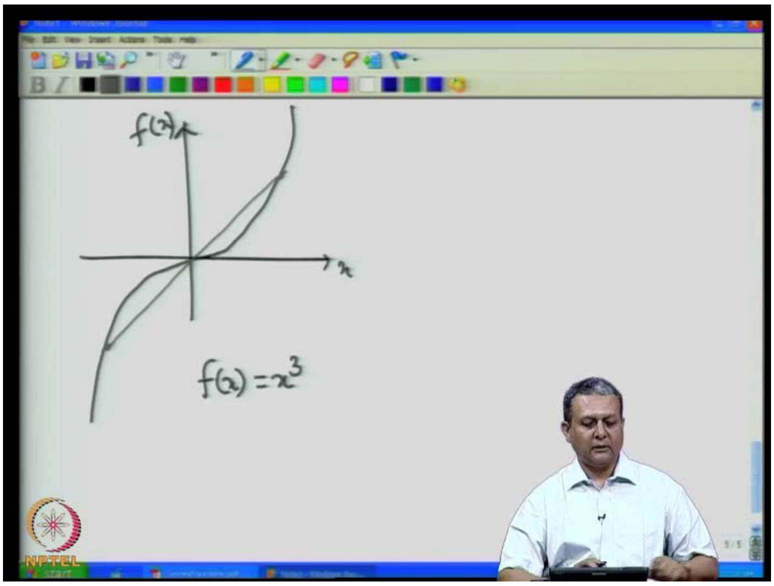


Now, this will look something like this. Now, if you take any two points on this graph of the function and draw a line segment joining those two, then you will say that the function always in between these two lines segments, in between these two points, the function always lies above this chord joining the two points. So, in this case, the function is concave. In fact, it is strictly concave function. So, let us look at some other examples. So, f x is equal to log x is a guess of a concave function on the set of positive real numbers.

(Refer Slide Time: 17:54)

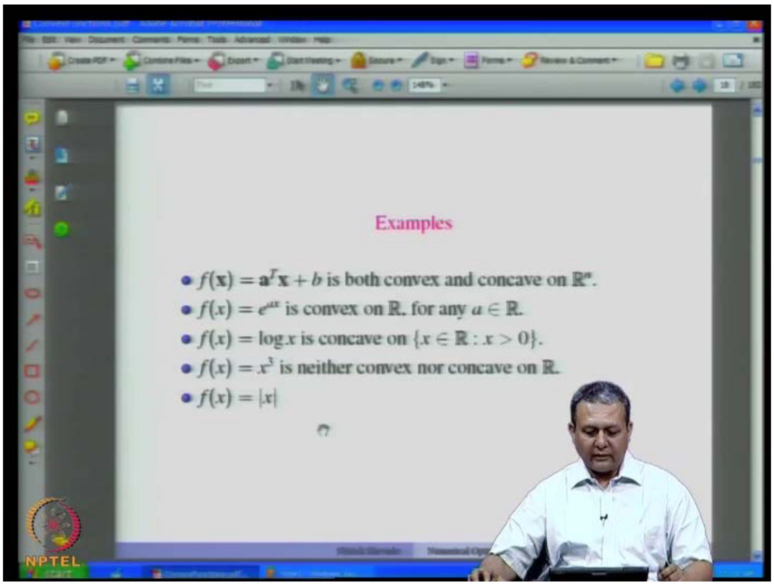


(Refer Slide Time: 18:05)

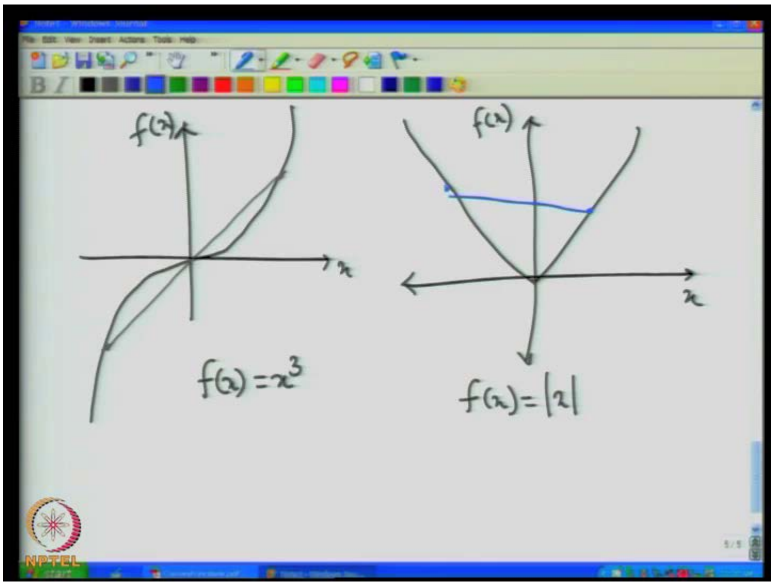


Now, if you take the function x equal to x q and if we try to draw the graph of the function, so you will see that the function is somewhat like this and you will see that now if you take any two points on the graph of the function and take a line segment joining those two, you will see that this function is neither convex nor concave, but suppose if you restrict ourselves only to the set of non-negative real numbers, so the domain of the function if you restrict to the set of non-negative real numbers, then you will see that the function is convex. So, these function f of x equal to x q is neither convex nor concave on the set of real numbers, now f x equal to mode x.

(Refer Slide Time: 19:07)

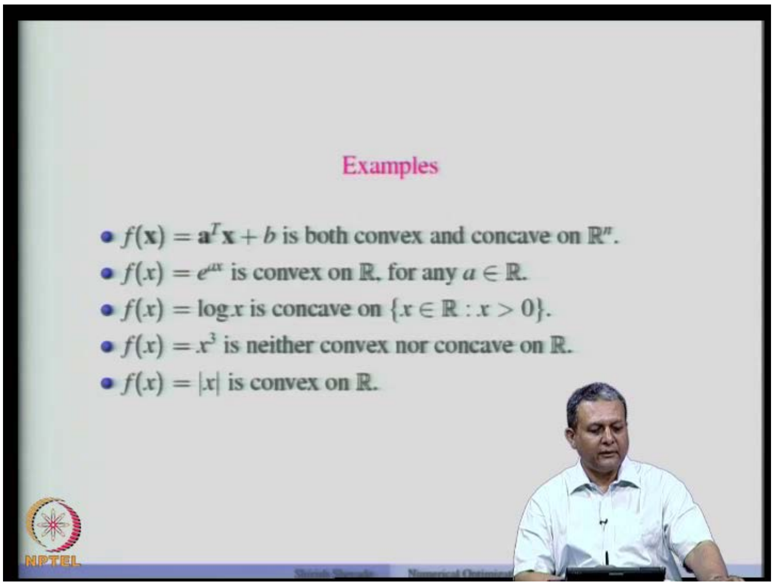


(Refer Slide Time: 19:18)



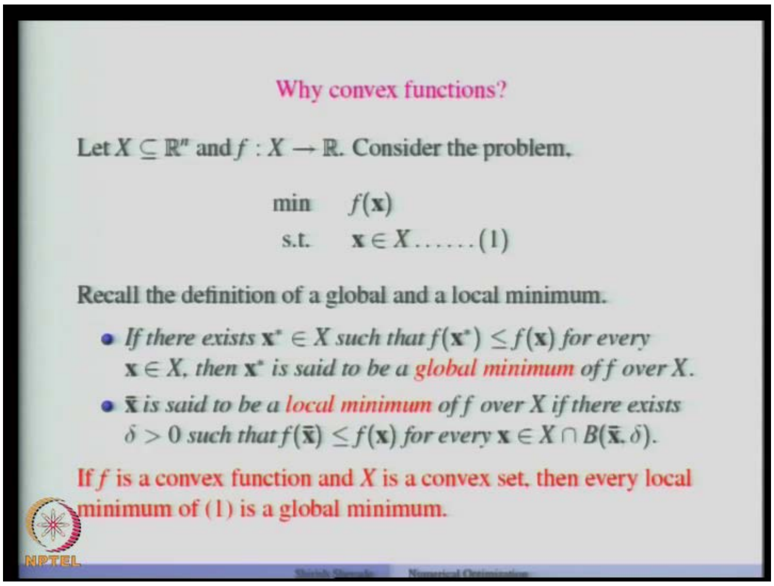
So, let us look at how this function looks like. So, this is f x equal to mode x. So, you will see that if we take any two points on the graph of this function, the line segment joining them always lies on or above the function. So, this function is a convex function. So, this shows that convex functions need not be differentiable. This is the continuous function. Now, one can also have some examples, where the convex functions need not be continuous, but remember that convex functions have to be continuous in the interior of the domain. There could be discontinuities at the (()).

(Refer Slide Time: 20:42)



So, these are some examples of convex functions. You can generate more examples of convex functions from the known convex functions. So, obviously the next question that would arise is that how we characterize a convex function, given a function how do we find out whether the function is convex. So, instead of trying to find out whether if you take any two points in the domain of the function and take a chord on the graph of the function joining two points, x f x 1 and x 1 f x 1 and x 2 f x 2, I am trying to find out whether the chord lies below the function or above the function. Instead of that, can we have a better way of characterizing a convex function?

(Refer Slide Time: 21:24)

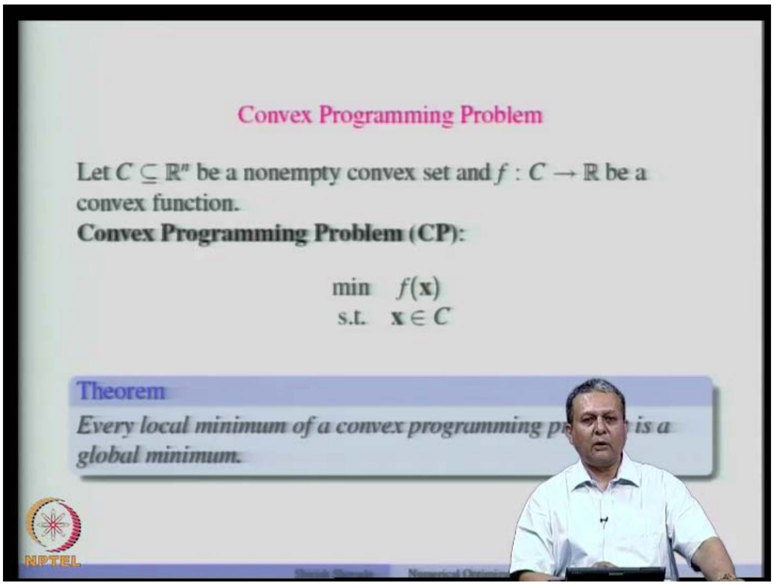


Now, before we go into those details, let us try to see why we worry so much about convex functions. Now, the convex functions play a very important role in optimization literature and has I mentioned that they have some nice properties, and we will try to see some of those properties of a convex functions. Let us consider a constrained optimization problem, where x is any subset of r n and f is a function from x to r and consider this problem where we want to minimize f of x subject to the constraint that x belongs to the set x. So, this is the general unconstraint constraint optimization problem.

Now, we know that there exist two types of minima. One is called a global minimum and one is called a local minimum. Now, a global minimum is a point x star belonging to x, such that the value of the function at x star is at least the value of the function at every other point in the domain. So, for any x belonging to the set x f of x star has to be less than or equal to f of x and then we said that x star is the global minimum of f over x. Now, as we saw earlier that it is very difficult to characterize a global minimum because to check whether a point is a global minimum, we need the knowledge of all the points in the set and then we need the knowledge of f of x for every point in the set x only. Then, we can characterize whether a point is a point, x star is a global minimum or not.

So, we also use the definition of a local minimum. So, a point x star is said to be a local minimum of f over x. So, if there exist a delta neighborhood of x x bar, such that in the delta neighborhood, this b x bar, delta is a delta neighborhood of x bar and that we take a intersection with respect to the set x because we are always worried about the feasible points. So, if we collect all that feasible points in the delta neighborhood of x bar, then if the value of the function at x bar is at least the value of the function at all the other points in the feasible neighborhood, delta neighborhood of x bar, then we say that x bar is a local minimum.

(Refer Slide Time: 24:43)

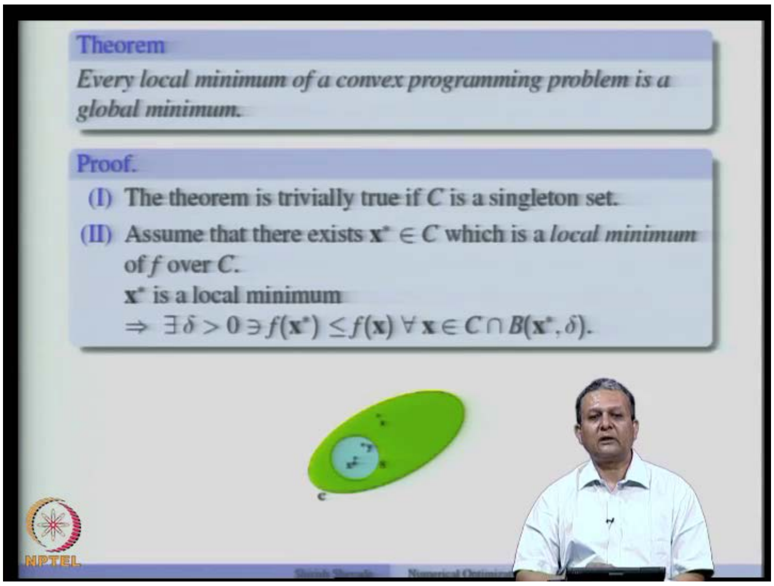


Now, the good thing about the convex function is that if f is a convex function and x is a convex set, then every local minimum of this problem is a global minimum. So, in other words, we do not have to worry about finding out different local minima for a convex programming problem, where the function is a convex. The objective function is to minimize this convex and the constraint set is of convex set. Now, such problems are called convex programming problems. So, let us formally define convex programming problems. So, let c be a non-empty convex set in r n and f be a function from c to r and let f also be a convex function. Now, what we are interested in solving a problem where we want to minimize f of x subject to the constraint that x belong to c.

So, when f is a convex function which we want to minimize and the constraint set is also a convex, then it is called a convex programming problem. Now, suppose f is a concave function and we want to maximize the concave function over a convex set, then that also is a convex programming problem because maximization of a convex function can also be written as a minimization of, a maximization of a concave function can be written as minimization of a convex function.

So, if a problem is of this type where we want to minimize a convex function over a convex set, this is called a convex programming problem and the important property of this convex programming problem is that every local minimum of a convex programming problem is a global minimum. So, we really do not have to worry about the problem of local minima as far as convex programming problems is concerned. So, this convex programming problem has lots of interesting applications in different areas of mathematics, engineering, science and so on.

(Refer Slide Time: 26:34)

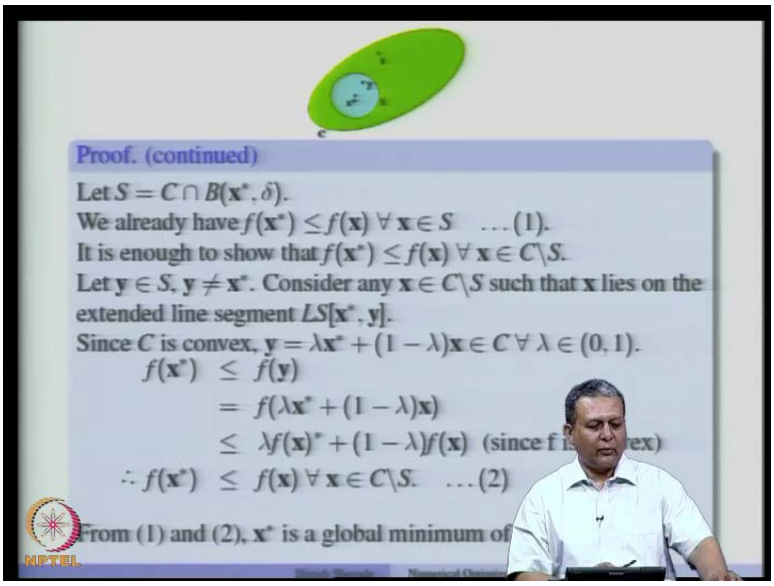


So, these problems are very important and we should know some of the important properties of convex programming problems. One of the properties is that every local minimum is of a convex programming problem is a global minimum. So, let us now try to prove this property of convex programming problem. Now, if the constraint set is the singleton set, then the theorem is obviously true because there is only one point in the constraint set and that point will always be a global minimum. So, the theorem is trivially true in such a case.

Now, so let us assume that there exist some x star belonging to c which is the local minimum of f over c. So, by the definition of a local minimum, what we have is that there exist a delta neighborhood of x star, such that f of x star is less than or equal to f of x for all x in the region of c intersection, the delta neighborhood of x star. So, the delta neighborhood of x star is shown here. Now, let us call this as a set s. Now, what we want to do is that by the definition of a local minimum, it is true that x star is a local mean in the delta neighborhood, where the rate of feasible neighborhood around x star.

Now, what we want to show is that f of x star is also a minimum function value that can be attained over all x belong to c. Now, we have shown that by the definition local minimum, it is the minimum in this region, but what about the remaining region, that is the region which is shown by the green colour here. So, if can show that f of x star is less than or equal to f of x for any x in this region, the remaining region, then we can conclude that x star is indeed a global minimum and let us see now how to do that.

(Refer Slide Time: 28:36)



So, let us call this region the feasible delta neighborhood around x star. Let us denote it by the set s. So, s is basically the intersection of c and the delta neighborhood of x star.

Now, what we need to show is that f of x star is less than or equal to f of x for all x in the set c, which are not in the set s. So, the green colour region is what we are interested in. So, if you take any x, we want to show that f of x star is less than or equal to f of x, but we already know that f of x star is less than or equal to f of x for all x belongs to s. So, let us consider a point in the delta neighborhood of x star and let us call that point as the point y and y is not equal to x star. Now, what we do is that we take a line segment joining x star and y and extend it to some point x in the region c minus s.

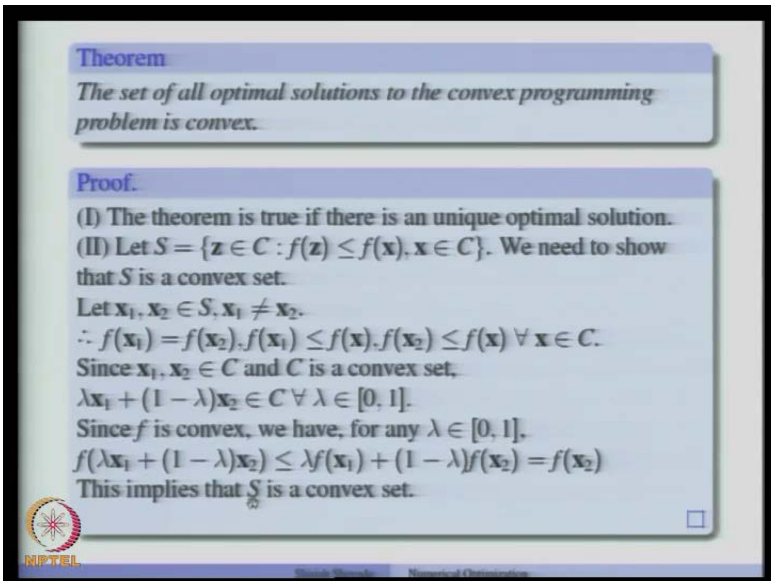
So, consider a point x on the line segment joining x star and y, and x belongs to the set s c minus s. Now, we want to see what can we say above f of x star and f of x and our aim is to show that f of x star has to be less than or equal to f of x. So, let us see how to do that. Now, note the set c is convex. So, x star is in the set c, x is in the set c. So, any line segment joining x star and x always lies in the set c. So, let us take any point y on the line segment joining x star and x. Let us exclude the points x star and x and take any point on that, open line segment joining x star and x.

Now, we know that f of x star is less than or equal to f of y. This is because y is chosen to be in the set s and by 1 f of x star is less than or equal to f of x for all x belongs to s. So, clearly f of x star is less than or equal to f of y. Now, this y we can write it as lambda x star plus 1 minus lambda x. So far, we have used the convexity of the set c. Now, we can use the convexity of the function f. Now, since f is convex, we take any x star and x in the domain take a line segment joining those two. So, by the definition of convexity f of lambda x star plus 1 minus lambda x is less than or equal to lambda f f x star plus 1 minus lambda f x because f is convex.

Now, remember that lambda is in the close interval 0 to 1. So, if you rearrange this, what we get is that on the left side, 1 minus lambda f x star is less than or equal to 1 minus lambda f x for all lambda in the open interval 0 to 1. Now, if you divide throughout by 1 minus lambda, what we get is f of x star less than or equal to f x for all x in the set c minus s. So, for all the points on the set c which are not in the set s, that is the points which are shown in the figure by green colour, we have shown that f of x star is less than or equal to f of x for all those points in the set c minus s, and by the definition of local minimum f of x star was always less than or equal to f of x for x belongs to x. So, if you combine 1 and 2, we will see that f of x star is less than or equal to f of x for all x belongs to c and which means that x star is indeed a global minimum of f over c, ok.

So, this is an important theorem which says that for every (()) programming problem, a local minimum is a global minimum. So, the next question that we would like to ask is that suppose convex programming problem has multiple local minima which also are now global minima. Because of this theorem, how are they positioned in the set in the convex set c? In other words, in a common terminology, we ask the following question are those global minima of a convex function scattered at different places in the set c or they always are combined together. So, we will try to answer this question.

(Refer Slide Time: 33:48)



So, this another important theorem in the theory of ha convex programming and the theorem says that if you collect a set of all optimal solutions to the convex programming problem, then the set of all those optimal solutions is a convex set. So, that theorem essentially tells us that the global minima of a convex programming problem, they are not scattered everywhere, but they always try to form a convex set. So, we will see how to prove this. Now, if the set c is a convex set and if we get unique optimal solution, then that set is always convex because we know that any singleton set is a convex set.

So, if we have only a unique solution, then the theorem is trivially true. So, there is nothing to prove there. So, let us try to prove in the case where we have multiple global minima. So, let us take all those global minima and put them in the set s. So, s is the set of all the points in the set c, such that f z is less than or equal to f x for all x belongs to c. So, this s is the set of all global minima of a convex programming problem. Remember that the function f is convex and the set c is also convex and we are trying to minimize objective function f of x which is a convex. Hence, the problem is a convex programming problem.

Now, what we have to do is that we have to show that the set s is a convex set. Now, to show that any set is a convex set, what we have to do is that we have to take two points, any two points in the set s and show that the line segment joining those two points entirely lies within the sets. So, if you can show that, then the set s becomes a convex set. So, for that purpose, what we do is that let us take two points, x 1 and x 2 in the set s and obviously, we take the points which are distinct and then take a line segment joining these two points.

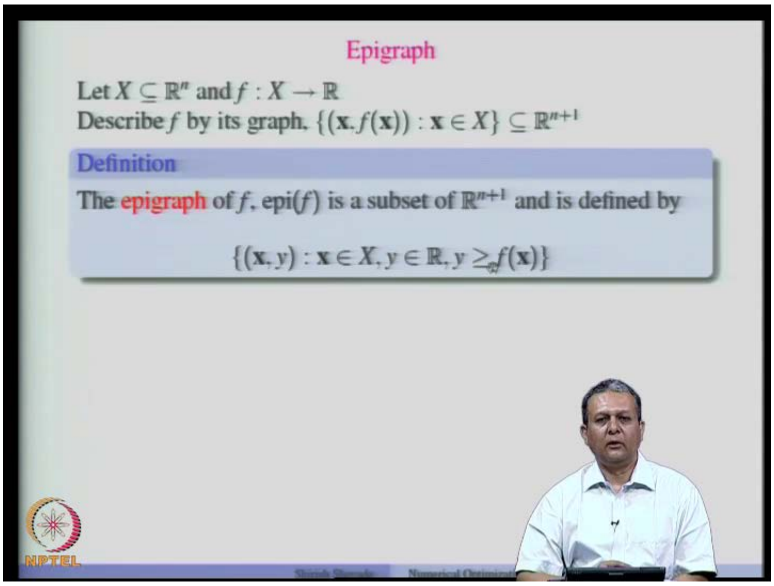
Now, since x 1 and x 2 belong to set s and they are we know that every local minimum of convex programming problem is of a global minimum, so we have f of x 1 and f of x 2 to be equal. So, since they belong to the set s, we can also say that f of x 1 is less than or equal to f of x and f of x 2 less than or equal to f of x for all x belong to c because both x 1 and x 2 belong to the set s. Now, since c is a convex set and x 1 and x 2 are any two points in the convex set, we can say that by the definition of convexity of a set, we can say that lambda x 1 plus 1 minus lambda x 2 that always belongs to the set s, set c for all lambda in the closed interval 0 to 1.

Now, we can use the convexity of the function. So, how do we use the convexity of the function? So, since f is convex, we can write f of lambda x 1 plus 1 minus lambda x 2 to be less than or equal to lambda f x 1 plus 1 minus lambda f x 2. This is by the definition of convexity of a function. Now, you will see that f of x 1 is less than or equal to f of x and f of x 2 is also less than or equal to f of x. So, the quantity on the left side is less than or equal to f of x 2 which is shown here. So, what does this mean? So, this means that f of lambda x 1 plus 1 minus lambda x 2 is less than or equal to f of x 2.

Now, this inequality cannot hold because in that case, what will happen is that will contradict the fact that f of x 2 is less than or equal to f of x for all x belong to c. So, it will contradict that this has to hold with equality. Now, what does that mean. That means that lambda x 1 plus 1 minus lambda x 2 also belongs to the set s for all lambda in the range 0 to 1 and therefore, this implies that that the set s is a convex set, ok

So, we have studied two important properties of a convex programming problem. The first property is that every local minimum of a convex programming problem is a global minimum and not only that, but the set of all global minima of a convex programming problem is a convex set. Now, given this two important results, now we will start studying more about convex functions. Now, one of the things that we would like to study is that how do we characterize a convex function. Now, to characterize a convex function, there is a nice way.

(Refer Slide Time: 39:27)



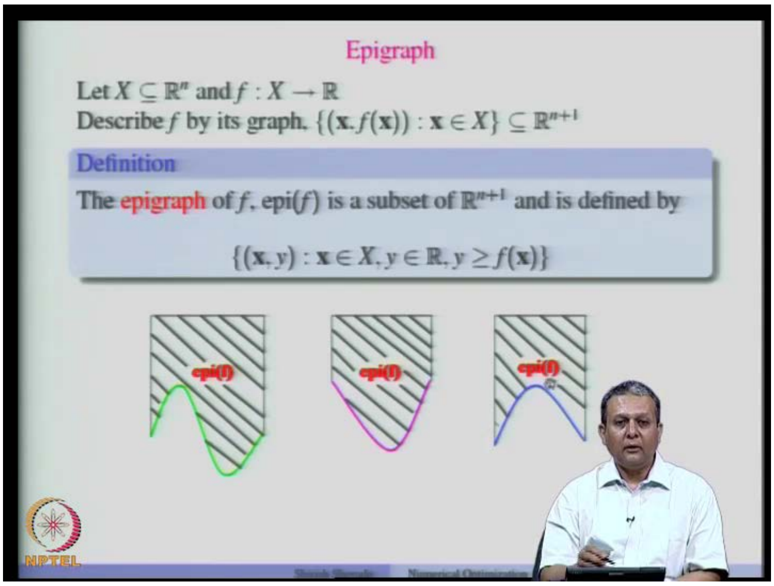
So, let us define a function f from x to r and describe the function by its graph. So, this graph is a subset in, then plus one dimensional space and it consists of the points x and x, f x, where x belongs to the domain of the function. Now, suppose if you describe the function by its graph like this, then the epigraph of a function is a subset of r n plus 1 and is defined as the set of all points x, y in n plus 1 dimensional space, so that x belongs to x and y is a real number and y is greater than or equal to f of x.

(Refer Slide Time: 40:22)

So, if you take a function s, this is the function, this is the graph of the function. So, I said the graph is a set of all points x, f x. So, this is the graph and the epigraph of the function is a set of points which are on or above the function. So, you will see that this is our unbounded set all the points for which y is greater than or equal to f of x, all the x, y where y is greater than or equal to f of x, the points on the curve as well as the points above the curve. So, this is called the epigraph of the function f. So, if we take a function which is like this, then if we consider all the points which are on or above the function, so this turns out to be epigraph of this function f. So, these are some examples of the epigraph of a function.

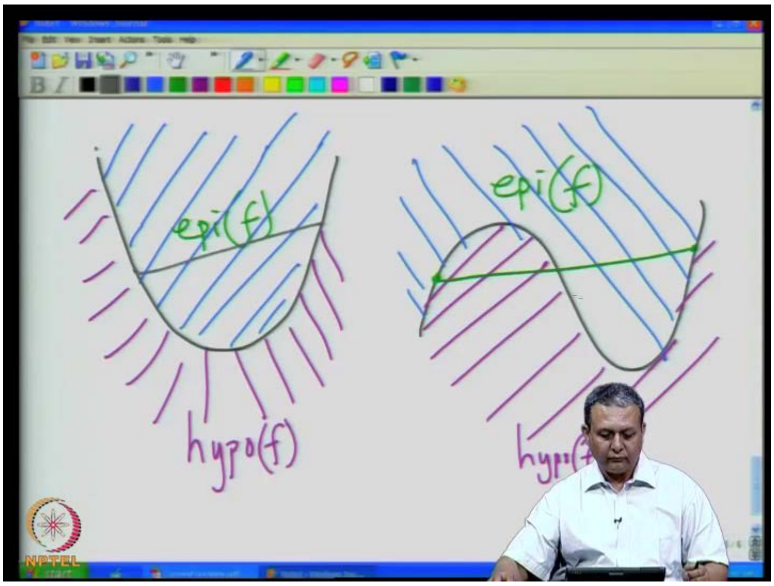
Now, along similar lines, one can also define. So, if we take the points which are below the function, then that is called the hypo graph of the function. Similarly, here we can draw the hypo graph to be like this. So, this is a hypo graph of the function. So, the points which are on or above the function, they constitute the epigraph of the function and the points which are on or below the function, they constitute the hypo graph of a function.

(Refer Slide Time: 43:15)



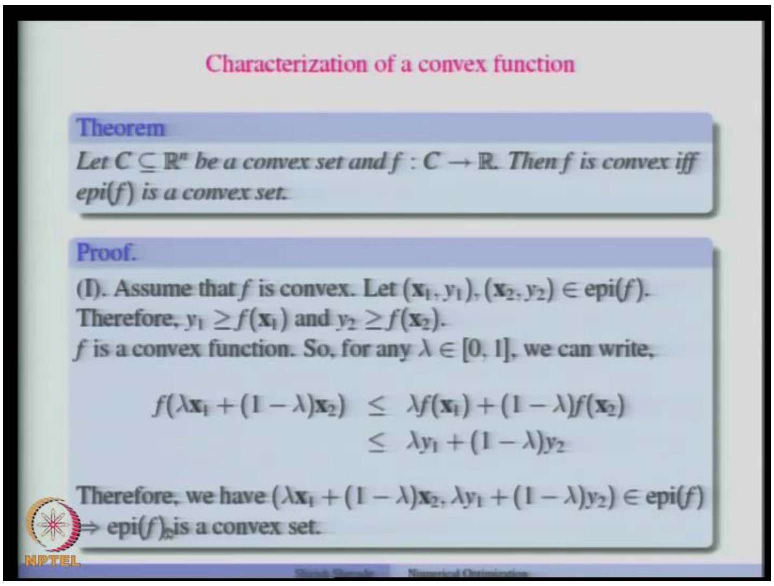
So, we can see some examples here. So, here I have drawn a subset of the epigraph because the epigraph is typically not bounded. So, I have just drawn a subset of the epigraph. Actually, you will see that all the points y, such that y greater than or equal all the points x y, such that y greater than or equal to f of x. So, you will get all the points even above this, but for continuance, I have just drawn the subset of the epigraph in this figure. Now, there are three different functions shown here and if you look at the epigraph of these three different functions, and especially if you look at the epigraph of this function, you will see that this function is convex and the epigraph of the function is a convex set. Now, that is not true in these two cases.

(Refer Slide Time: 44:24)



So, if you take a line segment joining two points in the epigraph of the set, you will see that the entire line segment does not belong to the epigraph of the set, while if we take a line segment joining any two points in the epigraph of this function, then you will see that the line segment entirely lies within the set. So, that means that if we have a convex function, then the epigraph of the function is a convex set and if you do not have a convex function, the epigraph of a function is not a convex set.

(Refer Slide Time: 45:08)



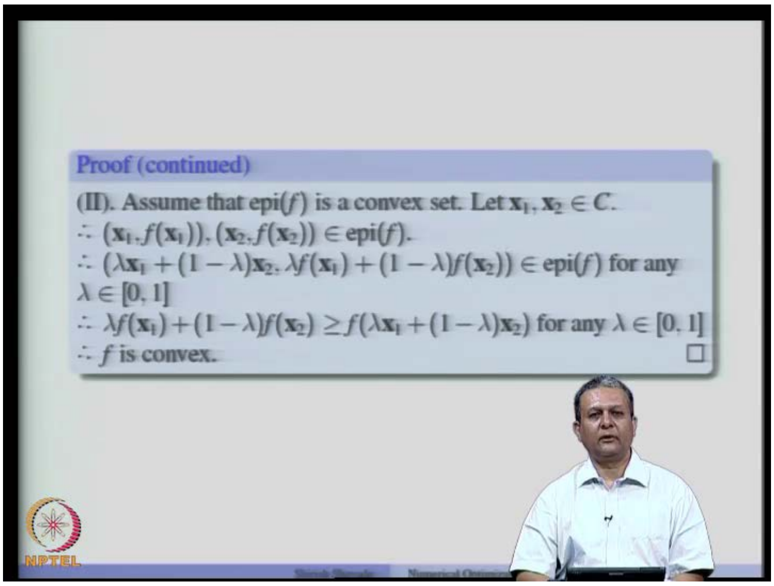
Now, let us try to give a formal theorem. So, suppose we have a set c which is a convex set and ha function defined on the set c are real valued function defined on the set c, then f is convex if and only if the epigraph of f is a convex set. We will try to prove this. So, let us assume that the function is convex. Remember that we have to give the proof in two parts. The first part is that we assume that f is convex and show that epigraph of f is a convex set, and the other part is that assume that epigraph of f is a convex set and show that the function is convex. So, let us first prove the part where we assume that the f is function of convex and prove that epigraph of that function is a convex set.

Now, to prove that epigraph of a function is a convex set, what we need to do is that we take two points in the epigraph of the function and then show that the line segment joining those two points belongs to the epigraph of the set. So, let us see how to do that. So, let us take two points, x 1 y 1 and x 2 y 2 in the epigraph. Now, since these two points belong to the epigraph of the function, we know that by definition of the epigraph of a function y 1 is greater than or equal to f of x 1 and y 2 greater than or equal to f of x 2. Now, we will use the convexity of the function. So, if you take any lambda in the close interval 0 to 1 because f is convex, that is what we have assumed f of lambda x 1 plus 1 minus lambda x 2 is less than or equal to lambda f x 1 plus 1 minus lambda f x 2.

Now, we know that f of x 1 is less than or equal to y 1, f of x 2 less than or equal to y 2. So, the right side is less than or equal to lambda y 1 plus 1 minus lambda y. Now, what does this mean? So, we have a point lambda x 1 plus 1 minus lambda x 2 which belongs to the set c, such that f of the function value at that point is less than or equal to some real number which means that the point lambda x 1 plus 1 minus lambda x 2, lambda y 1 plus 1 minus lambda y 2 does belong to the epigraph of the function by the definition of epigraph.

So, what we have therefore is that the point lambda x 1 plus 1 minus lambda x 2 lambda y 1 plus 1 minus lambda y 2 belongs to the epigraph of f, which means that the epigraph of f is a convex set because this holds for all lambda in the range in the close interval 0 to 1 and for any x 1 y 1 and x 2 y 2, belong to the epigraph of s. So, we have taken two points in the epigraph and shown that for any two points in the epigraph, the line segment joining the two points always lies in the epigraph of the set which means that the epigraph of f is a convex set.

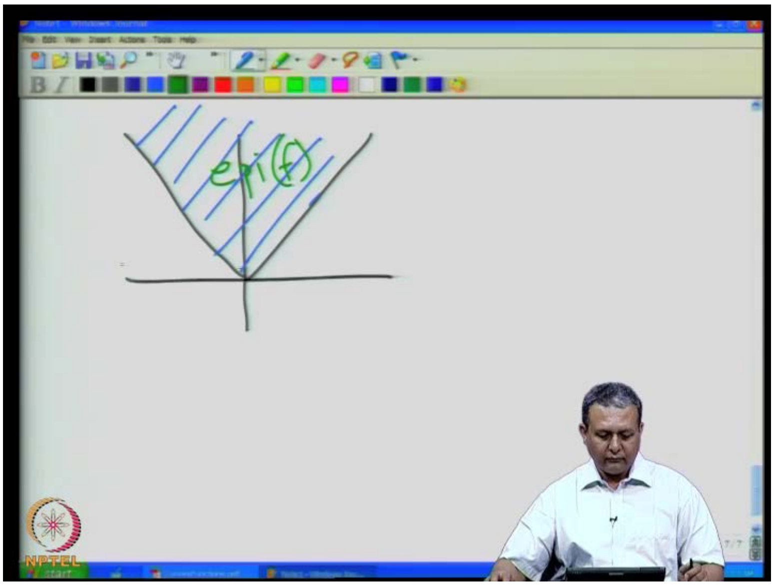
(Refer Slide Time: 48:32)



Now, we will prove the other part. So, let us assume that epigraph of f is a convex set and then we have to show that f is a convex function. Now, to show that f is a convex function, we have to show that it satisfies the definition of convexity of a function. Now, since epigraph is a convex set, let us take two points in the set c and let us consider the two points on the epigraph. So, these are two points actually on the graph of the function and the points on the graph also happen to be the points in the epigraph as well as the hypo graph. So, x 1 f x 1 and x 2 f x 2 always belong to epigraph of f.

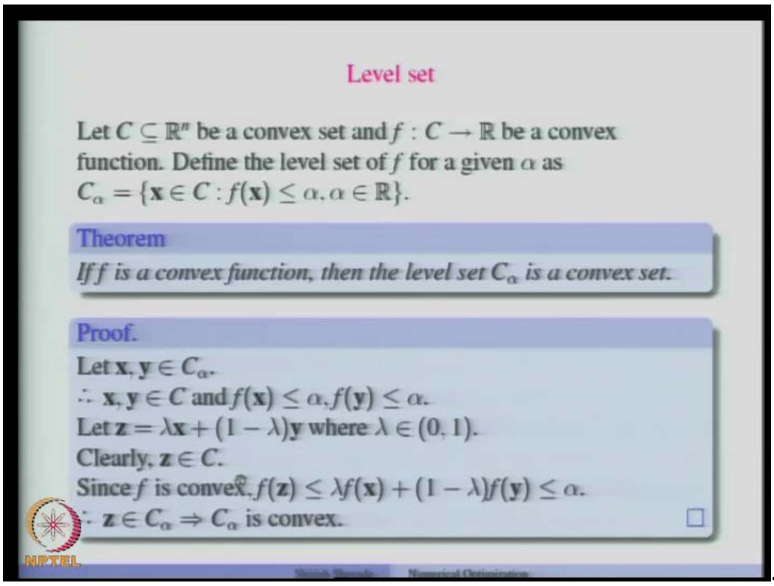
Now, we have assumed that epigraph of f is a convex set. So, the line segment joining any two points in the epigraph always belongs to the set. So, by this we can say that lambda x 1 plus 1 minus lambda x 2 and lambda f x 1 plus 1 minus lambda f x 2 always belongs to the epigraph of f for any lambda in the close interval 0 to 1. Now, by the definition of epigraph, what we have is that since this point belongs to the epigraph, what we have is lambda f x 1 plus 1 minus lambda f x 2 greater than or equal to f of lambda x 1 plus 1 minus lambda f x 2. So, this value which is a real number is greater than or equal to the value of the function at the point lambda x 1 plus 1 minus lambda x 2 for any lambda in the close interval 0 to 1 and therefore, this function is convex.

(Refer Slide Time: 50:42)



So, what we have shown is that the function f defined on a convex set is real valued function. We find on the convex set c is convex if and only if the epigraph of the function is convex. Now, if we look at some other function, so let us consider the function which is mode x and then let us try to draw the epigraph of this function. So, you will see that epigraph of this function is a convex. Now, one good thing about this characterization of epigraph, this characterization of convex function is that the function need not be different shape. So, even if the function is not differentiable, we can always find out its epigraph and check whether it is a convex set and find out whether the function is convex or not, but if the function is differentiable, it is analytically easy to characterize a convex function.

(Refer Slide Time: 51:42)



Now, before we look in to the differentiable look at the differentiable convex functions, let us look at what are called the level sets. So, let us consider a convex subset of r n and a real valued function defined on the convex set c. Now, the level set of f for a given alpha is determined as the set of all points in the set c, such that f of x is less than or equal to alpha, where alpha is a real number. So, since this set depends on alpha, we show the dependence of that set in the (( )). So, when we say c alpha, what we are interested in the set of all x in the set c, such that f of x less than or equal to alpha.

Now, we have theorem which says that if a is convex function, then the level set c alpha is a convex set for every alpha belonging to r. Now, it is easy to prove this theorem. So, suppose that we take any two points in the c alpha set and what we have to show is that if f is a convex function, then c alpha is a convex set. So, we assumed that f is a convex function and we have defined c alpha to be this way. Remember that while defining the level set, we do not need f to be a convex function. Now, x y belong to c and f x less than or equal to alpha and f y less than or equal to alpha, that is by the definition of c alpha because since x y belongs to c alpha, they clearly belong to c and f of x and f of y. Both are less than or equal to f of alpha.

Now, let us assume that f is a convex function. So, we take a point this side which is on the line segment joining x and y. So, x and y belong to c as well has c alpha. So, this line segment also belongs to both. So, clearly z belongs to a. So, any point on that line segment joining x and y belongs to the set c. Now, we assume that f is convex and by the definition of convexity of a function f z is less than or equal to lambda f x plus 1 minus lambda f y and since, f of x is less than or equal to alpha f of y less than or equal to alpha, what we have is f z less than or equal to alpha. So, for any line segment joining x y in c alpha, we get any point z on that line segment also lies in the set c alpha, which means that c alpha is a convex set.

So, if a convex function f, then the level set c alpha is a convex set. So, this is an important property which will be used to generate more convex sets from a convex set from convex functions, and we will see that sometime later, but remember that the converse of this theorem is not raw. So, if every level set c alpha of a function is a convex set that does not mean that the function is a convex function. So, I will leave it as an exercise to you to find out the functions for which the level sets are convex, but the function is not convex.

So far, we have talked about the convex functions and used epigraph character as a convex function. Now, suppose the function convex, the functions are also differentiable, then how do we characterize them to be convex functions, and if they are twice differentiable, then how do we characterize them to be convex functions. So, we will study those things in the next class.

Thank you.