Numerical Optimization
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Lecture - 7
Convex Sets (contd.)

So, hello welcome back to this series of lectures on numerical optimization. So, in the last class, we were looking at convex sets and the convex sets are defined as those subsets of R n where if you take a line segment in a set the line segment joining any two points of that set entirely lies within the set for example, suppose we take set like this.

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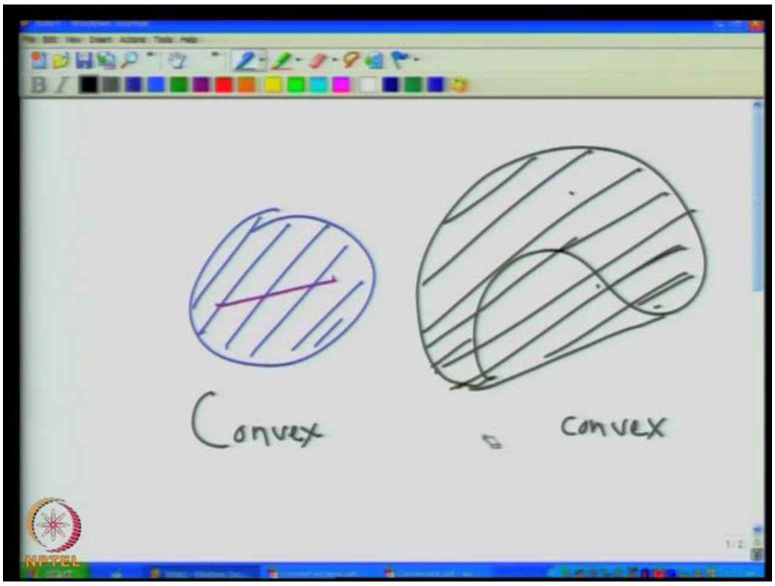

So, this set is convex set because if we take any line joining the two points that lies entirely within the set. On the other hand suppose, we take this set is convex on the other hand if we take a set like this where we do not include the interior of the set then such a set is not convex. So, it is important that the line segment joining any two points should lie entirely within the set for example, in the set on the right side. If you take a line joining these two points, you will see that only these two points the end points of that line segment lie within the set while the interior of that line segment does not lie within the set. So, it is important that any line segment joining any two points of the set should lie entirely within the set.

So, for example if we take a set like this and so we see that this set is not convex because if we take a line segment joining these two points that does not lie within the set , but we also saw that there are some ways to convexity our set , but that is called the convex hull of a set. So, for example if we consider this set which is not convex, now that set can be made convex by suppose we draw a line like this and then include the entire portion. Now, the new set which is formed now that becomes a convex set, so the convex hull of a non convex set is the convex set. So, in other words the convex hull is a set which is the smallest convex set which contains the given set, so we were looking at the definition of convex hull.

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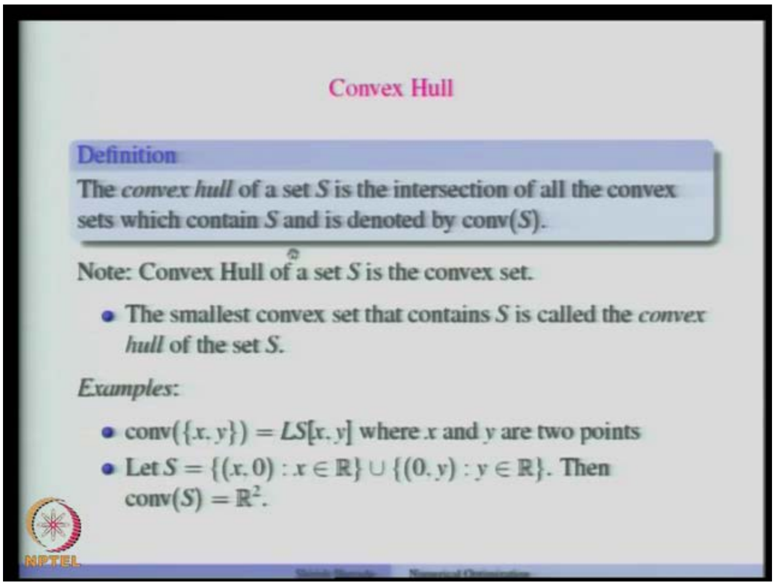

We saw that the convex hull of a set is the intersection of all convex sets which contain the set S and we are going to denote it by the symbol convex hull of the set. Now, by the definition of this convex hull it is a convex hull is a convex set because we are taking the intersection of all convex sets. And that is why we saw this result in the last class that the intersection of any collection of convex sets is the convex set. From examples of convex sets that we saw in the last class was that if you are given two points.

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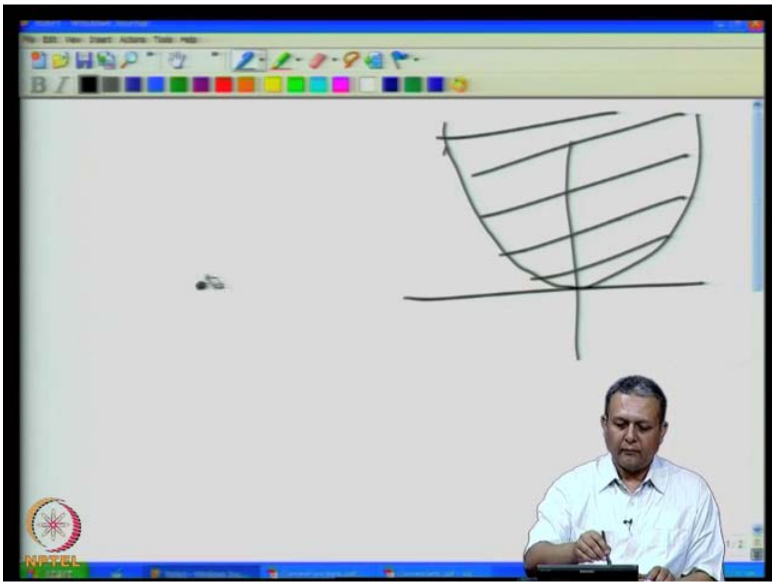

So, if you are given two points the convex hull of these two points is the line segment joining these two points. Now, if you are given a set in a two dimensional space which is like this then the convex hull of this set is the set of all points which are on or above this curve so this becomes a convex hull of the set, so we saw this in the last class. Now, in today's class we are going to look at some of the properties associated with this convex sets which could be used in deriving the optimality conditions of a non-linear or a linear programming problem. So, let us first look at a theorem, so we will call it as theorem C S I, so suppose S is a nonempty and closed convex set in R n.

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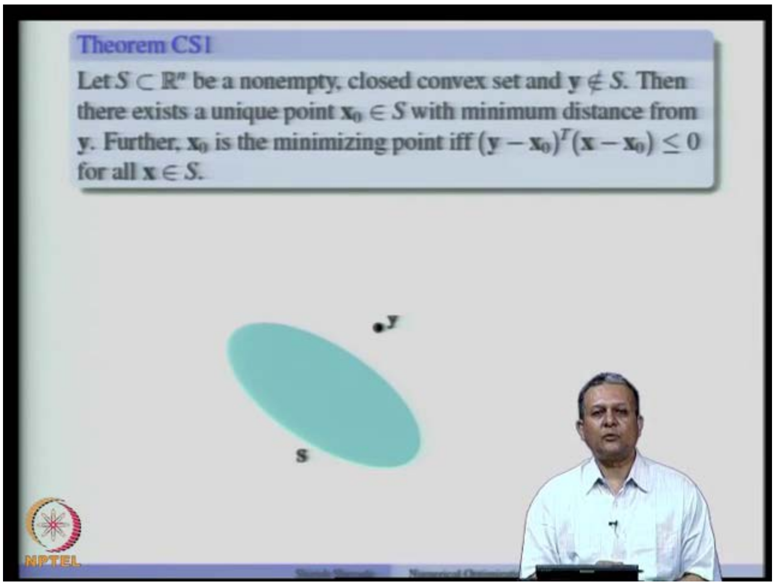

Let us consider a vector y or a point y which is not in the set S, so remember that S is a non empty and closed convex set then the claim is that there exists a unique point x naught which has a x naught belonging to S, which has the minimum distance from y. Then further x 0 is the minimizing point if and only if the condition y minus x naught transpose x minus x naught less than equal to hold for all x in the set S, so we will see this first the interpretation of this result. So, suppose we have a set S which is a non empty closed convex set in R n and let us consider a point y which is not in the set S.

So, the theorem says that there exists a point x naught in the set S which has a minimum or the least distance from the set y and not only that there exists a unique point not only that there exists a point, but that point is also unique. So, you cannot find any other point in the set S which is at the same distance as the distance between y and x naught.

So, this is a very important point that needs to be noted and further if we take that point x 0 and take any point x in the set S. If you take the two vectors y minus x 0 and x minus x 0 the dot product of two vectors those two vectors is non negative which means that they make an obtuse angle with each other, so we are going to see the proof of this theorem now.

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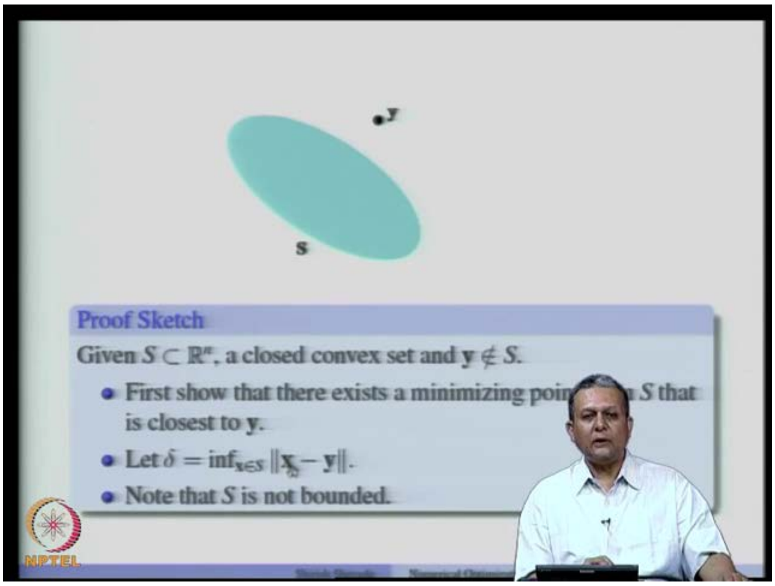

So, given the set S and the point which is a point y which is not in the set S we will give a proof sketch, it is very easy to write the proof, so we will give some important points that need to be considered while writing the proof. So, we are given a set S which is subset of R n and it is a closed convex set and y is not in S, now the first thing that we have to show is that there exists a minimizing point in x that is closest to y.

So, how do we show that there exists a minimizing point, so what we are interested in is that we are interested in finding out a point x in S such that the distance between x and y is the least. Remember, that I have written here in minimum because we do not know whether a point x exists in the set S so that is why we have written the minimum here.

So, let the minimum of norm of x minus y with respect to x belonging to S be denoted by delta so delta is going to be the this distance between the x minus y we still do not know such a x belongs to S exists or not. Now, let us look at this function norm x minus y, now norm x minus y as a function of x it is a continuous function. So, we are trying to minimize the continuous function, now look at the set S we have just said that the set S is closed convex set. Now, if you recall Weierstrass theorem my first theorem says that if you want to minimize a continuous function over a closed bounded set then the minimum and maximum exists in the set S.

Now, the set S which is given to us is closed, but it is not necessarily bounded and that is why we cannot use Weierstrass theorem directly to solve this problems although f is continuous the function norm x minus y is continuous the set x is not bounded, so we cannot directly use Weierstrass theorem. So, we have to modify this problem a little bit, so that we will try to minimize this function continuous function over a bounded set now to do that what we do is that let us assume that delta is the minimum.

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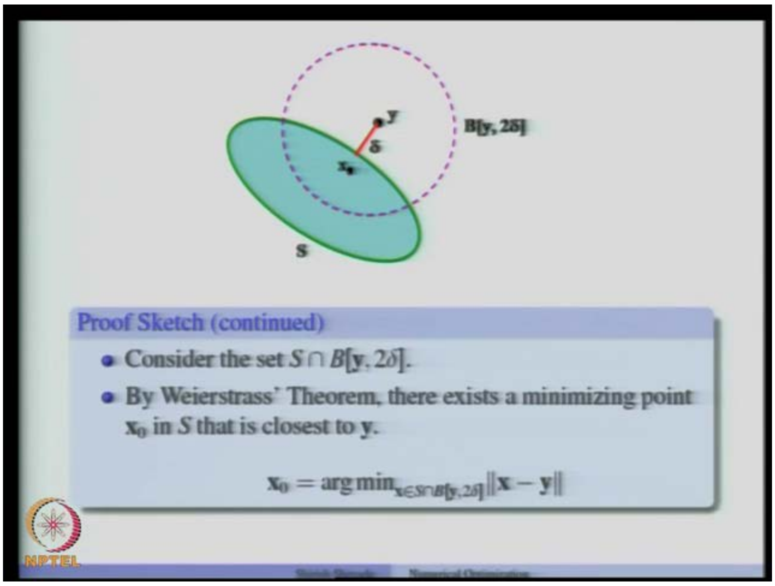

What we do is that around y taking or taking y as the centre we consider a ball of radius 2 delta centered at y remember that this is a closed ball, but I have not shown it in the figure to avoid the clutter in the figure. So, this closed ball basically is the boundary of the ball is shown and as well as the interior of the ball that is the closed ball.

Now, suppose if we consider the set S intersection the closed ball of radius 2 delta centered around y, now this ball is the closed ball S is the closed set. Now, this ball is a bounded set, now intersection of that ball with respect to S is the bounded set, so intersection of two close bounded sets will give us a close bounded set. So, it is this set that the we are talking about the intersection of S and then the closed ball. So, it is this part that we are talking about now we can use Weierstrass theorem because now we have a function norm x minus y which is the continuous in a continuous function in x and the consent set that we are talking about is closed and bounded.

So, by Weierstrass theorem there exist a minimizing point x 0 in S that is closest to y so this point x 0 is shown on here. So, this is closest to y and x 0 can be written as the R mean of norm of x minus y where x belongs to, now S intersection b y comma 2 delta, so that is the ball of radius 2 delta centered around y. So, such a x naught exists now by using Weierstrass theorem, so what we have shown so far is that there exists a minimizing point in the set S that is closest to y. Now, what is the guarantee that this point is a unique minimizing point?

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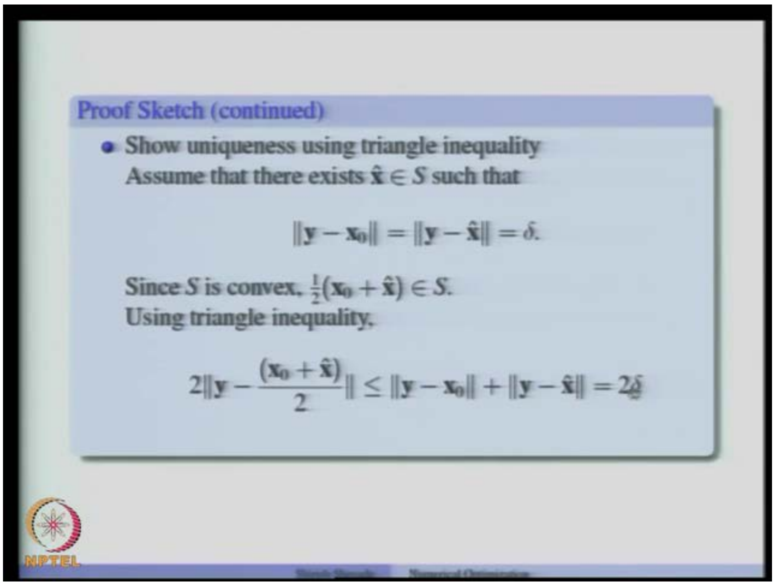

So, let us look at the how to show the uniqueness of this minimum point, so the idea is to use the triangle inequality. So, let us assume that there exists some x naught which belongs to S such that y minus the distance between y and x naught and the distance between y and x hat is same as delta. And then what we have to do is that we have to show that if such a thing happens then we may end up in a contradiction. So, the only thing that this is possible the only way this is possible is that x 0 coincides with x hat, so to do this we have to use a triangle inequality. Now, since S is the convex set that is given to us the convex combination of x 0 and x hat always lies in the set S.

So, we have lambda 1 x naught plus lambda two x hat always belongs to the set S when lambda 1 and lambda two are non negative and lambda 1 plus lambda 2 is 1 and in this case they are that is true so, we take this point and then we use the triangle inequality. So, if we use the triangle inequality what we get is the distance between y and x naught plus x hat by 2 is less than or equal to the distance between y minus y and x naught and the distance between y and x hat. Now, we know that the distance between y and x naught is same as the distance between y and x hat which is delta, so this is the right hand side becomes 2 delta.

So, what we have is that norm of y minus x naught plus x hat by 2 is less than or equal to delta now if norm of y minus x naught plus x hat by 2 is strictly less than delta then we get a quantization because we have proved that x naught is the minimizing point. So, if we find another point which is at a distance from y which is at a distance less than delta from y then we get a constant contradiction.

So, the equality here holds so which means that norm of y minus x naught plus x hat by two is equal to delta. So, which means that x naught and x hat should coincide, so if the strict inequality holds we get a contradiction. So, far we have shown that there exists a unique minimizing point in the set S which is closest to y.

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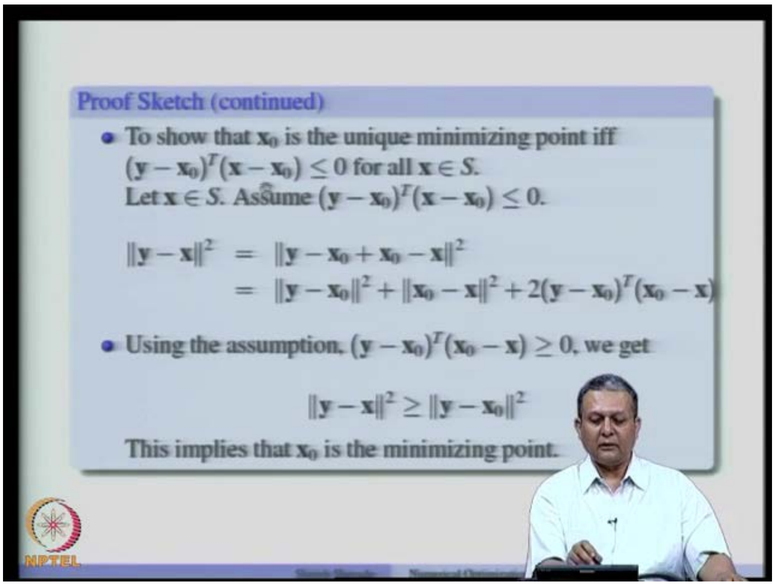

Now, we will prove the remaining part of the theorem now what we have to show is that if x naught 0 is the x naught 0 is the unique minimizing point if and only if y minus x naught transpose x minus x naught is less than equal to 0 for all x on s. So, let us remember that we have to show if and only if so we have to show it both ways. So, let us assume that some x belongs to S and the inequality y minus x naught transpose x minus x naught less than equal to 0 holds and then we have to show that x naught in such a case x naught is x naught is indeed a indeed the unique minimizing point.

Now, how do we show that, so let us take any x belongs to S and then try to find the distance of x from y and we have to show that norm of y minus x square is greater than or equal to norm of y minus x naught square for all x belongs to the set S. So, we have to introduce x naught in this equation, so we write a norm of y minus x square as y minus x naught plus x naught minus x square norm of y minus x naught plus x naught minus x square.

Now, we expand the right side, now if you expand the right side so the sums of the square of the first term plus the square of the second term the norm of the square of the norm of the second term and then two into the inner product of the two vectors. So, this is what is shown here now, so if you look at the second term x naught is the point which is in the set S and x is also in the set S, now, norm of x naught minus x square is always the non negative quantity. Now, if you look at the third quantity the inner product of y minus x naught and x naught minus x is always greater than or equal to 0, because of S remember that here we have used x minus x naught and here we have used x naught minus x.

So, the third term is always greater than or equal to 0, so these two terms are always greater than or equal to 0. So, which mean that norm of y minus x naught square is greater than or equal to y minus x naught norm of y minus x naught square for all x belong to the set S and which means that x 0 is the unique minimizing point for the is the unique minimizing point from y. Now, we prove the other way, now we assume that the x 0 is the unique minimizing point and show that this condition holds.

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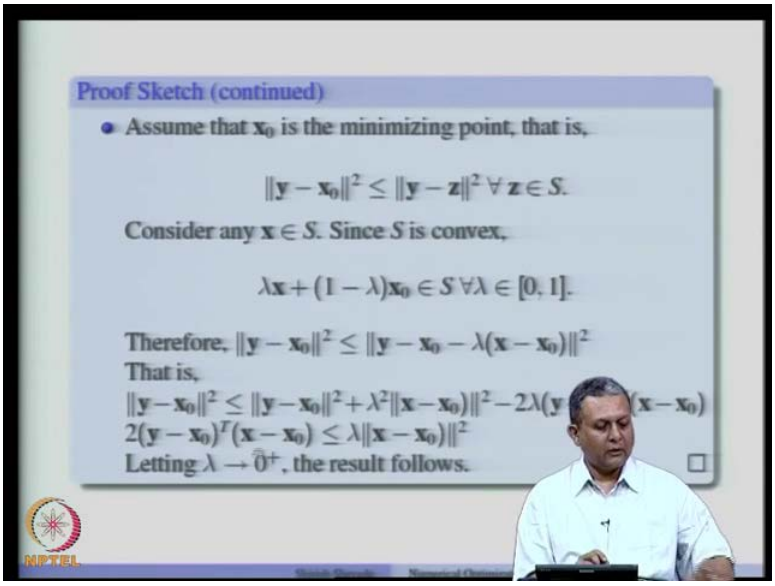

Now, x 0 is the unique minimizing point from y means that the norm of y minus x naught square is less that equal to y minus z square for all z belongs to S because x 0 is the closest point in the set S from y now. Now, we use the convexity of the set S, now since set S is convex if we take any x in the set x, I can say that lambda x plus 1 minus lambda x naught always belongs to the set S for all lambda in the closed interval 0 to 1. So, we can always say that for any x in the set S x 0 always in the set S, so the line segment joining those x and x naught always lies in the set. Now, we use this point and substitute it here, so what do we get so we get that norm of y minus x naught square is less than or equal to y minus x naught minus lambda into x minus x naught square.

So, we have just substituted this quantity here and rearranged the terms, so that we could write it as y minus x naught minus lambda into x minus x naught. Now, let us try to expand this, now if you expand this the right side what do we get so the what we get is that norm of y minus x naught square plus lambda square norm of x minus x naught square minus 2 lambda y minus x naught transpose x minus.

Now, these two quantities are the same, so they get cancelled and suppose if you bring in the third quantity on the left side and divide by lambda assuming that lambda is non zero then what we get is 2 into y minus x naught transpose x minus x naught is less than or equal to lambda norm of x minus x naught square. Now, if we take limits so letting lambda tending to 0 what we get is that y minus x naught transpose x minus x naught is less than or equal to 0. So, we get the result that if x 0 is the minimizing point then y minus x naught transpose x minus x naught has to be less than or equal to.

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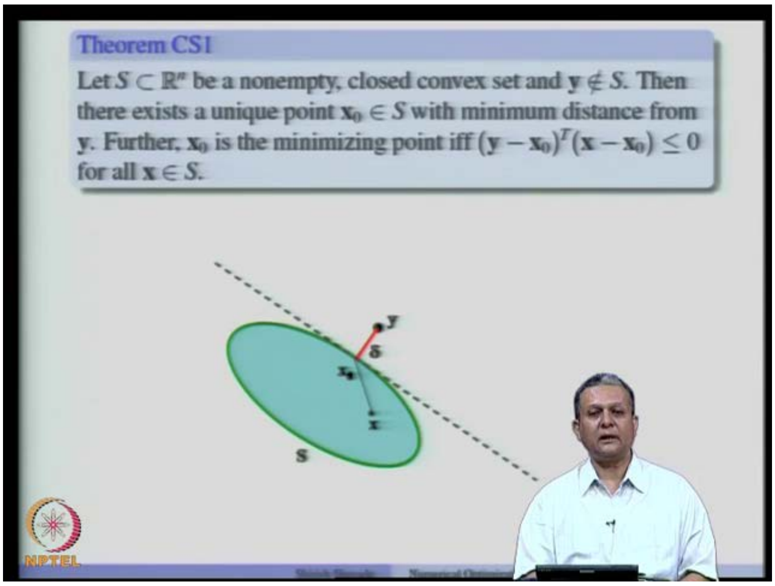

Now, geometrically what we have shown here is that given a set S and a point x naught and a point y which is not in the set S then the first thing we showed is that there exists a point x 0 in the set S, which is at a least distance or minimum distance from y and that point is the point x 0. Now, we also showed that such a point x 0 is always unique then we showed that if x 0 is the minimizing point then y minus x naught transpose x minus x naught less than or equal to 0. If y minus x naught transpose x minus x naught is less than or equal to 0 for all x belongs to S then x 0 is the minimizing point and it is easy to see it here.

Now, if we consider the vector of y minus x naught and then take any x in the set S and take the vector x minus x 0, we will see that the vector y minus x 0 always makes an obtuse angle with the vector x minus x 0. So, this is the interpretation of this result, now if we take a hyper plane which is perpendicular whose normal is the vector y minus x naught, so that hyper plane at x naught is shown here.

Now, you will see that this point the point y lies in one-half space of that hyper plane and then the entire set lies in another half space of that hyper plane. So, in some sense the hyper plane, we have shown here is trying to separate the set S and the point y which is not in the set S.

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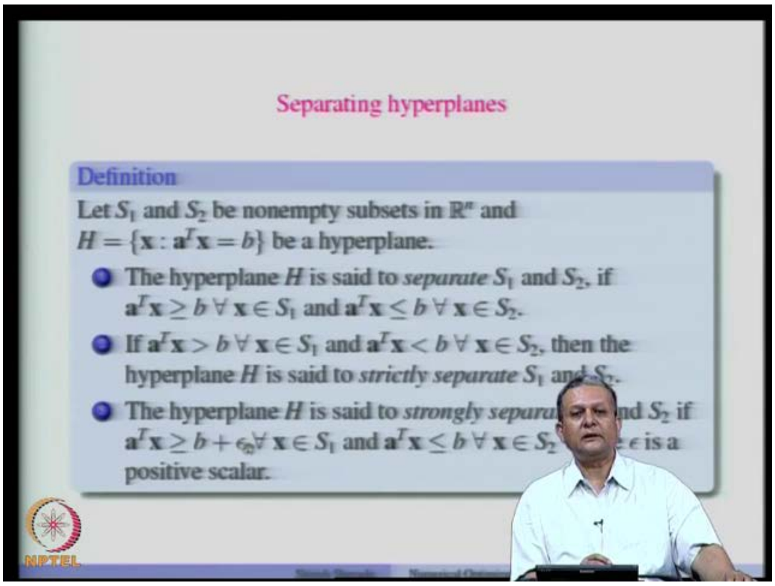

So, we will see more about this separating hyper plane. Now, so let us consider two non empty subsets in R n let us call them S 1 and S 2 and let h be a hyper plane is the set which is the set of points x such that a transpose x equal to b we saw in the last class that a is normal to the hyper-plane. Now, the hyper plane H is said to separate S 1 and S 2, if a transpose x is greater than or equal to b for all x in S 1 and a transpose x less than equal to b for all x in S 2. So, very soon we will give the geometrical interpretation of these results. Now, if the inequality in the definition is strict then we say that the hyper plane h is said to strictly separate the set S 1 and S 2.

So, notice the difference between the two definition that here the inequality is nor strict. So, a transpose x is greater or equal to b for all x in S 1 and a transpose x is less than or equal to b for all x in S 2 while here the inequalities is strict. Now, there is another notion which is called the strong separation, so we said that the we say that then hyper plane H is said to strongly separate S 1S 2. If a transpose x is greater than or equal to b plus epsilon for all x in S 1 and a transpose x less than or equal to b for all x in S 2, where epsilon is the positive scalar. So, we will see the meaning of these definitions.

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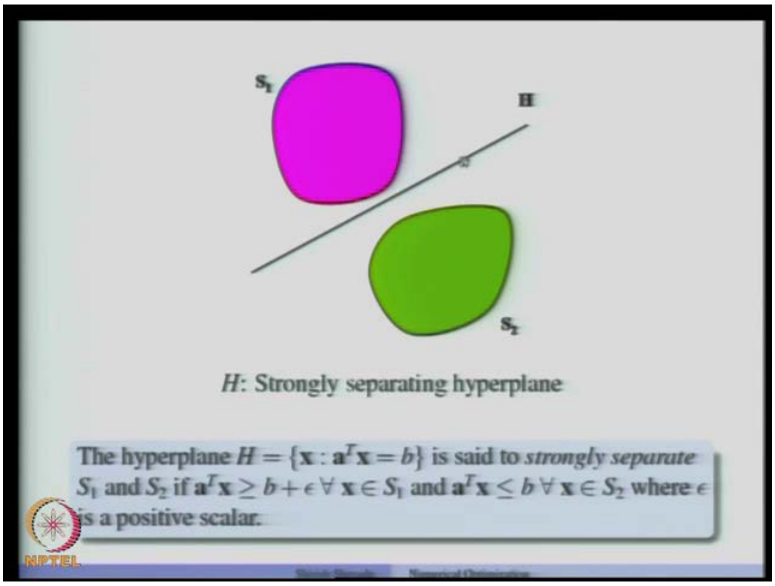

So, let us consider the two sets S 1 and S 2, this S 1 and S 2 they are touching each other, so there exists a point which is common to both the sets S 1 and S 2 now. Now, if we draw hyper plane H, which is like this passing through that point which is the intersection of S 1 and S 2 you will see that in one-half space of this hyper plane the set S 1 lies of course. I am talking about the closed half space and in the other closed half space the set S 2 lies, so such a hyper plane is called the separating hyper plane.

So, there exists one point which is the intersection of these two sets S1 and S 2 through which this hyper plane passes. Now, let us modify the sets S 1 and S 2 as open, so that is why the boundaries a boundary of these two set are shown by dotted lines, now they do not have any intersection, so if I take the same hyper plane. Now, we can say that it strictly separates S 1and S 2, so that means that if I take any x from S 1 we can say that a transpose x is strictly greater than b for all x in S 1and. Similarly, a transpose x is strictly less than b for all x in S 2, so such a hyper plane H, which is set of x that a transpose x is equal to b is said to strictly separate S 1and S 2.

So, far we have studied what are separating hyper plane that are strictly separating hyper planes, now what can we say about the strongly separating hyper planes. Now, here is an example of a strongly separating hyper plane, so you will see that we have sets S 1and S 2 which are away from each other they do not have anything in common even the boundaries do not touch each other then in such a case we say that h strongly separates S 1and S 2. So, which means that a transpose x is greater than or equal to b plus epsilon where epsilon is a positive scalar for all x in S 1and S a transpose x less than or equal to b for all x in S 2. So, you will see that this hyper plane h is the strongly separating hyper plane.

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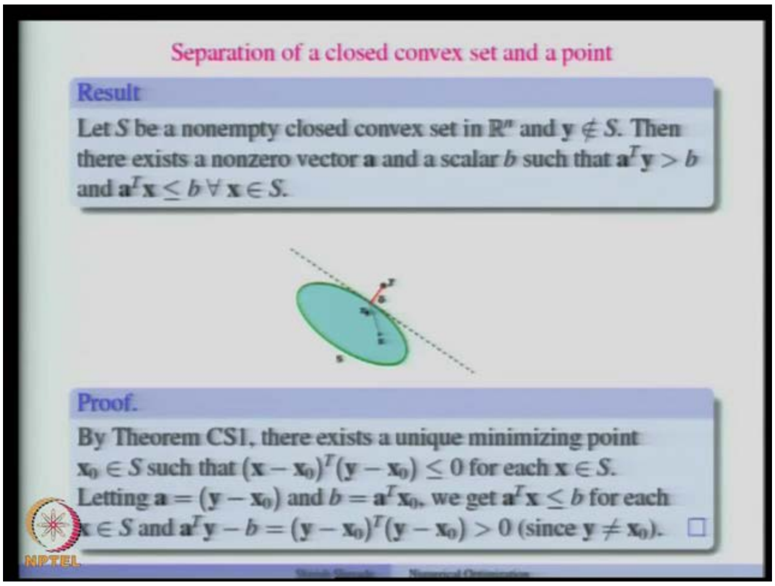

Now, with this notation we will now see the result which discusses about the separation of the closed convex set and a point. Now, suppose that this is a non empty closed convex set and let us take a point y which is not in the set S then there exists a non zero vector a and a scalar b such that a transpose y is greater than b and a transpose x less than or equal to b. So, which means that there exist a hyper plane, which separates a non empty closed convex set from a point y which is not in the set S, so we have seen this figure earlier.

So, we have set S which is non empty and closed and convex and we have point y which is not in the set S then the claim is that there exists non zero vector a and a scalar b such that a transpose y is greater than b. In other words there exists a hyper plane, which separates the two sets and we can easily see that this is this is one hyper plane which separates y and S.

So, if we take a vector a as y minus x naught and then choose b appropriately then what we get is the hyper plane passing through x naught whose normal is y minus x naught that clearly separates y and the set S. So, the proof for this is very easy by the theorem C S I there exists a unique minimizing point x naught such that x minus x naught transpose y minus x naught is less than or equal to 0. So, the idea is that we choose y minus x naught as a and a transpose x naught as b.

So, we have chosen y minus x naught as the vector a and a transpose x as b then we get that a transpose x is less than or equal to b for all x in the set S and a transpose is nothing but y minus x naught and since y is not equal to x naught since y is a point which is not in the set x. So, this quantity is always positive quantity, so what we get is a transpose y greater than b. Hence, the result follows, so if we are given a non-empty closed convex set in R n and y is as the point not in S then it is always possible to construct a hyper plane which separates the plane S and the point y.

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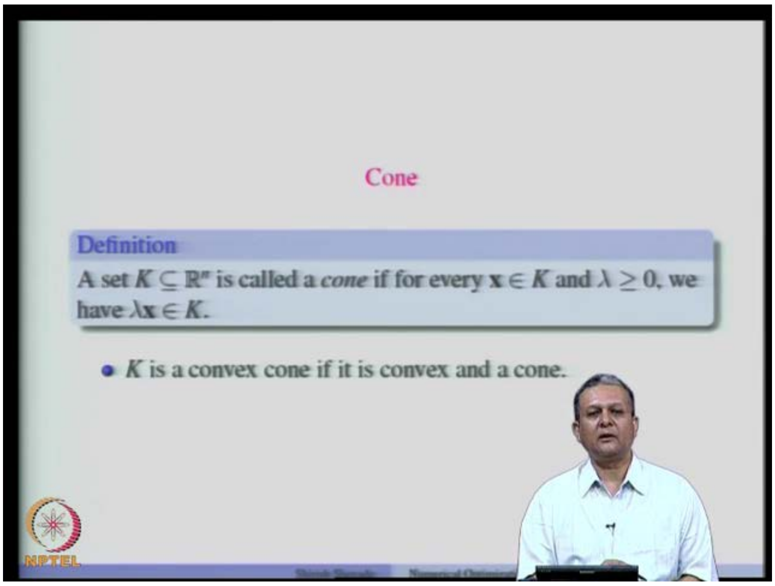

Now, let us look at the definition of cone a set k is a subset of R n is called a cone if you take any vector x in the set k and the scalar lambda, which is non negative then we have lambda x belongs to the set a and the one more definition about the convex cone. So, k is a convex cone if it both convex and if it is a cone, so first of all it must be cone, so which means that it should satisfy this definition.

So, what this definition essentially means is that you take any vector x in the cone then a nonnegative multiple of that vector should always belong to the cone. Now, remember that lambda is greater than equal to 0, so when you substitute lambda equal to 0 which means that 0 also should belong to cone the origin. So, in this course we will use the notation for the cone that the origin always belongs to the cone. Now, if we have a cone and if that is a convex set then we call that as the set as the convex cone.

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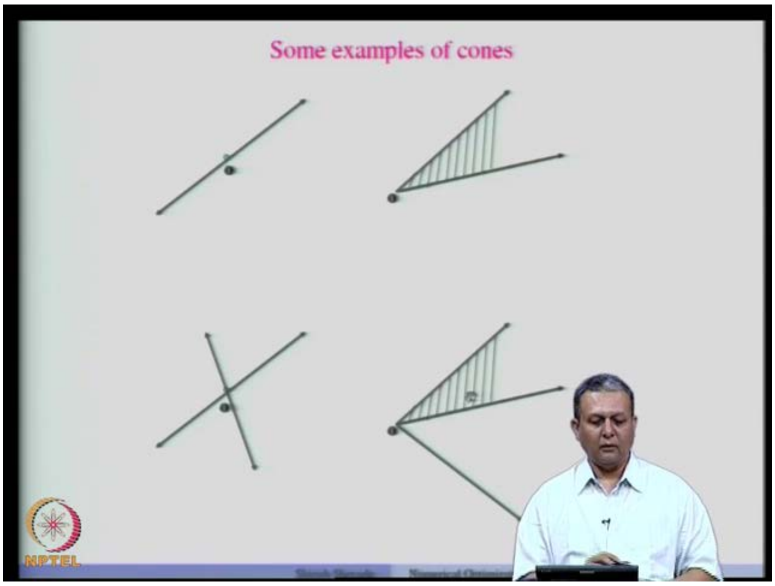

Now, we will see some examples of cones so in the left side you will see a line passing through the origin, now this is the cone because if you take any vector on this in this set and take a nonnegative multiple of that vector that always belongs to the set. So, if I take a vector here and take a non negative multiple of this. So, it would lie along that ray, starting from the origin.

And if I take a vector in this direction and take a nonnegative multiple of that it will lie along the ray, so this is a cone. Similarly, this is a cone, so if we take any point in this, on these arms of the cone in the shaded region and take that vector and take a non negative multiple of that vector that always lies in this set. So, this is an example these are some examples of cones, now here are some more examples.

So, in the first example what we have done is that we have added the one another line passing through the origin, now this also remains the cone because by the definition of cone any non negative multiple of any vector in that set always lies in the set, so this a cone. Similarly, this is also a cone we have just added one extra ray to the previously seen cone. So, any vector in this direction if you take a non negative multiple of that that always lies in this set right, but the difference between examples shown earlier and the two examples, which I have shown here is that these cones are convex while these cones are not convex.

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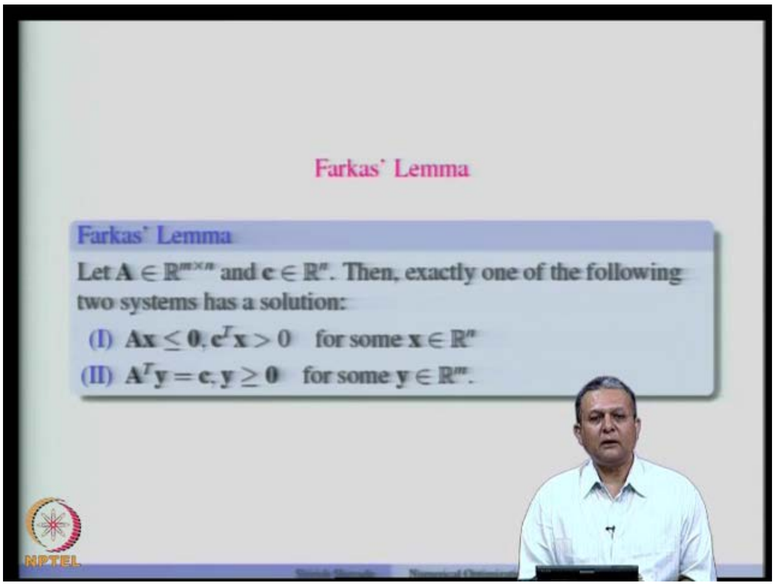

Now, we will look at a important result which is useful in deriving some of the optimality conditions that we are going to study later in the course and that result is called Farka's Lemma. So, both Farka's Lemma and its corollary they are very important in deriving optimality conditions for linear or non-linear programming problems, so let us first give the statement of the lemma.

So, suppose the a is m by n matrix of real number S and c is the vector in n dimensional space then Farka's Lemma states that exactly one of the systems has a solution and what are those two systems. So, one is the set of all x such that x less than or equal to x less than or equal to 0 c transpose x greater than 0 for some x in R n and the other one is a transpose y equal to c y greater than or equal to 0 for some y in R n. Now, before we prove this lemma, let us look at the geometrical interpretation of this.

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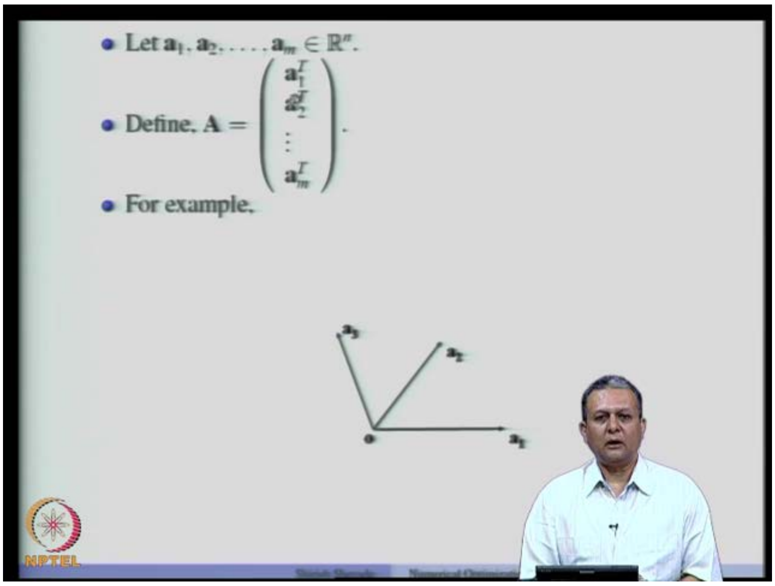

Let us consider m where n m n dimensional letters and let us denote then as a 1, a 2, a n. Now, define a matrix a where the vectors a the m vectors are stacked row wise, so the first row contains a 1 transpose, second row contains a 2 transpose and so on. Now, suppose for example, if we take three vectors a, 1 a 2 and a 3, so we can stack them in the matrix a, now if we look at Farka's Lemma.

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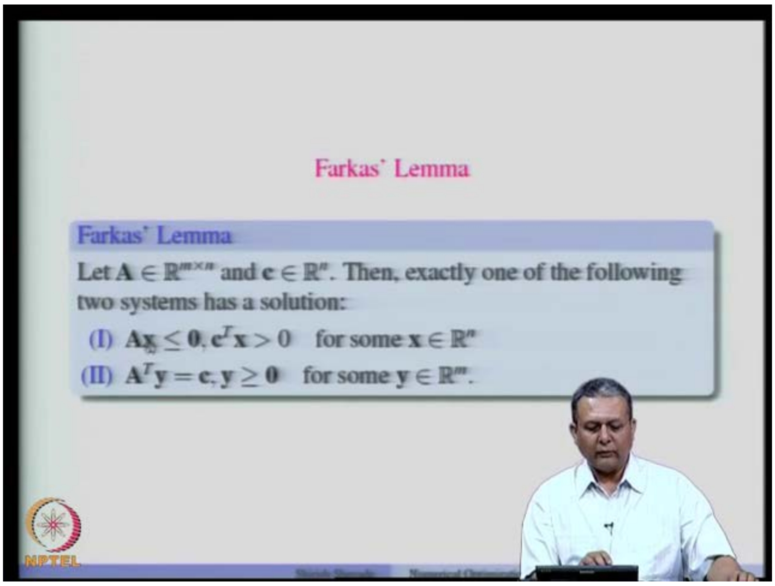

So, what we are interested in is the first system that a x less than or equal to 0 c transpose x greater than 0.

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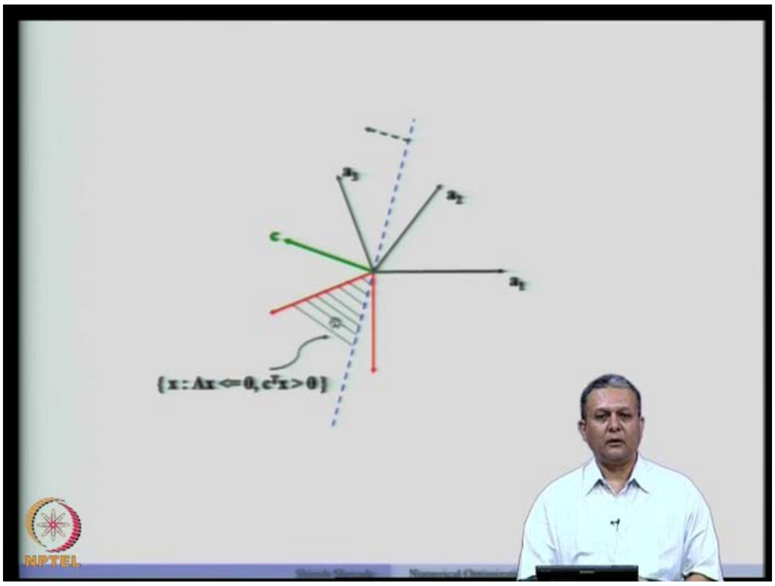

So, the first thing in the system one is that the set of all points x such that x less than or equal to 0. Now, since we have stacked the vector a 1, a 2, a 3 in the matrix row wise, so what we are interested in is the set of all x such that a 1 transpose x is less than or equal to 0, a 2 transpose x less than or equal to 0 and a 3 transpose x less than or equal to 0. So, a 1 transpose x less than or equal to 0 is the set of all vectors x which do not make an acute angle with the vector a 1. So, if we draw a in two dimensional space if we draw perpendicular vector to a 1, so one is this and the other one is this.

So, this is these are the perpendicular vectors and then collect all the vector S collect all the vectors which make an obtuse angle with a 1 and right or obtuse angle with a 1, so that becomes a half space. Now, we do the similar exercise for a 2 and a 3 and take the intersection of the half spaces, so then the intersection of those half spaces if we take finally, what we get is something like this.

So, this is the set of shaded region including then points on the arms of this cone and they form the set x such that x less than or equal to 0. So, all the points or all the vectors in this shaded region they do not make an acute angle with any of the vectors a 1 a 2 and a 3, so this was the first half of the system one. Now, the other half says that c transpose x greater than 0. Now, let us assume that the vector c is like this, so c is pointing like this, so c transpose x equal to 0 is the hyper plane, which is shown here the dotted line and the these part of the half space shows c transpose x greater than 0.

So, if we intersect the previously seen cone which was contained in the two arms of the red colored arms and intersect that with the set where c transpose x is greater than 0. So, remember that now we are taking about the open half space not the closed half space this is the open half space formed by the hyper plane c transpose x equal to 0. So, if you intersect them, if uses this intersection which is shown here.

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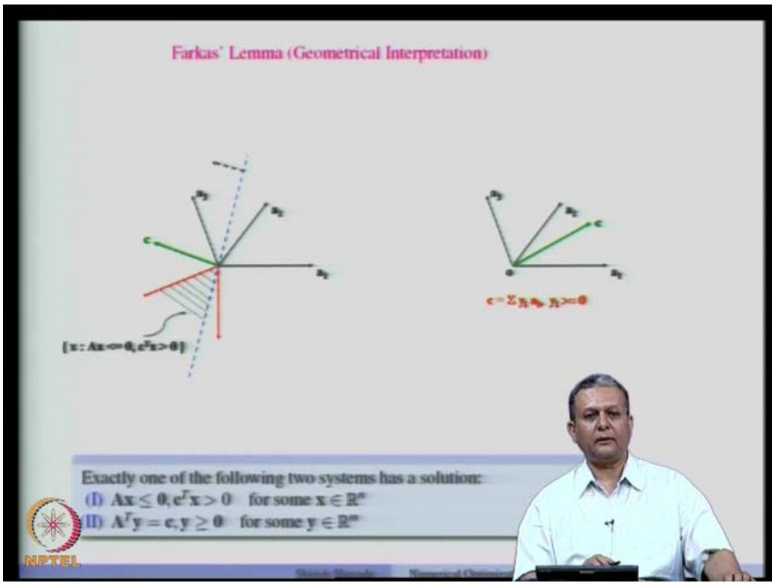

Now, what Farka's Lemma says is that either the system x less than or equal to 0 and c transpose x less than or equal to 0 for some x belongs to R n for some x in R n either that has a solution or if that does not have a solution then we can write c as a non negative linear combination of a. So, c can be written as sigma y where y is greater than or equal to 0, so either a x less than or equal to 0 and c transpose x greater than 0 for some x in R n or a transpose y equal to c and y greater than or equal to 0 for some y in R n. So, that means that either you find x which satisfies this or if you cannot do that then there exists some y such that we can write a transpose y equal to c and y equal to 0.

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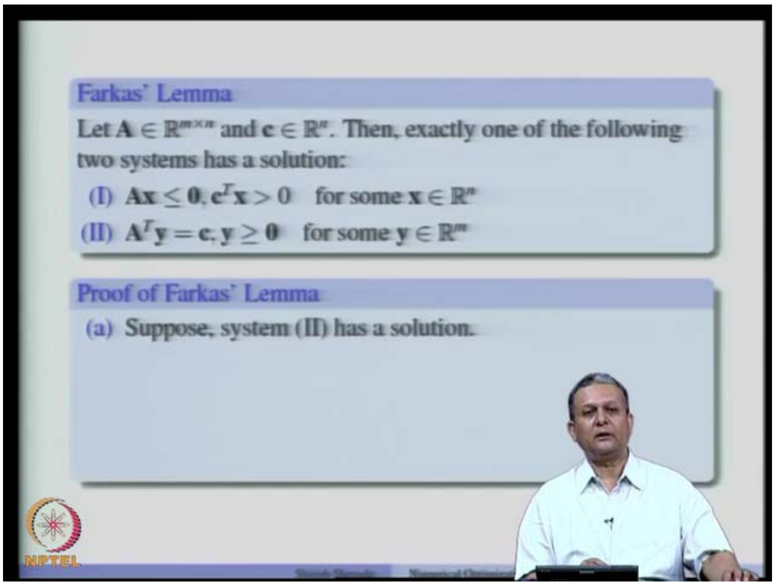

So, this is a very important lemma and we are now going to see the proof of this lemma. Now, what we have to show is that either of the two systems given here have a solution and not both. So, suppose is system two has a solution then we have to show that system one does not have a solution and suppose system two does not have a solution then we have to show that system one has a solution.

So, let us assume that system two has a solution, so system has a solution means that exists some y in R n such that A b transpose y equal to c and y greater than or equal to 0. So, there exists some non negative y such that c can be written as a transpose y. Now, let us assume that there exists a or let us take some x such that a x is less than or equal to 0 and see what happens to c transpose x. So, let us take some c x in R n such that x less than or equal to 0 and we are interested in finding out what happens to the dot product of c and x.

So, if we take c transpose x, now c transpose x is nothing but y transpose ax and we know that a x is less than or equal to 0 and the y is greater than equal to 0. So, y transpose x is less than or equal to 0, so if we consider any x such that a x is less than or equal to 0 the definitely c transpose x has to be less than or equal to 0. So, which means that we cannot find x where a x is less than or equal to 0 and c transpose greater than 0, so which means that system one has no solution.

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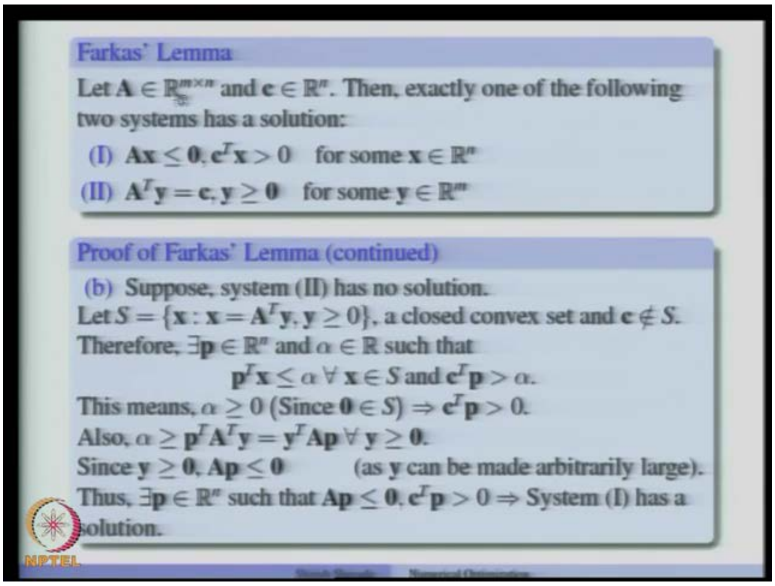

Now, let us look at the other part of the cone so far we studied if system one has a solution then system two system two has a solution then system one has no a solution. Now, let us consider the remaining part where system two has no solution, so that means that c cannot be written as non negative linear combination of a x or c cannot be written as a transpose y c where y is greater than or equal to 0. So, let us assume that, so let us collect all x such that the x can be written as non negative linear combination of the vectors of the rows of the matrix a or x can be written a transpose y where y is non negative.

Now, clearly S is a closed set and also a convex set intersection of closed convex sets is the closed convex set and then since system two does not have a solution c certainly does not belong to the set S. Now, what we have to do is that we have to now prove that there exists some vector x in n dimensional space such that x is less than or equal to 0 and c transpose x is greater than 0. Now, we have a closed convex set and a point c which is not in the set S, now can use the previously studied theorem to show that there exists, some vector p in R n and alpha in R such that p transpose x less than or equal to alpha for all p belongs to S and c transpose p greater than alpha.

That means there exists a hyper plane set of all x such that p transpose x is equal to alpha which separates the convex set and the point c which is not in the closed convex set. Now, if we look at the definition of S why we get equal to 0, so when y equal to 0 x becomes 0, so that means that the origin always belongs to the set S. Now, if you look at this inequality c transpose x less than or equal to alpha for all x belongs to S.

Now, if I since the origin belongs to S if I substitute x equal to 0 here what do we get is that alpha greater than or equal to 0. So, alpha is greater than or equal to 0 since the origin always belongs to the set and since alpha is greater than or equal to 0 and c transpose p greater than alpha, so c transpose p is always greater than 0. Now, what we do is that we will study the relationship between alpha and p transpose a transpose y, now alpha is always greater than or equal to p transpose a y because any x can be x in S can be written as a transpose y.

So, alpha is greater than or equal to p transpose x, so which means that alpha is greater than equal to p transpose a transpose y the reversing the order we can write this as y transpose a p this is true for all y greater than or equal to 0. Now, remember that y is greater than or equal to 0, so we can make y very, very large it is possible to do that.

Now, if that happens then a p is also greater than equal to 0 suppose a p is greater than 0 then y transpose, and y is very large then y transpose a p will become a very large quantity and we want alpha to be greater than or equal to y transpose and that may not be possible, if y is made very, very large. So, the only way that this inequality hold is that a p has to be less than or equal to 0 since y can be made arbitrarily large. So, we are able to get a vector p in R n such that a p is less than or equal to 0 and c transpose p is greater than 0, so which means that the system one has a solution.

So, we there exists a p in R n such that a p less than or equal to 0 and c transpose p greater than 0, so which implies that system one has a solution. So, either of these two things happen either if you are given a matrix A, so one can read the matrix a as the vector a 1 to a n and the vector c in R n then if you consider a system x less than or equal to 0 and c transpose x less than or equal to 0. If that does not have a solution then the vector c can be written as a non negative linear combination of the rows of the matrix A. So, this is a very important result which be useful and there is a corollary to that a result and that we will see now.

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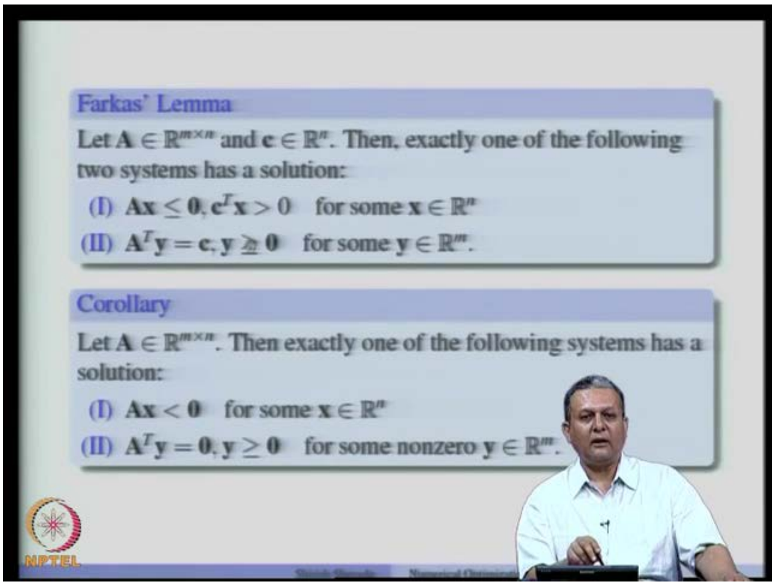

So, we just recall the Farka's Lemma where we had the matrix a and the vector c and the any either of these systems as the solution, now the corollary says that if you are given an m by n matrix then exactly one of the systems has a solution. So, either a x is less than 0 for some x in R n or a transpose y equal to 0 and y non negative for some non zero y in R n. So, remember that now y is non zero means that all the elements of y cannot be 0 at the same time, now this corollary does not use the vectors that we have used earlier. Now, if we look at the two systems the one given in the Farka's Lemma and the one given in the corollary we will see that there is some similarity between the two.

Now, here we had A x less than or equal to 0 while here we have a x less than 0 c transpose x greater than 0 is d naught there here. Similarly, A transpose y equal to 0 here we had a transpose y equal to c and y non negative. So, if we can put this system of it this system two systems of equation in the form like this then can use Farca's Lemma to prove the corollary.

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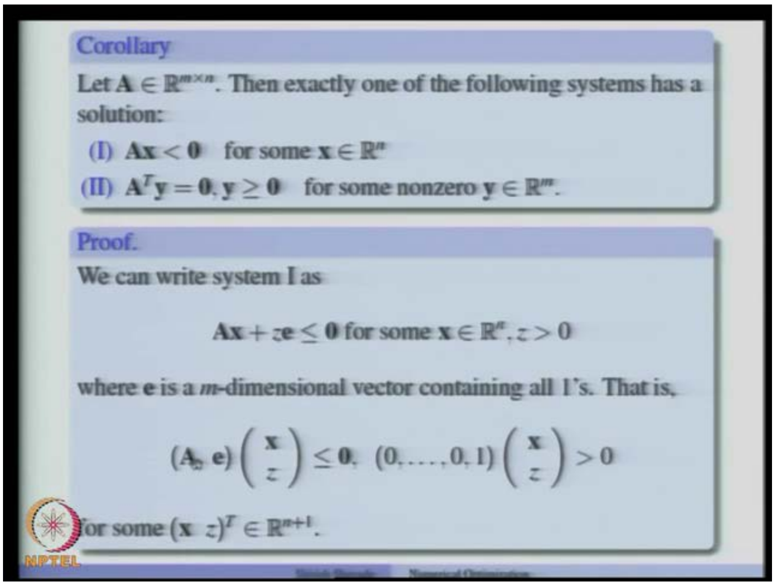

So, let us see how to put that in the form of Farka's Lemma, now this is the first system in the corollary a x less than 0 for some x in R n. Now, we can write this as, so we can add some variable we can take some constants z which is a positive constant and take a vector e is the vector m dimensional vector of all 1's. So, we can write a x less than 0 as a x plus x e less than or equal to 0 for some x in R n and z greater than 0. Now, this is now we can combine the vectors x and e to form a new vector and the we can write the we can combine x and z to form a new vector and write a new, write the same system in this form a e into x z less than or equal to 0.

Now, you will see that suppose we call these matrices a tilde and this x z as x tilde so we have a tilde x tilde less than or equal to 0 and let us call this vector the rho vector as c transpose c transpose x tilde greater than 0. Now, you will see that this is in the same form as the system one given in Farka's Lemma with the definition of x is now x appended by z and the definition of e is nothing but A appended by the vector e.

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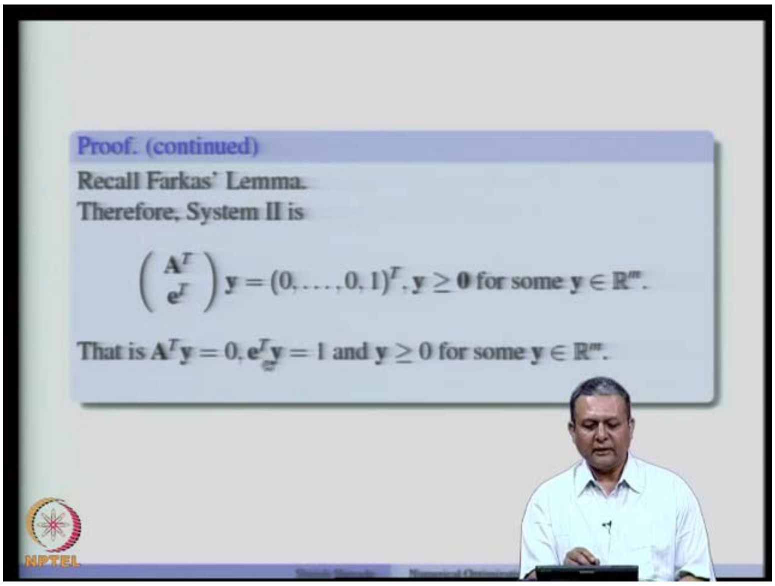

So, now system two also can using the same notation we can write the system two as this which in turn can be written as A transpose y is equal to y e transpose y equal to 1 and y greater than or equal to 0. So, the only thing that we have added in system is e transpose y equal to 1 because a transpose y equal to 0 and y greater than 0 was always there so the system two is of the form a tilde y equal to c y greater than or equal to 0.

Now, the only thing that we have is e transpose y equal to 1, so if using Farka's Lemma if suppose there exists the system two as the solution and we get y such that a transpose y equal to c and y greater than or equal to 0 we can always normalize that y to make sure that a transpose y equal to 1. Now, this is of the same form as what we wanted in Farka's Lemma, so the corollary can just be proved by easily rearranging the terms.

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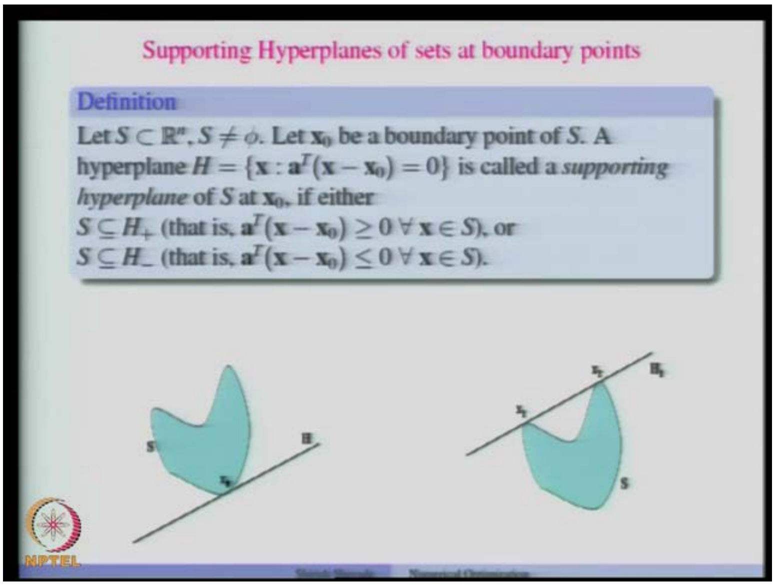

Now, let us look at the supporting hyper plane, so suppose the S is the non empty subset of R n and x 0 is the boundary point of S then the hyper plane edge where a is normal to the hyper plane and that passing through the point x naught is called the supporting hyper plane. So, that means that the set S entirely lies in the one of the closed half spaces of the hyper plane H, so either lies in H plus or H minus. Now, here are some examples so on the left side you will see that the set S is given and x 0 is the boundary point of S then there exists the hyper plane such that the entire set S lies on in one closed half space of this hyper plane.

Now, similarly, you can see that another supporting hyper plane for the same set s, now you will see that it is touching the set at the two points. Now, if you take this point we cannot draw a hyper plane which supports the set S because any hyper plane passing through this will intersect the set S and the set S will not entirely lie in the one of the closed half spaces. So, for such points it is very difficult to construct the supporting hyper plane, but we have a result which says that if S is the convex set then there exist a supporting hyper plane at in every boundary point of the set S.

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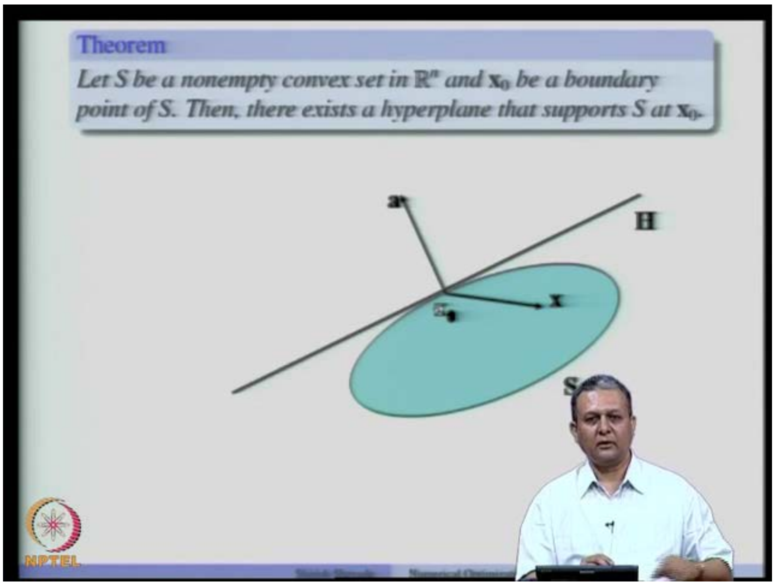

So, we have this theorem which says that S is a non empty convex set in R n and x 0 is the boundary point of S then there exists a hyper plane that supports S at x naught.

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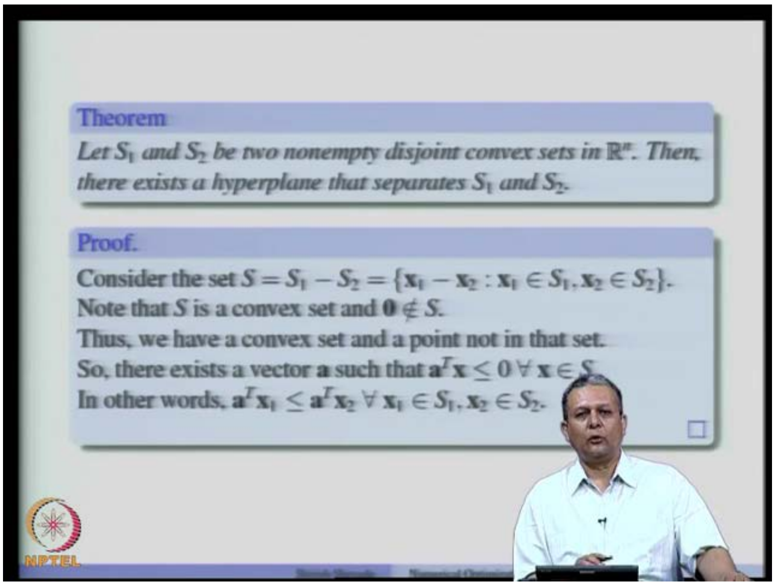

Now, we will see one more result that suppose S 1and S 2 are two non empty disjoint convex sets in R n then there exists a hyper plane that separates S 1and S 2, so we will see the proof of this. Now, we know that if we are given a convex set S and a point which is not in the convex set then there exist a hyper plane that separates the convex set from that point. Now, here we are given two non empty disjoint convex sets, now can we combine them to form a convex set and use that to prove this. So, let us how to do this, so let us consider a set S which is x 1 minus x 2, so which is the set of all x 1 minus x two such that x 1 belongs to S 2 and x two belongs to S 2.

Now, this set is the convex set one can easily prove that now further the origin does not belong to the set because we have disjoint convex sets, so they do not have any intersection the origin does not belong to the set S. Now, we have a convex set S and the point which is not in the convex set point origin which is not in that convex set, so there exists a by the earlier there exists a hyper plane that separates S and the origin. So, there exists a vector a such that a transpose x is less than or equal to 0 for all x belongs to S.

Now, if we replace x by x 1 minus x 2 what we get is that a transpose x 1 is less than or equal to a transpose x 2 for all x 1 in S 1and x two in S 2. So, that means that there exists a hyper plane that separates the two sets S 1and S 2, now under what conditions that each hyper plane will strictly separate S 1and S 2. So, I leave it as an exercise to you to find out the conditions under which the hyper for a given two non empty disjoint convex sets there exists a hyper plane that strictly separates x 1 and S 2.

So, this completes our discussion on the convex sets and as we will see in the next class that this convex sets will be used in defining what are called convex functions. Convex functions and minimization of convex function with respect to a convex set is called a convex programming problem, and that is very important in the optimization literature. So, we will study those things in the next class.

Thank you.