Numerical Optimization

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Lecture - 38
Lagrange Methods, Active Set Method

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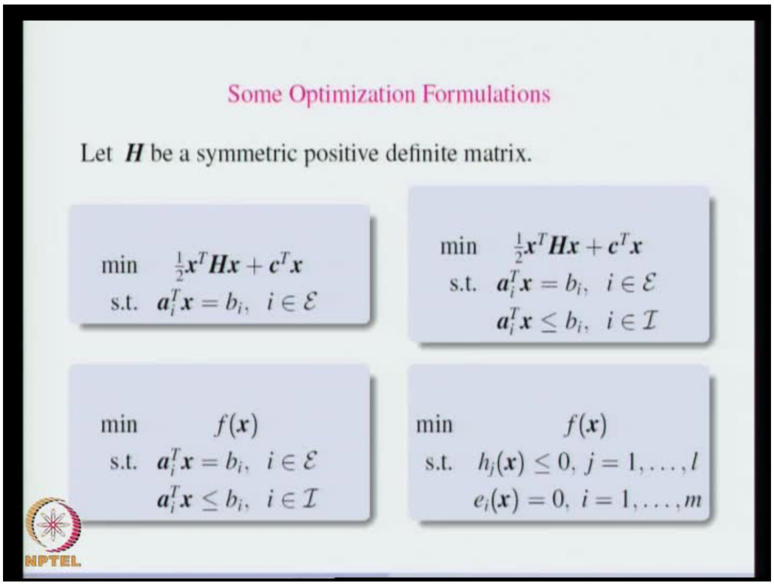

We have already, studied the theory of constraint optimization and in particular we saw that the Karush Kuhn Tucker or KKT conditions are necessary for a local minimum of a constraints optimizations problem, and the second order KKT conditions or sufficient under certain conditions. So, we make use of some of these ideas to device algorithms for solving constraint optimization problem. Now, will be looking at some formulations and in most of our discussion those formulations will be used to demonstrate the usefulness of some of the algorithms. Let us assume that H is a symmetric positive definite matrix, so one the first formulations that we are going study is a quadratic program with linear equality constraints.

So, the program is of form half of the x transpose x plus c transpose x to be minimized subject to the constraint that a i transpose equal to b i where i belongs to the set of equality constraints e. So, as I said that H is a symmetric positive definite matrix. So, in many occasions one comes across these quadratic programs and there are simple ways to solve these problems. Now, I must mention here that there is exist plenty of techniques to solve different constrained optimization problems and it is very difficult to cover each and every techniques in this course. So, we will select some of the techniques which are simple and easy to implement and demonstrate the usefulness and some of the problems.

Now, this is a simple quadratic program with linear equality constraint. Now, the ideas that we discussed here can be used to solve a quadratic program which has linear equality and inequality constraints. For the example, quadratic program with linear equality and inequality constrained could be of this form where, the script e n script i denote the equality and inequality constraint sets. So, the ideas we study here while studying the quadratic program with linear equality constraints will be useful in studying this problem formulation.

Now, we then move on to the problem formulation where the objective function is not necessarily a quadratic function. But general function f x but the constraints are of the same type the linear equality and inequality. You might recall that, when we studied unconstraint optimization we saw that Newton method can be used to solve unconstraint optimizations problems by approximating a given function by a quadratic function at current point. So, if we use similar ideas then the given function if it is twice continuously differentiable can be approximated by a quadratic function at a current point, and then one can use these ideas to solve the problem with respect to that approximation move on to the new point and then again repeat the procedure.

So, the ideas that study here could we useful in studying the optimization of general function subject to linear equality and inequality constraints and finally, we will we will solve a general constraint optimization problem or we will look at the some of the algorithms for the solving a general constraint optimization problem where objective is to minimize f x subject to some inequality constraints and some equality constraints and note that these constraints did not be linear as was the case in the earlier formulation. So, this is going to be the outline of our discussion for the rest of the course. If the function f is convex H j x is also convex and e i x are affine, then this program will be a convex program and there exists some efficient algorithms to solve convex programs. So, we will also see some of those methods.

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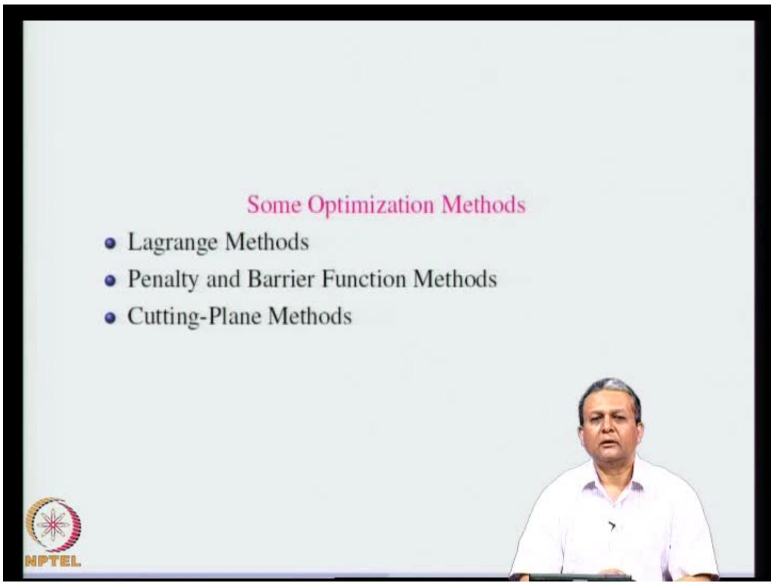

Now, there exist different optimization methods to solve this problem formulations and some of them are mentioned here. So, some methods are Lagrange the best methods the where they use lagrangian function to solve a given problem some functions some methods based are penalty or barrier functions and for convex programs their exist methods like cutting plane methods. Now, as I mentioned that their exist plenty of methods to solve constraints optimization problems and it is very difficult to cover every method. So, we will study some of the prominent methods that would be useful in solving constraints optimization problems.

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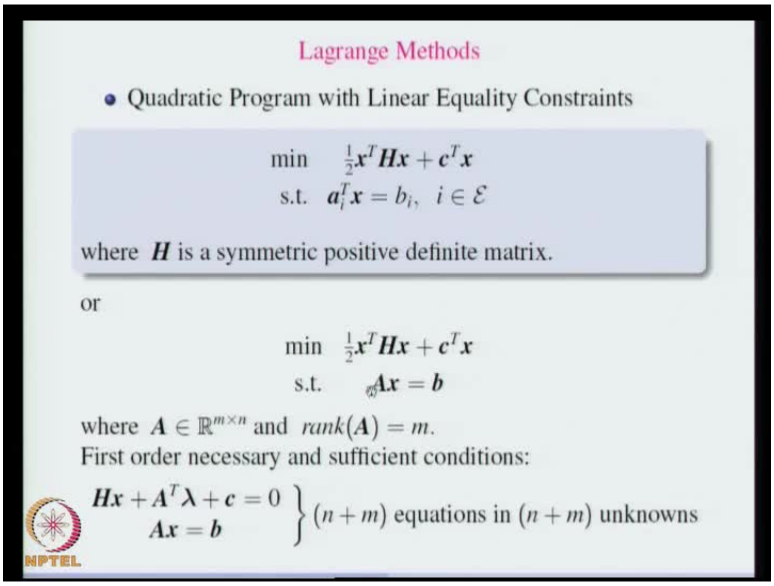

So, let us look at some of the methods which are based on the Lagrangian of the given problem. So, let as consider the program quadratic program with linear constraints so the objective is to minimize half x transpose H x plus c transpose x where h is a symmetric positive definite matrix and the constraints are linear equality constraints so a i transpose x equal to b i and e denotes the script e denotes the set of index set of equality constraints. Now, we can write this program in compact form where i think or by collecting the vectors a one transpose a two transpose up to a m transpose into the rows of matrix a. So, this matrix a will be a m by n matrix where m is the cardinality of the set e and the original program is just compactly written in this form and let us assume that the a the rank of the matrix a is m or m a is a full row rank of matrix.

Now, how do we use the lagrangian based ideas to solve this problem. Now, when we discussed KKT conditions we saw that for a convex program where status conditions are satisfied the first order KKT conditions are necessary and sufficient. Now, you may see that since the objective function is convex because H is a symmetric positive definite matrix the hessian matrix is positive semi definite positive definite. So, the objective function of convex the constraints are affine and we assume that latest condition holds that means that there is at least one point in the interior or the constraint set has non-empty interior so then the first order KKT conditions are necessary and sufficient. So, these conditions are obtained by writing by lagrangian of this problem, so the lagrangian will be the objective function plus lambda transpose b minus Ax.

Now, we intentionally use the multiple lambda here because in the course so far we have always used the lagrangian multiplier corresponding equality constraint the be mu but here we use lambda mainly because in when we discuss about quadratic programs with a linear equality and inequality constrained this notion of lambda will be useful so we continue using lambda for the equality constrained problem. So, if you write the lagrangian of the given problem and take the gradient of that with respect to x what we get h x plus a transpose lambda plus a equal to zero and it also should satisfy x equal to b. So, these are the first order necessary and sufficient conditions. Remember that lambda is a multiplier associated with the equality constraints and therefore, there are no sign restrictions on lambda.

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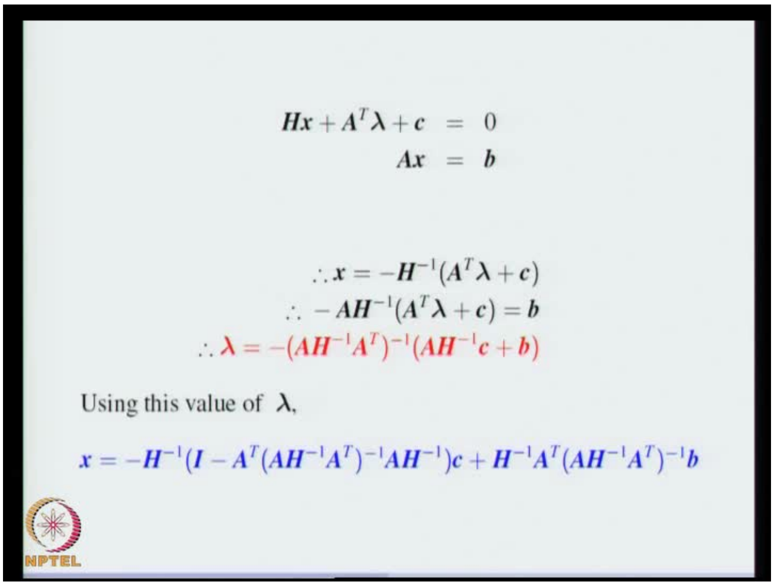

Now, these are n plus m equations in n plus m unknowns and so which can be solved easily. But in particular for this program since H is a symmetric positive definite matrix we can make use of that fact to write x in terms of lambda and then get an explicit value for x in terms of H H c a and b. So, let us see how to do that so we have this two equations and h is a invertible symmetric positive definite matrix so x can be returns has minus h inverse a transpose lambda plus c now this x should satisfy A x equal to b therefore, minus a h inverse a transpose lambda plus c equal to b now we can use this and the fact that is a full row rank matrix to find out the value of lambda. So, lambda will be the nothing but minus A H inverse a transpose inverse into A H inverse c plus b. So, we are able to get lambda in terms of A H c and b now, once we have that lambda we can plug in that value of lambda here to get x. So, using this value of lambda we get an expression for x which is shown here.

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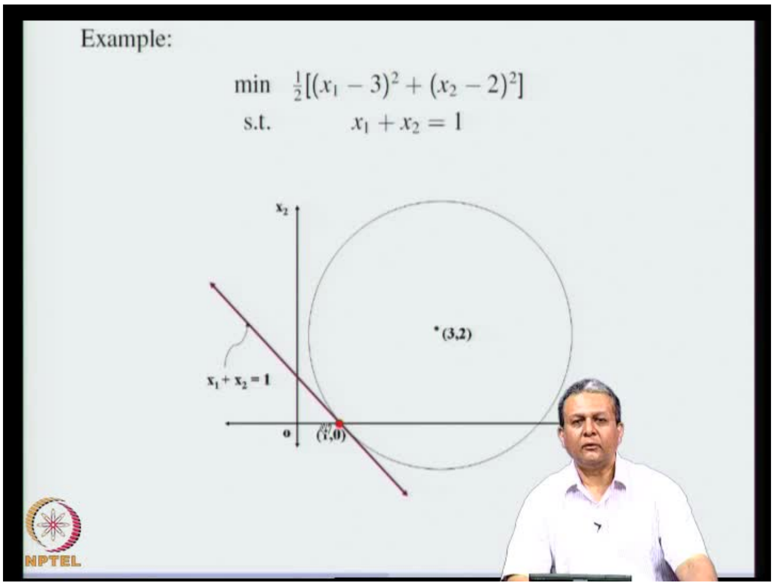

So, you will see that for a quadratic program with linear equality constraint of the form a x equal to b the optimum value is x equal to this. So, it very easy to find a solution of a quadratic program with linear equality constraint provided, the hessian of the objective functions is symmetric positive definite. So, that inversion of that matrix is possible let us see some examples which you listed this. So, here is a problem to minimize this objective function subject to the constraint that x 1 plus x 2 equal to 1. So, in other words we are interested in finding a circle of minimum radius which touches the line x 1plus x 2 equal to 1, and the circle is centered at point 3,2 that is a x coordinate x 1coordinate of the circle is 3 and x 2 coordinate of the circle circle center is two.

So, here is a scenario where is the point 3 2, and we are interested in finding out a circle centered at this point whose radius is as minimum as possible and which should touch the line x 1 plus x 2 equal to 1. So, you will see that at this point on the x axis x 1,axis this circle touches the line x 1 plus x 2 equal to 1. And this point is 1 0, which is the solution to this problem .If you reduce radius of the circle then we will not maintain the feasibility. Similarly, if we increase radius of the circle then new point there is the scope for the objective function to improve at a point where the circle intersects this line. So, this is will be the only point where the objective function will the list. Now, by writing down the hessian of this objective function and writing this constraint in the form x equal to b where a is a 1 by 2 matrix which contains both entries to be 1and b is a 1 by 1matrix with value one and if you use the formula that we earlier to find x then you can verify that this is indeed the global minimum of the given problem.

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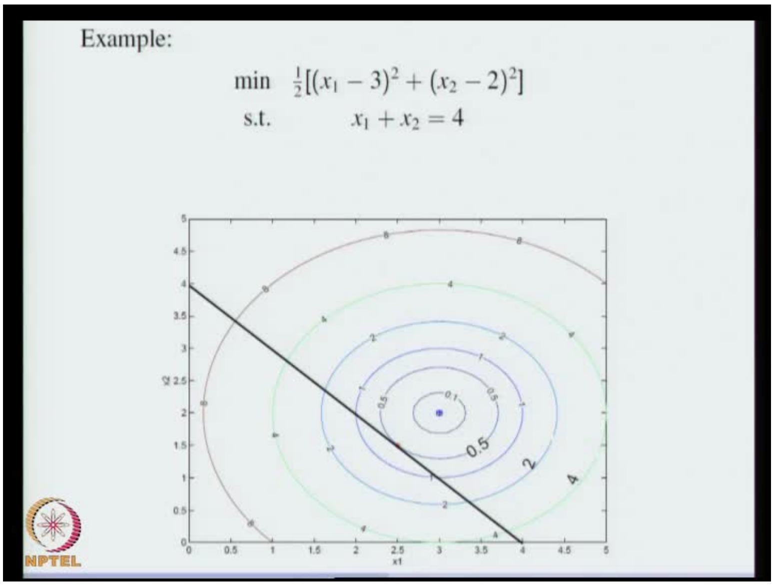

Now, let us see another example again this circle with the center goods coordinates are three and two we want to determine a circle of minimum radius which touches the line x 1plus x 3 equal to 4. Now, on the screen you will see the contours of circle contours of the circle which having center 3 2. This is the center of the circle, we will see contours with different function values. So, you will see that this is contour with objective function value point 1 then this is the contour with objective function value point 5 then 2 4 and so on. So, as we move away from the center the objective function value increases for example, this is the circle with objective function value eight now we are interested in finding out in that circle which touches the line in x 1 plus x 2 equal to 4 and the circle with minimum radius. So, this the line x 1 plus x 2 equal to 4 and we are interested in finding out the point or the x 1 an x 2 pair where, the circle touches the this line and as the least function value.

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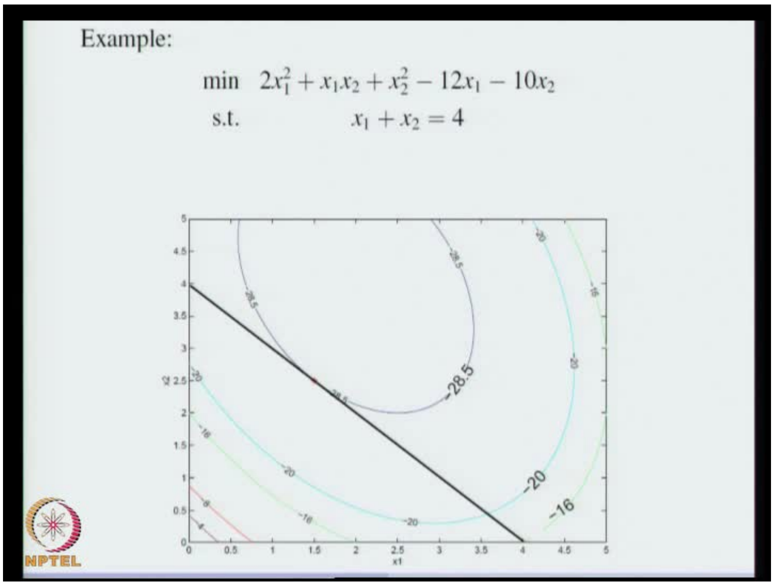

And, this is the point which that which is the point where a function value is minimum and it also satisfies this and this point is the point where x 1 coordinates is 2 point 5 and the y one x 2 coordinates 1 point 5. So, the solution to this problem is x 1 equal to 2 point 5 and x 2 equal to 1 point 5. Let us see another example, so here the objective function has elliptical contours the constraint set is still the same x 1 plus x 2 equal to 4. So, let us see how the contours look like. So, the contours of the objective function look like this. So, this is the contour with the objective function value minus 28 point 5 and then has one moves away the objective function value increases know among all contours that are shown here. This is the contour of the objective function which touches the given line x 1 plus x 2 equal to 4.

So, so the corresponding function value is minus 28.5 and the objective function value is achieved using this point. Where x 1 coordinate is 1 point 5 and x 2 coordinate is 2.5. So, this point is the global minimum of this objective function. Note that the objective function is convex. So, in every local minimum is the global minimum so we can say that this is a the global minimum of this objective function subject to the this constraint and a optimal objective function value is minus 28.5. So, these are the some examples related to the constraint optimization problem where, the objective function is a convex quadratic function with a positive definite hessian matrix and the constraints are linear equality constraints and we saw the simple formula which could be used to solve such problems.

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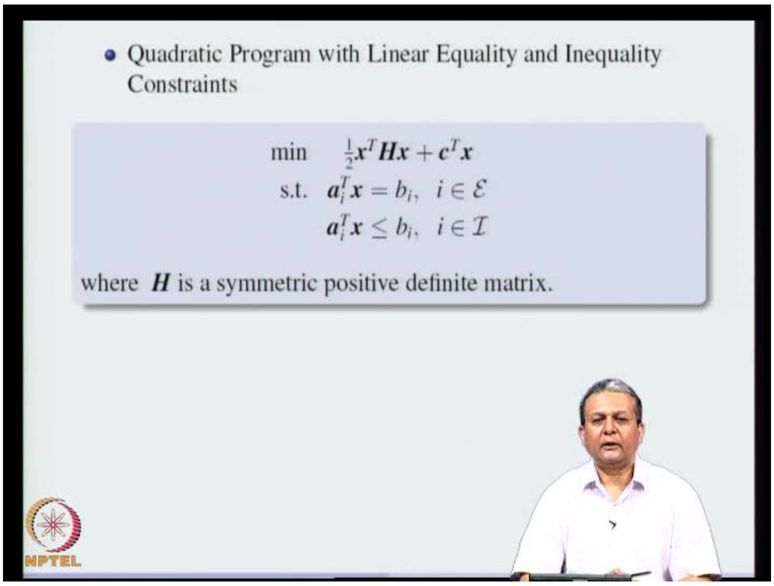

Now, we move on to the second problem where, we want to solve a quadratic program with linear equality and inequality constrained. So, in particular for a given H matrix which is a symmetric and positive definite matrix m is to minimize half x transpose H x plus c transpose x the x subject two a transpose x equal to b i, i belongs to the index set of equality constraints and a i transpose x less than or equal to b i where the index set comes from the script i.

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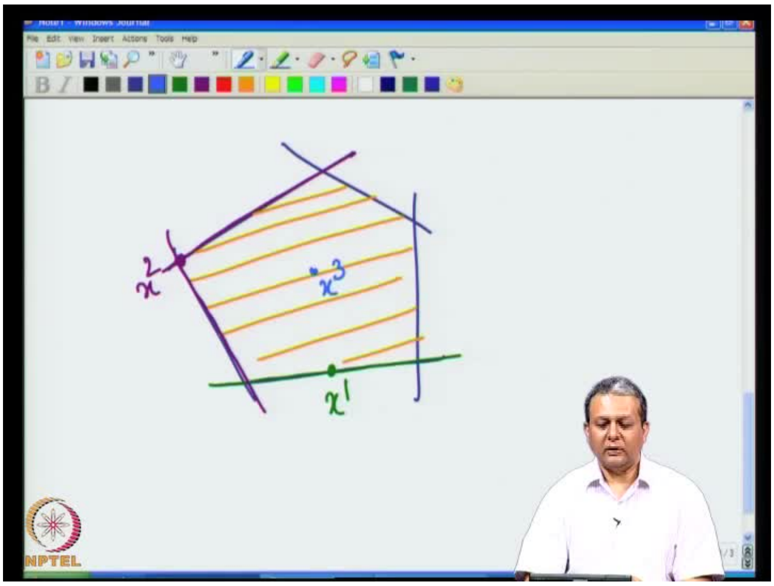

Now, we have set of inequality constraints added to problem. Now, how do we solve this problem that is our next question? Now, to solve this problem we require the concept of active sets and when we discussed about unconstraint optimization sorry when we discussed about constrained optimization theory, we saw the concept of active sets. So, suppose this set is a feasible set of the constraint set shown by the shaded region and if we consider a point x 1. Now, you will see that only one constraint is active at this point the other four constraints here are inactive inactive means the the corresponding constraint is satisfied with inequality and this is the only constraint which is satisfied with equality.

On the other hand suppose, we take this point you will see that this constraint and this constraint they are active. So, at this point these constraints which intersect at this point are active. So, at a point there could be multiple number of constraints which could be active. Now, if you take a point which is here, let as call this has x 3 this point was x 2. So, we will see that at this point none of the constraints is active. This point in the interior of the feasible region and none of the constraints are active, because all the constraints are satisfied at this point with strict inequality. So, this idea of these active sets is used in solving the quadratic programs with linear equality and inequality constraints and the resulting method is called active set method.

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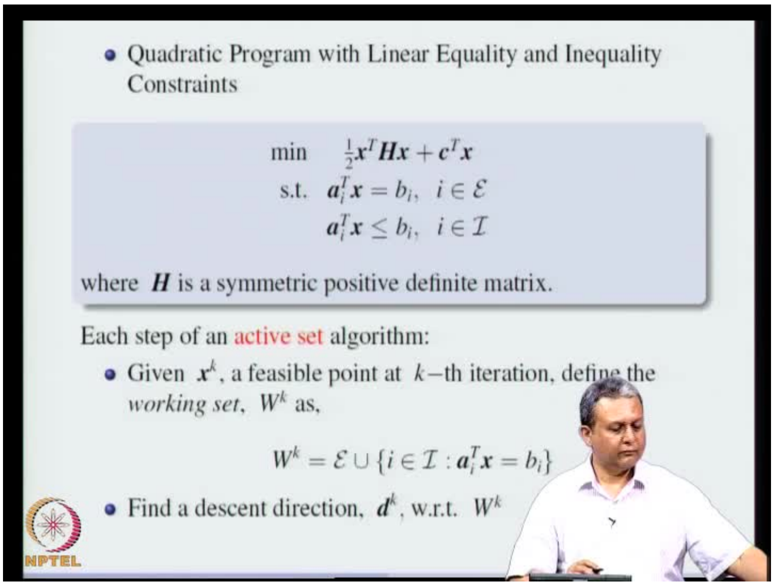

Now, let us look at how this active set algorithm works now given a feasible point what one needs to do is that, one needs to find out all the constraints which are active at a current point. Now, all the constraints which are equality constraint are definitely active at a current point. Now, in addition those equality constraints one has to check whether some of the inequality constraints are all also active and all such constraints put together form an active set or also called a working set. Given point x k since they are given x k of feasible point at the k th iteration of the algorithm then the working set associated with x k will be denoted by w k and it is defined as the set of all equality constraints and the set of all inequality constraints which are satisfied with equality.

So, we collect all the inequality constraints and find out which constraints are satisfied with equality as far as current point is constraint. So, a i transpose this should be x k that should be equal to b i. So, if the current point satisfies some of the inequality constraints with equality those are combined along with equality constraints to find out w k and that is going to be the working set at the iteration k. So, at every iteration the working set varies and the constraints which differ are the inequality constraint, the equality constraints are common for every feasible x k. So, one we find out the working set then the idea is to solve the quadratic program with respect to the working set and since the working set is the set of all equalities, then we can use some of the ideas that we studied, when we studied the optimization of the quadratic program with respect to linear equality constraints.

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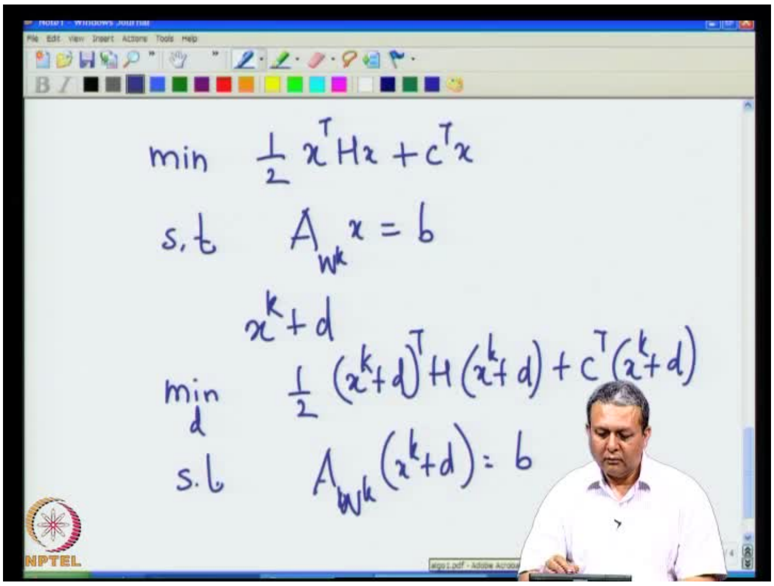

So, once you find the working set w k, the next step is to find a distinct direction p k with respect to w k now that direction can be found by solving another quadratic program. So, we have minimize half of x transpose h x plus c transpose x subject to A. So, A w k denotes the active constraints associated with the working set w k. Now, with the current point x k we are interested in finding the direction d, so what we have is what we want out is x k plus b such that any movement along the direction d at least a small movement along the direction d should feasible. So, we rewrite this problem as minimize half of x k plus d transpose H x k plus b pus c transpose x k plus d subject to A w k x k plus d equal to b.

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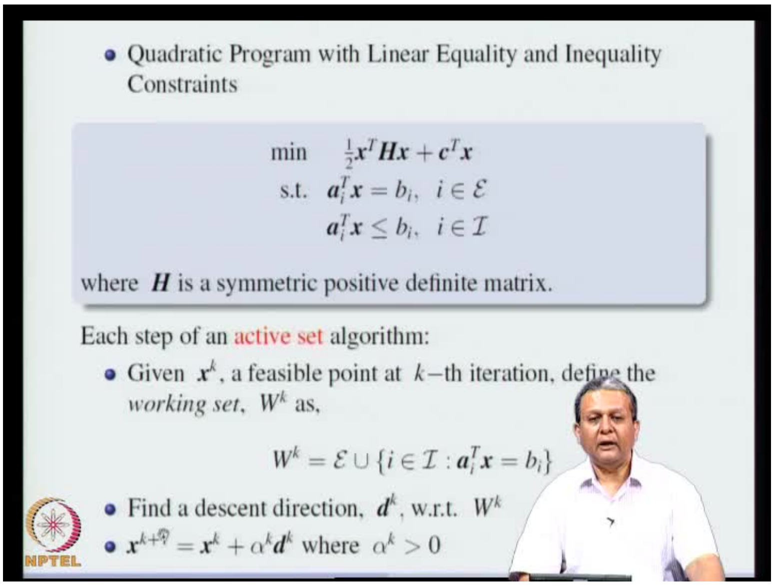

Now, this is the optimization problem with respect to t x k is constant so if we take out the constrained terms from away from this objective function what will be left with is half d transpose h d plus c transpose d plus h x k transpose d now H x k transpose d plus c transpose d is combined to write that function in terms of the gradient because the gradient of this function is H x plus c so the gradient act x x k is H x k plus c. So, the objective function becomes half d transpose h d plus g k transpose d now subject to a w k d equal to zero, because a w k x k x is always b and its two at x k. So, so once you find the distant direction d k you find out the steps size and such that be a x k plus 1 is also feasible.

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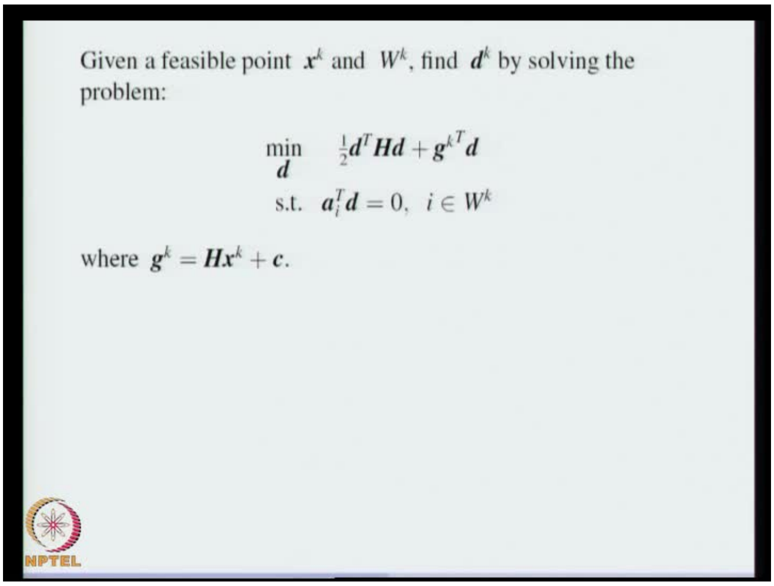

So, has I mentioned that given a feasible point x k and w k the associated program that we would like to solve with respect to the working set w is to find d k by solving this problem, where half of d transpose H d plus g k transpose d is minimized such that a r transpose equal to 0, where i belongs to the working set so the direction d is found by solving this problem and then the next step will be to get a step length and make a progress towards x k plus 1. So, that the objective function is minimized and this procedure will be repeated till one reaches an optimal point. Now, we will see more details about this in the next class.

Thank you.