Numerical Optimization
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Lecture - 35
Duality in Linear Programming

Hello, welcome back in the last class we discussed about simplex method to solve a linear program. And the first step of the simplex method was to get an initial basic feasible solution and many a times it is not available directly and therefore, one has to resolve to what is called a two-phase method. So, we started discussing about two-phase method in the last class.

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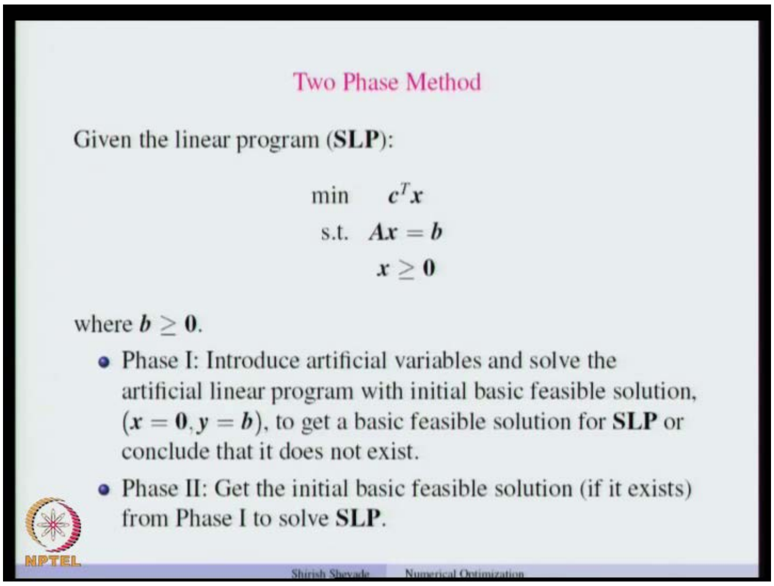

So, let us revisit the two-phase method that we discussed, so for a linear program minimize c transpose x subject to a x equal to b x greater than or equal to 0. If one cannot find sub-matrix of the matrix A, which is an identity matrix of size m, then one can introduce artificial variables and solve artificial linear program corresponding to the initial basic feasible solution x equal to 0 and y equal to b. So, the idea is to introduce a x plus i y equal to b and by introducing this artificial variables, we are introducing an artificial matrix, sub-matrix which is an identity matrix. And then we solve the artificial linear program to get a basic feasible solution for this program, if it exists, otherwise we, we conclude that the solution feasible solution to this program does not exist and the phase two we get the basic feasible solution obtained in phase one and start solving this problem using a simplex method.

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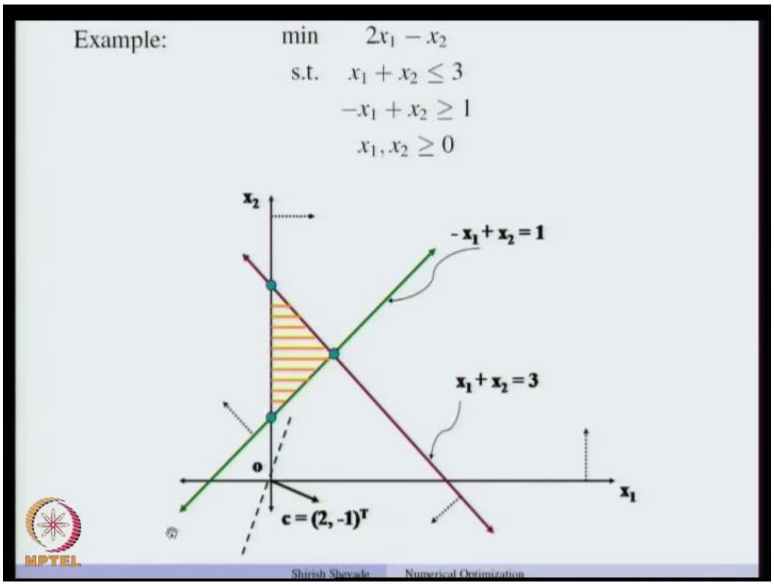

And we saw this example where we have this constraints x one plus x two less than or equal to x 1 plus x 2 less than or equal to 3 and minus x 1 plus x 2 greater than or equal to 1. And the matrix does not have a sub-matrix which is an identity matrix.

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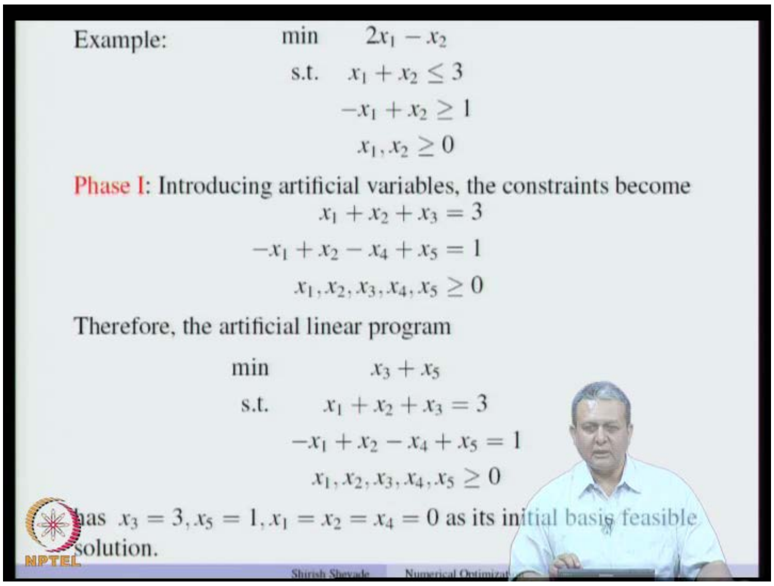

So, we introduced artificial variables. So, first we write the constraints as x 1 plus x 2 plus x 3 equal to 3 and minus x 1 plus x 2 minus x 4, x 4 is the surplus variable and we add an extra artificial variable we call it x 5. So, x 3 can also be treated as a artificial variable although it is a slack variable. And therefore, the artificial linear program becomes to minimize this artificial variable, some of the artificial variables subject to these constraints. And this artificial linear program has a initial basic feasible solution, which is x 3 equal to 3 and x 5 equal to 1 this is its initial basic feasible solution.

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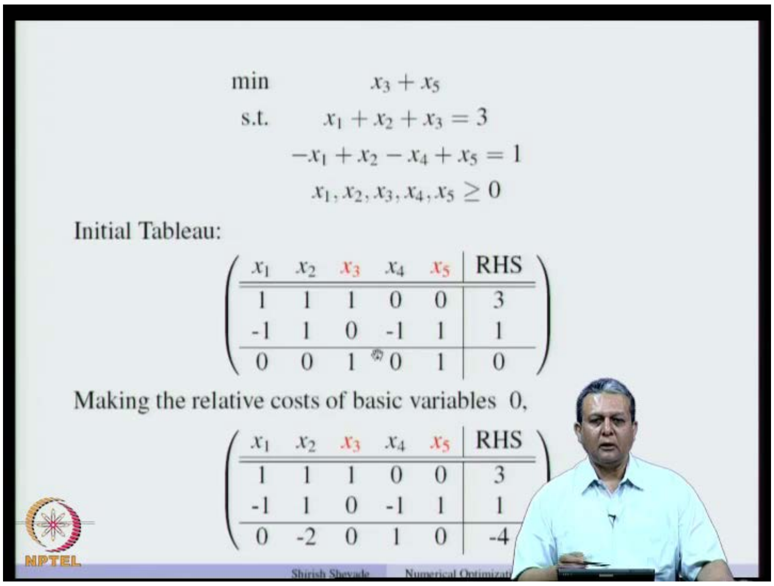

And by constructing the tableau and making sure that the relative cost of the basic variables are 0, we get this tableau.

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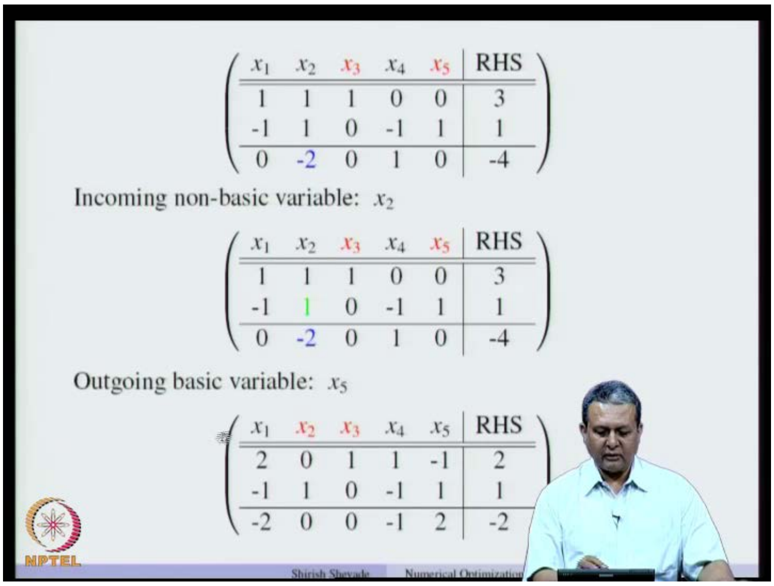

And then we saw that the incoming non-basic variable is x 2, because the corresponding relative costs in negative and based on the minimum ratio test the outgoing basic variable is the basic variable x 5. And by doing some matrix manipulations we can see that x 2, x 3 are the basic variables x 1, x 5, x 4 are the non-basic variables.

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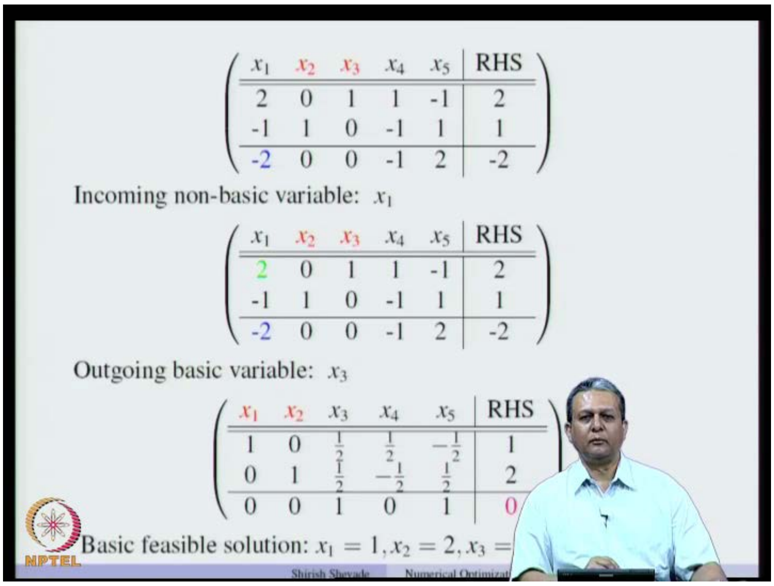

And we continue this procedure again by bringing in x 1 and if we use the ratio test then x 3 is the candidate which will leave out. And by again doing the matrix manipulations we get this final tableau for the artificial linear program where all the relative costs are non negative, most importantly at the current feasible point the cost is 0. So, which means the basic feasible solution does exist for the original system a x equal to b and that basic feasible solution is x 1 equal to 1 and x 2 equal to 2 and the rest of the variables are 0. So, we make use of this now to solve the original simplex program in phase two.

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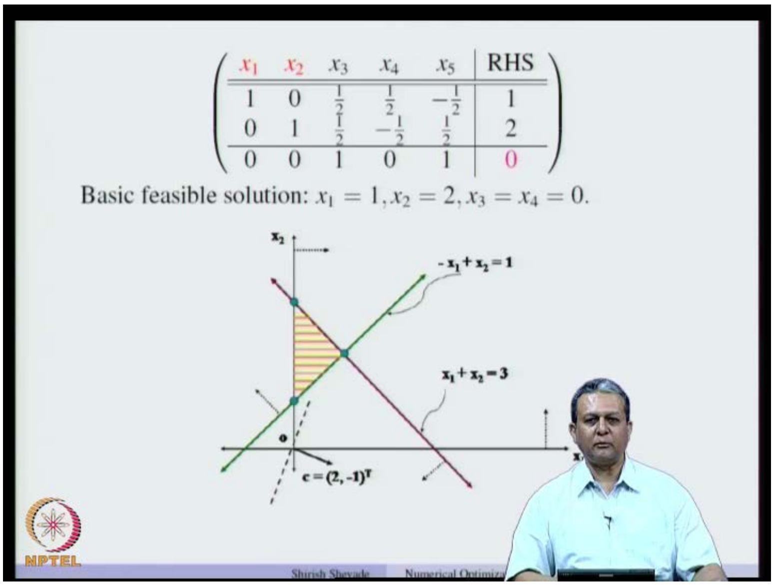

So, this is the current point that we have found, remember that this feasible set has three vertices and we have found one vertex which is a feasible point. Now, we start solving the original linear program using the simplex method with this initial basic feasible solution.

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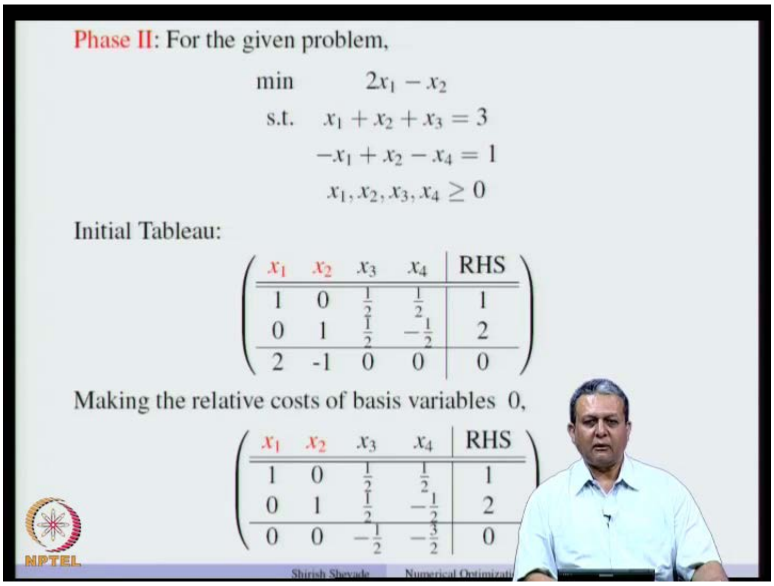

So, we have this initial tableau, now x 1 and x 2 are the basic variables the relative costs are not 0. So, we first make them 0 by multiplying the first row by minus 2 and the second row minus 1 plus 1 and adding them and this is what we get after making the basic cost of relative variable 0. And you will see that these two relative costs corresponding to the non-basic variables are negative and these are the candidate basic variables which can come into the basic variable set.

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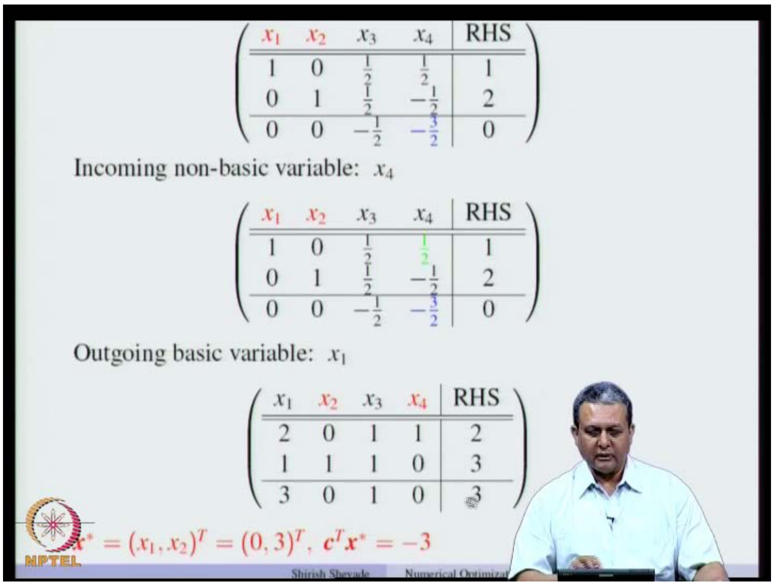

So, we choose the maximum negative among these two and therefore, x 4 becomes a x 4 will become the basic variable in the next iteration. And if we use the ratio test, so x 1 is the only candidate which can become non-basic. Now, now you will see that this obtained using the minimum ratio test and again by doing these matrix manipulations to make sure that x 4 becomes a basic variable. So, the entries related to x 4 should form a part of the identity matrix.

And therefore, by doing this matrix manipulations what we observe is that x 2 ad x 4 are the now the basic variables all the relative costs are nonnegative and the current objective function value is minus 3 x x 2 equal to 3 and x 4 equal to 2 this is the solution of the problem but, x 4 is a surplus variable, so we do not have to worry about that. So, x 2 equal to three and x 1 equal to 0 is the solution of the linear program and the optimal cost or the optimal objective function value is minus 3.

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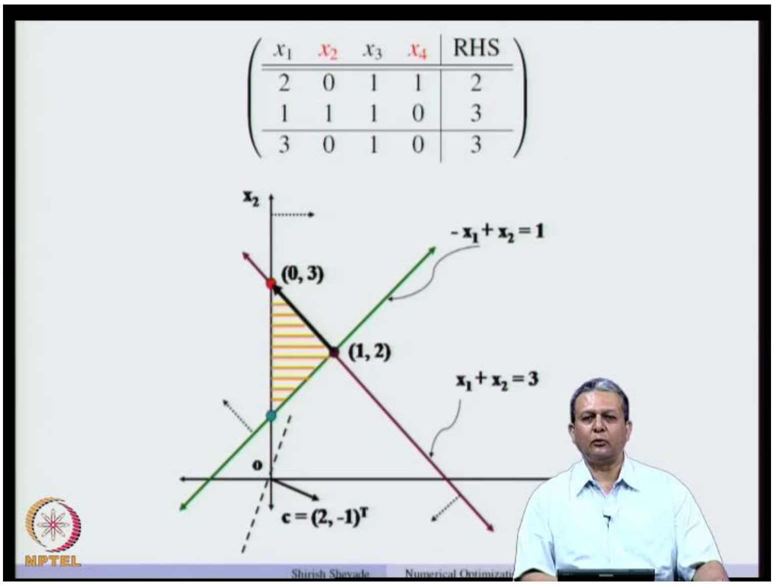

And we will see that the path the is traced with the simplex method we started with the point 1, 2 and move to the point 0, 3 at which point the objective function has optimal value. One can also see this; if you consider the vector the c vector as 2 minus 1, then this dashed line denotes the hyper-plane which is perpendicular or orthogonal to the vector c.

So, if we move this hyper-plane parallel to or if find the hyper-plane which is parallel to this and which supports this subset or this feasible set from below we will see that the minimum occurs at this point and therefore, this is truly a solution. Now, there could be occasions where the phase one of the simplex method may not give us initial basic feasible solution for the original linear program. So, in that case we cannot start the phase two and we terminate the algorithm.

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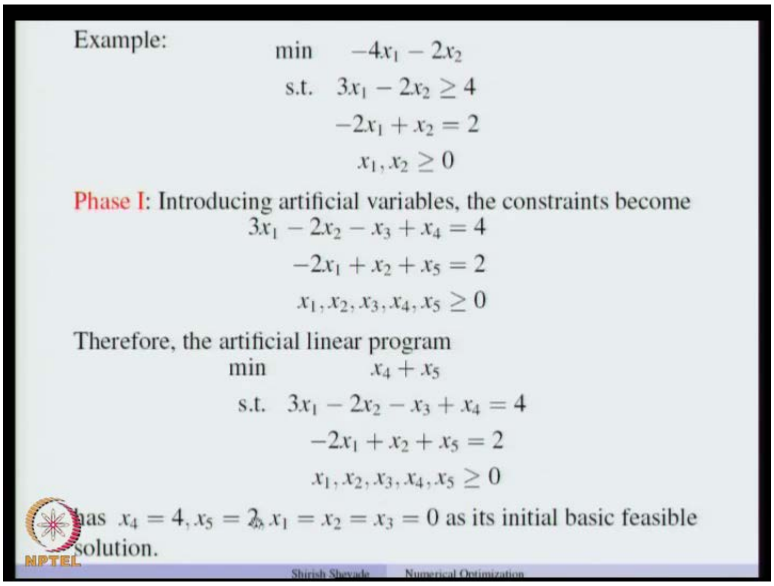

So, let us see an example, so consider this example where we want to minimize minus 4 x or minus 2 x 2 and subject to this constraint that 3 x 1 minus 2 x 2 greater than or equal to 4 and minus 2 x 1 plus x 2 equal to 2 and x 1 and x 2 are nonnegative. Now, one cannot directly find the identity sub-matrix of the matrix a, so we resolve to two-phase approach of simplex method.

So, the first phase is to introduce artificial variables and solve an artificial linear program. So, we introduce artificial variable, so first of all the variable x 3 is used as a surplus variable and we use the artificial variable x 4 here. And similarly, here the second constraint we use x 5 as artificial variable and therefore, our artificial linear program is to minimize x 4 plus x 5 subject to this constraints non-negativity constraints as well as the other two constraints. So, this is our artificial linear program now that has a basic feasible solution which x 4 equal to 4 and x 2 equal to 2.

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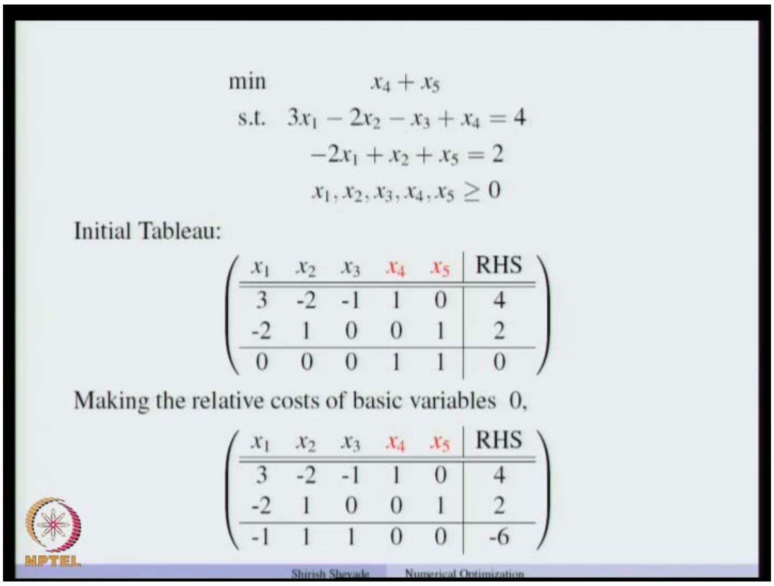

And we use this to write the initial tableau and making the costs relative cost of the basic variable 0, this is what we get as the initial tableau to start our simplex method. Now, you will see that there is a relative cost of the basic vector non-basic vector which is x 1 which is negative and therefore, that variable x 1 is a candidate to become a basic vector.

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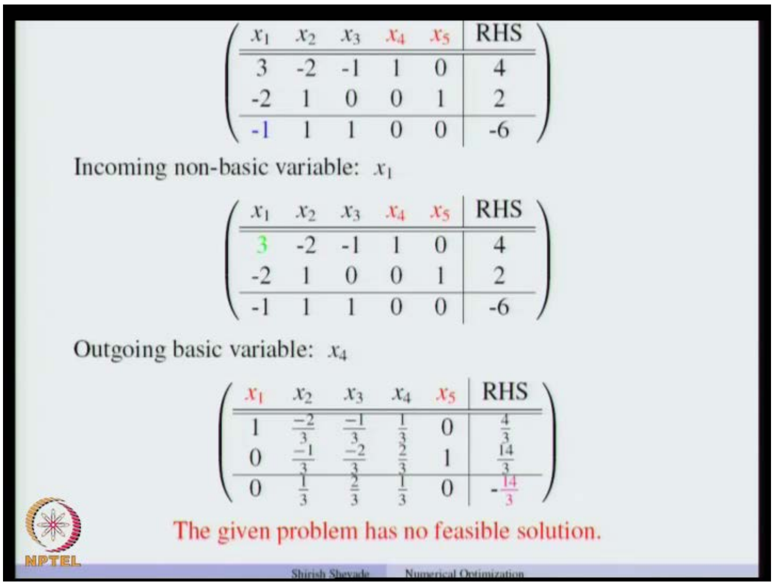

So, we want to introduce x 1 as the basic variable so; that means, we have to find out a variable which can give the basis and by using the minimum ratio test we will see that x 4 is the only candidate which can leave the basis. So, incoming the non-basic variable is x 1 and the outgoing basic variable is x 4 and the by making sure that now x 1 is the basis, x 1 is the part of the basic vector, which means that the corresponding entries in this matrix have to be such that all the basic variables should form the identity matrix.

And once we do this matrix manipulation we will see that all the relative costs are nonnegative. So, where x 1 is equal to 4 by 3 and x 5 is equal to 14 by 3 but, the optimal objective function does not have a 0 cost. So, in fact, the cost is finite which is 14 by 3 and therefore, the given problem has no feasible solution and therefore, we need not go to the phase two of the simplex method. So, by using phase one of the simplex method we have seen two examples where in one example we were able to get an initial basic feasible solution from the for the original program. While in the second example we concluded that the program does not have a feasible solution, so there is no need to go to the phase two of the simplex method.

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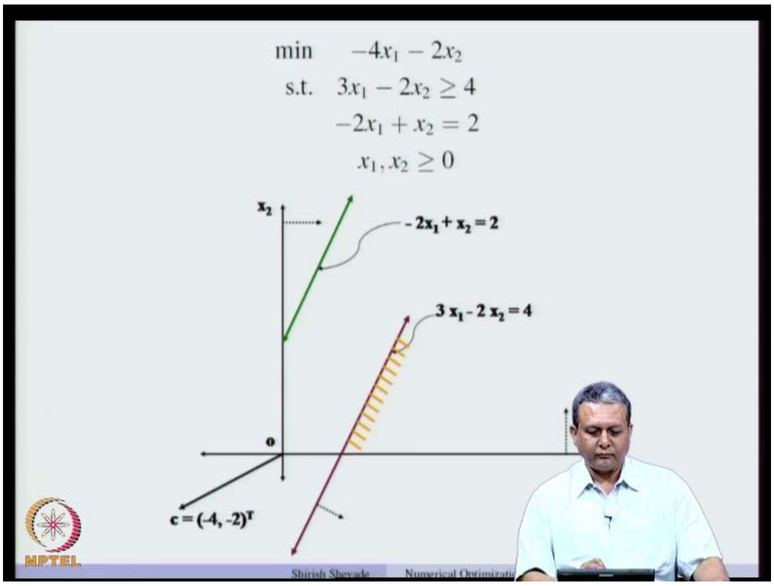

Now one can easily see that if we draw the constraints graphically then 3 x 1 minus 2 x 2 equal to 4 is the half space is the hyper-plane which is given here. And 3 x 1 minus 2 x 2 greater than or equal to 4 is half space shown by this arrow and the other constraint is minus x 2 x 1 plus x 2 equal to 2 which is shown here. Now, it is clear that they do not have anything in intersection or anything in common. Therefore, the constraint set is empty and therefore, we could use simplex method phase one to conclude that the original linear program does not have feasible point.

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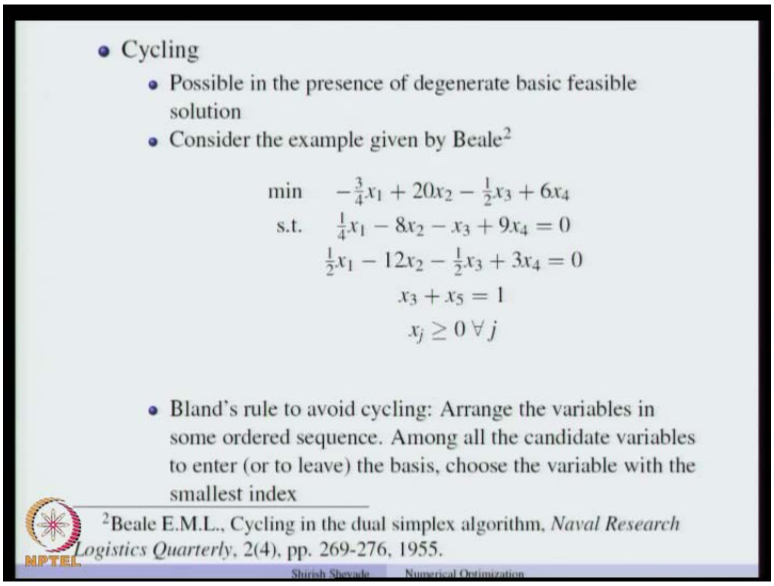

Now, there could be some occasion where simplex method can get into cycling where one repeats through the vertices without making any progress in the objective function. So, in his work in 1965 will gave an example where cycling in the simplex algorithm can occur, that typically occurs in degenerate cases. So, if the given problem has degenerate basic feasible solutions then for a particular extreme point there could exist different basis. And therefore, the simplex method keeps choosing different basis for the same vertex and thereby resulting in cycling.

Now, because of this cycling the there is no improvement in the objective function the simplex algorithm will not terminate. So, here is an example given by bill in his work in 1955, now one can write down the simplex tableau for this and one can see that because of the degeneracy in the basic feasible solution the simplex method will not terminate. So, only when the at every basic feasible point there is a non-degeneracy, then one can guarantee the every time when one moves from one vertex to another there is a decrease in the objective function.

And since there are only finite number of vertices or finite number of basic feasible solutions the simplex algorithm in the presence of non-degeneracy will terminate in finite number of iterations. However in the case of degeneracy there is no such guarantee and the simplex algorithm could result in cycling, there exist many more examples and you may have find them in literature or any, any text book on linear programming, where in the presence of degeneracy the simplex method results in cycling. Now, blank cave gave a rule to cycling, so his idea is very simple the idea is to arrange the variables in some ordered sequence and among all the candidate variables to to enter the basis choose the variables with the smallest index. Similar idea can be used for the basic variables which are leaving the basis, again one writes down all the candidate variables in the some order sequence and follow that ordering throughout.

So, if you do that then one can see that one can avoid cycling now this concludes our discussion on simplex method one important point that one has to note that is that simplex algorithm does visit each of the, does visit the vertices of the feasible set. And finally, if the solution exists finds a vertex which gives the optimal solution, now there could be situations where simplex method may have to visit all the vertices of the feasible set before it reaches the final solution.

And the problem is that the nature of the simplex method, that every time it, it visits the vertices only to freeze the solution. So, there exist different methods to avoid this situation where the complexity of the simplex algorithm can be very large. So, we will see those algorithms some time later in this course but, today we will discuss about duality theory in linear programming.

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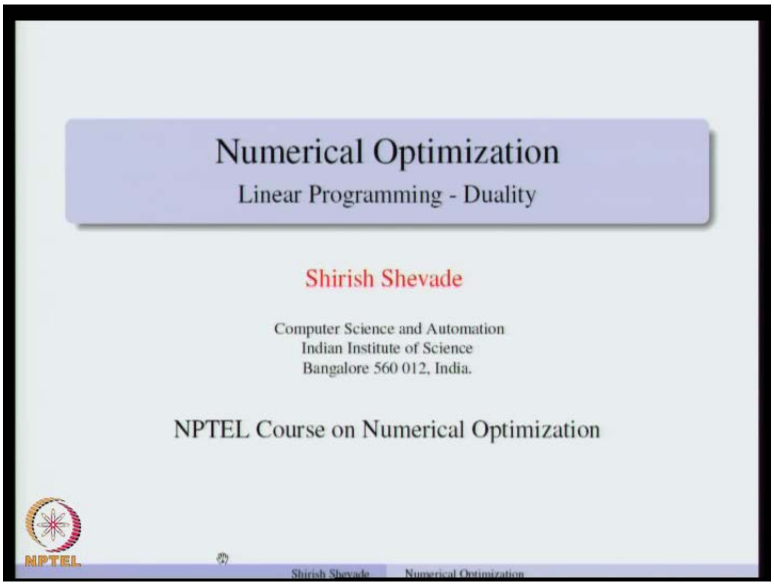

We have already studied duality theory in the context of general non-linear programs. So, the results discussed there also follow for linear programming case but, there are some special ideas that are useful in linear programming and we will discuss some of them.

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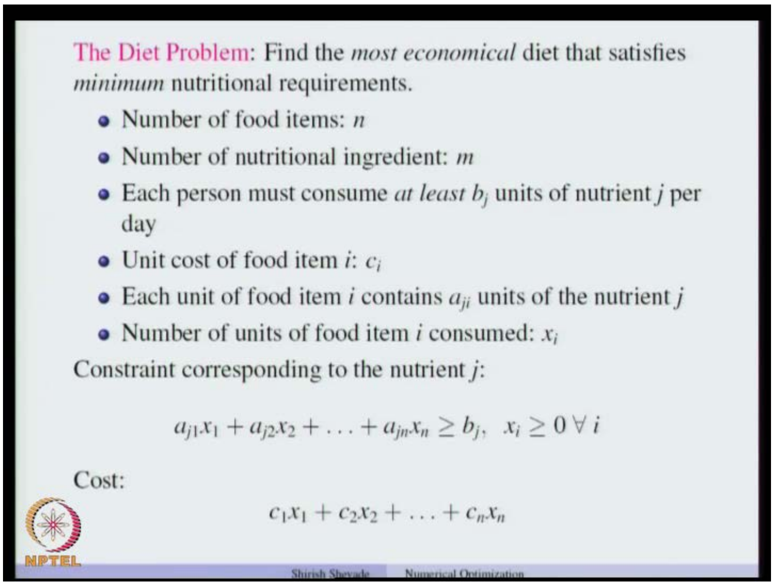

So, let us revisit the diet problem that we saw earlier, so the diet problem is to find the most economical diet that satisfies minimum nutritional requirements. So, here is the problem set we are given food items there are n such food items then the number of nutritional ingredients suggested by the dietician is m. And every person must consume at least b j units of the nutrients j per days the constraints imposed by the dietician.

The unit cost of the food item i let, let us denote it by c i and the unit of food item i, that contains a j i units of nutrient j and the number of units of food items i consume let us denote it by c x i. So, a person who has to follow this diet has to find the optimal price of the food items such that all the nutritional requirements are fulfilled. So, in order to fulfill all the requirements related to the nutrient j one needs to satisfy this constraint for the nutrient j. And the, the cost, cost of each food item is c i. So, if you consume x i units the food item i then the corresponding cost is c i x i, so the total cost is c one plus x 1 c 2 x 2 up to c n x n. So, this is the total cost which one would like to minimize subject to this constraint.

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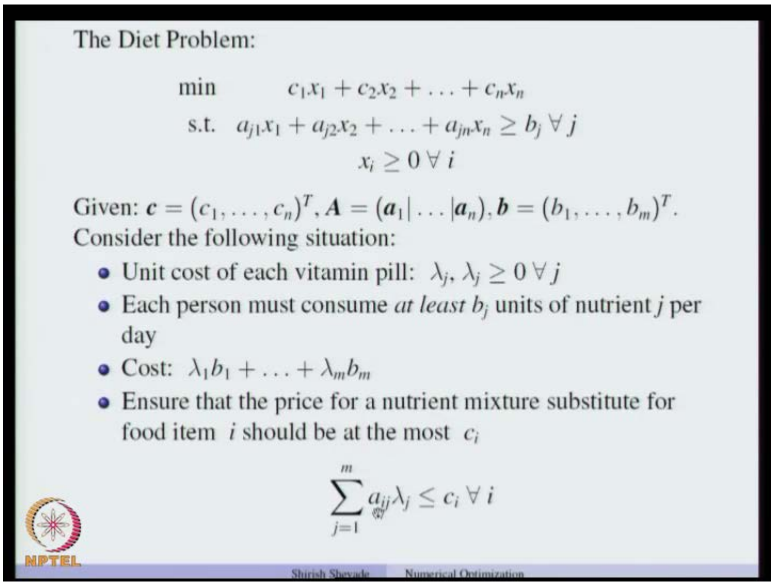

So, the diet problem is something like this now in this problem the c is given the a‘s are given and the b’s are given now imagine the situation, where an entrepreneur wants to sell vitamin pills, which are related to all the nutrients that a person is expected to consume. Now, the idea is that this, the mixture of vitamin pills will replace the food items, so let us assume that each vitamin pill lambda j the cost is nonnegative. Since each person must consume at least v j units of the nutrients j, the cost that the person has to pay for buying those vitamin pills is lambda 1 b 1 plus lambda 2 up to lambda m b m.

Now, the entrepreneur must make sure that the, the mixture of this vitamin pills which is going to replace each of the food items, the cost of this vitamin pills should not exceed the cost of the food item for every food item. So, this nutrient mixture for food substitute item i, if you take its cost that should be at the most the cost of the food item i. And in the constraint form it can be written as sigma a i j lambda j less than or equal to c i for all i.

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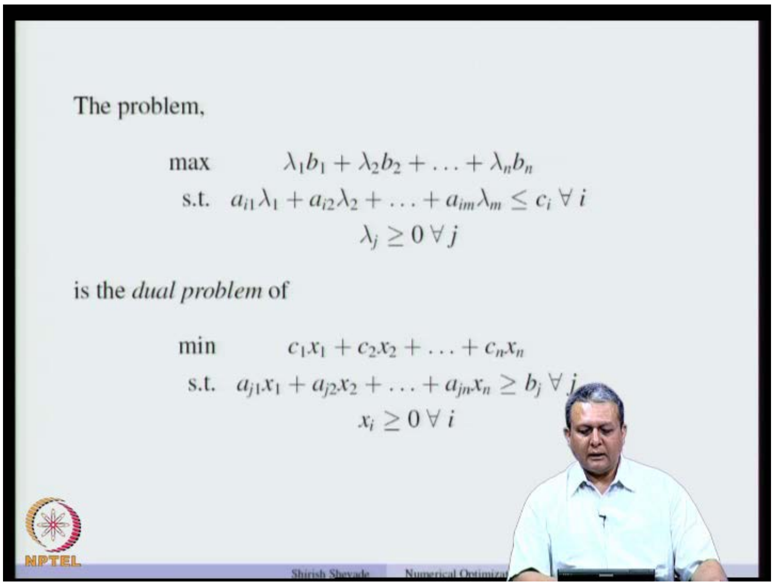

So, therefore the problem is to maximize the cost as far as the entrepreneur is concerned because he wants to make revenue by selling this vitamin pills. So, he would like to maximize the cost of the vitamin pills that is purchased by the customer and at the same time the entrepreneur also make sure, that the vitamin pills mixture cost should not exceed the food item cost. So, this problem that is solved by the entrepreneur is the dual of the original diet problem that a customer tries to solve.

Now, we would like to study whether it makes sense for the entrepreneur to solve this problem and make profit. So, these types of problems are called dual problems, you would see that one problem is a maximization problem while the other problem is the minimization problem. And if you have the constraints of the type less than or equal to while here the constraints are of the type greater than or equal to.

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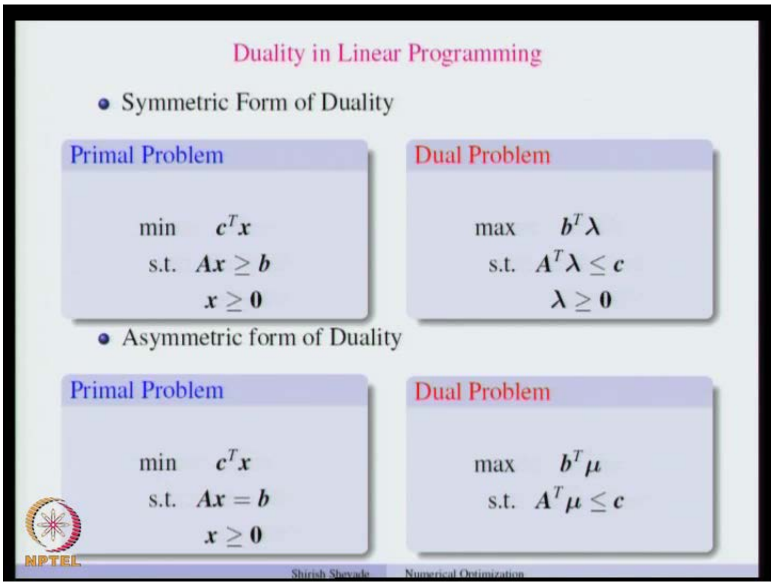

So, in some sense one problem is the dual of another and the we will see, this duality in linear programming now. So, as we have already seen that for the primal problem of the type minimize c transpose x equal to b subject to a x greater than or equal to b and x nonnegative that dual problem is maximize b transpose lambda subject to a transpose lambda less than or equal to c and lambda nonnegative. So, here x is the primal variable and here lambda is the dual variable, now you see the dual relationship between the primal and the dual problems.

We have minimization problem in the primal case and the maximization problem here the constraints of the type greater than or equal to and in the other problem the constraints are of the type less than or equal to. Note that both problems are linear programming problems and in both the cases the variables are nonnegative. So, this form of duality is called symmetric form of duality, there is nice structure symmetric structure involved in these two dual problems. Now, there could be some problems where we may not find this symmetric form of duality although there exist duality between the problems.

So, one such problem is the standard linear program minimize c transpose x subject x equal to b x nonnegative and the corresponding dual problem, which is maximize b transpose mu subject x transpose mu less than or equal to. So, in this problem you will see that there is a equality constraints here in the primal problem while here there is a less than or equal to constraint. And further the dual variable mu is unrestricted in sign while the primal variable is nonnegative here.

In fact, if you write the Lagrangian of this problem and assuming that Slater's condition holds one can write the dual of this problem which is equivalent to this, we will see that mu is the Lagrangian multiplier corresponding to the equality constraint. And the Lagrangian multiplier corresponding to the inequality constraint is observed in this inequality. So, therefore there exists different kinds of forms of duality for linear programs one is symmetric other one is asymmetric.

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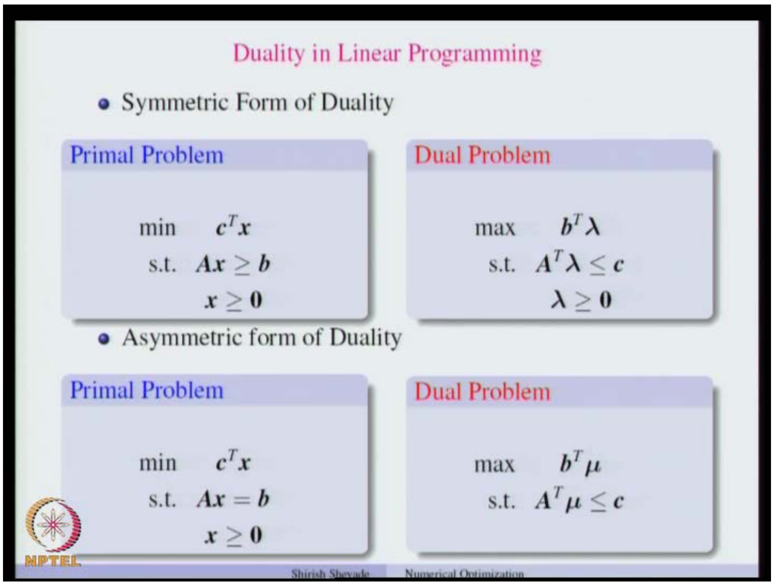

Now, let us consider the primal problem when it is dual problem, in the form of in the symmetric form of duality. Now the claim is that for the dual, for the linear programs the dual of the dual is the primal program. So, which means that if we treat this as the primal problem, then it is dual is the original linear program, now we already saw this result when we discussed the duality of general non-linear programs, where we treated linear program as a special case.

So, we will not prove this result again but, it is easy see this result by rewriting this program as a minimization program because it is given a program of this type we know how to write down the dual. Now, if you want to write down the dual first we want to write this program in the form of minimization program or which looks something similar to this. So, let us write this problem since we have maximization problem and we want it to be a minimization problem. So, we write its as minus minimum of minus b transpose lambda.

Now, for this problem all the constraints are of the type greater than or equal to therefore, we convert this constraints into the constraints of the type greater than or equal to by multiplying throughout by minus 1. So, we have minus a transpose lambda greater than or equal to minus c and we had a non-negativity constraint here and also non-negativity constraints here. So, that constraints remains as it is, now we compare these two programs now there is a minus sign here which we can take care of it after writing the final program. So, if we suppose exclude the minus sign for the time being now this c is replaced by minus b a replaced is minus a transpose and b is replaced by minus c.

Now, we know how to find the how to write the dual problem for this program. So, using similar ideas one can write the dual problem, so it will be maximize, now b transpose lambda. So, b was in the constraint of the type a x greater than or equal to b. So, corresponding to b we have minus c and suppose the dual variable is x. So, maximize c transpose a x a and subject a we can a transpose. So, minus a transpose will become a and similarly, b a transpose lambda less than or equal to c.

So, this minus b transpose will come in the constraint and by rewriting the constraint and using the fact minus max of minus c transpose x is minimum of c transpose x we can see that the dual problem of this problem is minimize c transpose x subject to a x greater than or equal to b and x nonnegative. So, you will see that this was our original primal problem and it is dual problem was here now by rewriting this dual problem in this form and then writing the dual problem. So, essentially this is the dual of this problem and the dual of this problem which is this is similar to the original primal problem. So, for linear programs the dual of the dual is the primal problem.

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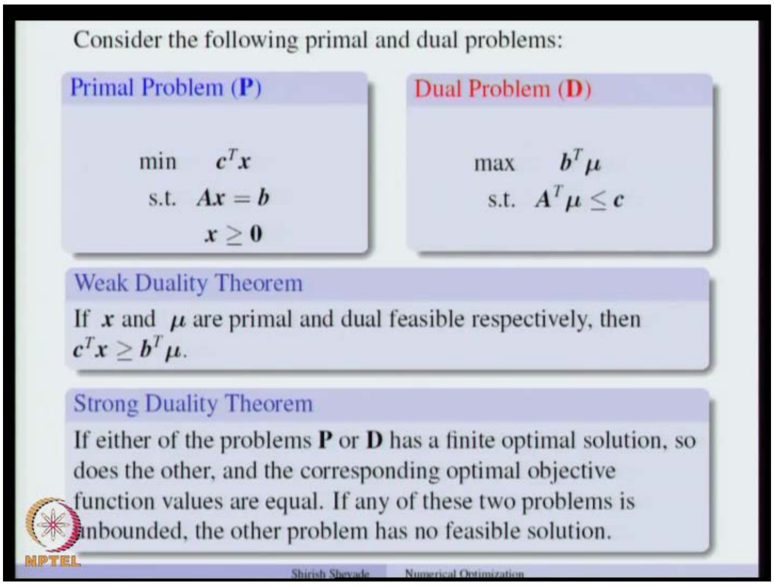

Now, let us consider the primal dual problems associated with standard linear programs. So, we have this is the standard linear program and the corresponding dual and we saw this results when we discussed the duality but, for the sake of completeness we will mention them here. Now, the weak duality theorem says that if x and mu are primal and dual respectively then; that means, x satisfies this constraints and mu satisfies these constraint. Then c transpose x is less than or equal to is greater than or equal to b transpose mu.

So, what it means is that if you take x and mu to be primal and dual feasible, then the primal objective function value c transpose x is always greater than or equal to the dual objective function value, which is b transpose mu. Now, it is easy to see that suppose we take mu to be feasible and x also to be primal feasible. So, mu is dual feasible. So, a transpose mu less than or equal to c, now if we take this vector c and find c transpose x.

So, c transpose x will be greater than equal to a x transpose mu because x is nonnegative and a x is nothing but, b therefore, b transpose mu less than or equal to c transpose x. So, it is easy to see this result we also saw this result earlier. So, the value of the dual objective function is always less than or equal to the value of the primal objective function and when are they equal at optimality and that is called strong duality theorem. So, strong duality theorem says that if either of the problems has already has a finite optimal solution then.

So, does the other and the corresponding optimal objective function values are equal, so at optimality. So, suppose x star is a optimal solution to this primal problem and mu star is an optimal solution to this dual problem nearly x star satisfies this constraint and mu star satisfies this constraint. But at optimality c transpose x star is equal to b transpose mu star. So, this is inequality will be replaced by equality only at optimality provided one of the problems has a finite optimal solution.

If any of the problems is unbounded then from this weak duality theorem you will see that this strong duality theorem does not hold in the sense that suppose primal is unbounded. So, c transpose x moves towards minus infinity and then you cannot have a quantity, which is less than minus infinity as as far as the dual objective function is concerned. Because we wanted b transpose mu less than or equal to c transpose x and similarly, if the case for the dual problem.

And thus any of the two problems is unbounded then the other problem has no feasible solution. So, this is a very important property, we will not prove this because we have already seen this earlier that this primal problem or the dual problems are convex programming problems. And if Slater's condition is satisfied then one can write the wolf dual and at optimality the two objective functions are same. So, we have seen this result earlier.

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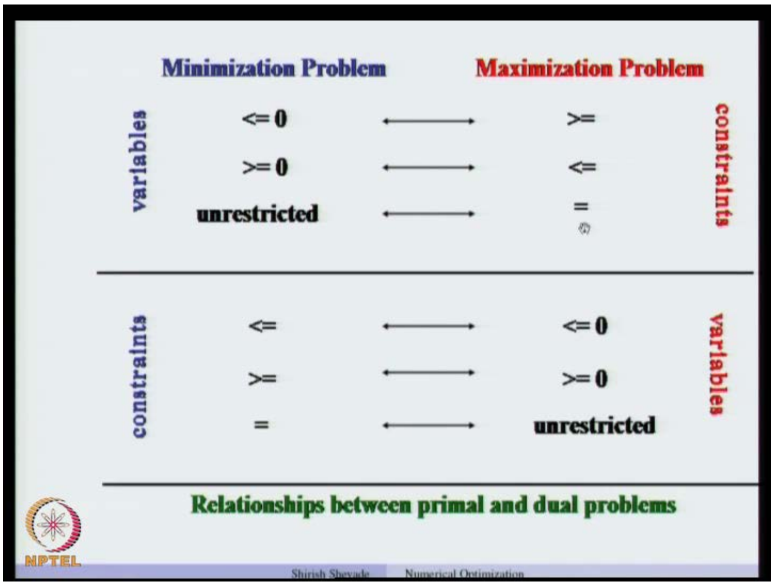

Now, this is just a quick idea about the relationship between the primal and dual problems. So, as we have seen earlier one problem is the minimization problem and the other is the maximization problem. Now, if one has variables in the minimization problem, which are less than or equal to type less than or equal to 0 type then the corresponding constraints in the maximization problem will be of the type greater than or equal to.

And if the variables are of the type greater than or equal to 0 then the corresponding constraints in the other maximization problem are less than or equal to type. If the variables are unrestricted in the minimization problem, the corresponding constraints in the maximization problem are of the type equal to; and similar relationships holds from the constraints to the variables in the maximization problem. So, you will see that there is a correspondence between the variables and the constraints in the two types of problems. So, in one case the variables are associated with the constraints in the, of the problem and variables in the other problem are associated with the constraints in the first problem. So, we will see some examples to illustrate these ideas.

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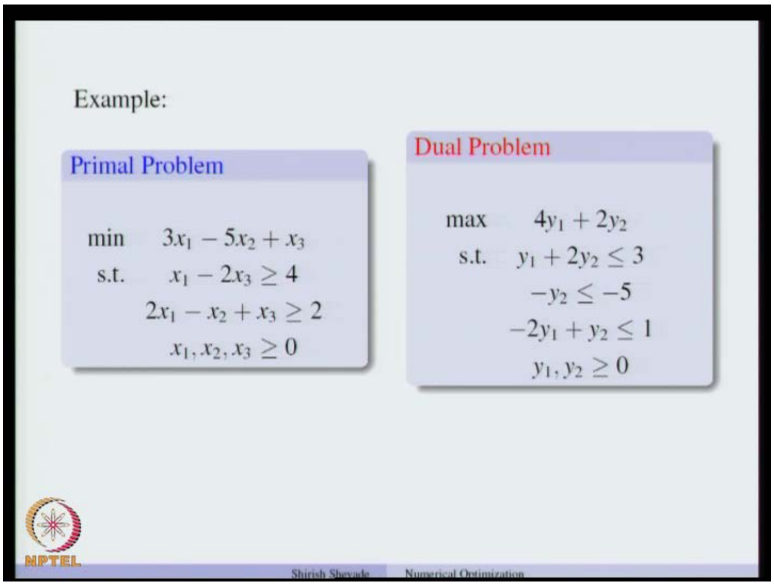

So, let us consider a primal problem to minimize 3 x 1 minus 5 x 2 plus x 3 subject to x 1 minus 2 x 3 greater than or equal to 4 and 2 x 1 minus x 2 plus x 3 greater than or equal to 2 and suppose we want to write down the dual of this problem. Now, there are two constraints here in this primal and associated with these two constraints will be two variables in the dual problem. So, let us call this variables as y 1 and y 2, so corresponding to the first constraint we have a variable y 1, corresponding to the second constraints we have the variable y 2.

Now the variables in the minimization problem are of the type greater than or equal to 0. So, therefore the corresponding constraints will be of the type less than or equal to in the maximization problem. So, we have the primal which is minimization then dual problem will be a maximization problem, the primal variables are which is the minimization problem.

The variables are of the type greater than or equal to 0. So, correspondingly the constraints in the dual problem will be of the type less than or equal to. Now, these constraints are of the type greater than or equal to. So, the corresponding variables will be of the type greater than or equal to 0, one can see from the, so from this figure. So, for minimization problem the constraints are of the type greater than or equal to, then for the maximization problem the variables will be of the type greater than or equal to 0. And if the variables for the minimization problem are greater than or equal to 0, the corresponding constraints will be of the type less than or equal to.

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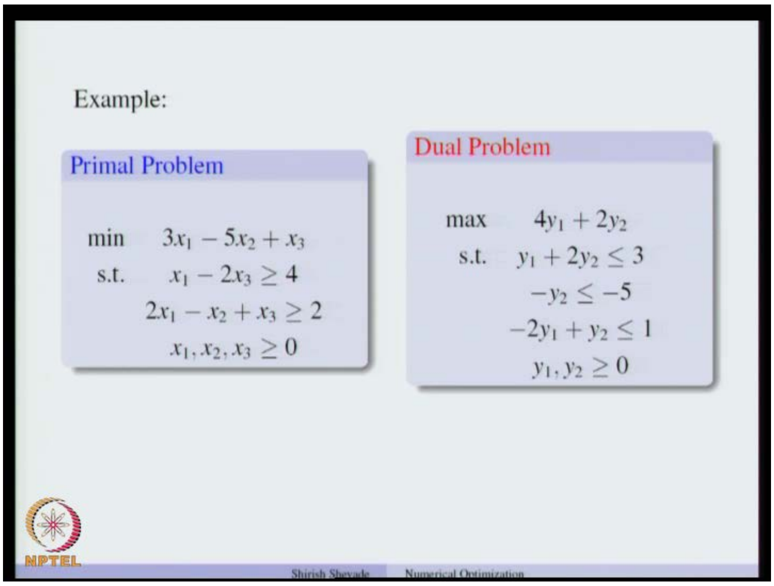

So, the dual problem of this primal problem becomes maximize. Now, there is a variable y 1 associated with this constraints variable y 2 associated with this constraint. So, y 1 plus 2 y 2 should be less than or equal to 3. So, x 1 is greater than or equal to 0. So, corresponding to x 1 the constraint will be of the type less than or equal to. So, y 1 plus 2 y 2, so y 1 plus 2 less than or equal to 3 then now there no term involving x 2 here. So, we go to the next constraint.

So, minus y 2 is less than or equal to minus 5 and then minus y 2 y 1 plus y 2 less than or equal to 1. Now, the constraints here are greater than or equal to type. So, the associated variables y 1 and y 2 will also be greater than or 0 type in the maximization problem. So, we have now a problem which is a dual problem of this problem, now what about the solutions of these problems. Now, you will see that the primal problem there is a term involving minus 5 as a cost associated with x 2 and note that all the variables x 1, x 2, x 3 are nonnegative.

So, if we if we can increase x 2 to a large number then and make sure that the other two terms 3 x 1 plus x 3 have lesser contribution compared to this, then it is possible to increase x 2 as much as possible and minimize the objective function as much as possible let us see how to do that. Suppose x 3 is said to 0 and we set x 1 to be greater than or equal to 4. So, choose any x 1 which is greater than or equal to 4. So, which remains feasible for this, now since x 3 is 0 we have to make sure that x 2 is less than or equal to 2 x 1 minus 2.

So, if we choose x 1 to be greater than or equal to 4 and x 2 to be 2 x 1 minus 2, then we will see 3 x 1 minus 5 x 2 becomes less than 0. So, we have satisfied all the constraints and by making x 2 very large I ma sorry by making x 1 very large x 2 can also be made to be equal to 2 x 1 minus x 2 and in together 3 x 1 minus 5 x 2 will be a quantity which is less than 0 we are assuming x 3 as 0. So, this primal problem is unbounded now what about the dual problem. So, let us write down this dual problem.

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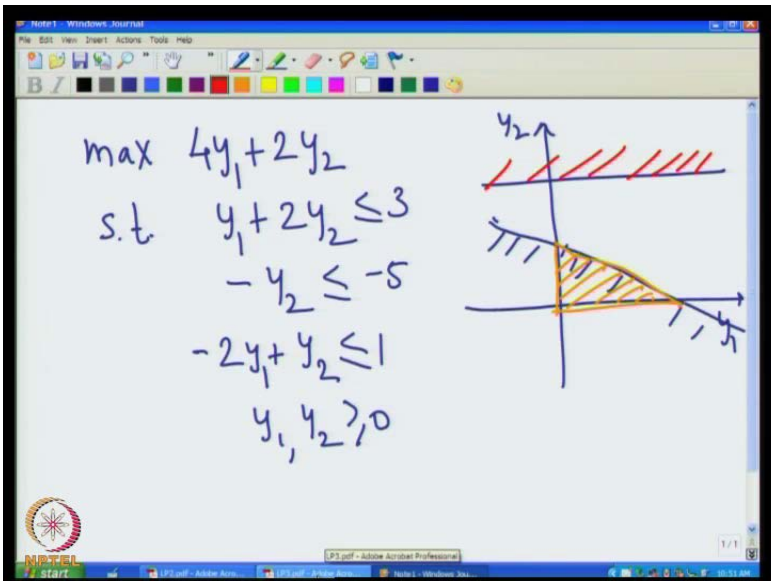

So, we have so we have the dual problem which is maximize 4 y 1 plus 2 y 2, so maximize 4 y 1 plus 2 y 2 subject to y 1 plus y 2 equal to 3 subject y 1 plus 2 y 2 less than or equal to 3 and minus y 2 less than or equal to minus 5 and minus 2 y 2 less than or equal to 1, now if we draw the so y 1 plus 2 y 2 is equal to 3. So, that will be, then we also have y 1 y 2 greater than or equal to 0 so; that means, that we are interested in this part. Now, if you look at the second constraint which says that y 2 greater than or equal to 5. So, y two greater than or equal to 5 means that it is this reason that one is interested in. Now, you will see that the two regions do not have anything in common and therefore, the dual problem in infeasible.

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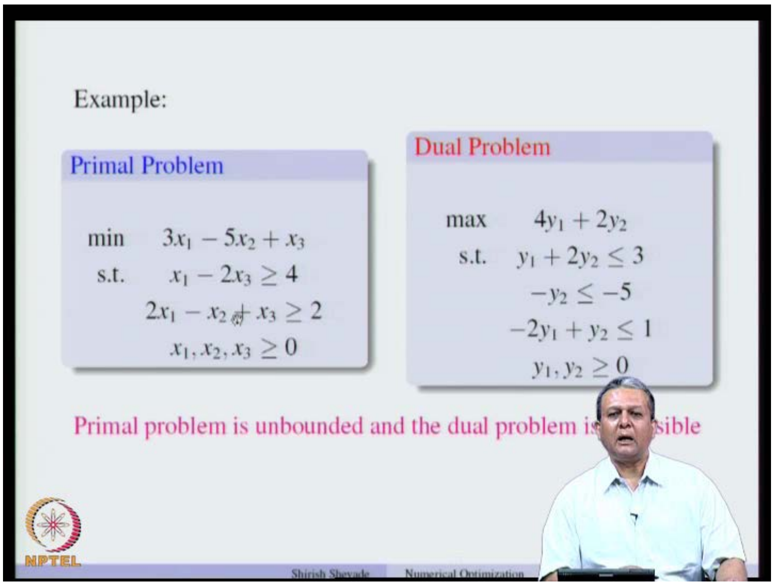

So, we have seen that the primal problem here is unbounded and the dual problem is infeasible. Now, if one problem is infeasible what happens to the other problem, the other problem sometimes also could be infeasible; in this case the dual problem is infeasible but, the other primal problem was unbounded. Now, there could be situations where both the primal and the dual are infeasible.

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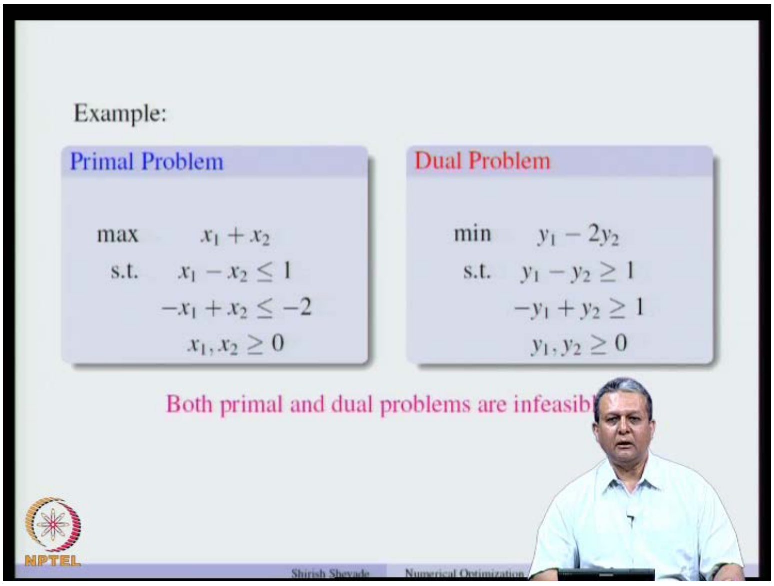

So, let us consider this case, so minimize x 1 plus x 2 subject to x 1 minus x 2 less than or equal to 1 and minus x 1 plus x 2 equal to minus 2 both x 1 and x 2 are negative, now note that this is a maximization problem. So, when we write the dual problem the dual problem will be the minimization problem, so again let y 1 and y 2 be the variables associated with the two constraints. So, the, the dual problem will be minimize y 1 minus 2 y 2 subject to the constraint that y 1 minus y 2 will be greater than or equal to 1, and minus y 1 plus y 2 greater than or equal to 1, and both y 1 and y 2 are nonnegative.

Now, if you rewrite this second constraint by multiplying throughout by minus 1, what we get is x 1 minus x 2 greater than or equal to 2, the first constraint says x 1 minus x 2 less than or equal to 1. So, these two constraints are such that they does not exist a feasible region for this problem, similar is the case for dual problem. So, if we multiply the second constraint by minus 1 what we get y 1 minus y 2 less than or equal to minus 1. So, the first constraints demands that y 1 minus y 2 should be greater than or equal to 1, second constraint demands that y 1 minus y 2 less than or equal to minus 1. So, these two constraints are such that the feasible region is empty. So, here is the situation where primal problem is infeasible as well as the dual problem. So, in the earlier example we saw that primal was unbounded dual was in feasible here both the primal and the dual problems are infeasible.

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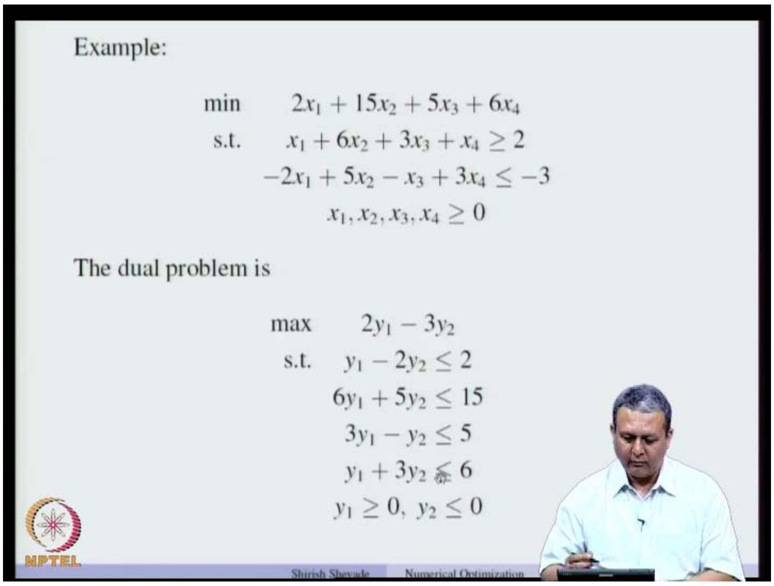

Now, let us see the application of duality to solve primal problems especially the case where the number of constraints is smaller than the number of examples, number of constraints is smaller than the number of variables. So, here we have only two constraints and four variables. So, this becomes a four dimensional optimization problem and there could be occasions as we saw earlier that the number of constraints m will be much less than the number of variables n. So, in such cases it may be a good idea to write down the dual of this problem.

Now, if you write down the dual of this problem it will be the maximization of 2 y 1 minus 3 y 2 where y 1 is the dual variable associated with this constraint and y 2 is the dual variable associated with this constraint. So, maximize 2 y 1 minus 3 y 2 subject to the constraint that y 1 minus 2 y 2 less than or equal to 2 and so on; and remember that there is one constraints which is of the type greater than or equal to 5. So, the corresponding variable y 1 in that other problem is greater than or equal to 0, the other constraint is of the type less than or equal to. So, the corresponding variable in the other problem is less than or equal to 0.

So, we have a dual problem like this where we have four constraints in two dimensional space. So obviously, some constraints are redundant but, important point to note that is that this is the problem in two variables and the, it is easy to find the solution to this problem graphically. So, compare this with the primal problem which was four dimensional optimization problem. So, it is not possible to find the solution of this problem using graphical method. So, one has to resolve to methods likes implex method and by looking at this problem you will see that, you will have to resolve to two phase simplex method. Where the first phase will be sued to find the artificial variables and the second phase will be used to solve the actual problem. So, let us solve this problem using graphical method.

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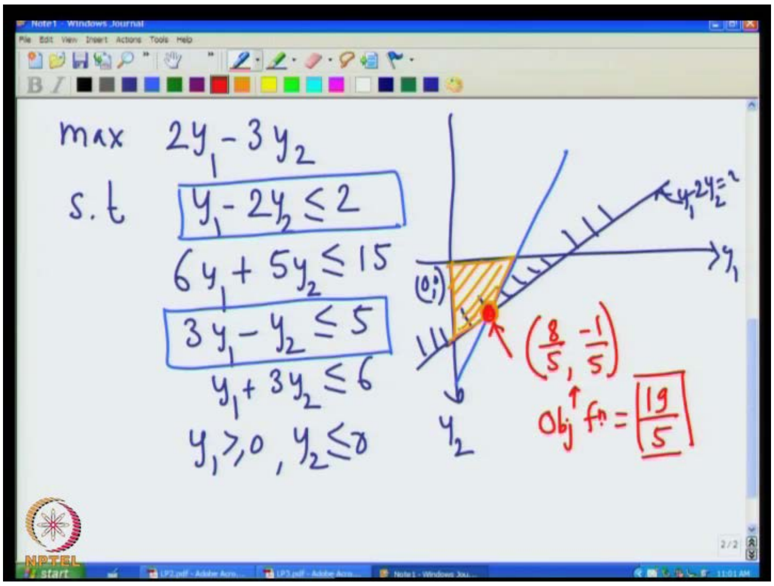

So, we have the problem maximize 2 y 1 minus 3 y 2 subject to y 1 minus 2 y 2 less than or equal to 2, 6 y 1 plus 5 y 2 less than or equal to 15, 3 y 1 minus y 2 less than or equal to 5 y 1 plus 3 y 2 less than or equal to 6. So, we have 3 y 1 minus y 2 less than or equal to 5 y 1 plus y 2, 3 y 2 less than or equal to 6 y 1 greater than or equal to 0 and y 2 less than or equal to 0. So, if we draw the feasible region the, we are interested in y 1 greater than or equal to 0 and y 2 less than or equal to 0.

So, if we take the first constraint, so y 1 minus 2 y 2 less than or equal to 2, so this is the first constraints y 1 minus 2 y 2 equal to 2 and we are interested in this part. And the we have another constraints which is important is 3 y 1 minus y 2 less than or equal to 5 and 6 y 1 plus 5 y 2 equal to 6 y 1 plus 5 y 2 equal to 15. So, we have four constraints here and out of which the important constraints for us will be this constraint and 3 y 1 minus y 2 is equal to 5. So, that constraint will be like this and the other constraints if we draw you will see that they are redundant.

So, let us not worry about those constraints and for feasible region is like this and this is the point where the solution lies. And this point is the intersection of y 1 minus y 2 equal to 2 and 3 y 1 minus y 2 equal to 5 and that is 8 by 5 and minus 1 by 5. So, one can draw the constraints to see that they are redundant and in fact we get the solution to be 8 by 5 minus 1 by 5 and if we calculate 2 y 1 minus 3 y 2 at this point. So, what we get is objective function at the point will be 19 by 5. So, so if we saw the dual problem we can quickly get the objective function value at optimality and which turn out to be 19 by 5.

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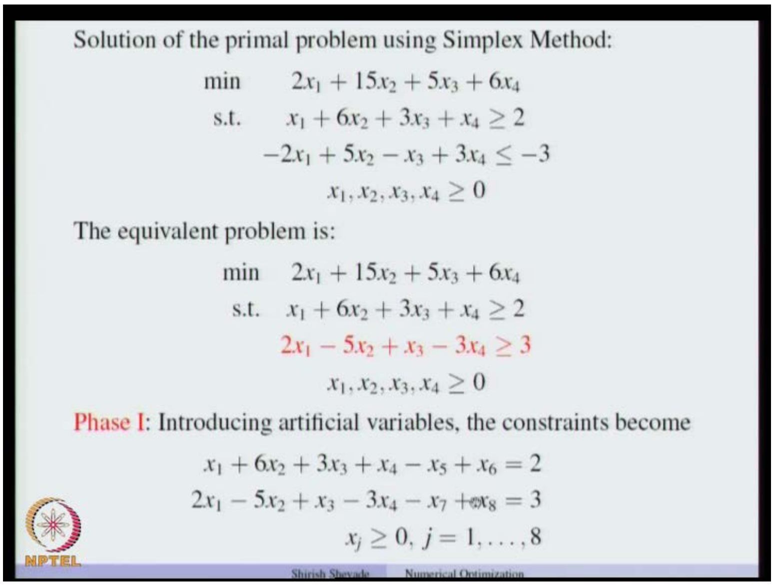

Now, instead suppose we decide to solve this problem using two-phase simplex method. So, so first step is to introduce artificial linear artificial variables, now before doing that, one has to see that the second constraint as a negative term on the right hand side. And throughout our discussion we assume that the b's are always positive, so in order to make this right hand side positive we have to multiply throughout this constraint by minus 1. And therefore, what we get is 2 x 1 minus 5 x 2 plus x 3 minus 3 x 4 greater than or equal to 3. So, now the entire right hand side is positive and now we are in a position to introduce surplus variables and artificial variables.

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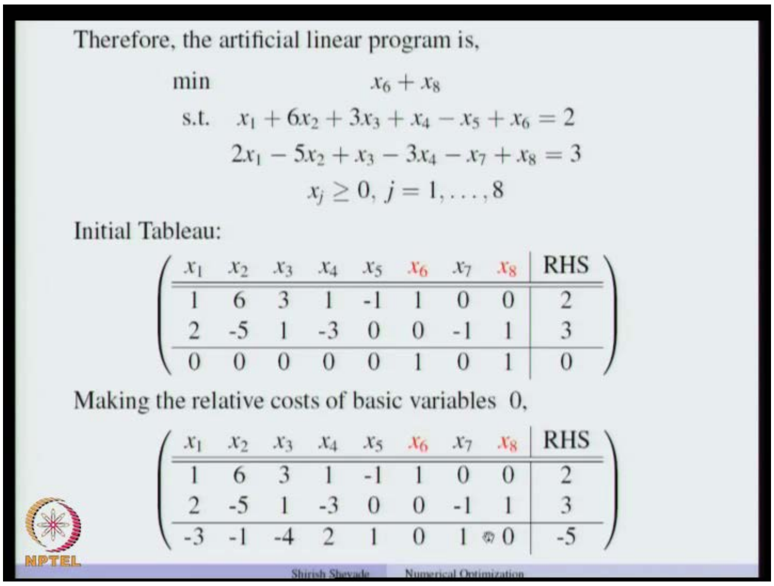

So, the surplus variables associated with the two constraints are x 5 and x 7 and are the artificial variables are x 6 and x 8. And therefore, we first solve the artificial linear program, which minimize x 6 plus x 8 subject to these two constraints and when we solve this problem we have the initial tableau where we have the relative cost of the basic variables to be positive. So, we make them 0 by multiplying the first row by minus 1, second row by minus 1 and adding them to the last row and this is what we get to start with and the current cost is 5.

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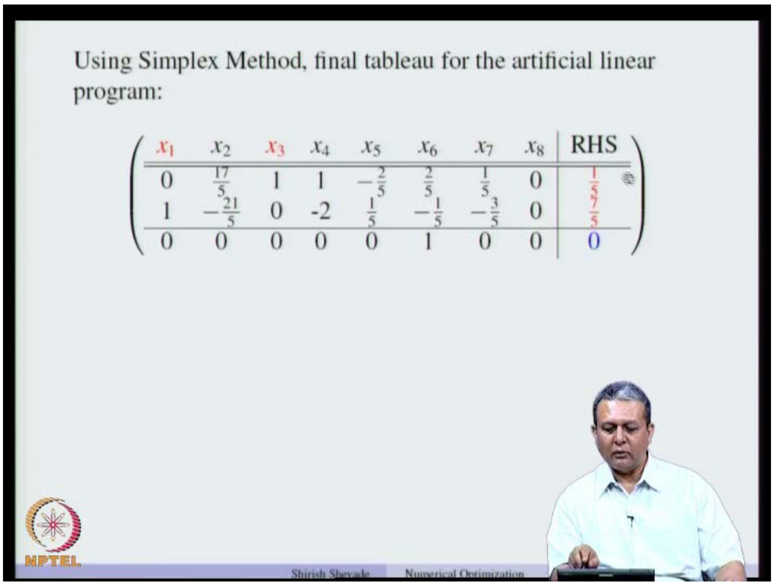

Now, if you keep using simplex method repeatedly the final tableau for the artificial linear program that we get is the basic variable x 3 has a value 1 by 5. And the basic variable x 1 has the value 7 by 5 and the objective function value 0, which means that the current problem is feasible. So, phase two of the simplex method, we will see in the next class.

Thank you.