Numerical Optimization
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Lecture - 34
Simplex Algorithm and Two-Phase Method

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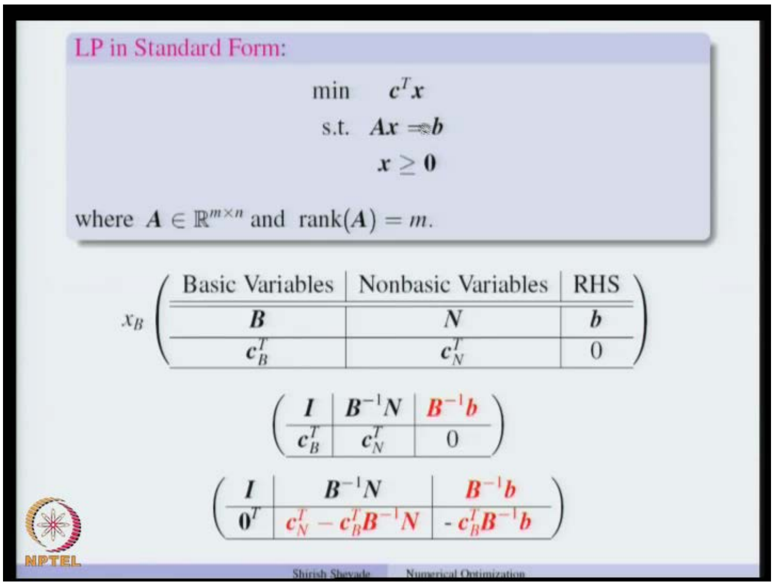

Hello, welcome back. So, in the last class we started discussing about obtaining a solution of linear program. So, in particular we consider the linear programming standard form which is minimize C transpose x subject to A x equal to b, x non negative. And we consider the matrix where the column associated with basic variables and the remaining variables are called non-basic. So, the sub-matrix of A associated with the basic variables will be denoted by B, and the remaining sub-matrix of A associated with non-basic variables will be denoted by N, and the right hand side will be denoted by b. And then the last row of this matrix indicates the cost associated with the basic and the non-basic variables.

Now, in the last class we saw that we wanted 3 pieces of information at every iteration, and that is the basic variable value which is B inverse b, and the current objective function value which is C B transpose B inverse b. And the relative cost of the lambda's, the Lagrangian multipliers corresponding to the non-basic variables; so which will be denoted by a vector C N transpose minus C B transpose B inverse N. So, when we consider this matrix and multiply the first m rows by B inverse, then we get the identity matrix here in the place of B and then B inverse N and B inverse B; the last row remains unchanged.

And, then what we do is that we multiply the first m rows by C B transpose and subtract it from the last row. So, we get 0 here and C N transpose minus C B transpose B inverse N and minus C B transpose B inverse B. So, you will see that B inverse B gives us the current basic variables C B transpose B inverse B. So, the negative of the entry in this cell gives us the current objective function value and C N transpose minus C B transpose B inverse N gives us the lambda's; lambda N. And at optimality what we want is, this lambda N should be nonnegative. So, this 0 corresponds to lambda B. So, these are the Lagrangian multipliers corresponding to the basic variables. And if we consider a nondegenerate solution x B is greater than 0; so at optimality we expect this lambda b's to be 0, are also called relative cost in the linear programming literature.

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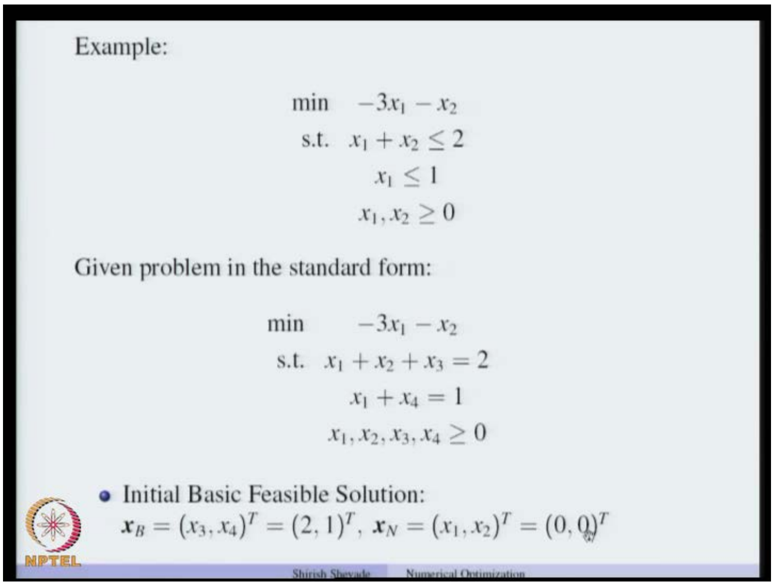

Now, let us consider example; then we will put this example in the standard form of the type minimize C transpose x subject to A x equal to b. So, the constraints which are inequality constraints can be converted to equality by introducing slack variables. So, x 3 and x 4 are the slack variables that we have included here and now we have the linear programming standard form. Now, from this constraint we can easily find an initial basic feasible solution which is x 3 equal to 2 and x 4 equal to 1 and x 1 x 2 are 0. So, x 3 and x 4 become basic variables, and x 1 and x 2 become non-basic variables. So, x 1 and x 2 is 0; so as you see here.

Now, if you construct the initial tableau or initial matrix associated with this, so there is a matrix associated with this set of constraints. So, whose first row is 1 1 1 0 and the second row is 1 0 1 1 and the right hand side is 2 1. And the basic variables are denoted by x 3 and x 4 in the red color. The objective function corresponding to the basic the parts corresponding to the basic variables is 0; cost corresponding to the non-basic variables is minus 3 minus 1. Now, before we start using this tableau one thing we have to make sure is that, the relative cost associated with the basic vectors has to be 0 and which is indeed the case here.

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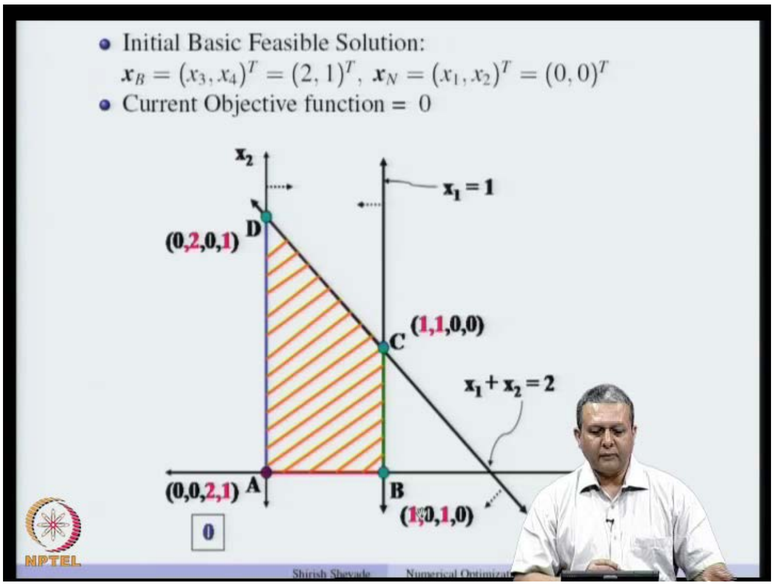

So, let us start considering the steps that we discuss in the last class. So, before we go into the details just want to mention that this is the current point where x 3 and x 4 are basic variables and their values are 2 and 1 respectively. And the non basic variables are x 1 and x 2 objective function is 0. And this was clear from this initial tableau that the basic variable x 3 has a value 2, x 4 has a value 1; the objective function value is 0 at the current point.

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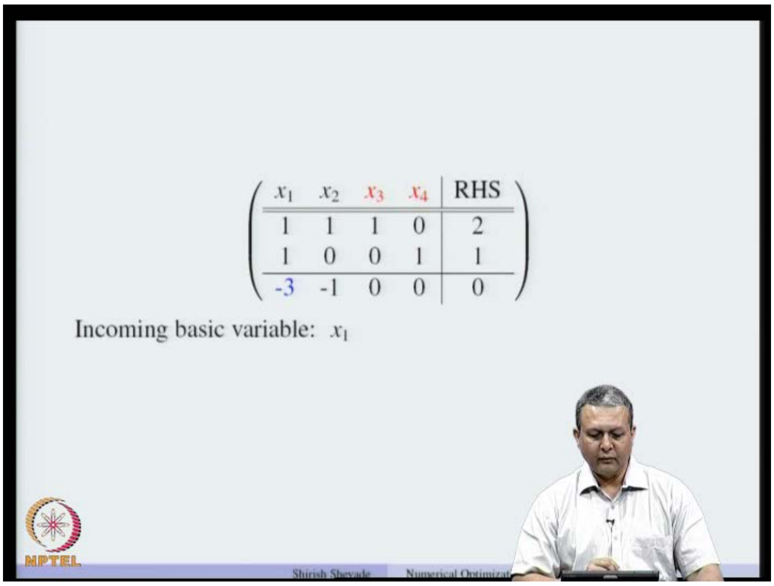

Now, now we will look at the last row and see whether the entries corresponding to the non-basic variables are nonnegative. Now, here the for both the non-basic variables the entries are negative. So, any of these variables is a candidate to become a basic variable. So, in this case we choose minus 3 to be the incoming basic variable. So, once we decide to bring in 1 variable, we have to decide a basic variable which should be made non-basic and as we discussed in the last class, we can use the ratio test. So, we collect all the elements in the particular column that we have selected which are positive. So, both the elements here are positive and then we find the ratios of this right hand side with respect to right hand side and this element. So, 2 by 1 and 1 by 1 are the two ratios that we have found and out of this 1 1 by 1 is the smallest.

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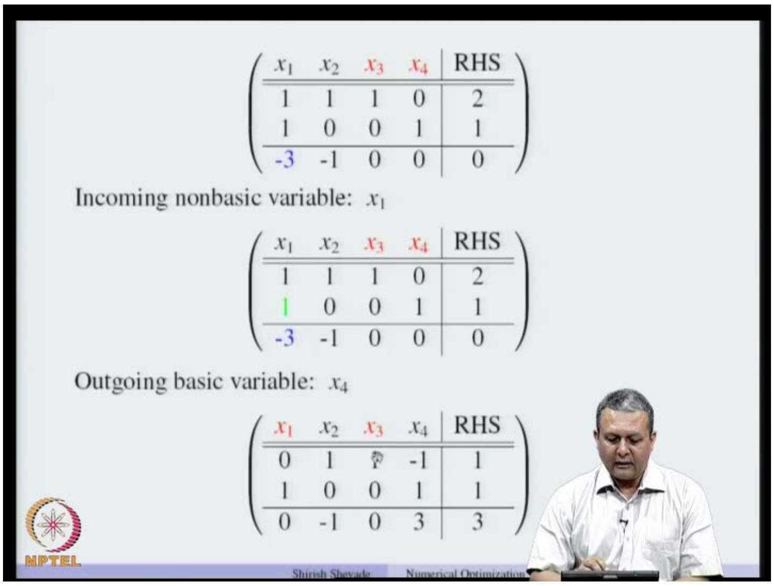

So, by using the minimum ratio test we decide that x 4 is the variable which needs to be made non-basic. So, that is shown by green color here. Now, once we decide to make a x 4 non-basic and x 1 basic; then in this column we should make sure that the remaining entries are 0 and this entry is 1. Now, in this case it is 1; so we do not have to worry about making this entry 1. So, what we need to do is that, in order to make this entry 0 we have to multiply the second row by minus 1 and add it to plus 1. And similarly, to make this entry 0, we have to multiply this second row by minus plus 3 and add it to this row; the last row; so that this entry will be made 0.

And, finally, what we get is 0 1 0 in this column and the x 4 column which was earlier 0 1 0, now becomes minus 1 1 3. Now, we will see that the matrix corresponding to the basic variables x 1 and x 3 is the identity matrix. And the basic feasible solution associated with this basis matrix is x 3 equal to 1 and x 1 equal to 1 and the current objective function value is minus 3. And remember that the non-basic variables x 3 and x 4; if you look at the entries in the last row, the entry corresponding to x 2 is minus 1, so which is negative. So, we have not yet reached the objective function value, the optimal objective function value. But note that we started from the objective function value of 0 and currently the objective function value which is the negative of the entry in this cell is minus 3. So, we have made a progress in the objective function by making x 1 basic variable and x 4 non-basic variable.

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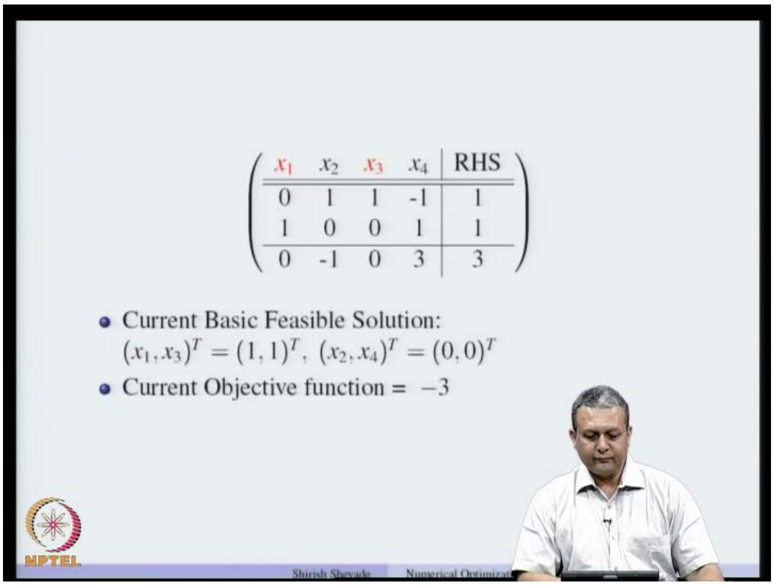

So, the current objective function value is minus 3 and the two basic variables are x 1 and x 3 and this is shown in this figure as point B. So, we started from point A and move to point B and in the process the variable x 4 which was basic is made non-basic. And variable x 1 which was non-basic at A was made basic and the objective function value decrease to minus 3.

Now, this is the current point that we have and the last row entry corresponding to a non-basic variable x 2 is negative. So, that means that x 2 is the candidate is the only candidate basic vector non-basic vector which can be made basic. So, incoming non-basic variable is x 2 and then we apply the ratio the minimum ratio test. So, in this column we collect all the entries which are strictly positive and find the ratio test, find the minimum ratio. Now, in this column you will see that only this entry is positive, the other entry is 0. So, obviously x 3 is the candidate because this row corresponds to the variable x 3; so x 3 is the candidate which should be made non-basic. And the next step is to make sure that all the entries in this column except this 1 are 0.

So, that is done by multiplying first row by 1 and adding it to the last row and what we get is the tableau which is like this. Now, x 1 and x 2 are basic variables; where x 2 is equal to 1, x 1 is equal to 1, the current objective function value is minus 4, negative of the quantity in this cell last cell. And the relative cost associated with the two non-basic variables x 3 and x 4 are strictly positive. And obviously the relative cost associated with the basic variables are 0; so it satisfies our optimality conditions. And therefore, we have reached the solution with the objective function value of minus 4 and x 2 equal to 1 and x 1 equal to 1. So, we can see in this figure that; so in the previous iteration we were at point B and then we made x 2 as our basic variable by making x 3 non-basic. So, when we move from B to C, the objective function value reduce further from minus 3 to minus 4 and we saw that is indeed a minimum.

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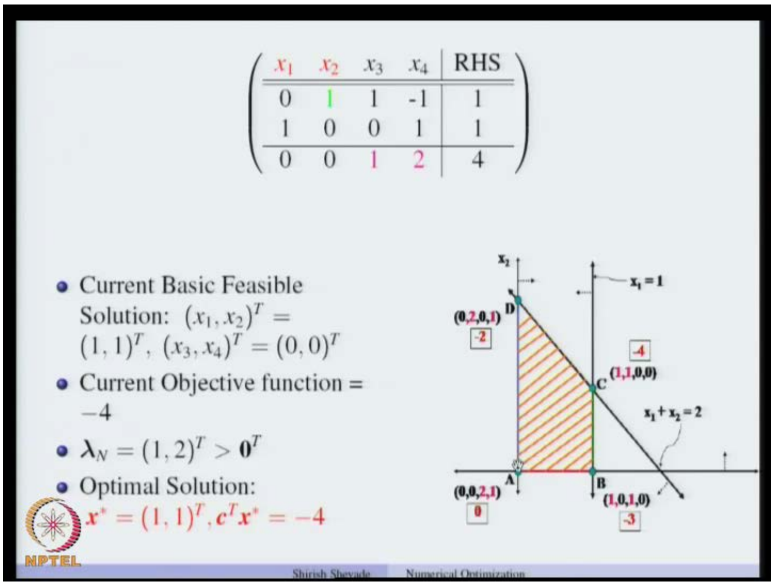

So, the starting from A we move to point B which is a neighboring vertex of A and then we move to point C which is the optimal point in this case. So, if you want to summarize the final tableau that we get gives us all the details. So, the current optimal the current basic feasible solution is x 2 equal to 1 and x 1 equal to 1, x 3 equal to 0 and x 4 equal to 0; those are the non-basic variables. Current objective function is minus 4 and lambda N is the relative cost associated with the non-basic variables, that is 1 and 2 which is strictly greater than 0. So, we have the optimal point.

So, optimal solution is 1 1 for x 1 and x 2 and the optimal objective function value is minus 4. So, as you will see from this figure that this is the point which is optimal and the path that our algorithm traced was from A to B and B to C. And there are 4 extreme points in this case. So, the optimal solution has to lie at one of the extreme points and we will see that this is the least optimal; this is the least objective function value. At point D the objective function value is minus 2 which is higher than minus 4. Now, it is also possible for the algorithm to follow path A to D and D to C and that amounts to choosing a different entering variable for the initial tableau.

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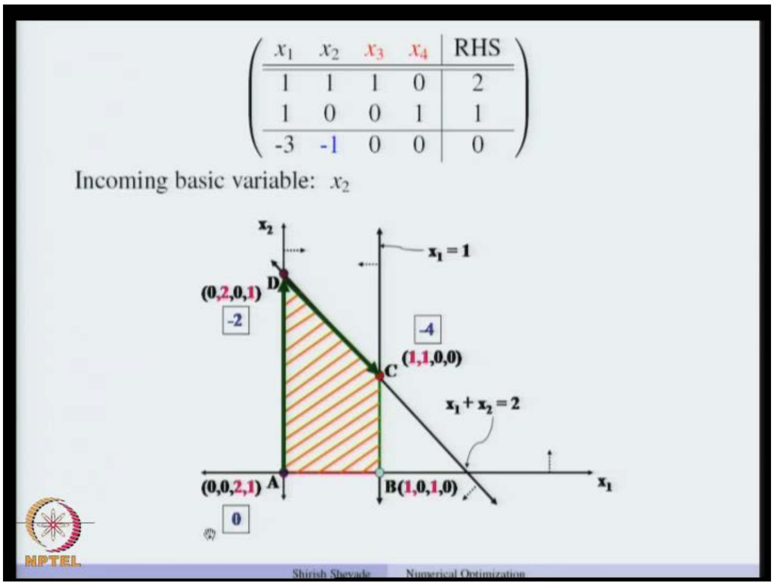

So, again we start with the point A and the initial tableau that we had considered was like this and initially we decided to make A x 1 as a incoming basic variable. Now, instead of x 1, we make x 2 as our incoming basic variable. Now, if you follow the same steps that we followed, we will see that from A one would move to point B and from B one would move to point C and from C one cannot go to any point because C is the optimum point in this case. So, there are different ways to reach the solution by following different paths. But then one thing that one has to keep in mind is that every time when we move from a basic feasible solution or a vertex to a neighboring vertex; the objective function value decreases if the solution, if the basic feasible solution is nondegenerate.

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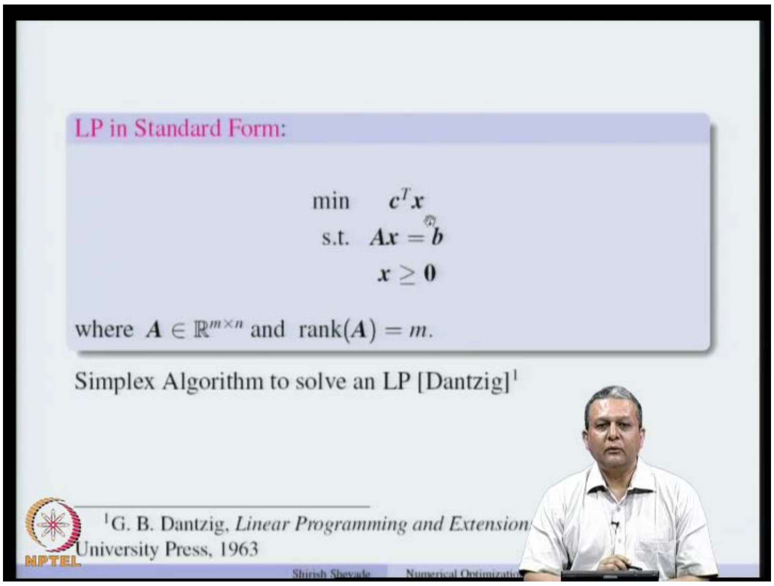

So, this steps what we saw earlier are part of the simplex algorithm to solve a linear program and these were proposed by Dantzig sometime in late 40s. And the basic algorithm which is also called simplex algorithm is also available in Dantzig's book on linear programming and extensions. So, let us consider the LP in standard form and give a simplex algorithm as proposed by Dantzig to solve this.

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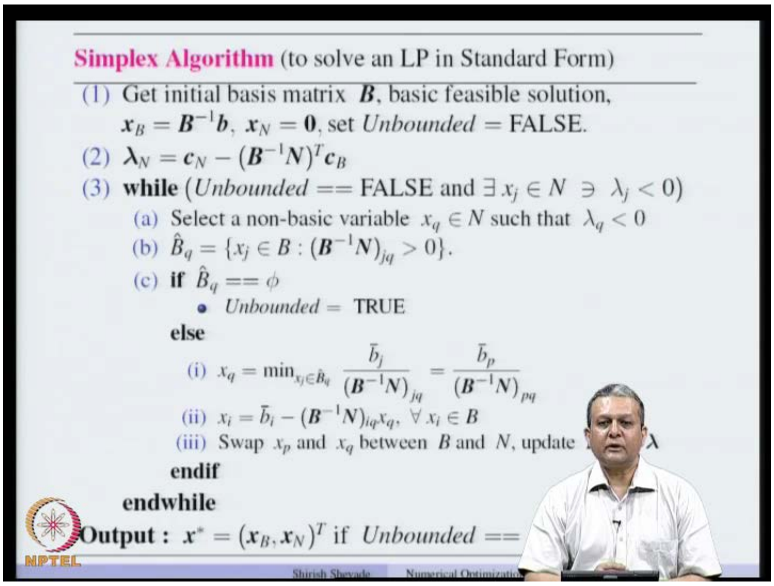

Now, one of the important steps of a simplex algorithm is to get the initial basis matrix and the basis vector set B. Once we have the initial basis matrix, we get the basic feasible solution which is x B equal to B inverse b and x N equal to 0. Now, the simplex algorithm is also used to find out whether the given problem is unbounded. So, initially we have no idea about the idea of the status of the problem; so we set the variable unbounded to be false and the variable will be made true if the problem is unbounded. So, given the basis matrix B, we get a initial basic feasible solution which is x B equal to B inverse b and x N equal to 0. Now, now we have to check whether this solution is optimal or not. So, for that purpose what we need to do is that, we need to find out lambda N and lambda N is nothing but C N minus B inverse N transpose C B. So, these are the relative costs associated with non-basic variables.

Now, as we saw earlier in the example as long as there is one lambda which is associated with the non-basic variables, if one of those lambdas is negative; that means that we have not achieved optimality. So, if the if the solution is bounded, that means that we have not set the flag associated with the boundedness of the solution to true. So, while unboundedness, unbounded is equal to false and there exists a non-basic variable x j such that lambda j less than 0; the lambda j is calculated here. Then that means that we have a chance to make a progress in the objective function or find out that the solution is unbounded.

So, the first step that we saw earlier was that to select a non-basic variable x q belonging to the set N; the set of non-basic variables, such that lambda q less than 0. And it is always possible to do that because we have already checked that there exists at least one x j in N such that lambda j is less than 0. So, we first select a non-basic variable to be made basic variable. And then the next step is to find out a basic variable which can be made non-basic and this can be done using ratio test. Now, for ratio test we need to first find out whether in the q th column of B inverse N matrix whether the entry is positive or not.

Now, as we saw in the last class if B hat q is empty. So, that means in that column q, the column we cannot find the n entering basic or leaving non-basic variable; leaving basic variable, then the problem is unbounded. So, if we cannot find x j which is in the basic variable set such that B inverse N j q is greater than 0; then it clearly shows that as x q increases all the variables, all the basic variables also increase in their values. And therefore, the solution becomes unbounded. So, if B hat q is a null set; then we set the unbounded flag to be true otherwise we find out the minimum ratio of B bar j and B inverse N j q and let that be associated with the basic variable p. So, that means that x p is the basic variable which will be made non-basic, while making x q as a basic variable.

Remember, that when x q earlier belong to the set N means that x q values earlier was 0 and now we are going to make it positive based on the minimum ratio test. And then we update all the basic variables. So, in that process x p will become 0; because we have already set x q to be B p bar by B inverse N p q. So, x p will be made 0 and then we swap x p and x q between the sets B and N and update the matrices B and the vector lambda. So that the relative costs associated with the basic variables are 0 and the matrix in the tableau corresponding to the basic variables is a identity matrix. So, that is why we need to update our basis matrix. And then this procedure is repeated; so next time if unbounded is not equal to true, so that is unbounded equal to false. Then there still exists some lambda associated with non-basic variables which is negative; then there is a scope for improvement in the objective function and then we move to the adjacent extreme point.

And, this procedure is repeated until either unbounded equal to true. So, which means that the problem is unbounded or we get a we get into a situation where lambda n's are all greater than or equal to 0 and we have reached the optimality. So, finally, as the solution what we get is x B and x N, if unbounded equal to false; otherwise, if unbounded equal to true, that means the problem is unbounded. So, this is a simplex algorithm as proposed by Dantzig sometime in late 40's and it is quite popular algorithm to solve linear programs.

Now, most of the steps in this algorithm are straight forward except that how to select a non-basic variable x q belonging to N; such that lambda q is less than 0. There could be many variables, non-basic variables belonging set N for which lambda q will be less than 0. So, how do we choose one non-basic variable from a set of non-basic variables which satisfy this property? So, that is one question that needs to be answered. The second question is that, how do we get the initial basis matrix B that may not always be readily available. So, how do we get the initial basic feasible solution? That is the second question that we need to answer. Now, the third question is that, will this algorithm terminate in finite number of equations or will the algorithm result in cycles? And these are the question we would like to answer now.

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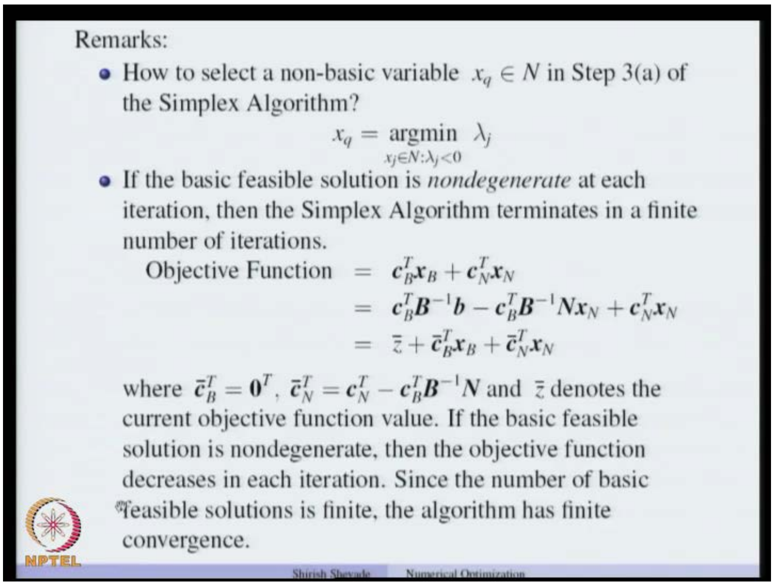

So, the first question is how to select a non-basic variable from the set N, if there are multiple variables? So, Danzig proposed a very simple idea that choose the variable which has the least value of lambda j. So, pick all the variables in the set N for which lambda j is less than 0 and then choose that variable which has the least lambda j value. So, this idea works fine. Now, the claim is that if the basic feasible solution is nondegenerate at each iteration, then the simplest algorithm terminates in a finite number of iterations. So, what we need to show is that if the solution is nondegenerate at every iteration, then by going to the adjacent point we increase; I am sorry we decrease the objective function value that is 1 part. And then since there exist only finite number of extreme points or finite number of basic feasible solutions. So, every time when we go to the adjacent, if we decrease the objective function; then in finite number of iterations our algorithm is going to converge. So, let us see that proof.

So, the objective function if we split it in terms of the basic and the non-basic variables, it will be C B transpose x B plus C N transpose x N. And x B is nothing but B inverse b at a given point. So, C B transpose B inverse b minus C B transpose B inverse N x N; remember that x N is 0. So, this two quantities are truly 0 but we have written them because we use them for explanation. Now, this first quantity is the current objective function z bar. And then we have C B bar transpose x B plus C N bar transpose x N; where C B bar is at 0 vector and C N bar is the vector C N transpose minus C B transpose B inverse N. Now, C B bar is 0 and x N is 0; so this last 2 quantities are 0 and the current objective function value at x is nothing but C B transpose B inverse b or z bar. Now, if we decide to make if any of the entries in the C bar N vector is negative; then as we saw earlier x N, one of the basic variables corresponding to that variables which is negative can be made basic by increasing that variable to a positive value.

And, therefore, once we increase that value to a positive value the objective function value will be because C B bar is any way 0. And since there exists a finite number of basic feasible solutions because we saw that if we have n variables and n constraints; then there are at the most n c m number of basic feasible solutions. So, since there number of basic feasible solutions is finite the algorithm is going to converge in finite number of iterations. So, under the conditions of nondegeneracy, the simplex the nondegeneracy at very iteration; the simplex algorithm terminates in finite number of iterations.

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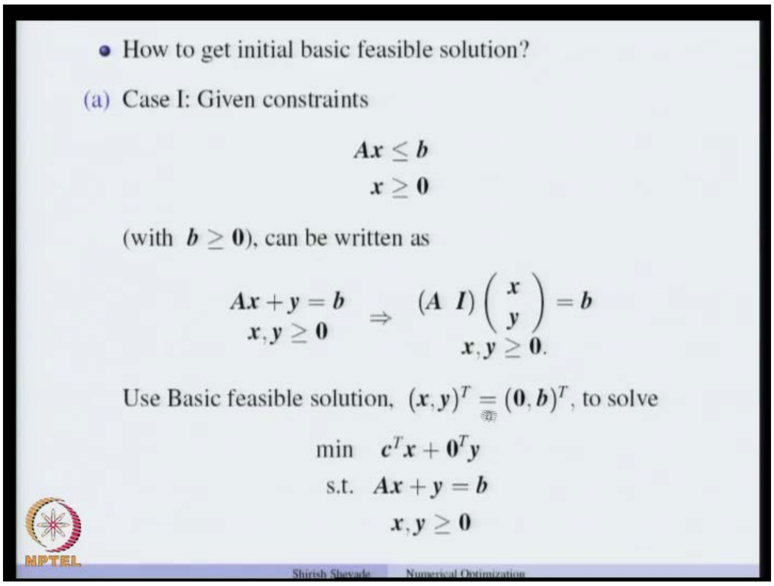

Now, the third important question that we need to address is how to get initial basic feasible solution? Now, for some problems it may be easy to get this initial basic feasible solution. While for some problems as we will see, we may have to solve another sub problem to get initial basic feasible solution. So, let us first see the easy case where it is the basic feasible solution can be determined quickly. Now, if the constraints are of the type A x less than or equal to b where x nonnegative; where only assume that b is nonnegative. And if in the initial constraints we are particular from component b is negative; then that can be easily that equation can be easily converted to the form A x equal to b, where b is nonnegative.

So, by using some matrix transformations matrix operations one can convert a given set of constraints to the form where b is nonnegative. Now, this constraint can be written as A x plus y equal to b; where x y's are nonnegative. Now, to get the basic feasible solution, what we need is that we need a sub-matrix of the constraint set matrix to be an identity matrix. So that the solution can be, the basic feasible solution can be easily obtained from the constraint set. And by introducing the slack variables we can see that the constraint set now will have a sub-matrix; which will be an identity matrix. So, this set of equations can be written in the form A I into x y equal to b; so this is nothing but A x plus I y equal to b and x y nonnegative. Now, the constraints set matrix if if you look at this matrix, now there is a sub-matrix which is a identity matrix.

So, in such a case it is easy to get a basic feasible solution. So, what we can do is that we can set A x equal to 0 and y equal to b. And since b is greater than or equal to 0 that means y is greater than or equal to 0. If x is set to 0 and then we get a basic feasible solution. So, one can use the basic feasible solution (x y) to be (0, b). So, for constraints like this, it is a straight forward thing to get a basic feasible solution. On the other hand so if we use this basic feasible solution, then one can solve the problem minimize c transpose x plus 0 transpose y subject to A x equal to B x y greater than or equal to 0. So, the in the objective function the costs associated with the slack variables are 0 and one can solve this problem with the initial basic feasible solution as x equal to 0 and y equal to b.

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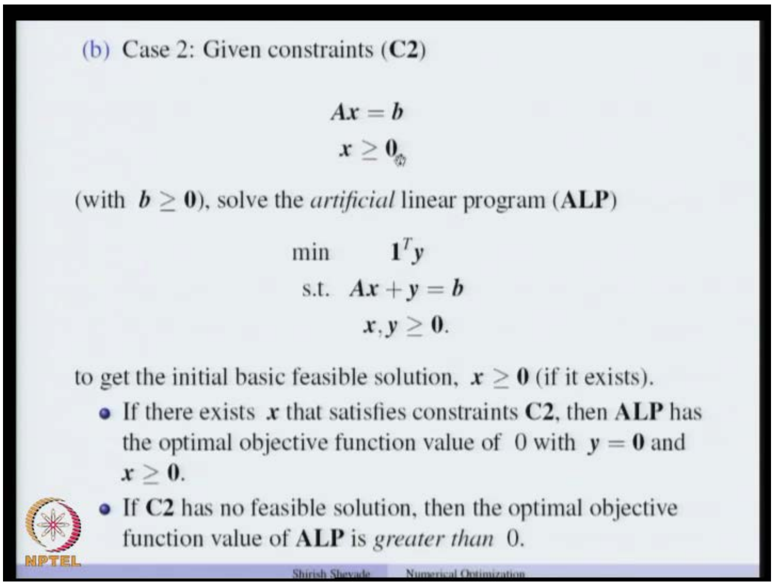

Now, on the other hand if the constraints which are given are A x equal to b and x nonnegative then it becomes difficult to get a initial basic feasible solution. Now, in this case it may be a good idea to introduce artificial variables. So, suppose y is used to denote the artificial variables then one solves an artificial linear program. So, minimize 1 transpose y subject A x plus y equal to b and x y equal to 0, so where 1 is a vector of a 1 is vector of all 1 of size depending upon the size of y. So, this is artificial linear program that one needs to solve and at the solution if the original system had a solution then the solution of this the optimal objective function value of this will be 0. So, which means that all y's are 0s and we get x which satisfies this.

On the other hand if the solution of this artificial linear program is positive then we can conclude that there is no basic feasible solution to the given set of constraints. So, if there is no identity sub-matrix associated with the constraints set matrix then it may a good idea to solve an artificial linear program to get either to get a basic feasible solution for this system of equations or conclude that there does not exist a basic feasible solution for the given system of equations or given system of constraints. So as I mentioned that there exists x that satisfies constraints then the artificial linear program has the optimal objective value of function 0 with all y's to be 0. And therefore, if a y's are 0 then we have as a solution of this program x where A x satisfies b A x equal to b and x is nonnegative. So, x the basic feasible solution x can be obtained by solving this if such a solution exists or if there exists some x which satisfies this constraint. On the other hand if c 2 has or the constraints have no feasible solution, then the optimal objective function of LP will be greater than 0 and in that case we can conclude that c 2 has no feasible solution.

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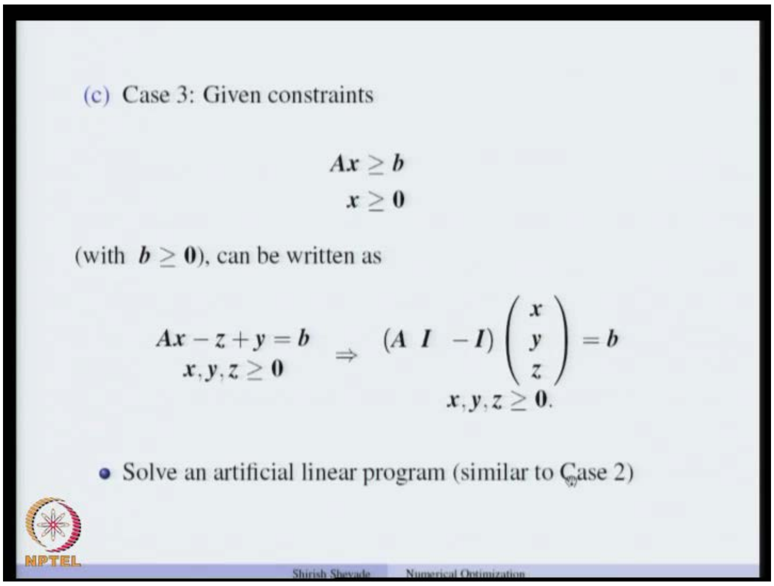

Now, if you are given the constraints of the type A x equal to A x greater than or equal to b, now this constraints can be converted to equality constraints by using surplus variables but then that those surplus variables will be associated with the negative of the identity matrix and therefore, we need again to use artificial variables and A x greater than or equal to B can be written as A x minus z plus y equal to B.

So, when we introduce surplus variables we wrote this Ax minus z equals to b but in order to get a initial basic feasible solution we add this artificial variables y. So, we have A x minus z plus y equals to b and x y z nonnegative and this can be written as a I minus I x y z equal to b and x y z nonnegative. now, if you look at the constraint set matrix which is consisting of sub-matrices A I and minus I, so A is associated with x I is associated with y and the negative of the identity associated with z. So, you will see that there’s a sub-matrix which is identity matrix and therefore, the variable now can be made a basic variable so x an d z become non-basic and y become basic with the value b and then we solve artificial linear program, as we did in the previous case to get a either a basic feasible solution for this constraints or conclude that there does not exist x which satisfy this. So, the rest of the steps are similar to the one's we discussed in the case of A x equal to b or the previous case.

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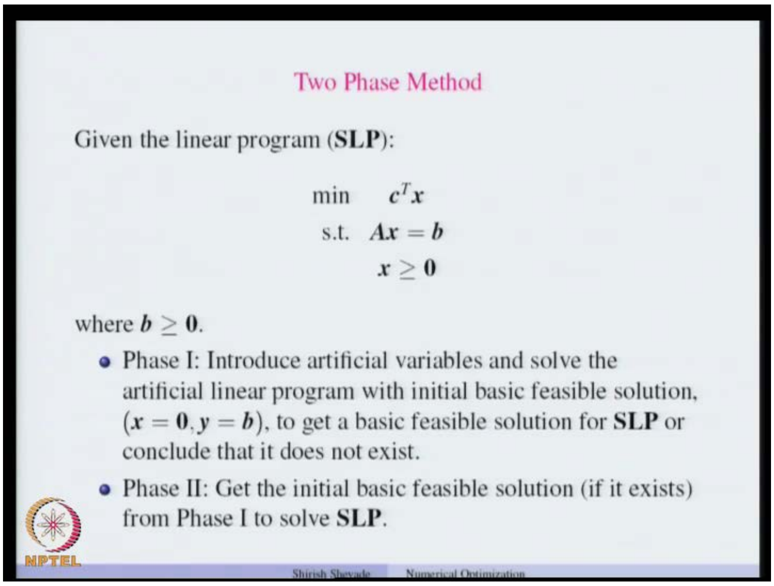

Now, this method is called two phase method. The first phase corresponds to finding a initial basic feasible solution for a given system of constraints or conclude that such a solution does not exist. And if the basic feasible solution exits in the first phase then use that to solve the original linear program. Remember that in the phase one to find out whether a basic feasible solution exits or not, we need to solve an artificial linear program and that can be solved again using simplex method. So, given the linear the standard linear program minimize c transpose x subject to x equal to B x nonnegative and b is also nonnegative. The phase one introduces artificial variables and solves the artificial linear program with the initial basic feasible solution as x equal to 0 and y equal to b. So, y is the artificial variable and that linear program solution is used to get a basic feasible solution for S LP if such a solution exits.

Otherwise one can conclude that the basic feasible solution for the given program does not exits. Now, after having obtained an initial basic feasible solution for this program through phase one then the phase two, the given SLP is solved. So, phase one solves the artificial linear program while phase two solves the actual linear program that we want to solve.

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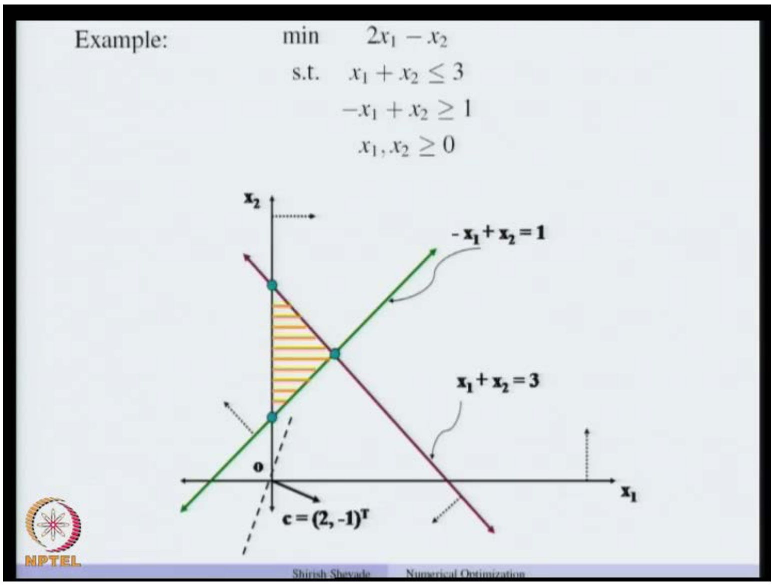

So, let us consider an example to show how this two-phase method works so let us consider a problem to minimize 2 x 1 minus x 2 subject to x 1 plus x 2 less than or equal to 3 minus x 1 plus x 2 greater than or equal to 1 and x 1 and x 2 nonnegative. Now, the first constraint is a is of the type less than or equal to inequality and the second constraint if of the type greater than or equal to inequality. Now in the first constraint we can add a slack variable and that itself can be made as a artificial variable in the second constraint. We need a surplus variable and also an artificial variable, so if you do that then we can get a identity matrix associated with the constraint set matrix. But before we go into the details let us look at the feasible set and the constraints set we have. So, the first constraint is that x 1 plus x 2 less than or equal to 3 so x 1 plus x 2 equal to 3 is the line which is shown in the x 1 x 2 space and x 1 plus x 2 less than or equal to 3 which means that we are interested in the half space pointed by this arrow.

Now, the second constraint is minus x 1 plus x 2 greater than or equal to 1. So, this is the equation of the line minus x 1 plus x 2 equal to 1 so the line cuts the x 2 axis at 0 1 and minus x 2 plus x 2 greater than or equal to means the half space pointed by this arrow. Further we have the nonnegativity constraints as usual. So, x 1 greater than or equal to 0 corresponds to this half space and x 2 greater than or equal to 0 corresponds to this half space, the vector c is 2 comma minus 1 so which points in this direction. So, this plane is the hyper-plane which is the, whose normal is the vector c and hyper-plane passes through the origin. So, when you saw geometric interpretation of linear programs we saw that we have to get a hyper-plane which is parallel to this and where the function value is the list. So, by looking at this figure we can conclude that, so this hyper-plane if you start if you place it here place a hyper-pane parallel to this passing through then as we move in the c direction the optimal the objective function value decreases and finally, then we move when we reach this point, we will see that the objective function value is the least. So, therefore, this turns out to be our optimal point using graphical method so let us verify this using the simplex algorithm that we have seen.

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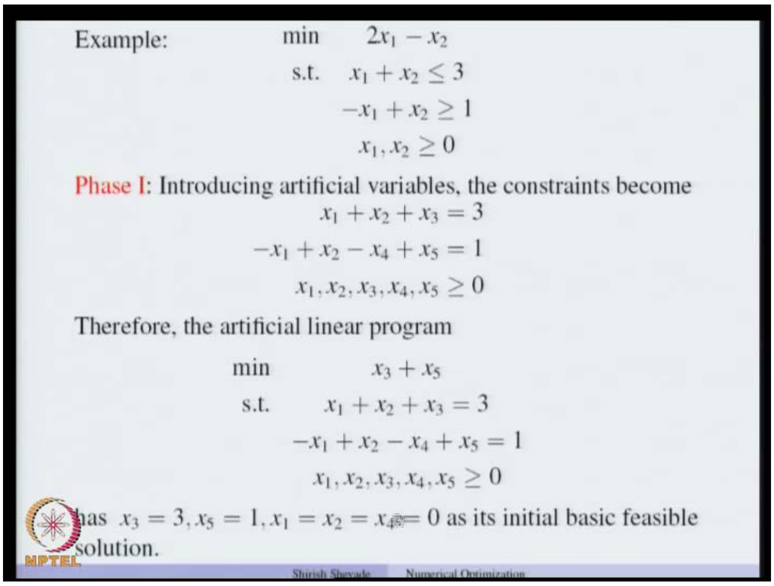

Now, let us look at the constraints. So, we introduce the artificial variables so we introduce the artificial variable x 3 and also x five so x 4 becomes a slack surplus variables x 3 can also be called the slack variable. But the important thing is that we can get a identity a sub-matrix in the constraint set matrix associated with the variable x 3 and x 5 and that gives us the basic feasible solution for the artificial linear program. So, for the artificial linear program the variables x 1 x 2 and x 4 are 0 and x 3 equal to 3 and x five equal to 1. So, the artificial linear program is the sum of the artificial variables, so which is x 3 and x five sum of x 3 and x 5 and subject to the same constraints.

And, we can use the simplex method to solve this artificial linear program and that is what we will do now. So, as I said x 3 equal to 3 and x five equal to 1 is a basic feasible solution and the corresponding non-basic variables are x 1 x 2 and x 4 which are all set to 0 and that is the initial basic feasible solution for the artificial linear program.

Remember that we first saw in the phase one. We first solve this artificial linear program and in the phase two, if we are able to get the solution in phase one.

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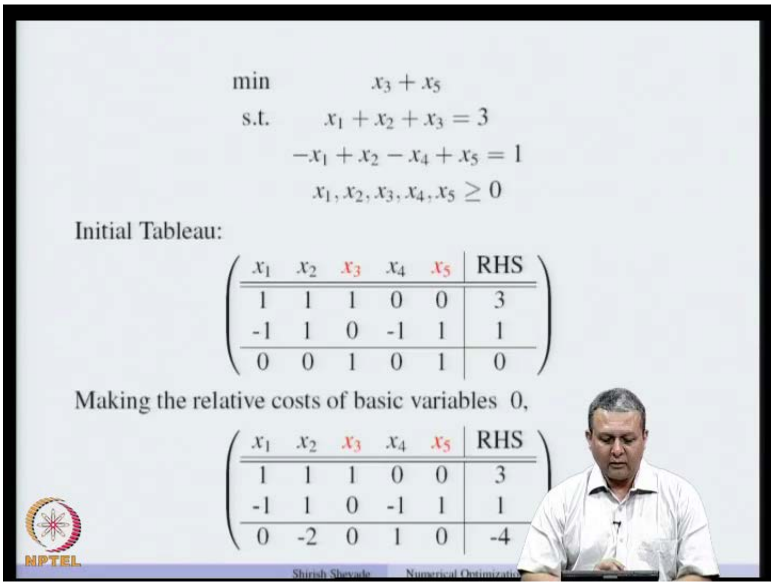

If we are able to get a solution which is feasible for this constraint set in phase 1 then we move on to phase 2. So, as is the case in our earlier examples we first construct a simplex tableau associated with this set of constraints x 3 and x 4 are our basic variables and x 1 x 2 and x five are non-basic variables x 3 equal to 3 and x 5 equal to 1. Now, the first step is to make the relative cost of the basic variables 0, now here the relative cost of the basic variables is 1. So, we first make them 0, so that is done by multiplying first this row by minus 1 this row minus 1 and then adding the 2 to the last row so if you do this operation you will see that the basic variables.

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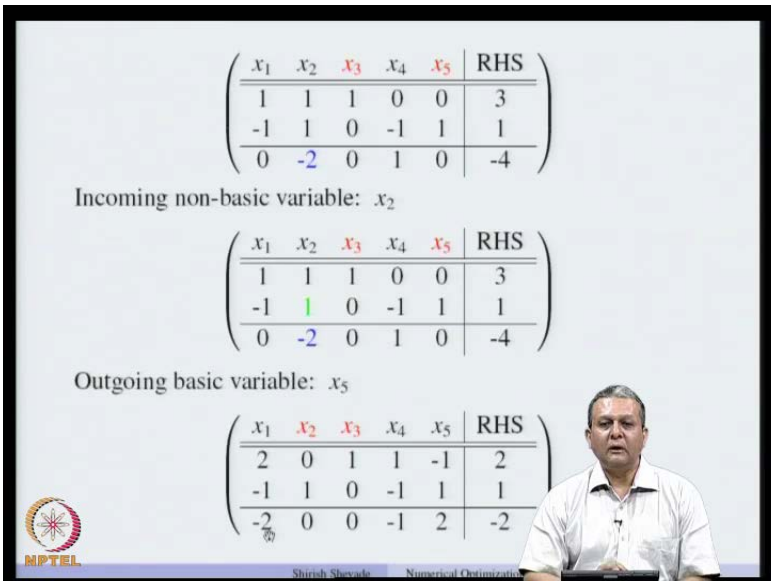

X 3 and x five the relative costs are 0 and we get a initial point where the objective function is 4. Now, you will see that for the non-basic variables x 1 x 2 and x 4 the relative costs are 0 minus 2 and 1, so clearly not all relative costs associated with non-basic variables are nonnegative so that means that x 2 is a candidate for becoming basic vector basic variable. So, this is shown by this blue mark here, that this cost is negative and x 2 becomes a candidate to become a basic variable.

Now, once again x 2 becomes a candidate to become a basic variable we need to find out which is the outgoing basic variable or which basic variable can become non-basic and that depends on the ratio test. So, we pick all the elements in this column which are positive in this case both are positive and take the ratio so 3 by 1 by 1, so this the second row which corresponds to the basic variable x 5 is the candidate to become non-basic, so x five will which is the existing basic variables will become non-basic in the next iteration. And the remaining step is to make all the entries in this column zeros so that x 2 can become basic variable along with x 3 and this is shown here, now earlier the objective function value was 4 now to has come down to 2 and x 2 and x 3 are now basic variables x 3 equal to 2 and x 1 equal to 1 and the relative cost associated with non-basic variables are minus 2 minus 1 and 2. So, again which are not strict not nonnegative and therefore, there is a scope to improve the objective function further.

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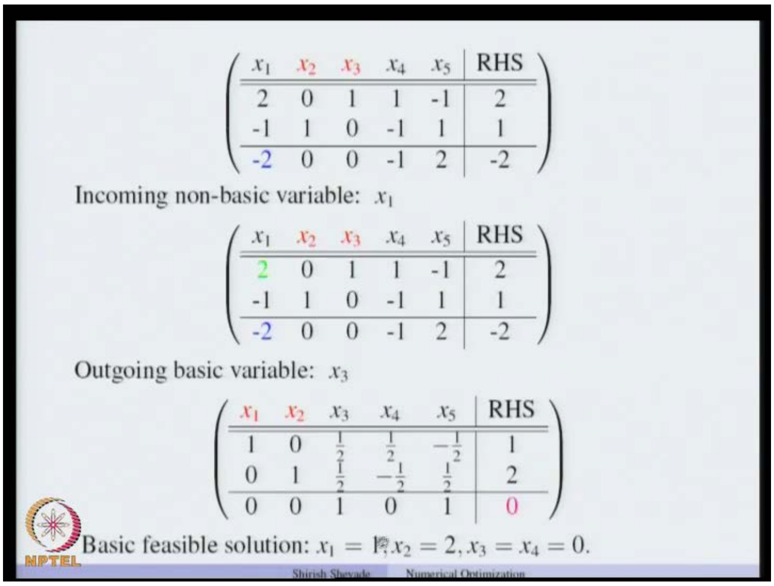

Now, suppose we choose x 1 to be entering basic variable into the current basic variable set and then we have to look at the outgoing basic variable from the existing basic variable set and that is again based on the ratio test. So, in this case only this first rows entry is positive so we take this ratio and obviously that is the minimum 1. So, the first row corresponds to the basic variable x 3, so x 3 is the basic variable which is which will lead the existing basis to make a entry for x 1 to become a basic variable. Now, what we do is that we first make sure that at this point the variable has a at the this point the tableau has a value 1. So, this is done by dividing the entire row by 2 and therefore, the x the new basic variable x 1 will have a value which is 1.

And, then once we make this entry in the tableau to be 1 we have to make sure that the rest of the entry is in that column have to be made 0, so that x 1 will represent the basic variable. Note that by making the rest of the entries 0, first of all we make sure that the columns in the first the columns the first m entries in this column, they are associated with the identity matrix and the last entry when we make it 0 we make sure that the relative cost associated with that basic variable is 0. So, the so this first column associated with x 1 now becomes a column with the identity matrix in the simplex tableau. This entry is made 0 which means that the relative cost associated with that variable is made 0. So, these are all done using simple matrix operations and you’ll see that the current objective function value is 0.

So, x 1 and x 2 are basic variables x 1 equal to 1 x 2 equal to 2 the non-basic variables are x 3 x 4 x five and the corresponding relative costs are nonnegative. So, which means that x 1 equal to 1, and x 2 equal to 2 is the basic feasible solution for the original linear program, the very fact that we have got the optimal objective function value of 0 for the artificial linear program shows that the initial basic feasible solution exists for the original linear program and that basic feasible solution is x 1 equal to 1 and x 2 equal to 2. So, this completes the phase 1 of the simplex method. So, the basic feasible solution for the original program is x 1 equal to 1, x 2 equal to 2, and x 3 x 4 are 0.

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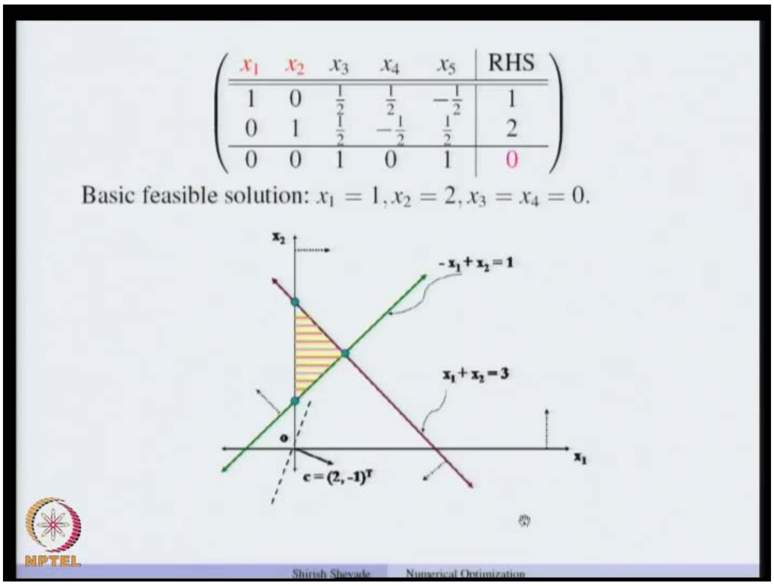

So, x 1 equal to 1 and x 2 equal to 2 corresponds to this point and this is going to be our starting point for our next phase. So, from the original set of constraints it was very difficult to find out what is what will be the initial basic feasible solution for the given problem. But by using phase one of the simplex method and finding out that optimal objective function value of phase one of the artificial linear program was 0. And therefore, that gave us the initial basic feasible solution for the original problem and therefore, we start with this point and then use phase 2 of simplex method to solve the original linear program.

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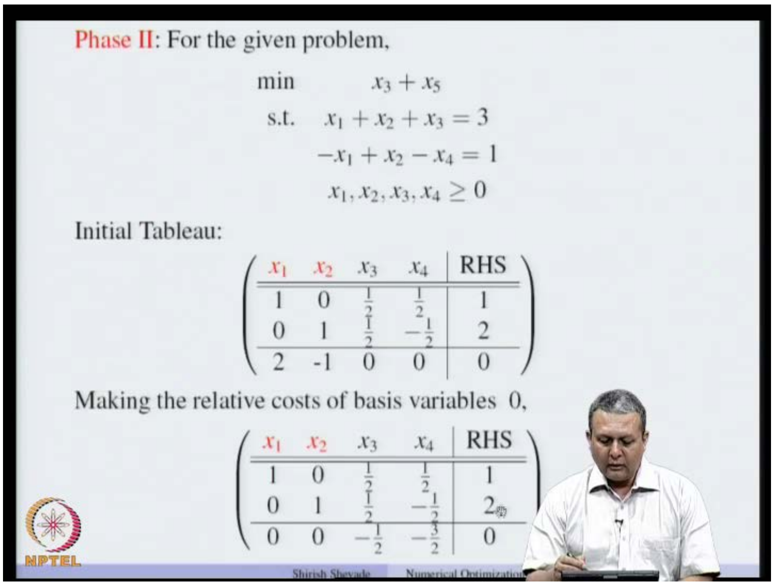

Now, now for this problem the initial tableau was I am sorry this should be 2 x 1 minus x 2 that should be the objective function and for this objective function the initial tableau was x 1 equal to 1 and x 2 equal to 2. Now, as you will see that the these are the basic variables the relative costs of this basic variables are not 0 so first, we make them 0 by using the matrix operations. And now you'll see that the x 3 and x 4 are the non-basic variables and the corresponding relative costs are negative. So, that means there is scope to improve the objective function. Now, note that the current objective function is 0.

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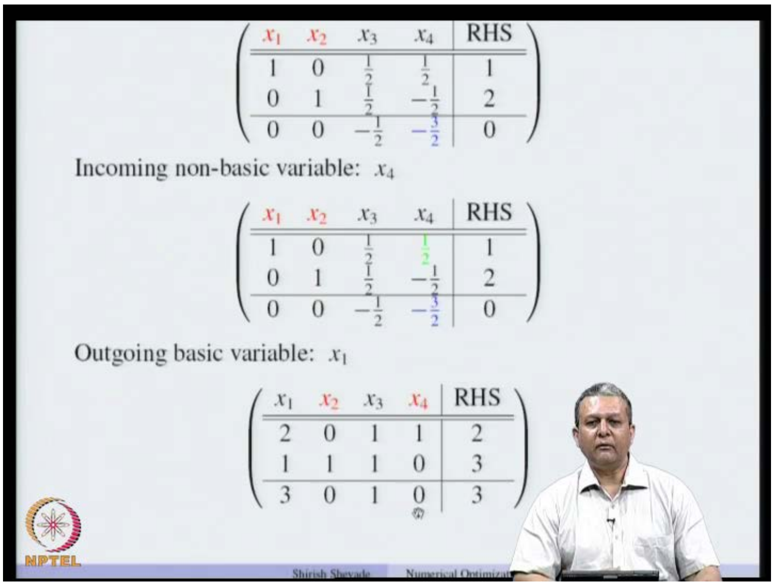

So, suppose we decide to use x 4 as a incoming basic variable now x 4 is the incoming basic variable because the incoming cost of x 4 is negative then we have to find out the outgoing basic variable so using the ratio test. So, in this case the entry corresponding to the variable x 1 is the only positive entry in this column and therefore, x 1 is the only choice for making it non-basic therefore, x 1 will be made non-basic by bringing in x 4 into the basic vector set now again doing the same matrix operations you'll see that x 2 and x 4 are the basic variables x 4 equal to 2 x 2 equal to 3 and associated with the non-basic variable x 1 and x 3, the costs are 3 and 1 which are positive. So, that means that we have indeed found a solution for where the objective function value is minus 3.

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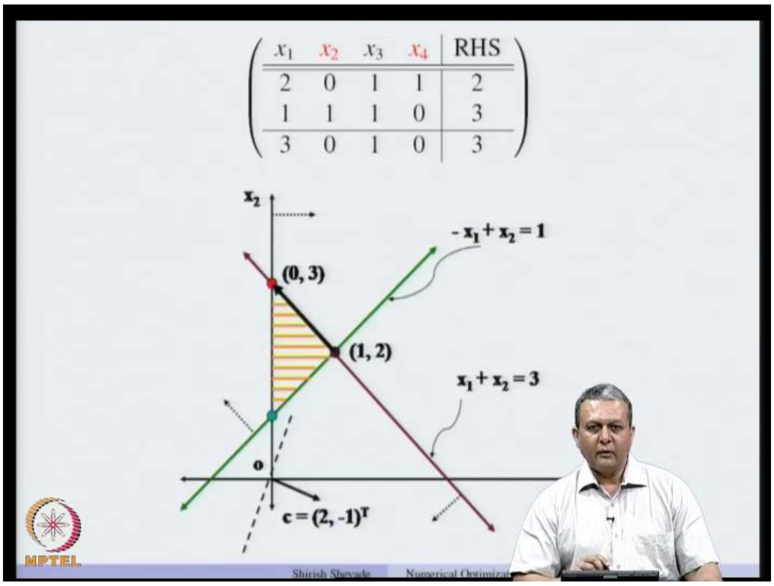

And, this is shown in this figure, where we stared from 1 2 in this phase 2 of the simplex method, and then we took a step 0 3 to the neighboring vertex 0 3 and at this point the objective function which is 2 x 1 minus x 2 it is value is minus 3. And that is the least among all the 3 vertices. So, if the initial basic feasible solution is not available, we have seen that phase one can be used to get a basic initial feasible solution by solving an artificially linear program and then use that in phase two to solve the original linear program. It may so happen that in phase one, we will, we may not be able to find the basic initial feasible solution and we will see an example of that in the next class.

Thank you.