Numerical Optimization

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Lecture - 32
Basic Feasible Solution

Hello, welcome back to this series of lectures on numerical optimization. In the last class we discussed some of the properties of solution of a linear programming problem. In particular we saw that for a linear program, the solution always lies on the boundary of the feasible set. And when the constraint set is compact, then the solution also lies at an extreme point or vertex. We also saw that any linear program can be written in the standard form where the objective is to minimise C transpose x subject to the constraint A x equal to b and x non-negative.

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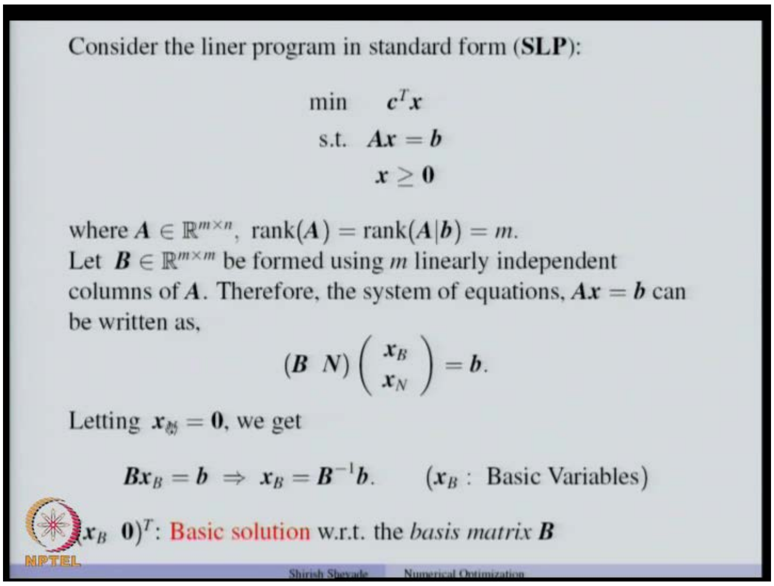

So, we will continue working on the linear programs in standard form. And in today's lecture we will find a way to characterize an extreme point of a constraint set. So, let us consider linear program in standard form; minimize C transpose x subject to A x equal to b, x greater than or equal to 0 where A is m by n matrix and rank of A is m and rank of A is equal to rank of A appended with b indicates that the system of equations x equal to b is consistent. Now, let us take m linearly independent columns from A and form a matrix B. Now, since the rank of A is m we can always get m linearly independent columns from the matrix A. And without loss of generality we assume that these columns are the first m columns of the matrix A.

So, this is still A x equal to b can be written as B and N which are the parts of the matrix A and then the vector x is split into two parts; x B and x N. Therefore, A x equal to b is written in this form which if we expand we get B x B plus N x N equal to b. Now, if we let x N is equal to 0 in that in this system, then what we get is B x equal to b. And since B contains m linearly independent columns, B is invertible; so we can write x B to be B inverse b and x B is called the basic variable. And in particular x B and x N are the two components of the vector x and if you set x N to 0 and find x B using this; that is called a basic solution to the system A x equal to b.

Now, the word solution here is misnomer; in the sense that we are not talking about the solution of the linear program in the standard form which is shown here. But we are only talking about the solution of the system of equations A x equal to b. So, in some text books this is also referred to as basic point rather than the basic solution to avoid any confusion. But we will continue to use the word basic solution because that has been the common practice in many text books or in the literature. So, this basic solution is associated with the basis matrix B and hence it is called the basic solution. Now, if the basis matrix changes the basic solution also changes in particular the first part or the first component of the x B changes. Note that we are letting x N to be 0; so this part remains the same, only the first part changes.

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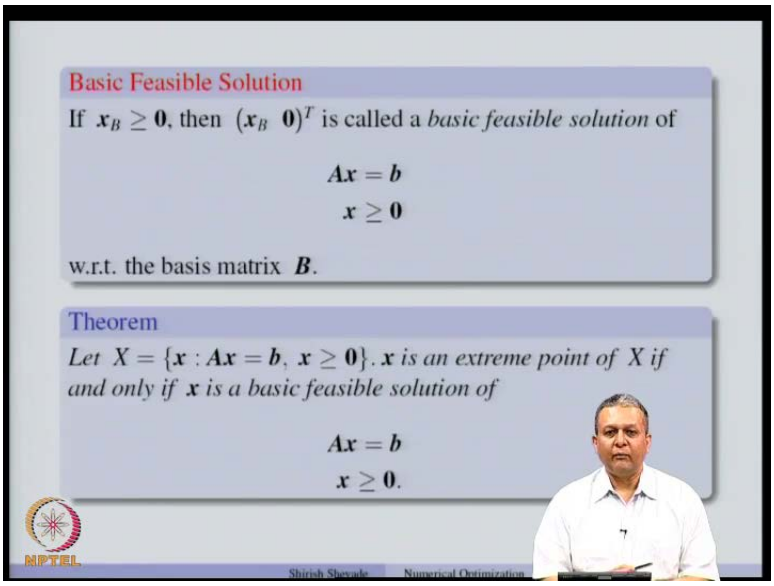

Now, let us look at basic feasible solution. So, we get the solution x B 0 and if in addition to that x B is also nonnegative; then it is a it is called the basic feasible solution, the system A x is equal to b and x nonnegative with respect to the basis matrix B. So, the component x N is always 0 and then the component related to the basis matrix is nonnegative. So, x B is called the basic variable and x N is called the non-basic variable. So, B is the matrix associated with the basis variable and N is the matrix associated with non-basic variable. So, the columns of the matrix A which are associated with B are called the basic variables. So, we denote them by the set B and the columns associated with the matrix N we call them as non-basic variables and denote by the set N.

Now, let us look at an let us look at an important theorem which gives the correspondence between the extreme point and the basic feasible solution. So, let us assume that x the said X is the set of all x s that A x is equal to b and x nonnegative and x is an extreme point of X; if and only if x is a basic feasible solution of A x equal to B and x equal to x greater than or equal to 0. So, what it means is that if we are given an extreme point of x, then that extreme point is also basic feasible solution of the system. And if we are given a basic feasible solution of the system of equations, that is also an extreme point; so this an algebraic way to characterize an extreme point of x. And as we saw in the last class that an extreme point or the solution of a linear program lies at an extreme point if the set x is compact. And therefore, it is enough to look at the extreme points of the constraint set to get the solution of a linear program. And although this method uses a way to find out the solution of a linear program, we do not have any algebraic characterization of extreme points. So, this theorem gives us an algebraic characterization of an extreme point. So, let us first prove this theorem.

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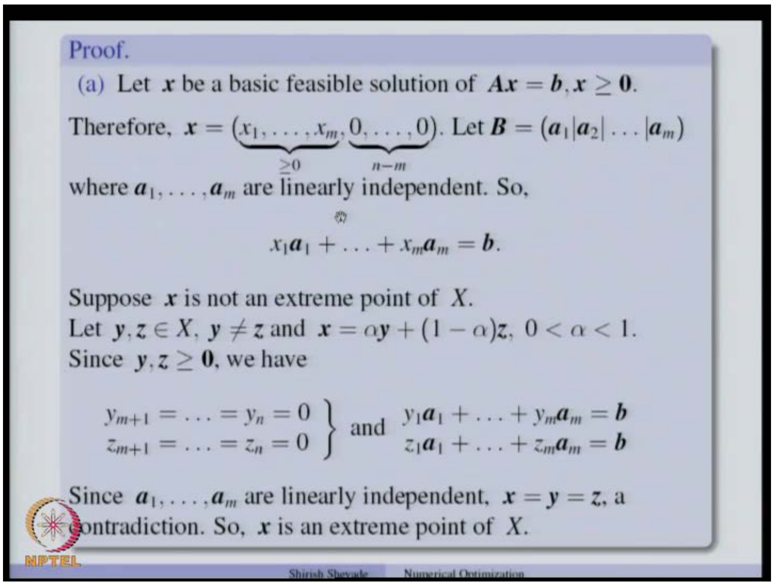

So, we assume that x is the basic feasible solution of A x equal to b and we want to show that x is an extreme point of the set x; set x which is obtained by this system A x equal to b, x nonnegative. So, to show that it is an extreme point, we have to show that; that point cannot be represented as a strict convex combination of any two other points and we prove this by contradiction. Now, since x is the basic feasible solution of A x equal to b, we know that x has two components corresponding to the basis B and corresponding to the non-basic matrix N. So, without loss of generality, we assume that the first m components of x are associated with the basis matrix b and they are nonnegative. And remaining n minus m components of x are 0.

Now, let us choose the m linearly independent vectors from the matrix a and matrix B which contains a 1 to a m as it columns and a 1 to a m are linearly independent. And these are the these are the components of the matrix B associated with the first m components of the vector x. And therefore, we can write x, we can write b as combination of a 1 to linear combinations of the vectors a 1 to a m; so we have x 1 a 1 plus x 2 a 2 up to x m am equal to b.

Now, let us assume that x is not an extreme point. So, if x is not an extreme point x can be written as a strict convex combination of two different points. So, let us take those points as y and z. So, if x is not an extreme point, let y and z belongs to the set X; so set X is the set formed using this A x equal to b and x nonnegative. So, y and z belong to the X set X and y not equal to z; such that x is a strict convex combination of y and z z. So, x equal to alpha y plus 1 minus alpha z; where alpha is in the open interval 0 to 1, so which means that it is a strict convex combination of y and z. Now, y and z belong to the set X; so that means that y and z are nonnegative. So, all the entries of the vectors y z are nonnegative. And here we are assuming that x is written as alpha y plus 1 alpha z; so alpha is a positive fraction and 1 minus alpha is also positive fraction. And therefore, if x has this n minus m components 0, then the corresponding n minus m components of y and z also have to be 0.

So, since y and z are nonnegative the last n minus m components of y and z are 0. And therefore, if they are 0 then we can write y 1 a 1 plus y 2 a 2 plus y m a m equal to B and z 1 a 1 plus z 2 a 2 plus z m a m equal to b. Now, a 1 to a m are linearly independent; so there exists a unique combination of unique nonnegative combination of a 1 to a m which uses the vector bb. And therefore, y is equal to z and which implies that whatever we assumed here was not correct and therefore, x has to be an extreme point of x. So, since a 1 to a m are linearly independent, we have y equal to z and therefore, x equal to y equal to z, which is a contradiction. Because we assumed that y not equal to z and x is strict convex combination of y and z which is a contradiction. And therefore, x is an extreme point of the set x. Now, we can show the other part by assuming that x is an extreme point of X and then show that x is a basic feasible solution.

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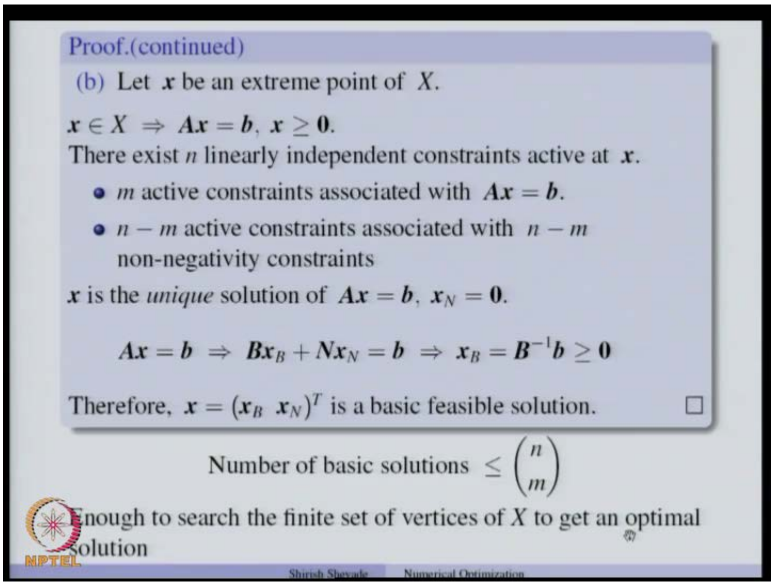

So, let us assume that x is an extreme point of the set X. Now, since it is an extreme point, we know that there are n linearly independent constraints which are active at the extreme point. Now, x belongs to the set X which means that A x equal to b and x nonnegative and being an extreme point there exists n linearly independent constraints which are active at x. Now, out of these n active constraints, the m constraints come from the equality that A x equal to b. So, there are m equality constraints; so all the equality constraints are active at a feasible point and extreme point is also a feasible point in our case. So, there are m active constraints associated with A x equal to b and the remaining n minus m constraints have to come from the non-negativity part which x greater than or equal to 0. So, there are n such constraints, inequality constraints; out of them n minus m have to be active at the extreme point A at the extreme point. So, n minus m constraints come from this inequalities and m constraints come from equalities; this together form the n active constraints at the current point x.

Now, this n minus m constraints, nonnegativity constraints which are active; which means that for them the value of the component of x is equal to 0 and they correspond to the x N part of the variable x and the x B part comes from the m active constraints. And therefore, x is the unique solution of A x equal to b and x N equal to 0. And therefore, A x equal to b which implies B x B plus N x N equal to b and that implies x B equal to B inverse b and that has to be nonnegative; because we have assumed x is a feasible point.

So, we have x B which is nonnegative, x N which is 0; so these two components of x, x B and x N together indicate that x is an extreme point.

The n minus m components are 0 and the m components associated with the basis matrix B are nonnegative and therefore, x can be written as x B and x N as two parts and it is a basic feasible solution. So, this is an important result which gives us the characterization of an extreme point. So, if the constraint set is convex, then the solution of a linear program if it exists lies at an extreme point and an extreme point is characterized by basic feasible solution of the type x B, x N; where x B is nonnegative and x N equal to 0. So, it is finding an extreme point just amounts to finding a basic feasible solution at that particular point. Now, there are m possible ways of choosing a there are m m columns that we have to choose from n. And n choose m is the number of possible ways of choosing those m columns from the set of n columns.

So, therefore, m choose m is an upper bound on the number of basic feasible solutions that we can have; because not every time the m columns that one chooses are linearly independent. So, therefore, at the most m choose n is the number of ways of choosing m columns from the set of n columns. And since there is a correspondence between an extreme point and a basic feasible solution, we can say that there are finite number of extreme points for a given linear program. And therefore, any algorithm which looks at only the extreme points of a constraints set will converge in finite number of iteration because only finite number of extreme points exists for a linear program with compact constraint set. So, the number of basic solutions is n choose is at the most n choose m and therefore, that since that number is finite it is enough to search this finite set of vertices X to get an optimal solution of a linear program.

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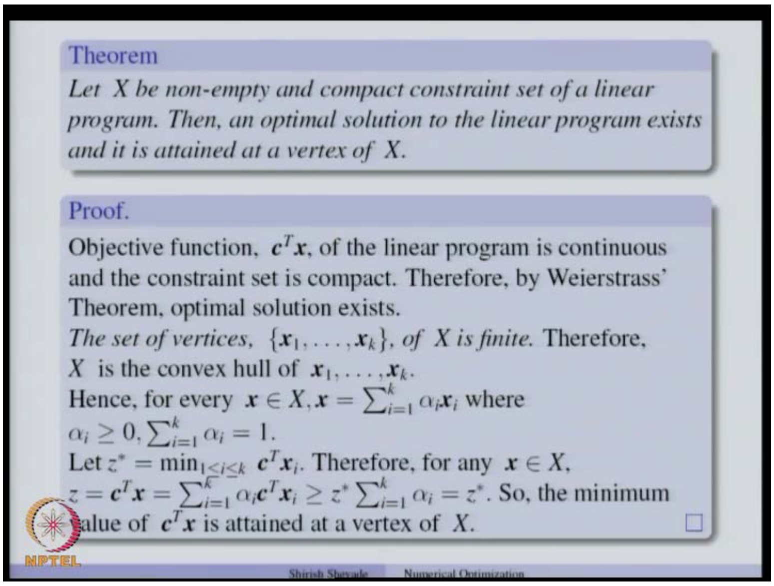

Now, we will formally show that if the constraint set of a linear program is compact and nonempty, then an optimal solution to the linear program does exist and it is attained at a vertex of that constraint set. Now, first of all we note that the objective function C transpose x of a linear program is continuous. And the constraint set we have assumed that it is nonempty and compact. So, therefore, if you want to minimize a continuous objective function over a compact constraint set by Weierstrass theorem we know that the solution exists.

So, optimal solution exists due to Weierstrass theorem in this case. Now, we have already seen that there is a one to one correspondence between the extreme points or the vertices of a constraint set and basic feasible solution. And increase in the number of basic feasible solution is finite the number of vertices of the constraint set is also finite. So, let us denote this set of vertices as x 1 to x k; there are k vertices. So, since the constraint set is a convex set any point in the convex set can be written as a convex combination of this k vertices or x is the convex hull of this k vertices. And any x in the set X can written as sigma alpha to sigma x i, where alpha is nonnegative under some.

So, let the optimal objective function value be z star and z star is minimum of C transpose x i are going from 1 to k. So, therefore, for any x which belongs to the set X, z is equal to C transpose x. And C transpose x can be written as any point x can be written as sigma alpha x i; where alpha's are nonnegative and sigma alpha is 1. And this quantity is greater than or equal to z i star into sigma alpha i and that is nothing but z i star since sigma alpha is 1 alpha i's are nonnegative, because of our definition. So, at any point x which belongs to the set X, since it belongs to the convex combination of the vertices; the optimal objective the objective function value at x which is z is always greater than or equal to z star which is the objective function value at the vertices. So, therefore, the minimum value of C transpose x is attained at a vertex of x. So, from this theorem it is clear that we need to search only over the set of finite vertices of a linear program to find out an optimal solution if it exists.

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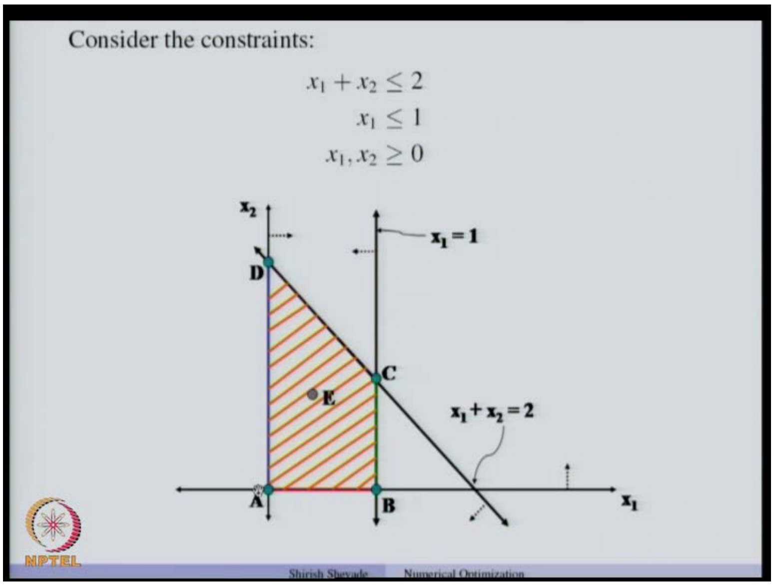

Now, let us consider the constraints x 1 plus x 2 less than or equal to 2, x 1 less than or equal to 1, x 1, x 2 is greater than or equal to 0. Remember, that every vertex corresponds to an extreme point or a basic feasible solution and that is not necessarily an optimal solution of a linear program. Now, let us denote this set of constraints using figure. So, x 1 plus x 2 equal to 2 is the constraint is shown by the line, x 1 equal to 1 is another constraint and then the intersection of all this x 1 plus x 2 less than or equal to 2, x 1 less than or equal to 1 and x 1 x 2 nonnegative is shown here. And A B C D are the vertices of this feasible region. So, if you want to minimize any objective function, linear objective function with respect to this constraint set; it is enough to look at the objective function values at A B C and D. So, what we show now is the correspondence between the extreme point and the feasible solution basic feasible solution.

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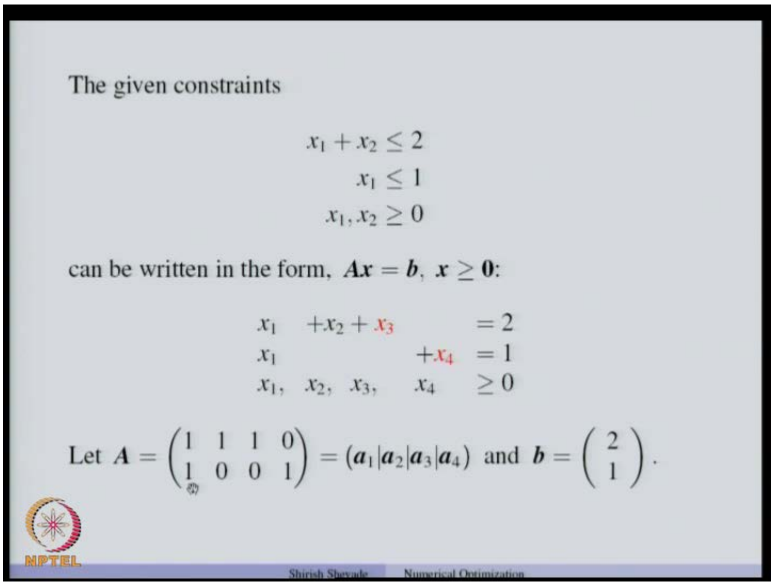

Now, to show that we first write this set of constraints in standard form, where we have only equality constraints and nonnegativity constraints. So, this set of constraints can be written in the form A x equal to b, x greater than or equal to 0 by adding slack variables for the first two constraints. So, these slack variables are x 3 and x 4, for the first two constraints and now we have 4 variables and all are nonnegative.

So, first we bring the given set of constraints to the form A x equal to b, x greater than or equal to 0 and then we form the matrix A, the vector B as the columns of the columns related to different variables. So, if you take the first constraint; so if you take the coefficients of x 1 x 2 x 3 and x 4, so those are 1 1 1 and 0, so those are included here. And the second constraints the variables are x 1 and x 4 and the corresponding coefficients are 1 0 0 and 1; so those are written here. And let us denote them as a 1 a 2 a 3 and a 4; the 4 columns of the matrix A will be denoted by a 1 a 2 and a 3 and a 4. The right hand side will be denoted by b; so b is equal to 2 and 1. Now, finding the basis matrix from the set A amongst to choosing n linearly independent columns. Now, in this case the rank of A is 2; so we can we choose at the most we can choose 2 linearly independent columns from the matrix A. So, we can choose say first and second column, first and third column they are linearly independent, first and fourth column they are linearly independent.

So, suppose if we start with second column; then we cannot choose the third column because second column and third column are not linearly independent. But instead we can choose second and the fourth column or third column and the fourth column; that will be that will form the basis matrix b. And once we have the basis matrix b the suppose we choose the first 2 columns. So, each column is associated with a variable; so first column is associated with the variable x 1, second with x 2, third with x 3 and fourth with x 4. So, if we choose the first 2 columns as our basis matrix b which means that x 1 and x 2, the associated variables form the basic variables and x 3 and x 4 become non-basic variables. So, this way we can get different combinations of basic and non-basic basis variables and those will correspond to different vertices of the feasible region.

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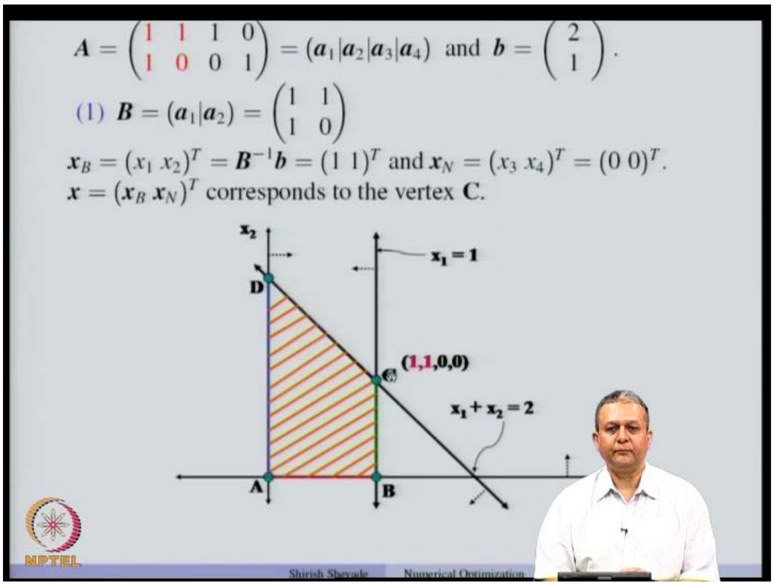

So, we will see that now. So, let us choose the first 2 columns of the matrix A. Now, these 2 columns are linearly independent. So, if we choose the first 2 columns of the matrix A and form the matrix B as 1 1 and 1 and 0; then x B will be x 1 x 2. And x 1 x 2 are the basic variables associated with the basis matrix B. And x B can be written as B inverse b. So, if we take the inverse of this matrix and multiply that by the vector B, what we get is 1 1. Now, the remaining two variables are the non-basic variables; so x 3 and x 4 are the non-basic variables, their values are 0. And we can denote this in the figure as the basic variables are 1 1 and the non-basic variables are 0 0.

So, note that in this representation the first entry denotes the x 1 variable, the second entry denotes the x 2 variable and so on and last entry denotes the x 4 variable. So, the basic variables are shown in the magenta colour and these are the x 1 and x 2 basic variables whose values is 1 1 and is obtained using B inverse b. So, this basic variable, this basic feasible solution x which is 1 1 0 0; because now it is a 4 dimensional vector. So, this corresponds to the vertex c.

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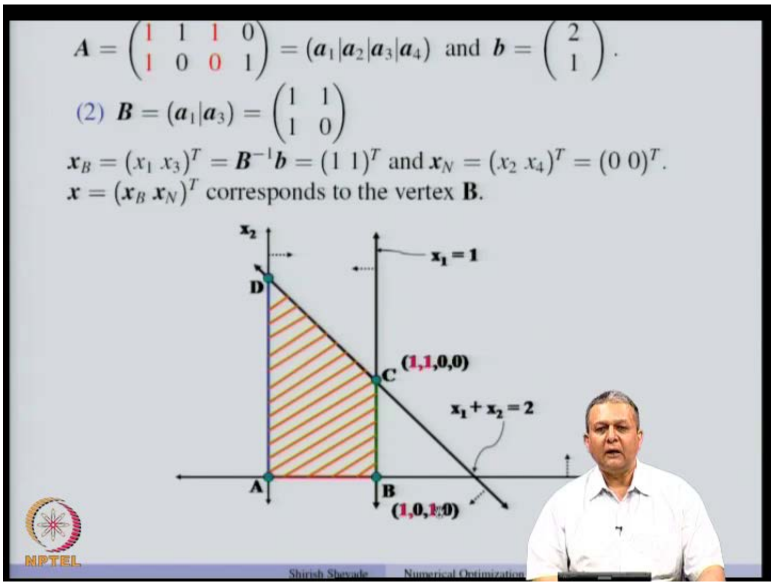

Now, similarly, if you take the first column and the third column of the matrix A; they are linearly independent. And therefore, we can form the matrix B using a 1 and a 3 and once we formed the matrix B, if we find out B inverse b that will gives us the values of the basic variables associated with this basis matrix. So, let us see. So, B is 1 1 and 1 0; then x B so x 1 and x 3 are the basic variables, x 3 and x 4 are the non-basic variables. So, x B is x 1 x 3 and that is nothing but B inverse b and that is equal to 1 1. And x N as usual these are the non-basic variables; so their values are 0 and 0 and that will be represented in the constraint set figure as vertex B. So, you will see that x 1 and x 3, the magenta colours coloured numbers are basic variables; they have the value 1 1 and then the x 2 and x 4, they have the value 0.

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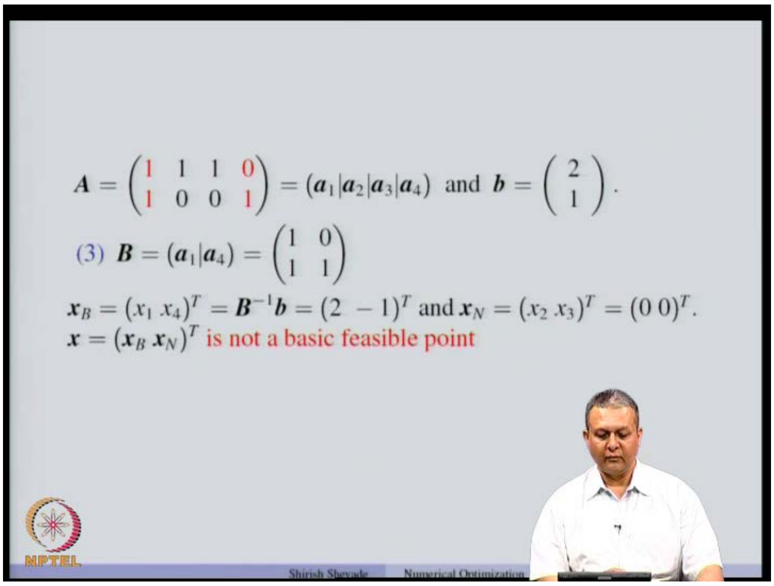

Now, we can do this exercise. So, if you take x 1 and x 4 as our basic variables; they clearly these 2 columns are linearly independent. So, if we can form the matrix B using a 1 and a 4. So, x 1 and x 4 are the basic variables and then their values are obtained using B inverse b. And if we do find out B inverse b and we get 2 and minus 1 as the components x 1 and x 4. And x N will be x 2 and x 3; the components corresponding to the non basis vectors and that will be 0. Now, you will note that this entry which is minus 1 is not possible in our constraint set because all the components of x have to be nonnegative. So, this combination of basis vector does not lead to a feasible point of the set X. So, this given x which is a combination of x B and x N is not a basic feasible point as far as the given constraint set is concerned. So, not every combination of linearly independent columns would give us a basic point which is also feasible. In this case we do not get feasibility of point.

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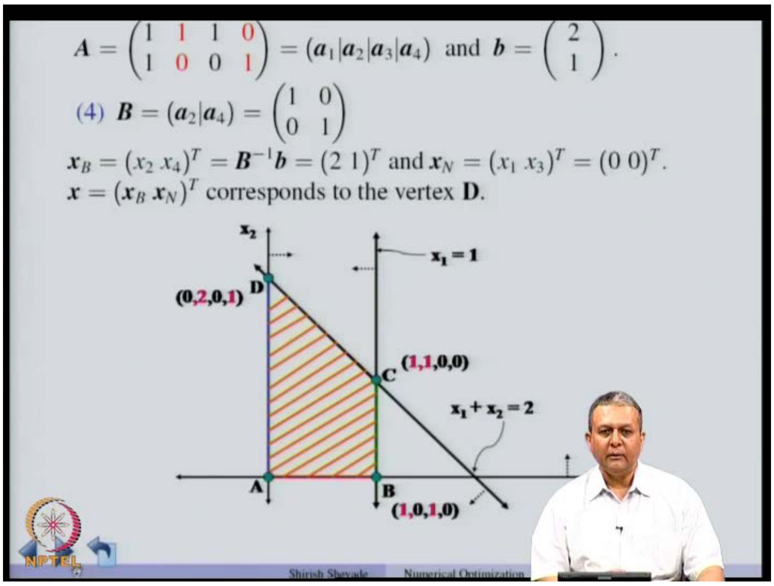

Now, if you repeat that exercise, so if you consider columns 2 and column 4 as our basis matrix and that means that x 2 and x 4 are the basic variables and x 1 and x 3 are the non-basic variables. So, if you take the B inverse b, we get the values of x 2 and x 4 and that turns out to be 2 1 and x 1 and x 3 are non-basic variables. So, their values are 0 and that will be indicated by the vertex D. So, x 2 and x 4 correspond to the basic variables and x 3 and x 1 correspond to the non-basic variables.

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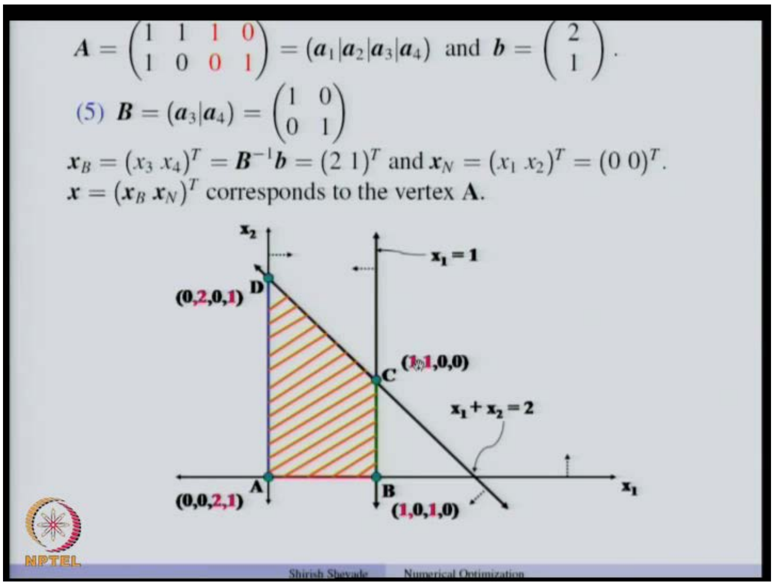

Now, similarly, by choosing x 3 and x 4 as our basic variables one can find out the representation in terms of the vertex. So, this corresponds to the vertex A where the basic variable is 2 1 and the non-basic variable basic variable are x 3 x 4 whose values are 2 1, non-basic variables x 1 x 2 whose values are 0 and that is shown here. Now, now if you look at this figure you will see that, if you consider vertex C, x 1 and x 2 are basic variables and x 3 and x 4 are non-basic variables. And if you consider the adjacent vertex of the vertex C which is suppose B, then you will see that x 1 and x 3 are basic variables. So, the variable x 1, basic variable x 1 is common between B and C. And in in the case of the vertex C, x 2 was basic variable (( )) x 3 is a basic variable.

So, between the adjacent vertices there are some common basic variables. Now, if you consider C and D we will see that x 2 is a common basic variable in this case. And here is a basic variable, while here x 1 is not a basic variable but instead x 4 is basic variable. So, when we move from one vertex to the adjacent vertex, one basic variable and one non-basic variable gets swapped. Same is true when we consider say A and B and A and D. So, in A x 3 and x 4 are basic variables; while if you move from A to D, x 4 still remains a basic variable. But then x 3 and x 2 they get swapped; so x 3 becomes a non-basic variable at D, while x 2 becomes a basic variable and which was non-basic at A. So, similarly, if we consider vertices adjacent vertices A and B, we will see that x 3 is a basic variable in both the cases while in the case of A, x 4 was the basic variable; while in case of B, x 1 is the basic variable. So, x 1 and x 4 they get swapped between the basic and non-basic variables case, as far as the vertices A and B are concerned.

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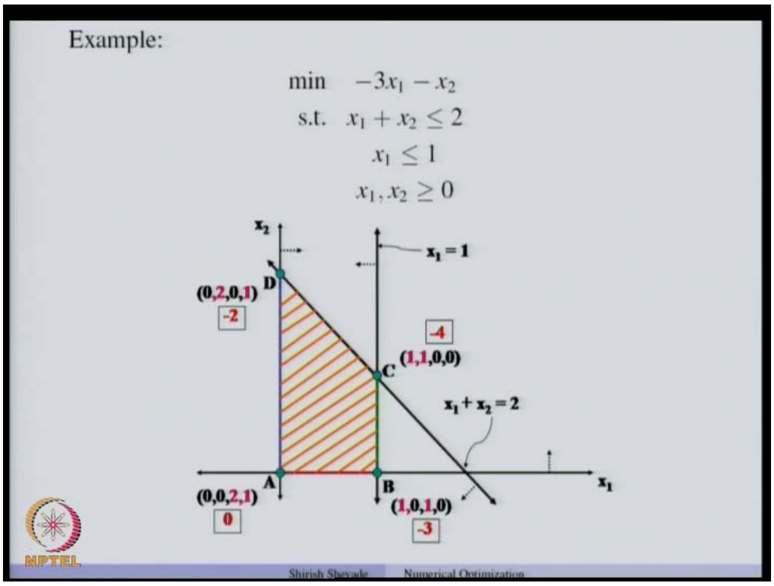

Now, if you look at the objective function values at the vertices because we are mainly interested in getting the optimal solution of a linear program and we are mainly interested in looking at only the vertices of the constraint set. So, we have seen one point earlier that vertices there is swap of basic variable and non-basic variable. So, suppose if we start from a vertex of a feasible set and then move to the neighbouring vertices by swapping appropriate variables, we can see that we can decrease the objective function. So, suppose if we start from A and by swapping A x 4 and x 1 we go to the vertex B.

Now, you will see that the objective function value which is shown in the boxes adjacent to the variable values. So, objective function value is 0 at point A becomes minus 3, at point B. So, that means that we have decrease the objective function value. Now, if you keep doing this experiment again and again every time going from a vertex to the neighbouring vertex, such that the objective function value decreases. And since there exist only finite number of vertices, we will reach a point where we cannot decrease objective function value further.

For example if we are at B, now one way to move is to point C; where the objective function value decreases. Because from B if we move towards the point A, the objective function value the objective function value increases and we do not want that; because we want to minimize the objective function. So, from B we can move to the point C by swapping the second and the third variables from the basis and the non-basic vector set and we got to the point C. At this point the objective function value is minus 4. Now, that is that value is the least among the objective function values at the 4 vertices. So, suppose we are at a point C, then if we go to any adjacent vertex the objective function value increases. Because if you move from C to D the value of objective function value becomes minus 2, on the other hand if you move from C to B the objective function value becomes minus 3.

So, this is a point where the objective function value is least and we cannot make any progress as far as the objective function is concerned. Now, this is to irrespective of any starting point. So, for example, if you start from point D at which the objective function value is minus 2; then in this case D has two neighbours, A and C. If we move from D to A, objective function value is going to increase. So, the only alternative would be that is left is to move from D to C. And if you move from D to C the objective function value is going to decrease. And from C as we saw earlier, the objective function value cannot decrease further. So, this point this vertex is an optimal point. So, this gives us an idea about how to solve a given linear program.

So, the first step is to write the linear program in standard form. And then find the objective function value at the current point and see by swapping with basic and non-basic variable can we improve the objective function. Now, if you repeat this procedure again and again every time where we find out the variables which need to be swapped between the basic and non-basic (( )); so at to move to the adjacent extreme point. And if this procedure is repeated; since there exist only finite number of vertices, we will certainly reach a solution if solution exists. Note that, sometimes linear program could have an unbounded solution. So, we should be able to find out the whether the linear program has a unbounded solution and if not, if it does not have a unbounded solution then the solution exists. And we should be able to find (( )) visiting only the extreme points. And the algorithm becomes very simple that one simply has to go to the adjacent extreme points which gives the which gives the decrease in the objective function value. So, now we will see that algorithm for solving a linear program.

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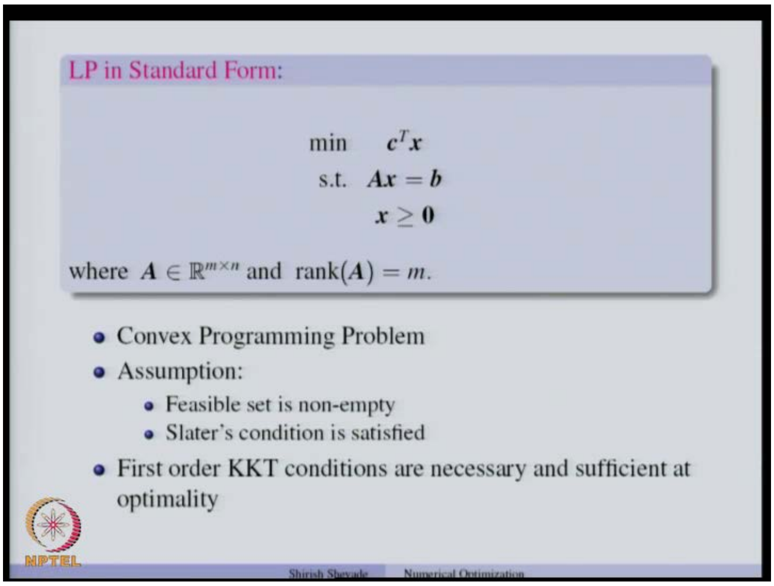

So, let us consider the linear program in standard form which is minimize C transpose x subject to A x equal to b, x nonnegative and A is a m by n matrix and rank of A is m. So, A has m linearly independent rows.

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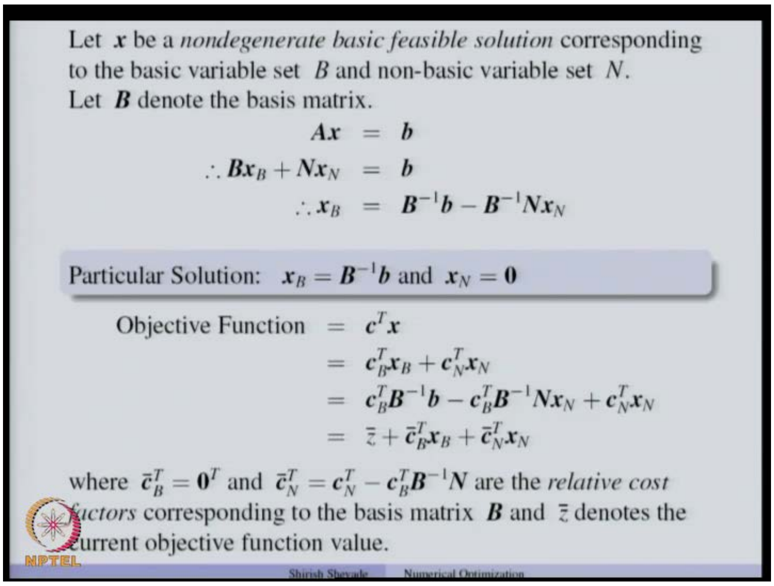

So, let us assume that x is nondegenerate feasible solution corresponding to the variable set B and the non-basic variable set N. So, what I mean by nondegenerate is that the x B is greater than 0. Now, the (( )) phase B will be used to denote the basis matrix. So, as the difference between the basic variables set B; so which contains the variables x 1 to x n and the remaining n minus m form the non-basic variable set, and B denotes the basis matrix which are the m linearly independent columns of the matrix A. And without loss of generality we assume that these columns are the first m columns of the matrix A.

Now, we have A x equal to B which can be written as B x B plus N x N equal to m associated with the basis matrix B and the non-basic matrix N and x B and x N are the basic and non-basic variables respectively. Now, since B is the basis matrix the columns of B are independent, linearly independent; so we can invert B and write x B as B inverse b minus B inverse N x N. Now, this is a general solution of the system A x equal to B. Now, in particular by letting x N to be 0, we get the solution x B equal to B inverse b and x N equal to 0. So, this is a particular solution which is also feasible if x B is nonnegative. Now, let us look at the objective function value at the current point.

So, C transpose x is the objective function; now that can be split in to two parts to every vector C and every the vector x can be split in to two parts corresponding to the basic and non-basic variables. So, C transpose x can be written as C B transpose x B plus C N transpose x N. And x B is nothing but B inverse b minus B inverse N x N; so we plug-in that value here. So, this becomes C B transpose B inverse b minus C B transpose B inverse N x N plus C N transpose x N. Now, if we consider the particular solution x B equal to B inverse b and x N equal to 0, then C B transpose x B is nothing but C B transpose B inverse b. So, that is the value of the objective function which we are going to denote by z bar. So, z bar is the current objective function value at x and we can write this equation the right hand side as C B bar transpose x B plus C N bar transpose x N.

Now, so there is no term involving x B. So, we assume that C B bar transpose is 0 and C N bar transpose is nothing but C N transpose minus C B transpose B minus inverse N. In the linear programming literature these are called the relative cost factors. So, relative cost factors denote how much change is expected, if a particular vector becomes a basis vector. So, we will see a different interpretation of this from the theory that we have seen earlier in term of a Lagrangian multipliers. So, as we know that the linear program is a convex programming problem. So, the set of solution from a convex set we make some assumptions. One of the assumptions we make is that the feasible set nonempty and this other important assumption that we make is that the Slater’s condition is satisfied, so which means that the feasible set has a nonempty interior. So, there exists at least 1 point which lies in the interior of the feasible set. So, under this condition under Slater’s condition, we know that for a convex programming problem the first order Karush Kuhn Tucker or KKT conditions are necessary and sufficient and sufficient at optimality. And we make use of that to give an algorithm to solve a linear program.

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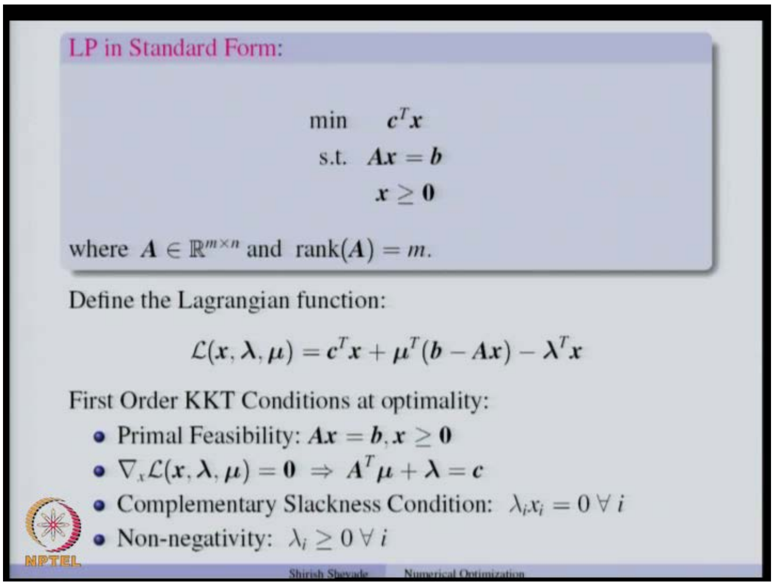

So, let us look at the Lagrangian function associated with a given linear program. So, there are some equality constraints and some inequality constraints. So, the Lagrangian is a function of Lagrangian multiplier associated with the equality constraints. So, we are going to denote those Lagrangian multipliers by mu and the Lagrangian multipliers associated with the inequality constraints we are going to denote them by lambda. So, the Lagrangian is the objective function plus mu transpose b minus A x minus lambda transpose x.

And, since the KKT conditions are necessary and sufficient at optimality for this convex programming problem, let us write down those conditions. So, the first order KKT conditions at optimality are the primal feasibility. So, the feasibility of x with respect to this program which is also primal program; A x equal to b and x nonnegative. Then the second condition is that the gradient of the Lagrangian with respect to x should be 0. So, if you take the gradient of the Lagrangian with respect to x, what we get is C minus A transpose mu minus lambda that equal to 0. So, which means that A transpose mu plus lambda equal to C. So, this condition should be satisfied at optimality and the complimentary slackness condition which is lambda x i equal to 0 for all i.

So, the complimentary slackness condition is important. So, we will see its use sometime later. So, this KKT conditions give us an idea about the algorithm. So, if you move from a extreme feasible point to another extreme point, we know that these conditions will be satisfied. So, we just have to make sure that this other two conditions are satisfied at the optimality. And there is another important condition that we want to keep in mind is that the nonnegativity of the Lagrangian multipliers. So, lambda i has to be nonnegative. So, this is a very important condition which can be used for stopping our algorithm. So, we will see that now.

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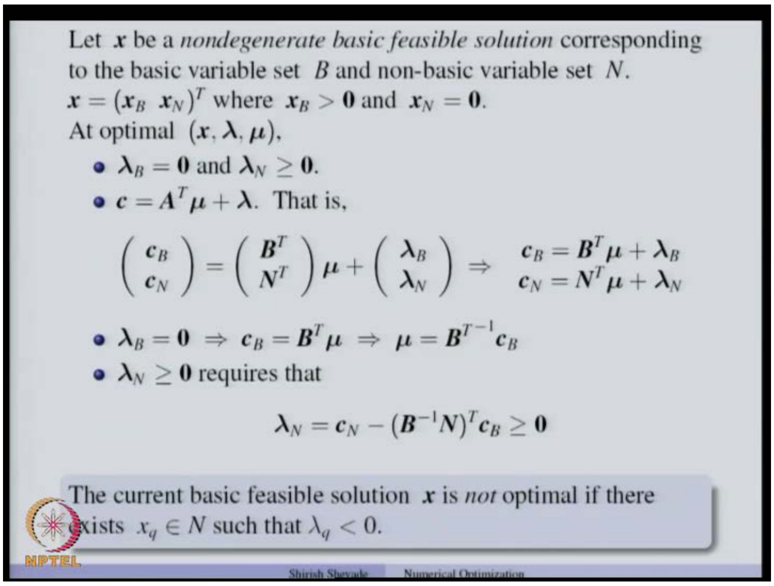

So, let x be a nondegenerate feasible solution corresponding to basic variable set B as we saw earlier. And the non-basic variable set N and x equal to x B x N where x B is greater than 0 and x N equal to 0; this is basic feasible solution. Now, at optimal x lambda mu, what we want is that lambda B should be equal to 0 and lambda N not equal to 0. So, if the current x lambda mu equal to 0, we know that x B greater than 0 and x N equal to 0. And in order to satisfy the complimentary slackness condition what we want is that the Lagrangian multipliers corresponding to the basic variables have to be 0 at optimality. And the Lagrangian multipliers corresponding to the non-basic variables are nonnegative because of the complimentary slackness condition.

The gradient of Lagrangian with respect to x equal to 0 implies that A transpose mu plus lambda equal to C. So, if we expand this in terms of C B and C N; so what we get is that

C B is equal to B transpose mu plus lambda B and C N is equal to N transpose mu plus lambda N. Now, lambda B is 0; so C B is equal to B transpose mu. Now, that gives us a way to find out mu; because B is known, C B is known and B is invertible. So, that can be used to find mu. So, C B is equal to mu transpose mu plus lambda B and C N is equal to N transpose plus mu equals lambda N. And lambda B equal to 0 implies mu is equal to B transpose B inverse C B. So, we get the variable mu at a given point which is B transpose C inverse B. Now, remember that mu's are the Lagrangian multipliers associated with equality constraints; and therefore, mu's are unrestricted in sign.

So, at a given point x which is feasible; we can use this equation to find mu. Now, if the current point x is optimal, what we want is that the Lagrangian multipliers corresponding to the basic variables have to be 0. And Lagrangian multipliers corresponding to the non-basic variables have to be nonnegative. So, if we assume that corresponding to the basic variables lambda B equal to 0; then what we should get is lambda N to be nonnegative. And if you do not get lambda into be nonnegative; that means, that the point is not an optimal point. Because we know that at optimality this conditions have to be satisfied; this conditions are necessary and sufficient. So, lambda N nonnegative; it requires that C N minus B inverse N transpose C B. So, if you plug-in this value of mu in this equation; then we can write lambda N to be C N minus B inverse N transpose C B and that should be greater than or equal to 0. And you would notice that this quantity it is nothing but the relative cost.

Another explanation for this is that as we had seen earlier in our discussion on Lagrangian multipliers. The one of the interpretations of Lagrangian multipliers is that they indicate the rate of change of objective function. So, if the Lagrangian multiplier is a negative; that means, that there is a scope for improvement in the objective function in this case. And that is what we will utilize while determining our algorithm. So, at optimality what we want is that C N minus N transpose C B greater than or equal to 0 which is nothing but the relative cost associated with the non-basic variables has to be nonnegative.

So, this is an important point that the current basic feasible solution x is not optimal if there exist x q belonging to N such that lambda q is less than 0. Because for a given x we can find mu and for a given x B if you set lambda B to be 0; then what we need to find out is lambda N. And if lambda N turns out to be negative for some variable non-basic variable x; so that clearly shows that that non-basic variable can be made basic and the objective function can be decreased for further. So, this condition which is lambda N greater than or equal to 0 plays a very important role in deciding which non-basic variable can be made basic. Or in other words which non-basic variable can be increased from 0 to some positive value by certain amount, so that the objective function value decreases.

Now, as we have seen earlier that making a non-basic variable basic requires that one of the existing basic variables has to be made non-basic; so that we can move to the adjacent vertex. And how do we select of basic variable to be made non-basic and then how do we move from one vertex to the adjacent vertex will be the part of our discussion in the next class.

Thank you.