Numerical Optimization
Prof. Shirish K. Shevade
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 31
Geometric Solution

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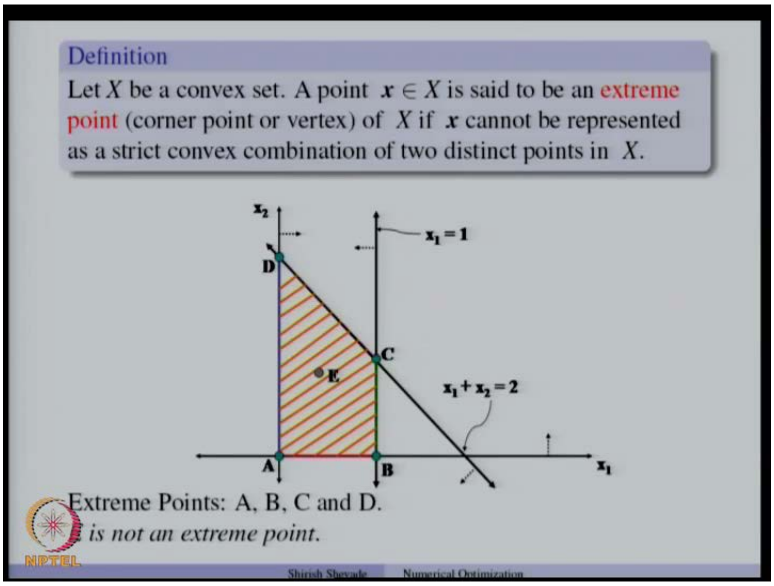

In the last class we started discussing about linear programming problem. So, linear program is a mathematical program where the objective function is linear and the constraints are linear in the variable. So, we started discussing about some of the properties of the constraint set, and we looked at what are called the extreme points of the constraint set. So, in this figure the shaded region is the constraint set and we define extreme point as for a convex set X, a point x is an extreme point. That is also called the corner point or the vertex of X, if x cannot be represented as a strict convex combination of two distinct points in X. So, for example, in this figure the points A, B, C and D are the extreme points; well the point E is not an extreme point.

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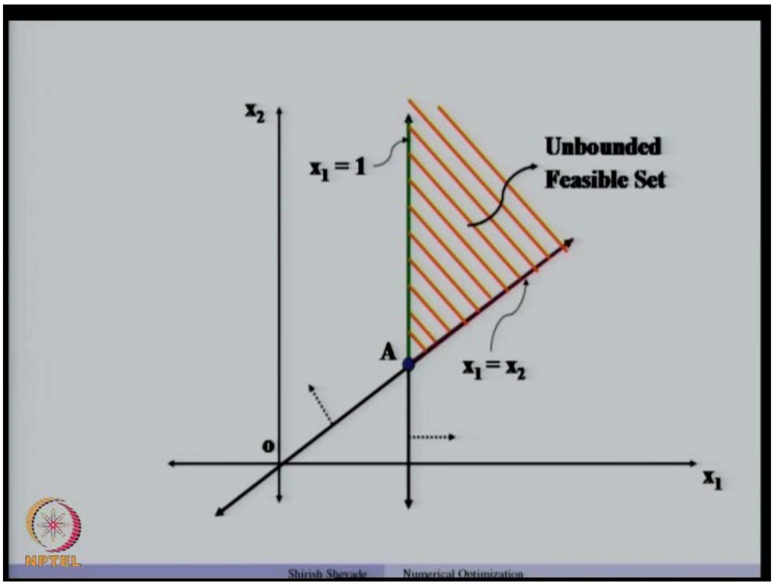

Now, here is another example of constraint set. Remember that the constraint set in a linear programming problem is a convex set and the objective function being a being a linear function is also a convex set. So, the linear program is a convex programming problem. Now, the constraint set could be bounded or unbounded; so here is an example where the constraint set is unbounded and this has only one extreme points. Sometimes if the constraint set is just the half space associated with the hyper plane, then that constraint set does not have any extreme point or a vertex. So, a typical linear program may have 0 vertices or one vertices or more than one vertices.

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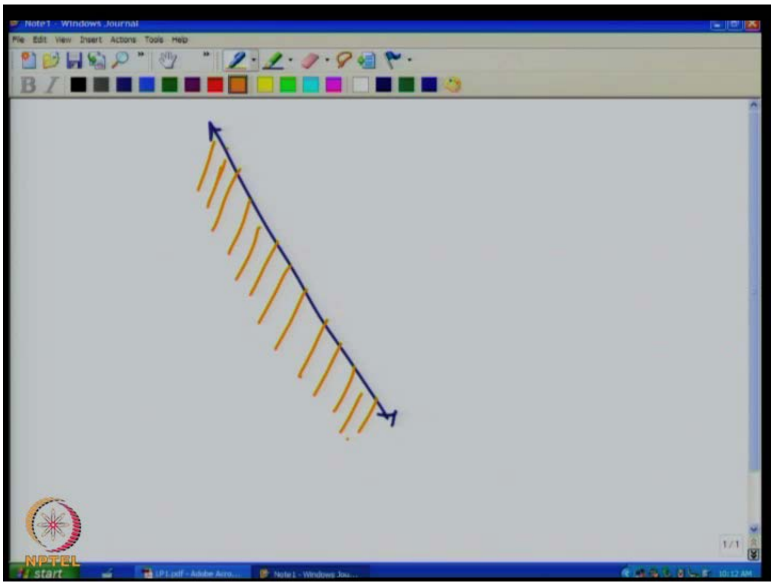

So, we saw that; if we have the constraint set which is the half space shown by the shaded region then this constraint set does not have a vertex. So, the only extreme point in this case is the point A.

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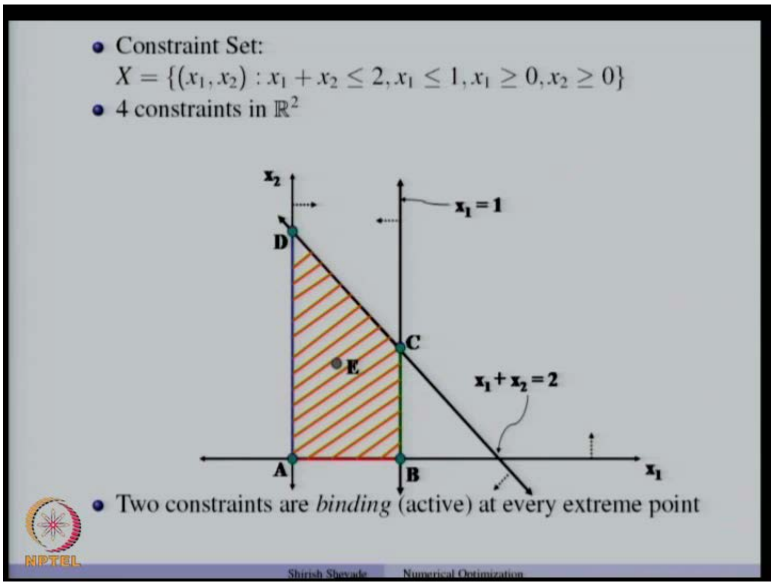

Now, let us see more about these constraints sets. So, consider the constraint set as a pair of points (x 1, x 2) such that x 1 plus x 2 less than or equal to 2, x 1 less than or equal to 1 and x 1 and x 2 both non-negative. Now, we have 4 constraints in this set X. Now, if you take a point extreme point of a vertex, you will see that in this two dimensional space only two constraints are active at the extreme points. For example, if you take the point C; then, the constraints x 1 plus x 2 is equal to active, is equal to 2 is active and the constraint x 1 equal to 1 is active and C is the intersection of these two constraints.

Similarly, if we take the point A, then the last two constraints are active and the other two constraints are inactive. On the other hand, if we take a point like the point E which is in the interior of the set, we will see that none of the constraints is active at this point. On the other hand, if we take a point on the line segment joining set the points B and C, then only the constraint x 1 equal to 1 is active, while the rest of the 3 constraints are inactive. So, we can conclude that in this case when we have 4 constraints in two dimensional space at the extreme points, we have 2 constraints which are active. Of course, there could be a situations where more than 2 constraints are active at an extreme point in this two dimensional case. But then one of those, some of those constraints will be redundant.

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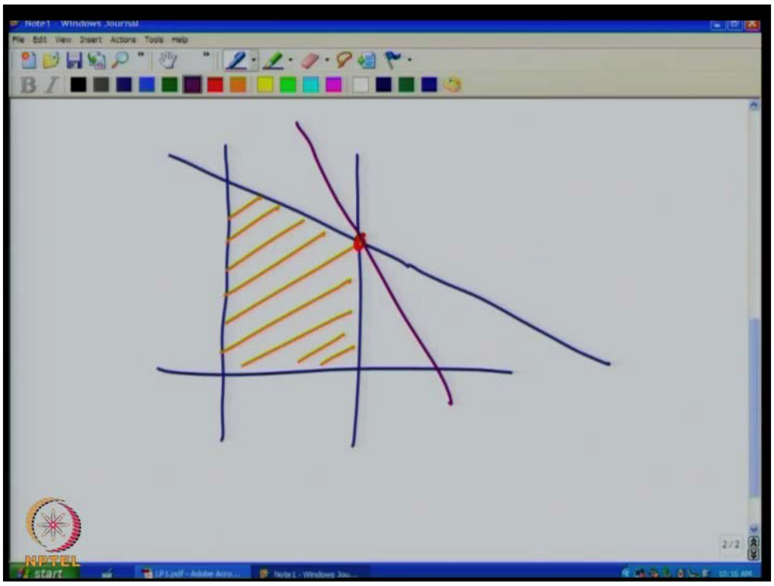

For example, if we consider the similar situation. So, suppose this is our constraint set, and if you are considering this point, then this constraints and this constraints are active at this point. Now, one could have an extra constraint which which is like this. Now, you will see that this constraint is redundant constraint. So, when we have 3 constraints which are active at a extreme point in two dimensional space, one of the constraints can be treated as redundant constraint. So, in effect there will be 2, only 2 constraints which are active. So, one can think of it like this; that if we take a take the hyper planes corresponding to those active constraints and take the normals to the hyper planes. Thus, the two normal directions are linearly independent in this case. In the example which is shown here, we have only 2 constraints active at all the extreme points. And then fewer than 2 constraints are active at other points. So, on the line segment joining the vertices we have only 1 constraint which is active. And in the interior of the constraint set we have constraints which are active.

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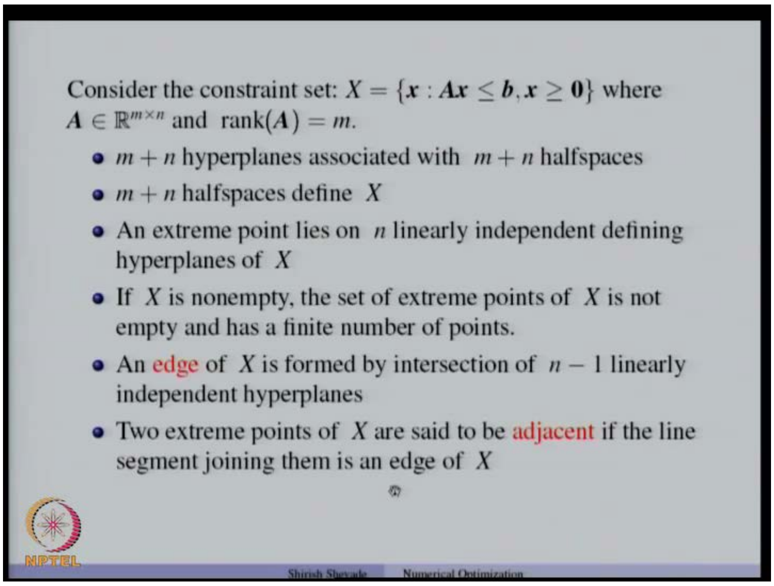

Now, suppose if you consider the constraint set of the form set of all points such that A x less than or equal to b and x greater than or equal to 0; where A is a m by n matrix of real numbers and rank of A is m. Then, there are m plus n hyper planes associated with the m plus n half spaces. So, remember that there are m constraints associated with A x less than or equal to b and n constraints associated with x greater than or equal to 0. So, in all we have m plus n constraints associated with the constraint set X. So, m plus n constraints means that we have m plus n hyper planes associated with those m plus n half spaces.

Now, these m plus n half spaces together define the set X and an extreme point lies on n linearly independent defining hyper planes of X. So, out of these m plus n half spaces, we will be interested at an extreme point on n linearly independent hyper planes. Remember that x is an n dimensional space; so at an extreme point we have an n linearly defining hyper planes of x. And if X is nonempty then the set of extreme points is not empty and this set of this set X has finite number of extreme points.

So, this is a very important observation. So, we will prove this fact related to a general constraint set sometime later. But remember that if X is nonempty then the set of extreme points is also nonempty and further the set has finite number of extreme points. Now, let us define a edge of the set X. Now, as we saw in the previous case that on the line segment joining the two vertices, the number of constraints which are active was n minus 1. So, edge is formed by the intersection of n minus 1 linearly independent hyper planes. And two extreme points of X are said to be adjacent, if the line segment joining them is an edge of X. So, we will take some example to illustrate this.

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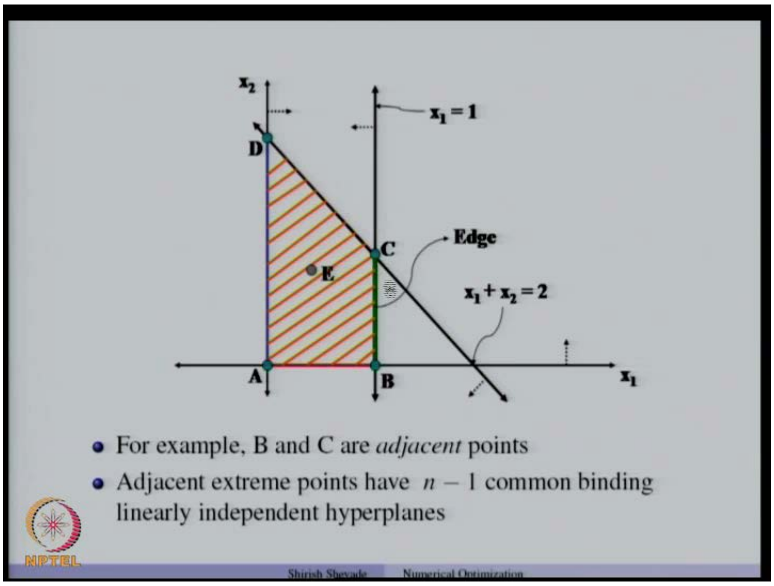

So, the line segment joining the two vertices B and C is called the edge or in other words the edge connects the two adjacent extreme points. So, the points B and C are adjacent extreme points or adjacent vertices. Similarly, points C and D are adjacent vertices and the line segment C D connect connecting the two vertices C and D is also an edge. Now, you will see that, under edge there is only one constraint active in this two dimensional space. So, the edge is typically defined by n minus 1 linearly independent hyper planes that define the constraint set X in n dimensional space. So, adjacent extreme points have n minus 1 common binding linearly independent hyper planes. So, those are the hyper planes which are active associated with the edge.

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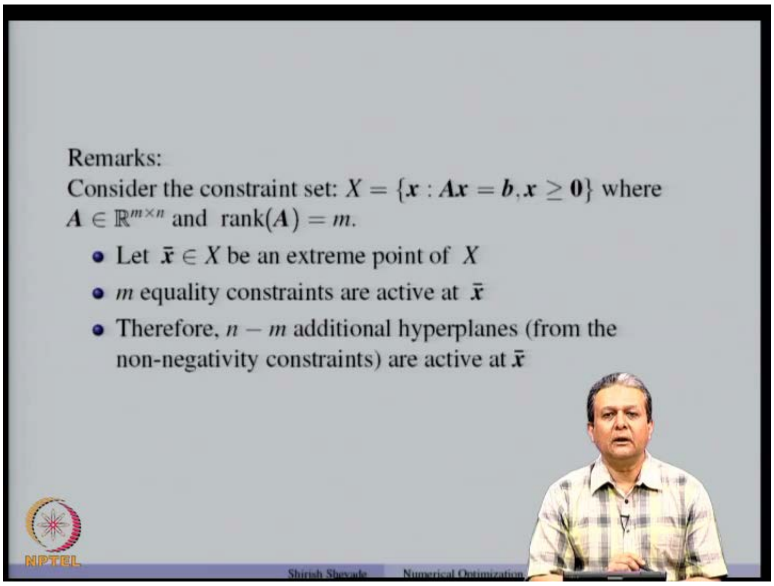

Now, let us see some remarks about the equality constraint set like the set of all X such that A x equal to b, x greater than equal to 0; where again A is m by n matrix of real numbers and the rank of the matrix A is m. Now, let us take an extreme point of x and let us call it as x bar. Now, since x bar belongs to X, it has to satisfy all the equality constraints; because at every feasible point all the equality constraints are active. So, there are m equality constraints which are active at x bar. And since it is an extreme point, we know that there are n linearly independent hyper planes defining the extreme point x bar. So, out of n m constraints have come from the equality constraints; so the remaining n minus m active constraints come from this part, x greater than or equal to 0. So, that is n minus m addition planes coming from x greater than or equal to 0 are active at x bar. So, out of n constraints which are represented by x greater than or equal to 0, n minus m are active. That means, that corresponding to those variables the value of x is 0.

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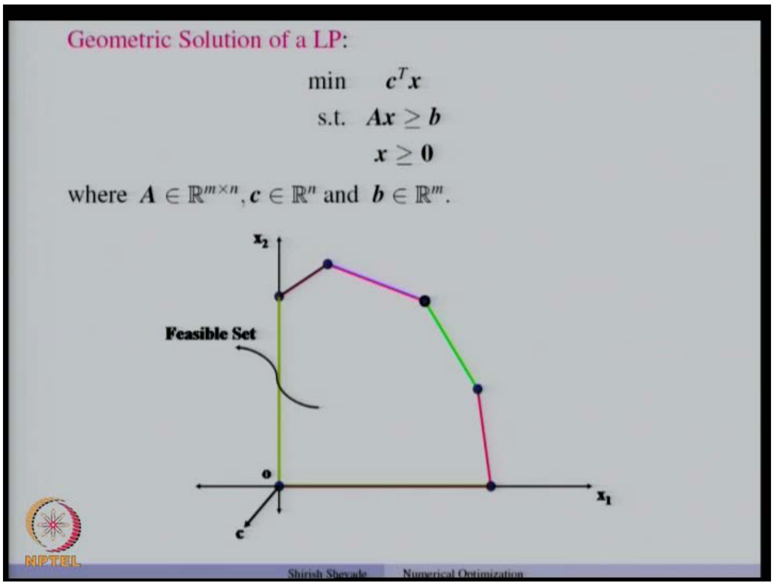

Now, let us look the geometric solution of LP. So, to understand how to solve a given linear programming problem, the geometric solution gives us a good idea and that can be used to device efficient algorithms to solve a linear program. Now this geometric solution typically is useful when a linear program is in two dimensions. So, we will consider such cases and show how the geometric solution of a given linear program can be opt. So, let us consider a program which is of the type minimize C transpose X subject to A x greater than or equal to b, x greater than or equal to 0; where a is m by n matrix, c is a n dimensional vector and b is a n dimensional vector.

Now, let us assume that the constraints are such that we have a nonempty feasible set. So, for example, here is a two dimensional case where we have two variables x 1 and x 2 and the constraint is a set of half spaces defined by this constraints. So, there are 4 constraints of this type and then two nonnegative constraints corresponding to x 1 and x 2. So, these together form a feasible set. So, the interior of this polygon is the feasible set. To avoid any notational, to avoid any clutter in the figure I have not shown it as a shaded region, but the entire boundary as well as the interior of this polygon denotes this feasible set. And assume that this vector C is as shown here and our aim is to minimize C transpose X subject to the constraint that x belongs to the feasible set x.

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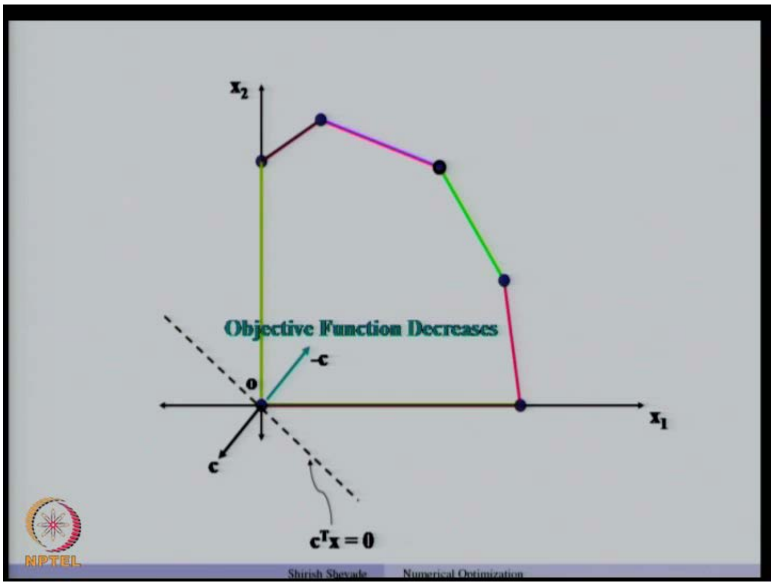

So, let us see how to do this. Now, we know that c is pointing in this direction; so the objective function decreases when we move along the direction minus c. Because in this half space, if you consider the hyper plane which is perpendicular to c, so that will be denoted by C transpose X equal to 0. So, in the direction where C is pointing, C transpose X is greater than 0 and in the direction of minus C transpose X is less than 0. So, obviously the objective function decreases when we move along the direction minus C. So, what we need to do is that take the hyper plane which is parallel to this hyper plane C transpose X equal to 0. And keep moving that hyper plane along the direction where the objective function decreases. And we maintain the contact with the set the feasible set X. And there will be some point where the function value will have minimum.

Now, there may be situations where minimum may not exist. So, such problems are called unbounded problems, and we will not worry about those cases at this point of time. Let us assume that the minimum does exist and let us see how to find out the one.

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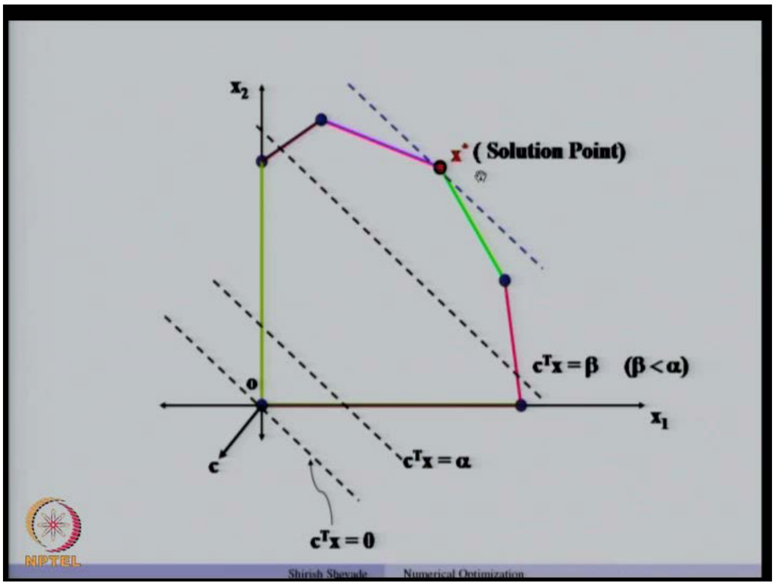

So, here is the hyper plane which is parallel to hyper plane C transpose X equal to 0. That means, that the normal to the hyper plane is still C but the intercept on the axis are different. So, here is the equation of a hyper plane C transpose X equal to alpha and another hyper plane C transpose X equal to beta. And as we have indicated earlier that the objective function value decreases then we move along the minus C direction. So, obviously beta will be less than alpha. So, if we keep moving this hyper plane further, we may come across some point where we get the minimum of the objective function with respect to the constraint set.

So, if you move this hyper plane further; then, we will hit the solution point of the given problem. Now, remember that if we move this hyper plane further, we will not satisfy the constraints set X. So, this point appears to be the solution point of the given problem, minimize C transpose X subject to this constraint set. Now, there may be situations where the solution point may not be unique. So, let us see an example.

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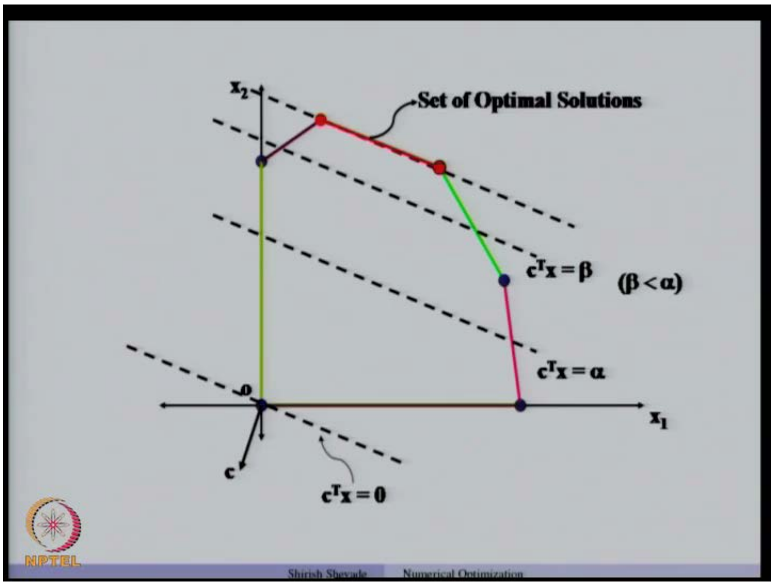

So, let us consider a different C vector different from the one we saw earlier and the constraint set is still the same. And this is the hyper plane C transpose X equal to 0. So our method should start constructing hyper planes which are parallel to this hyper plane, and that has a nonempty intersection with the feasible set. So, if you start constructing those hyper planes; so obviously C transpose X equal to alpha is this hyper plane and C transpose X equal to beta is this hyper plane and beta will be less than alpha. And if we go on moving this hyper plane further, we will see that we get the solution. But this time the solution is not unique. So, what I mean by the non uniqueness of the solution is that there exists different X which will give us the same objective function value. So, the objective function value, optimal objective function value is always unique. But the set of X which gives these optimal solutions may not always be a singleton set. So, here is an example where we have infinitely many optimal x's which give us the same optimal objective function value; that is C transpose X star.

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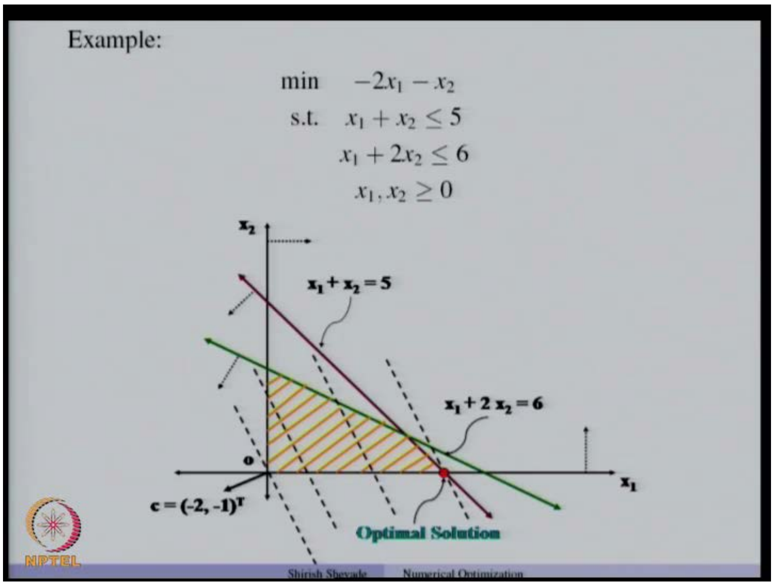

Now, let us consider an example. So, we have problem minimize minus 2 x 1 minus x 2 subject to the constraint x 1 plus x 2 less than or equal to 5, x 1 plus x 2 less than or equal to 6 and x 1 and x 2 are nonnegative. So, let is draw the constraint set; so x 1 plus x 2 equal to 5 is this line and x 1 plus x 2 less than or equal to 5 is the half space denoted by this arrow. Similarly, x 1 plus 2 x 2 equal to 6 is this line and the set x 1 plus x 2 less than or equal to 6 is denoted by this arrow. Now, the intersection of these two half spaces along with the fact that x 1 and x 2 are nonnegative. So, this indicates that x 1 is greater than or equal to 0 and this indicates the half space where x 2 is greater than or equal to 0. So, the intersection of these 4 half spaces gives us the constraints sets which is shown by the shaded region.

Now, let us take the C vector which is minus 2 and minus 1 which is shown here. So, if you move along the direction minus C the function value decreases. So, for that purpose let us first take the hyper plane which passes through origin and which has normal as vector C. So, this is this hyper plane which is shown by the shaded line. So, the idea is to move this hyper plane in the direction of minus C; such that the hyper plane each of those hyper planes has an nonempty intersection with the feasible set. And we keep moving it till we get, till we cannot move further and at that point we get the minimum.

So, the value of the objective function at any point on this at any feasible point on this hyper plane will be greater than the value of this objective function at any feasible point on this hyper plane. So, if you keep on moving like this, we will reach a situation where we cannot move that hyper plane further without violating the constraint. And at this point we stop and we will see that this is an optimal solution; in fact this is the only optimal point in this case. So, you will have noticed that the optimal solution always lies at the boundary point, if the optimal solution exists. There may be linear programs where the optimal solution may not exists. Those linear programs are unbounded linear programs and whenever we say that the optimal solution exists, we mean that the optimal solution is finite.

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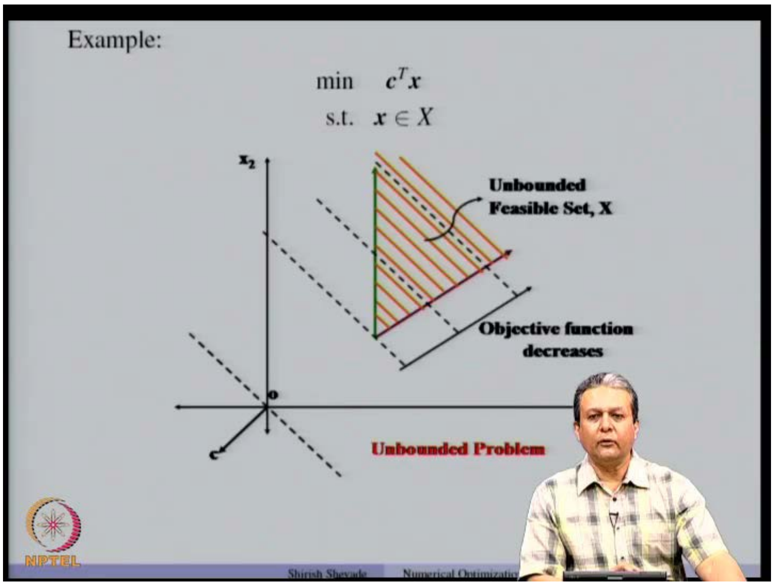

Let us consider a general case minimize C transpose X subject to x belongs to X where X is convex set. Now, in this case the set is unbounded and convex and assume that the C vector points in this direction. Now, C transpose X is equal to 0 is this hyper plane and we want to move in the direction of minus C. So, if you consider any feasible point and start moving in the direction of minus C. Suppose, we start from this hyper plane and then start decreasing the value of the objective function; finally, we will reach a point where we cannot decrease the objective function value further without violating the constraint. So, this turns out to be the optimal solution of the given problem.

So, the example illustrates that even if the constraint set is unbounded in this case, does exist the optimal solution. On the other hand, if we take the vector C to be pointing in this direction and if you want to minimize this objective function C transpose X subject to x belongs to X then and if you start from this, then in this direction the objective function decreases. And the constraint set is unbounded, so the minimum will not exist. So, such a problem is called unbounded problem. So, I repeat that; whenever we say that the optimal solution exists for a linear program, we mean that the optimal solution or optimal objective function value is finite. Note also that as we have seen that the number of optimal points, the number of optimal x's could be either 0 as in this case when the problem is unbounded or 1 where we get the optimal solution at the vertex or the number of optimal x's could be infinitely many. An entire edge of the set X can be part of the set of optimal solutions. But the objective function value if it exists is always unique.

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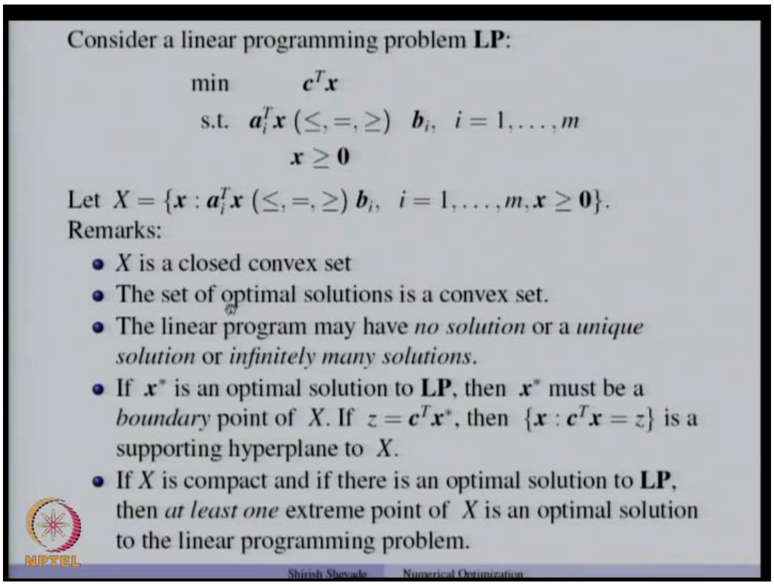

So, let us consider a linear programming problem where we minimize C transpose X subject to a i x. It will be either less than or equal to b i or equal to b i or greater than or equal to b i. So, we have either types of constraints in either constraint set; less than or equal to type, the equality type or greater than or equal to type. But whatever is the case ,the constraints are always linear in x and assume that there exists nonnegativity conditions on x or there are nonnegativity constraints on x. So, let us assume that or let us denote the constraint set by X. So, capital set X, X is the set of all x's such that a i transpose x less than or equal to or equal to or greater than or equal to b i depending upon the case. There there are n such constraints and we have nonnegativity constraints. Now, we have already seen that each constraint denotes or close half space and therefore, this constraint set X is a intersection of closed half spaces. Now, each half space is a convex set; so we have intersection of closed convex sets and therefore, X is closed convex set.

We have seen this fact earlier when we discussed about this convex set that intersection of an arbitrary collection of convex sets is a convex set. So, here the each constraint set is also closed and therefore, the intersection of closed convex sets is a closed convex set. Now, the objective function is also a convex function and therefore, we have the convex programming problem. So, every linear program is a convex programming problem and as we have already shown that the set of solutions or set of optimal solutions are for convex programming problem is a convex set.

Now, we have seen so far by examples that typical linear program may have no solution. Now, there are different cases related to this; one possibility is that the constraint set is unbounded and the problem becomes unbounded because of the nature of C. So, remember that there could be situations where the constraint set is unbounded but that there exists a unique solution which is bounded. So, if the constraint is bounded there and the vector C is such that the optimal objective function value may not exist for a linear program; then, we say that the problem is unbounded in such a case no solution exists.

The other possibility is that the set x itself is empty, so then obviously there is no solution to the linear program. We also saw some cases where there exists a unique solution and we also noted that the unique solution does exist at the vertex or an extreme point of the set X. And there is a possibility that a linear program may have infinitely many solutions. Now, we also note that if x star is an optimal solution to LP, then x star must be a boundary point of X, which means that x star cannot be a solution an optimal solution to LP and is in the interior of the set.

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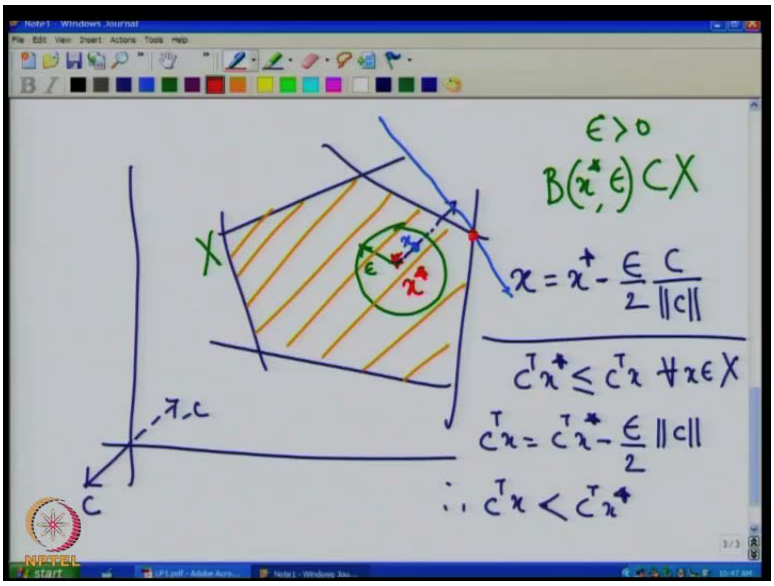

Now, let us quickly see why this is not possible. So, let us assume that and the constraint set is this set and let us assume that the solution lies in the interior of the set. Now, if x star is in the interior of the set and x star is an optimal solution to the linear program. Since, the point lies in the interior we know that there exists a circle of radius epsilon around x star such that, that circle the ball of radius x star radius epsilon around x star is contained in X. So, there exists some epsilon greater than 0; such that a ball of radius epsilon around x star it is contained in the set X.

So, let us assume that the vector C is pointing in this direction. So, that means that if we move along the minus C direction we decrease the objective function. So, let us take a point x, x to be x star minus epsilon by 2 along the direction C by norm C. So, in this direction, in the negative C direction we take a point. So, suppose this is a negative C direction. We take a point which, so we will take a point x. Now, remember that x star we have assumed that x star is an optimal to a linear program. So, therefore, C transpose X star is less than or equal to C transpose X for all x belongs to X; because x star is optimal. And suppose x star lies in the interior, so there exist a ball of a radius epsilon around x star which contains in X. So, let us take a point x in that ball. Now, let us see what happens to C transpose X. So, C transpose X is equal to C transpose X star minus epsilon by 2 on C; because this will be C transpose C which is norm C square divided by norm C it will become norm C. Now, epsilon is positive; so remember that epsilon is positive.

So, norm C is positive; therefore, C transpose X will be strictly less than C transpose X star. So, that means that there exist a point x in the set X such that C transpose X is less than C transpose X star. And this contradicts the assumption that x star is a minimum. Because if x star had been minimum C transpose X would have been less than or equal to C transpose X and we get the contradiction. And therefore, x star cannot lie in the interior of the set X; so it has to lie on the boundary point of the set X. Now, if x star is a solution, let us take z to be C transpose X star. Then, the set X such that C transpose X equal to z is a supporting hyper plane to the set X.

So, if you take along the minus C direction the objective function decreases and finally, we get a plane in this case, a line which is supporting hyper plane to the set X and this point turns out to be the minimum point. So, we will see that at the optimal point or the optimal x, the supporting hyper plane C transpose X equal to C transpose X star is a supporting hyper plane for the set X and it supports from below. Now, here is a important result which says that if X is compact, so not only that x is close but x is also bounded. So, if X is closed bounded convex set or compact convex set and if there exist an optimal solution to LP, then at least one extreme point of X is an optimal solution to the linear programming problem. So, we will prove this result later on but what is important about this result is that if you see the previous statement, we said that if x star is an optimal solution to LP; then x star must be a boundary point. Now, there may be infinitely many boundary points on the set X. But then if x is compact, then we need to search only for the extreme point or the vertices of the set X to get an optimal solution to LP.

So, finding solution of an LP becomes an easy task because what one has to do is just to search over the set of extreme points. So, although this procedure is inefficient but it gives us a way to solve the given LP in a simple way just to search over the set of extreme points. Now, the natural questions that arise are whether there exists finitely many extreme points for a constraint set and how do we characterize these extreme points algebraic. So, we will start looking at those things.

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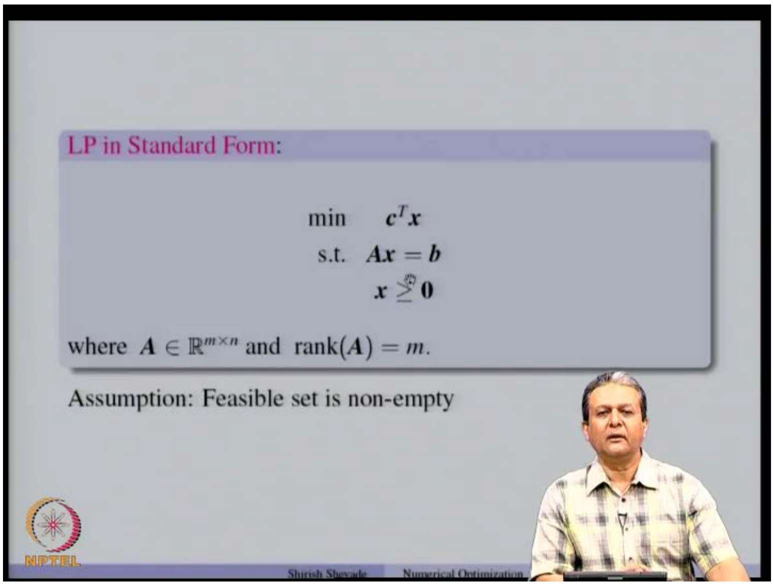

Now, before we go into the details let us look at the linear programming standard form. Now, different text book use different notations for standard forms. But in this course, we will follow this notation; where we define the linear program in standard form as minimize C transpose X subject to A x equal to b, x greater than or equal to 0. So, the objective function is linear; more importantly the constraints are of the type A x equal to b, so that means that we have all equality constraints. There are no less than or equal to less than or equal to, type constraints in this A x equal to b and all the variables are nonnegative.

So, this is the only constraint which is inequality constraint that the variables are nonnegative. So, when we have a program in this form, we call it a linear program in standard form. So, there are m equality constraints; so which means that A is m by n matrix of real numbers and rank of A is equal to m. Now, we also assume that the rank of A and the rank of A appended with b is same; in other words b lies in the columns space of A. So, that means the system of equations that we have is consistent and the feasible set is nonempty. So, this is what we assume for linear programs. Now, the important importantly any linear program can be written in this form. So, let us see how to this can be done.

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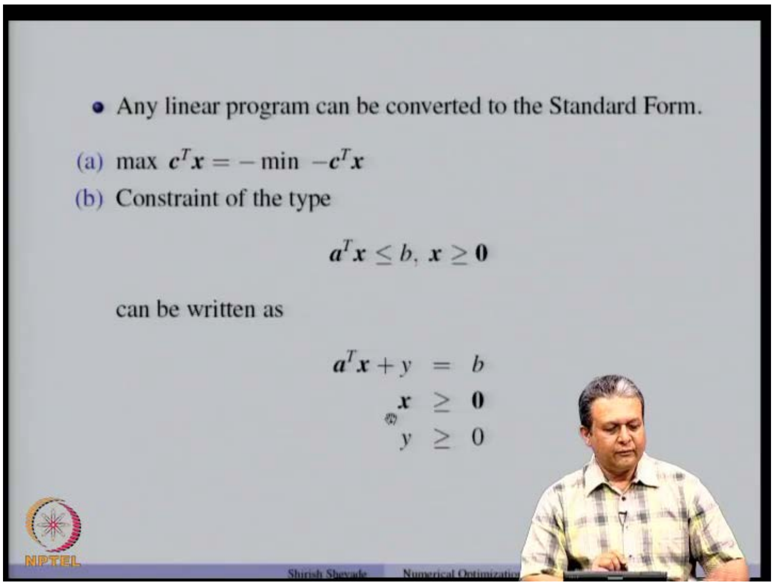

So, this is a very important observation. Now, there are different possibilities that instead of minimization we may have a maximization problem or there could be some constraints where there is a less than or equal to inequality or there could be some constraints where there is a greater than or equal to kind of inequality. Further, the nonnegativity constraints may not always be there. But despite all that we can always convert a linear program, any linear program to this standard form and let us see how to do that. So, maximization problem can be always be written as a minimization problem and we have seen that earlier also. So, maximize C transpose X can be written as minus minimize of minimization of minus C transpose X. So, therefore, even if any linear program has a maximization problem, we will we can always convert it to a minimization problem.

Now, if the constraint is of the type a transpose x less than or equal to b and x greater than or equal to 0; so we have a inequality constraint. And that can be converted to the equality constraint problem by introducing what are called slack variables. So, this constraint can be written as a transpose x plus y equal to b; no, this y is a nonnegative number. Therefore, the given constraint which is of the type lees than or equal to is converted to an equality constraint by introducing a slack variable y the other constraint remains the same.

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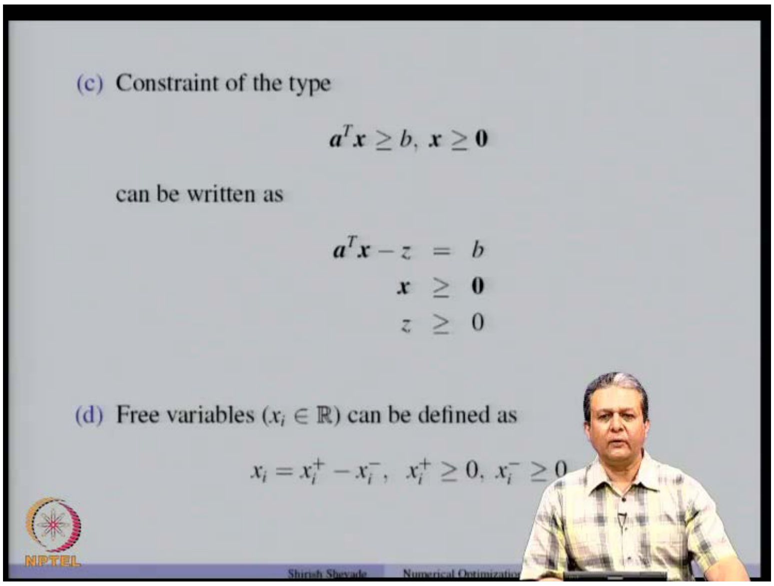

Similarly, if we have constraint of the type a transpose x greater than or equal to b and x is nonnegative; that can be converted to a transpose x minus z equal to b. So, this z is called the surplus variable and z is nonnegative. So, any inequality of the type a transpose x greater than or equal to b can be written as a linear equality constraint by introduction of surplus variable; and of course, that surplus variable is nonnegative. So, thus any constraint of the type a transpose less than or equal to b or a transpose greater than or equal to b can be written as an equality constraint by having some slack or surplus variables depending up on the case.

Now, the only question that remains is that what about the variables which are unrestricted in sign? So, the free variables; for example, x i belongs to r they can be defined using some extra variables. So, for example, x i can be written as a difference of two nonnegative variables. So, x i plus and difference of x i plus and x i minus; where x i plus is nonnegative and x i is also x i minus is also nonnegative. So, any free variable can be written as difference of two nonnegative variables and this is what is shown here. So, let us take an example to illustrate this before moving further.

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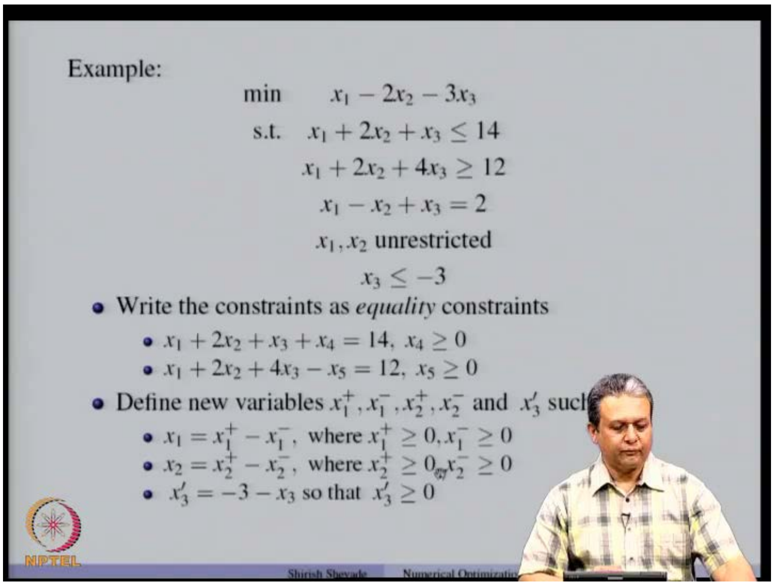

So, let us consider a problem to minimize x 1 minus 2 x 2 minus 3 x 3 subject to x 1 plus 2 x 2 plus x 3 less than or equal to 14. x 1 plus 2 x 2 plus 4 x 3 greater than or equal to 12. x 1 minus x 2 plus x 3 equal to 2 and x 1 and x 2 are unrestricted. So, which means that x 1 and x 2 can be any real numbers and x 3 is less than or equal to minus 3. Now, given this program; suppose we want to convert it to the linear program in standard form. Now, we have already seen that the constraints of the type less than or equal to can be converted to the equality type constraint by introduction of slack variable. And the constraints of the type greater than or equal to can be converted to equality constraints by introduction of surplus variables. So, we will need one extra variable for this constraint, one extra variable for this constraint and those two variables are nonnegative; so that is not a problem. The third constraint is equality type; so we do not have to worry about that.

But now the question is that we have some variables which are unrestricted and then there is a constraint in x 3 to be less than or equal to minus 3. So, how do we convert all this variables x 1, x 2 and x 3 to the variables which are nonnegative? So, let us see that. So, let us first write the constraints as equality constraints by introduction of slack or surplus variables depending up on the case. So, if we take the first constraint; then, we have to add a slack variable to make it a equality constraint. So, x 1 plus 2 x 2 plus x 3 plus x 4 is equal to 14 where x 4 is nonnegative; so x 4 is a slack variable. Now, similarly, for the second constraint we can remove the surplus variable to make it a equality constraint. So, x 1 plus 2 x 2 plus 4 x 3 minus x 5 equal to 12; x 5 nonnegative.

Now, third constraint will remain as it is; so we do not have to worry about that. And now we have to worry about the variables. So, since x 1 and x 2 are unrestricted, what we can do that? We can define the variables x 1 plus and x 1 minus and x 2 plus and x 2 minus; such that x 1 can be written as the difference of x 1 plus and x 1 minus, where x 1 plus and x 1 minus are both nonnegative. Similarly, for the variable x 2 because it is also restricted in sign. So, x 2 is equal to x 2 plus minus x 2 minus where x 2 and x 2 plus and x 2 minus are again nonnegative.

Now, we have the variable x 3 less than or equal to 3 and certainly we cannot use the variable x 3; because our standard form demands that all the variables have to be nonnegative. So, let us define a new variable. How do we get that variable? So, we can rewrite this constraint x 3 less than or equal to minus 3 as minus 3 minus x 3 greater than or equal to 0. And therefore, if we define a new variable called x 3 dash such that x 3 dash equal to minus 3 minus x 3. Then, naturally x 3 dash will be greater than or equal to 0. So, define x 3 dash to be minus 3 minus x 3; so that x 3 dash will be greater than or equal to 0.

So, note that the slack and the surplus variables that we have added are nonnegative. Then, we have added x 1 plus and x 1 minus for the variable x 1 and x 2 plus and x 2 minus for the variable x 2; such that all these 4 variables are nonnegative. And then we transformed x 3 to x 3 dash which is defined as minus 3 by minus x 3; so that x 3 dash is greater than or equal to 0. So, these 2 constraints are written in the form of equality constraints; this constraint anyway is a equality constraint, so we do not have to worry about it. And then the two variables x 1 x 2 converted to or we used some extra variables which are nonnegative in nature. And the variable x 3 was transformed to x 3 dash; this is also nonnegative. So, we have equality constraints and all the variables are nonnegative.

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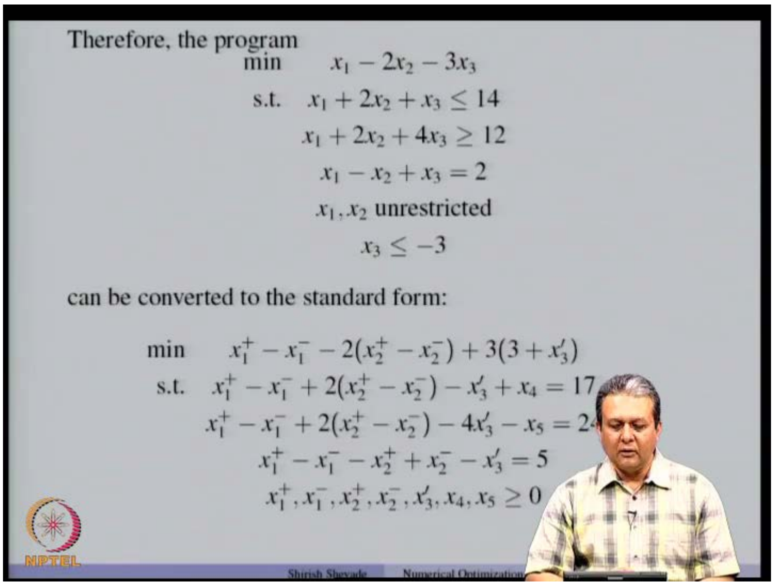

And, therefore the program which was given to us can be written in standard form and as so wherever x 1 is there we replace x 1 by x 1 plus minus x 1 minus, x 2 is replaced by x 2 plus minus x 2 minus and minus 3 minus x 3 will be written as x 3 dash. So, one can verify that the given linear program now can be written in terms of the newly defined variables. And more importantly all the constraints which are given here are equality constraints and all the variables which the linear program uses are nonnegative. So and thus using this or some other ideas any linear program can be converted to the linear program in the standard form. Of course, by introducing some more variables the number of variables has increased in this case but nevertheless we got the linear program in standard form where the constraints are of the type A x equal b and x greater than or equal to 0. So, for most of our discussion on linear program we will concentrate on the linear program in standard form and that is why it is important to convert any linear program to the linear programming in standard form. So, that the theory that we discuss about the solution of linear programs in standard form holds or can be used efficiently.

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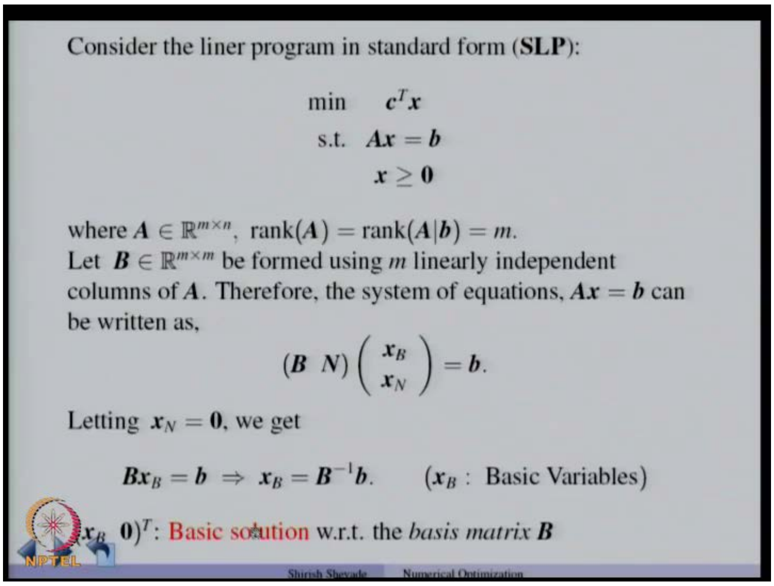

So, now let us consider the linear programming standard form we will call it as SLP; the standard linear program where we want to minimize C transpose x subject to A x equal to b and x nonnegative. And A is a m by n matrix and rank of A is equal to a appended with b and that is m. So, A is a full row rank matrix and the rank of A is m. So, this also indicates that b lies in the column space of A and the system of equations x equal to b is consistent. Now, we have m equations and n unknowns and the rank of the matrix A is m. So, we can always choose m linearly independent columns of A to form a matrix b. And since rank of A is m, we can always choose those m linearly independent columns of A.

So, in other words the matrix A is divided into two parts and without loss of generality we assume that the first m column of the matrix A are linearly independent columns. And therefore, the matrix b is formed using those columns and the remaining n minus m columns will correspond to a new matrix called n. So, in other words the system of equations x equal to b can be written as (( )) split into two matrices, two sub matrices B and N; B is of the size m by n and N is of the sizfe m by n minus m. And remember that columns of B are linearly independent and as I mentioned earlier that without loss of generality, we can assume that the first N columns of a are linearly independent.

Now, corresponding to those linearly independent columns are the variables which we are going to denote by x B. And corresponding to the remaining n minus m columns there are n minus m variables which we are going to denote by x N. So, the system A x equal to b can be written as like this. And now if you expand this it becomes B x B plus N x N equal to b. And suppose we let x N to be 0, then what we get is B x B equal to b. Now, note that B contains linearly independent columns and there are N such columns design by n by m matrix; so B is invertible. So, B x B is equal to b means that x B equal to B inverse b. Now, such x B is called the basic variable of the set of basic variables. So, x B denotes the basic variable associated with the basis matrix B. And since we have let x N to be 0, so x B and 0 uses what is called the basic solution associated with the basis matrix B. So, the variables corresponding to the linearly independent N columns are called the basic variables. And the solution x B equal to B inverse b along with x N equal to 0 is called the basic solution associated with the basis matrix B.

Now, this basic solution helps us in characterizing the vertex of a feasible set provided that the solution is feasible. So, we will see more about basic feasible solution and the characterization of vertices in the next class.

Thank you.