Numerical Optimization
Prof. Shirish K. Shevade
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 30
Linear Programming Problem

Hello, welcome back. Our next topic is Linear Programming. Linear Programming is a very well studied topic in optimization literature and the theory and applications of linear programming are very rich. So, let us start discussing about linear programming.

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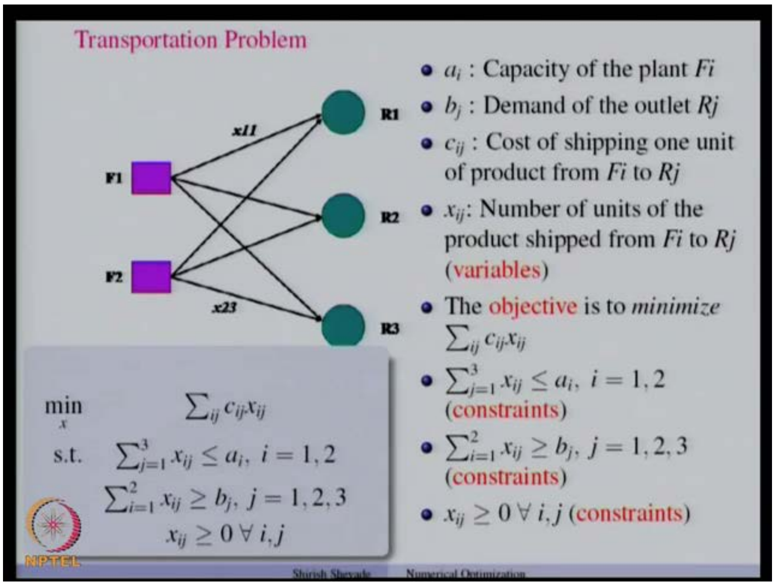

You may recall that in one of our earlier lectures, we discussed about transportation problem. So, there are 2 factories which produce some commodity and that is transported to the retail outlets. So the factories are f1 and f2 and the retail outlets are r1, r2, and r3. Now, every plant has a capacity of producing some a i units of that particular commodity and every retail outlet has a daily demand. And, let us denote that demand by b j corresponding to the outlet R j. Now, let us assume x 11 units are transferred or transported from the factory 1 to the retail outlet 1. And, let us denote by c ij; the cost of shipping 1 unit of product from the factory i to the retail outlet j. So, the variables in this optimization problem are x ij. And, the objective function is to minimize the cost involved in shipping the finished product from the factory to the retail outlet.

So, the objective function is to minimize sigma c ij x ij; and that is not enough we need to satisfy some constraint; that the amount shift from the factory i cannot exceed the capacity of that plant or factory. And, the amount received by each of the outlets should be at least equal to the demand of that outlet. So, we need to satisfy the constraint that sigma x ij over all j is less than or equal to a I; and sigma x ij over all i is greater than or equal b j. And, moreover the number of units that gets shift from the factory to the outlet are non-negative. So, we can write the program as minimize cij xij subject to the constraint that; sigma xij less than or equal to ai and sigma xij greater than or equal to bj and xij are nonnegative. Now, 1 good thing about this objective function is that it is linear in terms of the variables. Also if we look at the constraints the constraints are also linear in terms of the variables.

And, further there are some extra constraints which are nonnegative constraints. So, depending upon the problem these constraints may or may not be there. So, when the objective function is linear in x the constraints are also linear in terms of the variables then such a program is called a Linear Program. So, transportation problem is a classic example of a Linear Programming Problem.

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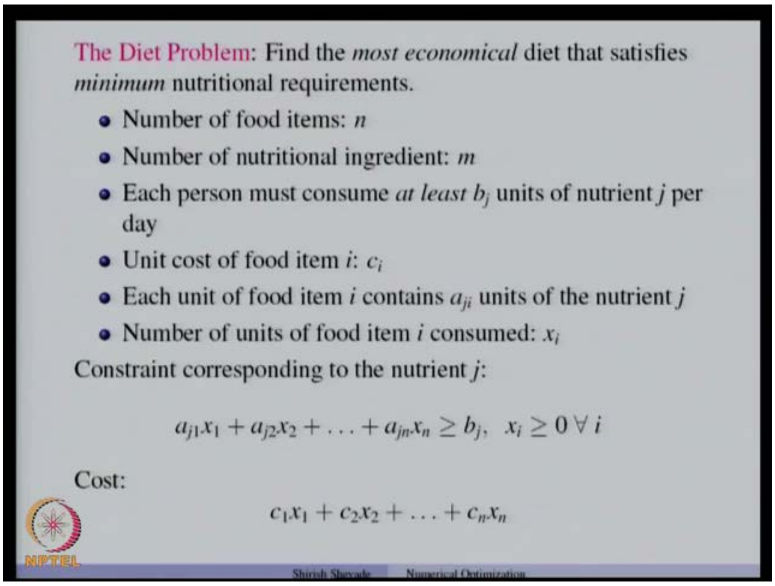

Let us look at another example which we use in our daily life this is called the famous Diet problem. The problem is that we want to find out the most economical Diet that satisfies minimum nutritional requirements. So, let us look at the problem description in detail.

Suppose, that there are n food items available in the market; and the number of nutritional ingredients that we are supposed to consume is m. Now, there is a requirement that each person must consume at least bj units of nutrient j per day. So, that requirement is suggested by the dietician. Now, there is cost associated with each of the food items and let us denote that cost by ci. So, the unit cost of the food item i is nothing but ci. Now, moreover every food item contains some units of nutrient. So, let us assume that each food item i contains aij units of nutrient j. And, let us assume that the number of units of food item i that we consume is xi.

So, given this data our aim is to find out what is most economical diet that satisfies the minimum nutritional requirements? So, minimum nutritional requirement is that we should consume at least bj units of nutrient j. So, this condition needs to be satisfied and at the same time we want to find out the most economical diet. So, if you consume xi units of food item i and the corresponding unit cost of food item is ci. Then, what we are interested is to minimize the cost? So, let us see more about this problem.

Now, corresponding to the nutrient j we need to satisfy the constraint that aj1 x1 plus aj2 x2 up to ajn xn greater than or equal to bj. Because we should consume at least bj units of the nutrient j per day. And, each food item contains each unit food item i contains aji nutrients of j. So, if you consume x1 units of food item I, x1 units of food item 1, x2 units of food item 2 and xn units of food item n. Then, we need to satisfy that for the j th nutrient this condition is satisfied. And, that should be satisfied for all j going from 1 to m. So, this is the constraint that we need to satisfy.

Now, that takes care of the minimum nutritional requirements, but we also have to find out what is the most economical diet? So, that means that the cost of food items that we buy should be minimized. So, the cost is equal to c1 plus x1, c2 x2 up to cn xn. If xi is the number of units of food item i consumed and ci is the unit cost of food item i.

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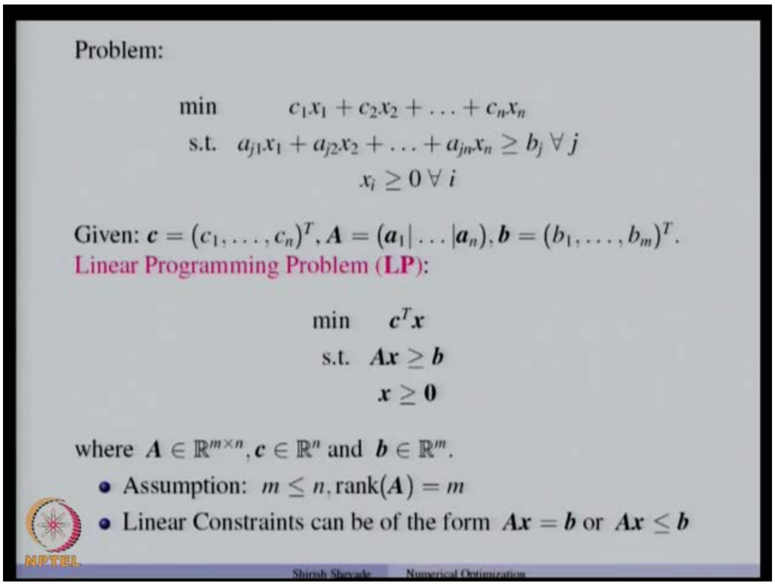

So, the problem becomes minimize c1 x1 plus c2 x2 up to cn xn subject to aj1 x1 plus aj2 x2 up to ajn xn is greater than or equal to bj for all j. So, this corresponds to the nutrient j and this constraint needs to be satisfied for all nutrients that we need to consume. And there are there are non negativity constraints on the variables x. So, you will see that the objective function is linear in terms of the variables x; the constraints are linear in terms of the variables x. So, this is another interesting example of a linear programming problem. Now, there exist many more examples of linear programming problems in our daily life. Now, let us try to this linear programming problem in compact form.

So, we are given a vector c containing n components c1 to cn and the matrix A whose columns are a1 to an. So, each ai vector each ai vector in each column is a m dimensional vector. And, suppose let us assume that b1; b is composed of b1 to bm. So, this program can be written as minimize c transpose x subject to the constraint Ax greater than or equal to b and x greater than or equal to 0; where A is a m by n matrix, c is a n dimensional vector and b is a m dimensional space. Now, this is called a linear programming problem. As you can see that the objective function is linear in x; the constraints are also linear in x and there are some non negativity constraints.

As I mentioned earlier in some applications these constraints may not be there. Now, let us make some assumptions which are very reasonable that m is less than or equal to n.

So, when we talk about this constraints the number of constraints is less than or equal to the number of variables. This is a reasonable assumption and we also assume that the rank of the matrix A is m which is less than or equal to n. Now, if there are more constraints then n then there could be some redundant constraints or the system of equations may not be consistent. It depends on how the constraints are represented.

So, if the redundant constraints are there they can be always reduced or eliminated. And, finally we have a situation where the rank of the matrix A is m and m is less than or equal to n. Now. another important point is that the linear constraint which are of the type Ax greater than or equal to b; they can be of the form Ax equal to b or Ax less than or equal to b. So, we could have equality constraints or inequality constraints in a linear programming problem. Now, let us see more about the constraint set of a linear programming problem.

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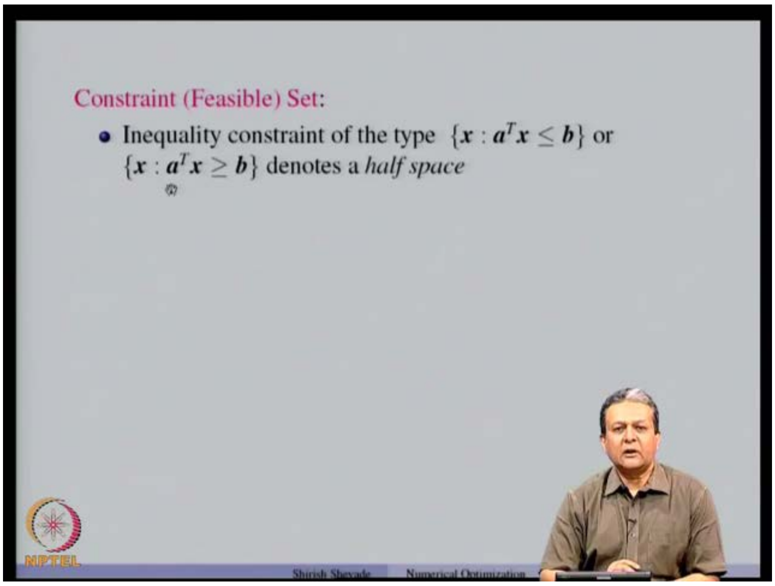

The constraint set is also called the Feasible set. Now, if you take inequality constraints of the type the set of all x such that a transpose x less than or equal to b or set of all x; such that a transpose x greater than or equal to b that denotes a half space.

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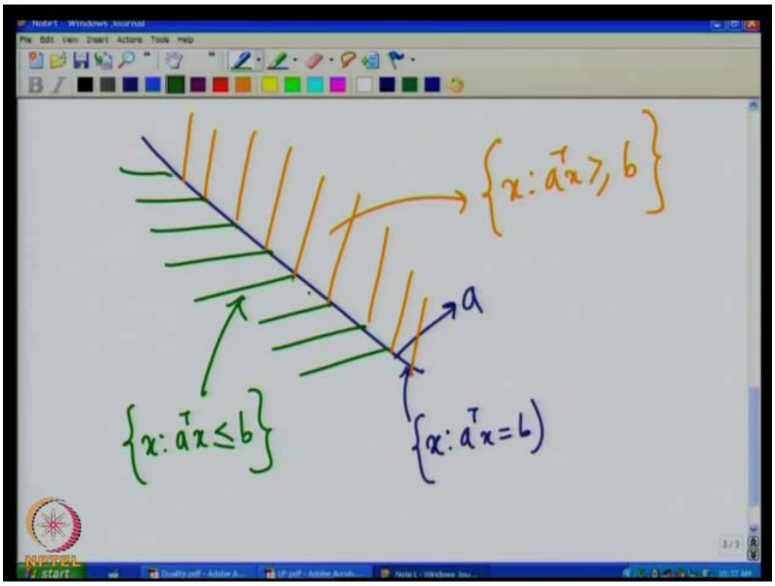

So, if we have in 2 dimensional space a line and this is the set of all x such that a transpose x equal to b. So, then the set a transpose x. So, this half space is the set where a transpose x is greater than or equal to b. So, that includes this set as well as the hyper plane. And, similarly the other part this will be the set of points x such that a transpose x less than or equal to b. So, this denotes another half space. So, both the half spaces contain the hyper plane and as you can see that each of this half space is a convex set.

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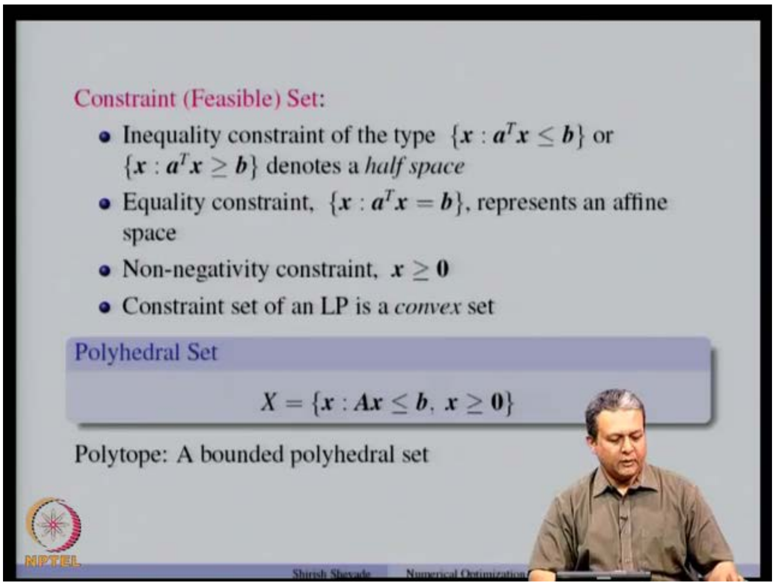

So, inequality constraint of this variety denotes a half space. Now, if you have equality constraint of the type x such that a transpose x equal to b. We know that it is a hyper plane. So, it representation of affine space and that also is convex set. So, every constraint in a linear programming problem is represented by a convex set. Now, if we combine all the constraints together that represents a intersection of collection of convex sets. And, we already know that the intersection of an arbitrary collection of convex sets is a convex set. Moreover, we have sometimes the non negativity constraint again they can be written in the form x greater than or equal to 0. So, this set again is a convex set. Therefore, the constraint set of a linear programming problem is a convex set.

Now, let us denote the set of the type Ax less than or equal to b and x greater than or equal to 0 as a polyhedral set. So, such sets are called polyhedral sets which are obtained using the intersection of half spaces. Now, if you have a bounded polyhedral set then that is also called polytope. So, we are going to denote a bounded polyhedral set by polytope.

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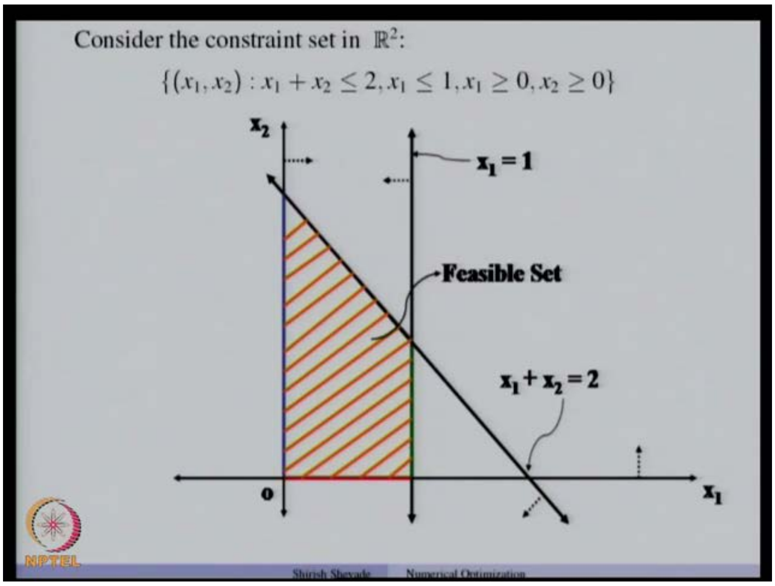

Now, let us look at the constraint set by taking some examples. Now, consider this constraint set in R2; the set of all x1 x2 such that x1 plus x2 is less than or equal to 2. x1 less than or equal to 1 and both x1 and x2 are nonnegative. Now, here is a diagram which represents the constraint set which is given here. Now, let us look at the first constraint. So, first constraint says that x1 plus x2 is less than or equal to 2. So, in the x1 x2 space let us first take the line x1 plus x2 equal to 2 and x1 plus x2 less than or equal to 2. So, we are interested in this half space which is indicated by this arrow. So, the half space corresponding to this hyper plane which is indicated by this arrow; is the region corresponding to the first constraint set or first constraint equation or first constraint inequality.

The second inequality x1 less than or equal to 1. So, first let us take the line which is x1 equal to 1. So, this is the line x1 equal to 1 and we are interested in all those x1 is less than or equal to 1. So, that half space will again be divided by this arrow. So, we are interested in this half space which is represented by this arrow corresponding to the line x1 plus x2 equal to 2 and this arrow corresponding to the line x1 equal to 1. Further we have non negativity constraints which are x1 nonnegative and x2 nonnegative. So, x1 nonnegative means that all the points in the right side of the x2 equal to 0 axis. And, x2 greater than or equal to 0 means all the points which are above x1 equal to 0 axis.

So, if we take the intersection of the half spaces formed by this 2 lines; and the first quadrant where x1 and x2 both are nonnegative. So, we get the feasible region or the feasible set which is shown by the shaded region. Now, you will see that this set is a convex set.

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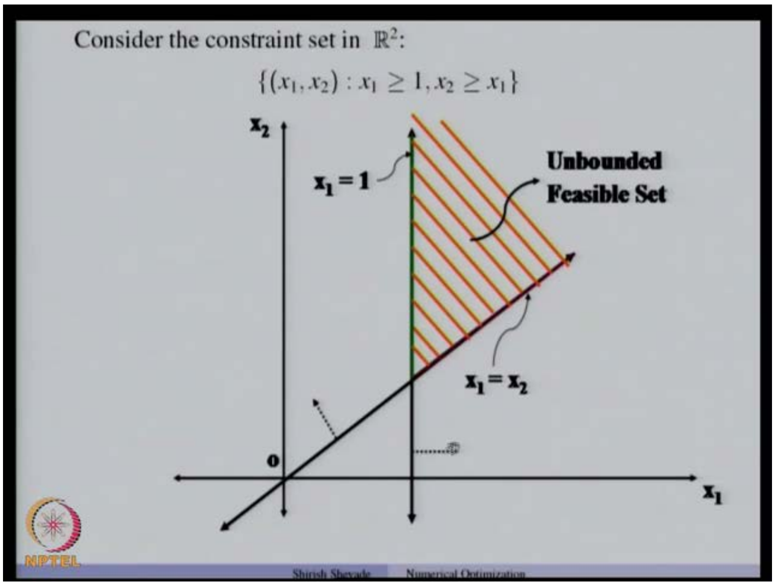

Let us take another example. So, here is another constraint set formed by x1 x2 such that x1 is greater than or equal to 1 and x2 greater than or equal to 2. Now, note that there are non negativity constraints on x1 and x2, but as you will see from the example that those constraints would have been redundant. So, even if those constraints were there they would have been redundant.

So, let us look at the x1 x2 space. So, x1 greater than or equal to 1. So, this is the line which is x1 equal to 1 and we are interested in all those x1's which are greater than 1 or greater than or equal to 1. So, the half space is indicated by this arrow. Now, the other constraint x2 greater than or equal to 1. So, first we draw the line x1 equal to x2 and the half space that we are interested is indicated by this arrow. So, if you take the intersection of this 2 half spaces that is shown by the shaded region in this figure. Now, you will see that this constraint set or feasible set is unbounded. Further you will also see that x1 and x2 both are nonnegative in this case.

But even though they are not part of our original constraint set; the constraint set in this case is such that x1 and x2 both are nonnegative. So, this is an example of an unbounded feasible set. So, the feasible set in the earlier case we saw that it is a convex set which is bounded. Here, the feasible set is also convex set, but it is unbounded. Now, there could be situations where feasible set can be a singleton set can be unbounded.

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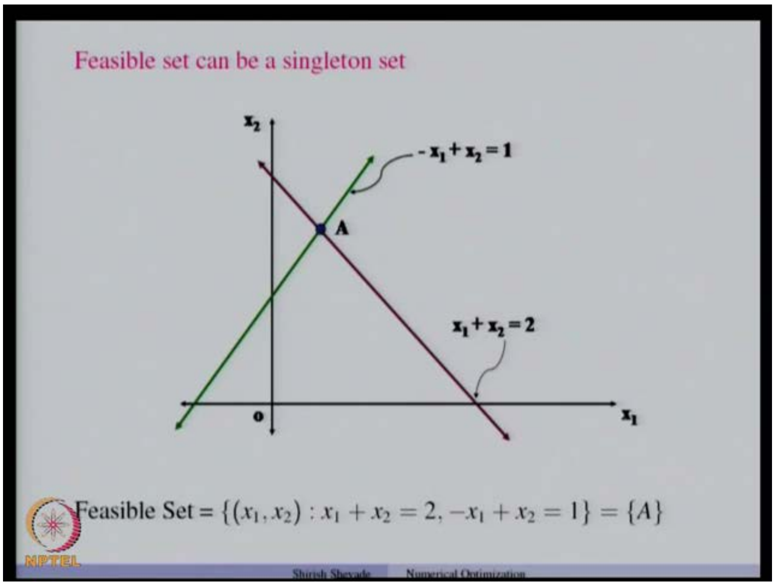

For example, suppose the feasible set is an intersection of 2 affine sets. And, the affine set the 2 affine sets in this case are x1 plus x2 equal to 2 and minus x1 plus x2 equal to 1. Now, the intersection of these 2 affine sets is a point. So, if we have a feasible set which is x1 plus x2 equal to 2 and minus x1 plus x2 equal to 1; the set of all x1 x2 will satisfy this is only the point A. And, therefore the feasible set can also be a singleton set, but for most of our discussion we will not worry about this kinds of problems. Because when the feasible set is singleton there is no need to worry about finding a solution to the problem. Because since the feasible set is a singleton set that itself is a solution to the given optimization problem. So, this case is not interesting.

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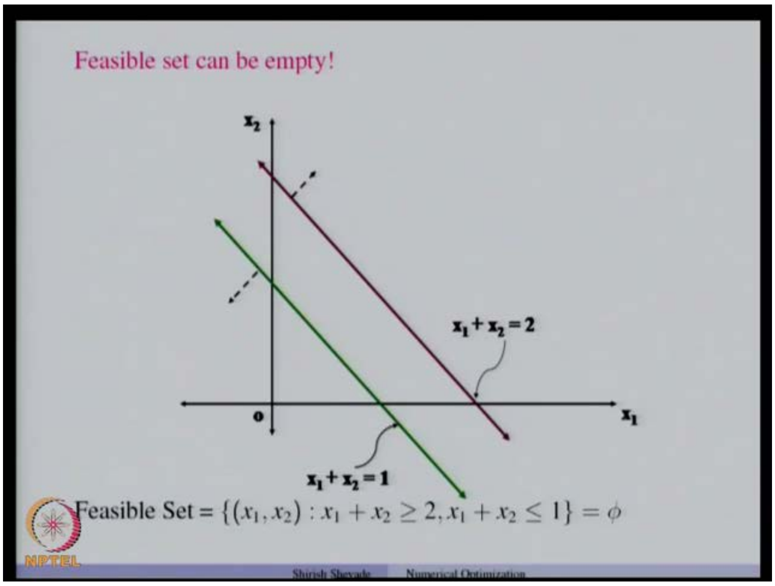

Similarly, there could be cases where feasible set can be empty. For example if we consider feasible set of a type of x1 plus x2 greater than or equal to 2; and x1 plus x2 less than or equal to 1. So, x1 plus x2 greater than or equal to 2 is the region or half space indicated by the arrow which corresponds to x1 plus x2 equal to which corresponds to which corresponds to x1 plus x2 greater than or equal to 2. So, this half space is the region that corresponds to x1 plus x2 greater than or equal to 2. And, other half space is denoted by this arrow which indicates x1 plus x2 less than or equal to 1.

You will see that there is no intersection between these 2 half spaces. And, therefore the feasible set here is a null set. So, again in our discussion will not worry about in this case also because then the problem or the optimization problem cannot be solved; as the feasible set is empty. So, we will be mostly interested in those cases where the feasible set is n1 empty and non singleton. It may be bounded or unbounded but our main focus will be only on those cases.

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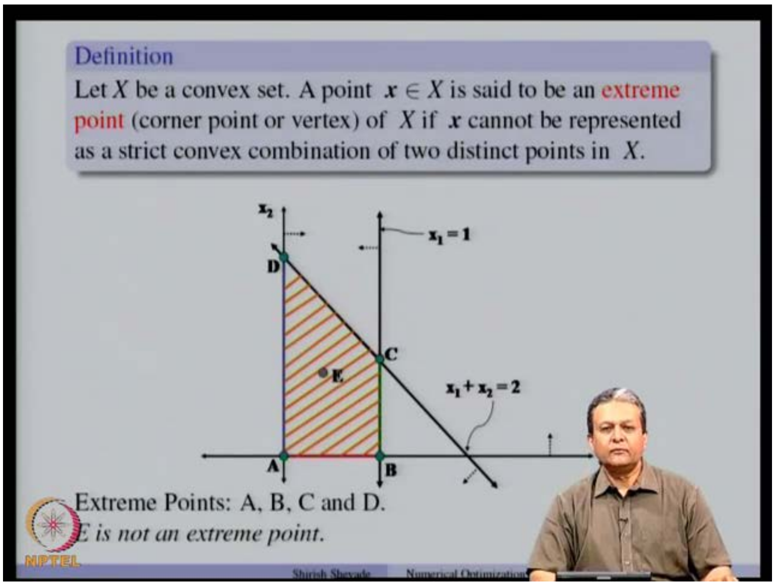

Now, let us look at some more definitions. Before seeing how to solve the linear programming problem? So, assume that x is a convex set; and any point x of that convex set x is said to be an extreme point which is sometimes also called a corner point of vertex of x. If that point x cannot be represented as a strict convex combination of 2 distinct points in x. So, if a point belonging to the given convex set cannot be represented as a strict convex combination of 2 distinct points in x. Then, such a point is called an extreme point or a corner point or a vertex of x. So, we will illustrate this using some examples.

Now, let us consider the same feasible set that we had seen earlier. Now, this feasible set is a convex set. Now, let us take a point A. This point A cannot be represented as convex combination of strict convex combination of any 2 distinct points in the set x; same is true for the points b c and d. But if you look at the e e can always be represented by convex combination of any 2 points in the set. So, what we need to do is that we need to draw a line segment whose end points are in the set x. And, that line segment passes through the point e.

So, this point e can be represented as a linear combination of convex combination of 2 distinct points of the set x. And, that is not true for a points a b c and d. And, therefore these 4 points form the vertices or corner points or extreme points of the feasible set which is indicated here. While e is not an extreme point; so a b c and d are extreme points and e is not an extreme point.

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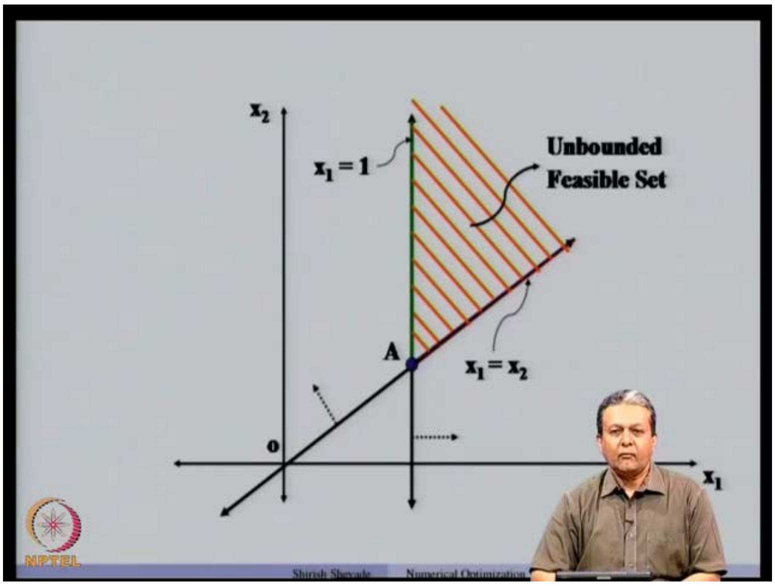

Now, here is another example of unbounded feasible set. And you will see that the point is only extreme point of this set. There are no more extreme points as far as this set is concerned. There could be instances where the constraint set may not have an extreme point or corner point.

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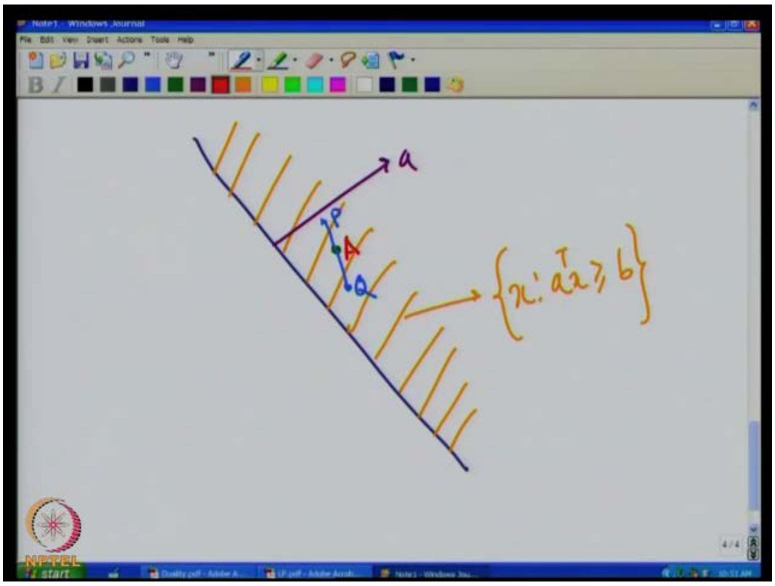

For example, so if our constraint set is… So, these constraints set does not have a extreme point because if you take any 2 points in this constraint set; if you take any point in this constraint set that can always be represented as a convex combination of other 2 points. For example, we can take a line segment let us call it as p q. So, the point A can be written as a convex combination of p q.

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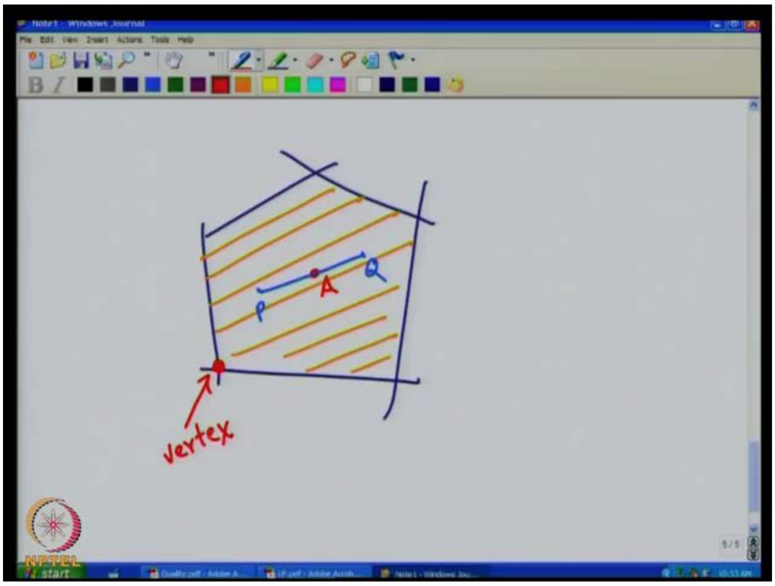

So, if you consider a set like this which is convex set and this is the feasible set that we are interested. If you take a point in the interior of this set let us call this point as A. We can always draw a line segment p q; which is the set in the given set which passes through A. So, that a can be represented as a strict convex combination of p q. On the other hand if you take a point which is here. You will see that we cannot express this point as a convex combination of any 2 points of the set. So, this is a vertex or an extreme point.

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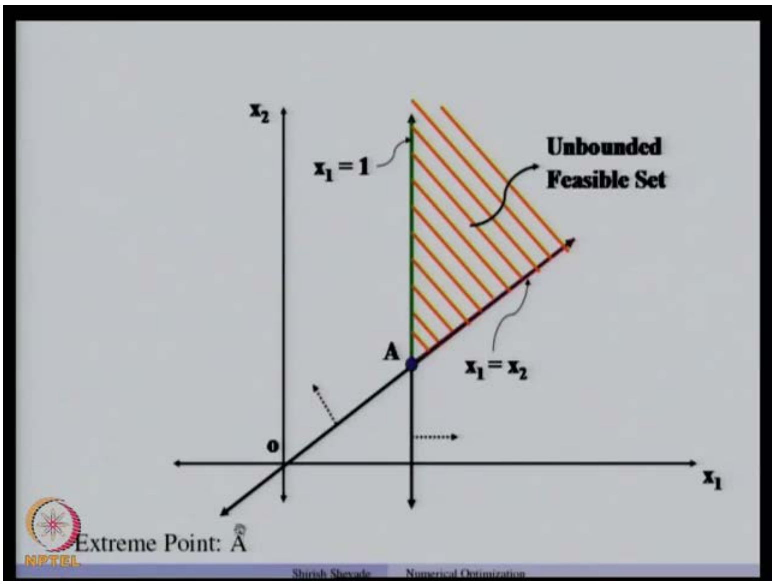

So, this has an extreme point A.

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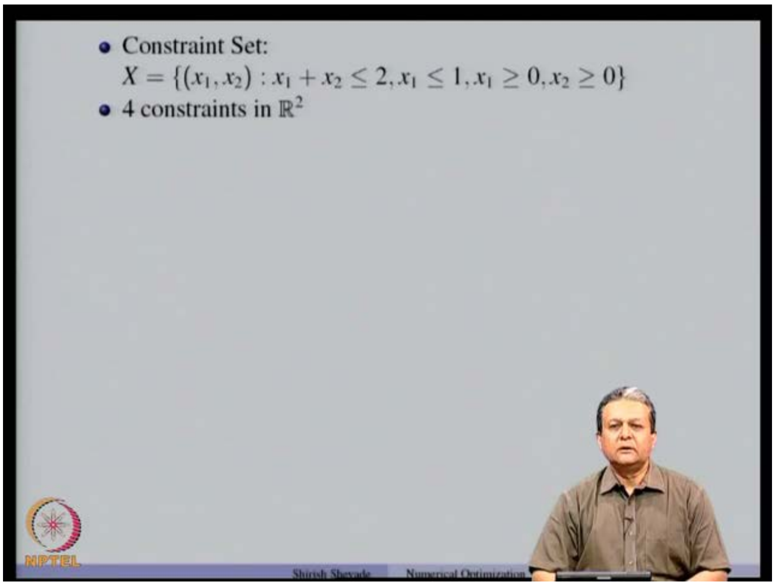

Now, let us see more about the geometry of the constraint set. And, how to get the geometric solution of a linear program in the next class?

Thank you.