**Numerical Optimization**
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**Lecture - 28**
**Geometric Interpretation**

In the last class, we started studying about duality theory, in particular we looked at two players zero sum game, and we saw that under the saddle point conditions, the game is in equilibrium. Now, we wanted we want to extend those ideas to a non linear programming problem.

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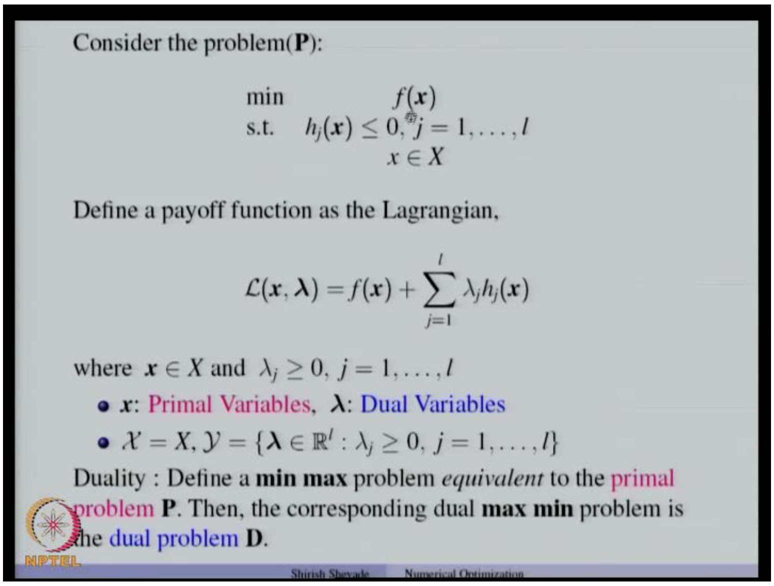


So, we first consider the problem, which is of the type minimize f(x) subject to h j(x) less than or equal to 0. So, we have inequality constraints later on we will see how to extend these ideas to a general non linear programming problem. So, we defined out payoff function as a Lagrangian function, which is f(x) plus sigma lambda j h j(x). So, the x are called the primal variables and the lambda's the Lagrangian multipliers are called the dual variables. So, the idea in duality is that, if you define a min max problem, which is equivalent to the original problem then the corresponding dual problem is a max min problem. So, so the corresponding dual problem is a max min problem.

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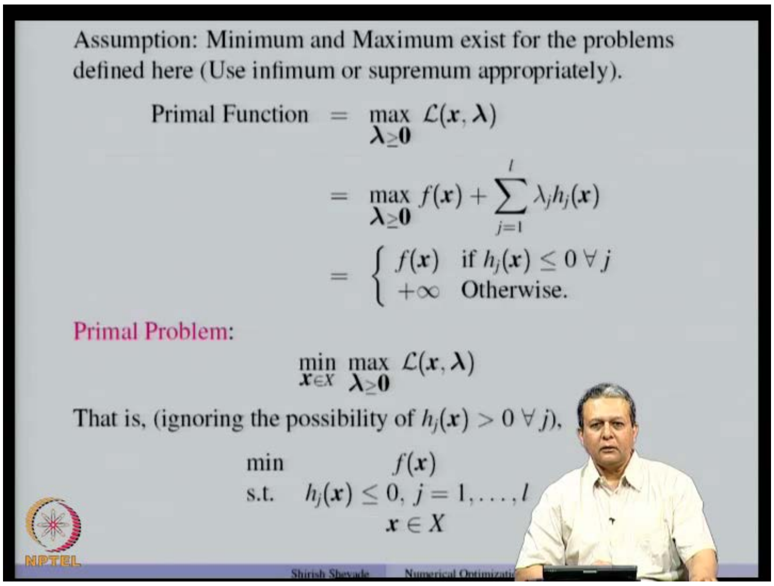


And one assumption that we will make throughout this discussion on duality theory is that minimum and maximum exist for the problems, which are different here. And otherwise one has to use infimum or supremum. So, the primal function is max of the Lagrangian subject to the constraint at lambda greater than or equal to 0. And that function is f(x), if h j(x) is less than or equal to 0, because that is the maximum value that f (x) can achieve, if h j(x) is less than or equal to 0. And of course, we are assuming that lambda's are non negative and the maximum values plus infinity. Otherwise so if we ignore this h j(x) greater than 0 possibility, then the primal problem becomes minimize f(x) subject to h j(x) less than or equal to 0 x belongs to x, so which is same as our original problem.

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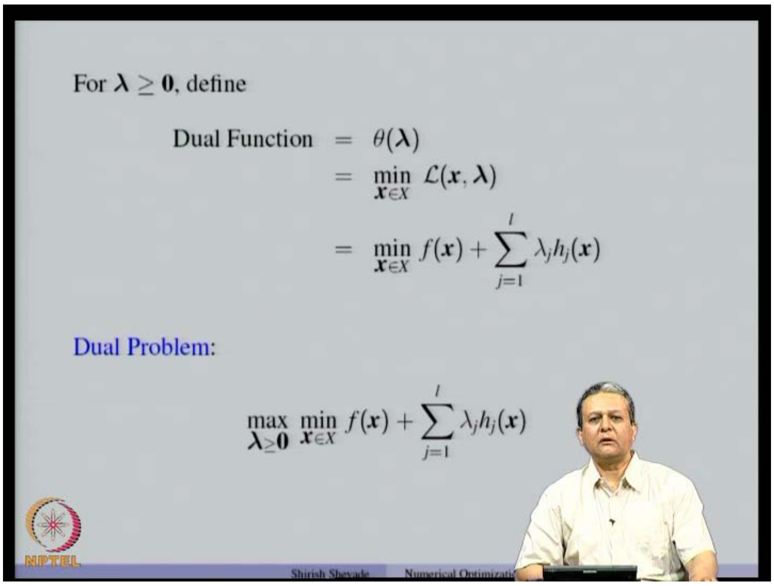


And similarly, one can define a dual function, which is theta lambda and that is nothing but minimum of the Lagrangian function, and then the idea is to maximize this theta lambda. So, maximize with respect to lambda the minimum of the Lagrangian function with respect to x. So, you will see that, the dual problem is the max min problem, max min of the Lagrangian and primal problem is the min max of the Lagrangian. And in this discussion, we have used Lagrangian function as a payoff function.

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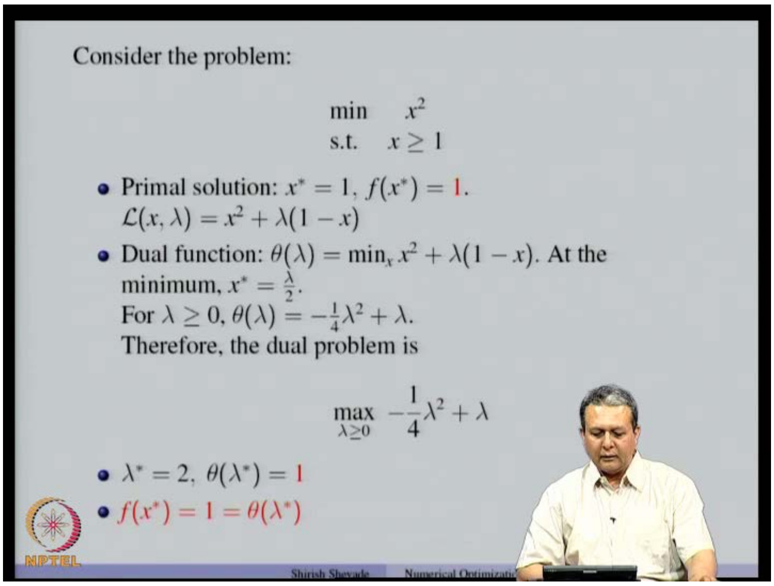


Then, we saw one example.

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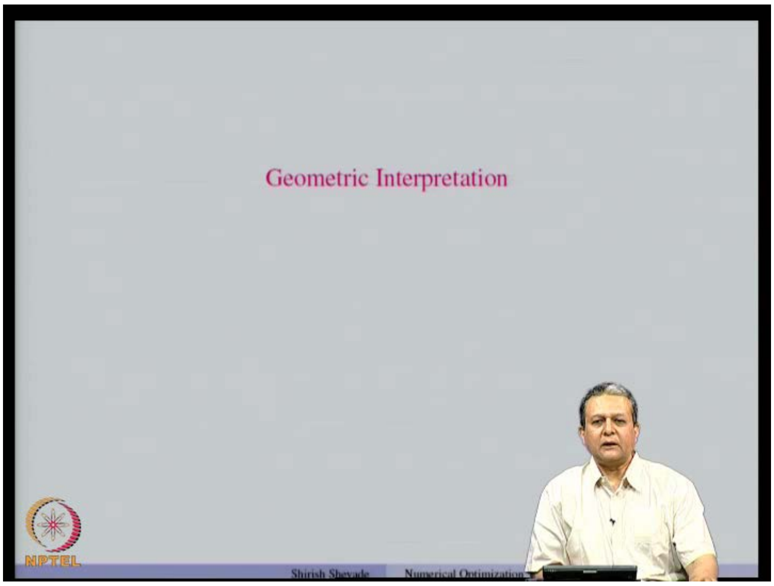


So, in today's class, let us look at the geometric interpretation of duality.

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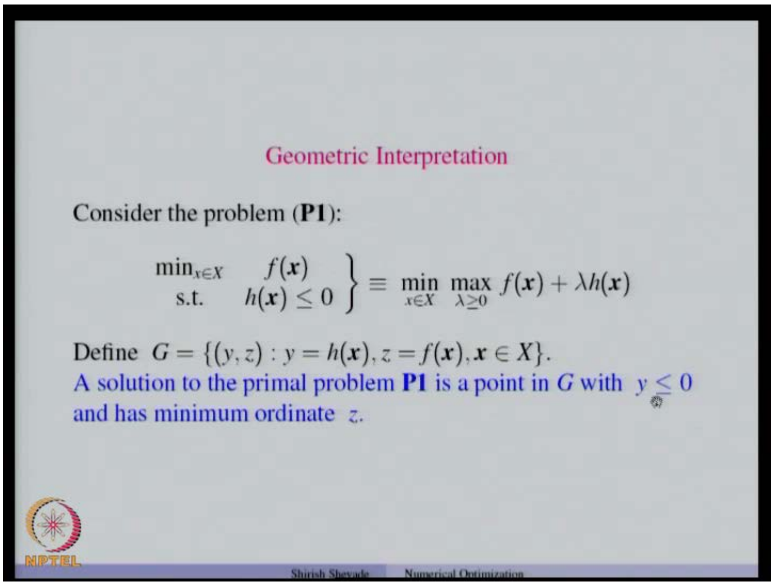


So, to see this geometric interpretation, let us look at a simple problem, where we want to minimize a function f(x) subject to a single inequality constraint, which is given here as h(x) less than or equal to 0. And of course, x belongs to the set x that is, another constraint, we saw that this problem can be written as a min max problem. So, the Lagrangian of this is f(x) plus lambda h(x), where lambda's are non negative. And we saw that, the Lagrangian the primal function is max of f(x) plus lambda h(x), where lambda greater than or equal to 0.

And we want to minimize this with respect to the set x, so this is our primal problem. Now, let us define a set G, which is the set of all y, z, such that y is equal to h (x) and z is equal to f (x), where x is in the original constraint set x. So, we take every x from the original constraint set x and the final mu set by using h(x) as the abscissa and f(x) as the ordinate.

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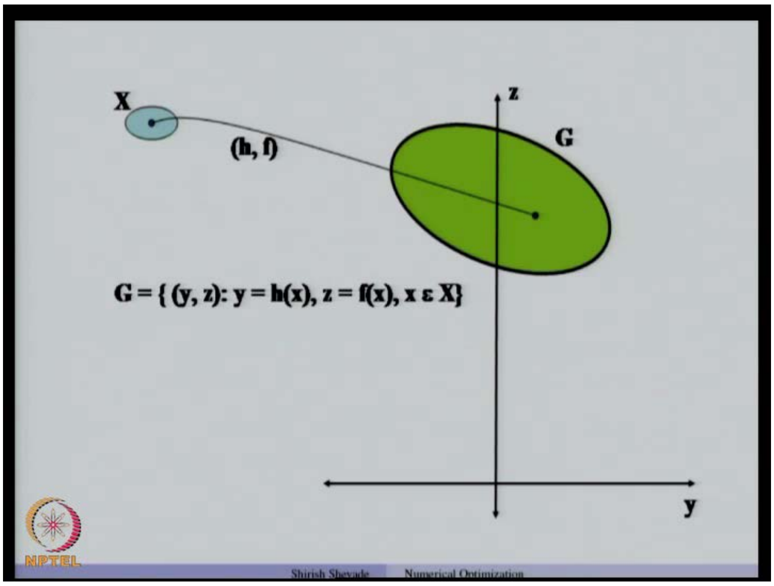


Let us see one example, so this is our original set x and we use the mapping h of x f of x to transform every point or to map every point in the original space to a new point in the space, which is called the y, z space. So, this is our y, z space so y is nothing but h of x and z is nothing but f of x and this entire set x is mapped to this set, which is denoted by capital G in the y, z space. So now, what we want to do is that, we want to find an optimal solution to the given problem in this y, z space.

In other words so what we are interested in looking at is the point, which satisfies h(x) less than or equal to 0 or in other words y less than or equal to 0 and we have to find the point, which is minimum f of x so that, this minimum z. In other words, we are interested in finding the point, in this space or in this set such that, y is less than or equal to 0. So that means, this part, so that means, the entire set G is not a feasible set, only some part of it is feasible.

And which has the point, which is having y less than or equal to 0 and which has minimum ordinate z that will be a solution to the given primal problem. So as I said that, we are interested in finding a point in G with y less than or equal to 0, because our primal problem constraints is h(x) less than or equal to 0 and y is nothing but h(x). So, point in G with y less than or equal to 0 and has the new ordinate z.

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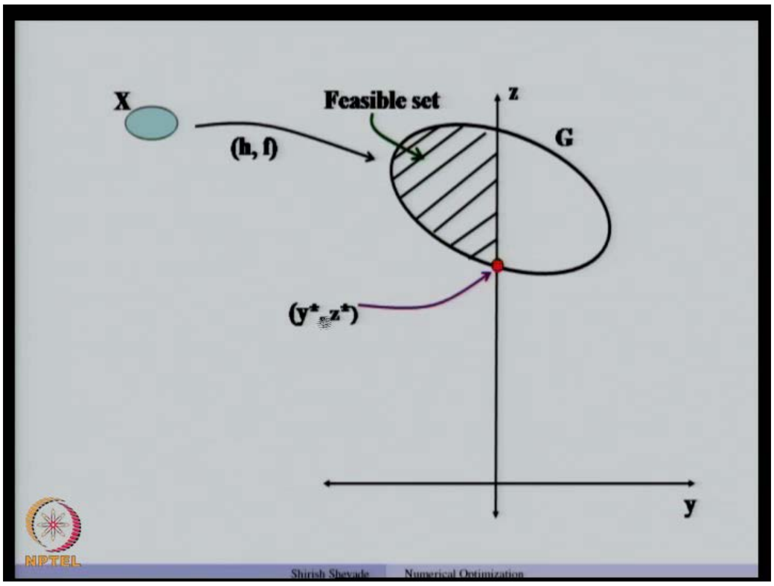


So now, the shaded portion of the set G is the feasible set, because that satisfies y less than or equal to 0. Now, among all this points, which are feasible than to find out the point, which has the least ordinate and that point is this point. So, let the coordinates of this point be y star, z star.

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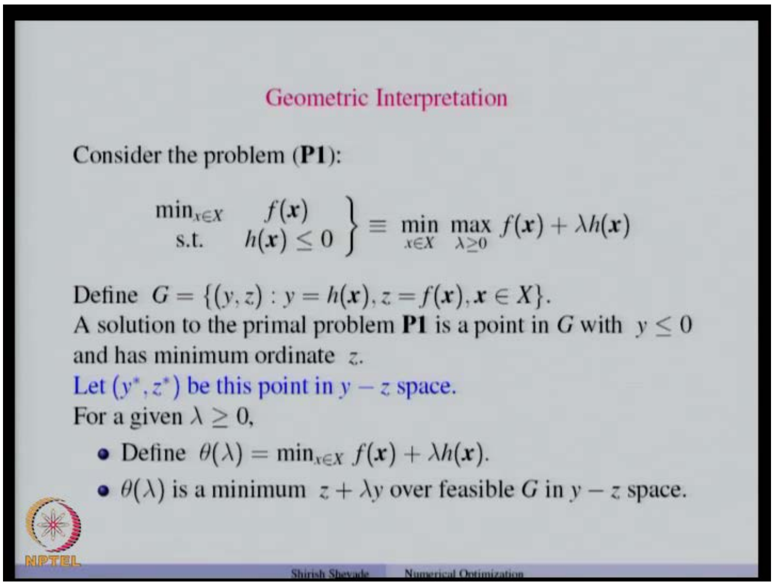


So, y star, z star is a point in the y z space, where y star is less than or equal to 0 and the ordinate is the least, so z star is less than or equal to any z in the feasible y z space. Now for given lambda, which is non negative, let us define a function called theta lambda, theta lambda is nothing but minimum f of x plus lambda h(x), where x belongs to x. Now, if we look at the corresponding problem in the y z space, so in the y z space, what we are interested in is minimizing z plus lambda y.

So, in other words, theta lambda gives us the minimum z plus lambda y over the feasible G in the y z space. So, so compare this to the the original problem was in x, now we are looking at the problem in y z space. And what we are interested in is minimizing z plus lambda y over feasible G in y z space remember that, we are given lambda, which is a non negative quantity.

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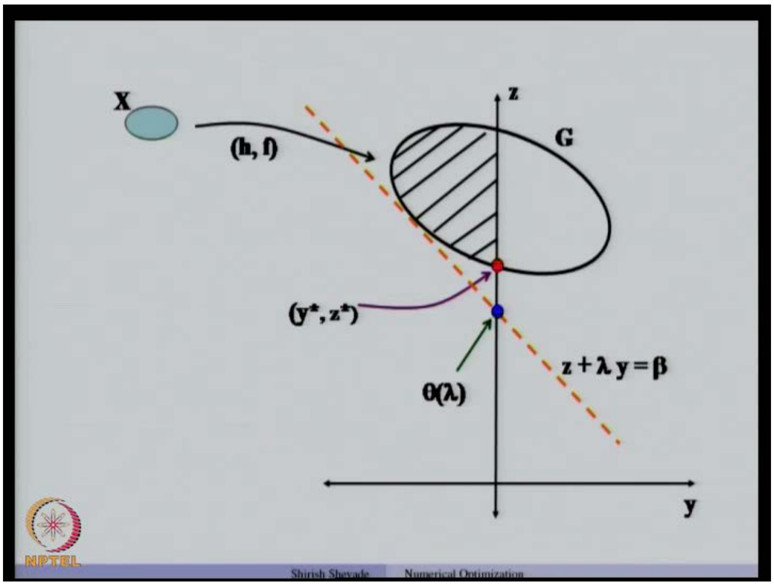


So, let us consider the same problem that, we saw earlier so for a fixed lambda, so here is the line, which has a intercept on the vertical axis as alpha. So, the equation of this line is z plus lambda y equal to alpha or in other words, this is a line with slope minus lambda and intercepts the vertical axis at alpha. Now, theta lambda will be a point for a in this y z space such that, it is gives the, for a given value of lambda, it supports this feasible region and gives the least value of the ordinate.

So So, for the same lambda, we look at all possible values of the variable alpha. So remember that, as z plus lambda y moves in this plane, the line moves in this plane, we get different values on the intercepts on the vertical axis or the z axis. Now, among all those intercepts, which one gives us the minimum value, which which value of this variable gives us the minimum. So, you will see that, this is the point, which gives us the least value of the ordinate for a given lambda and remember that, we have to maintain the feasibility.

So, this line should support the feasible region, so among all those lines, which support the feasible region, which gives us the least value of the ordinate. So, so this point is the intercept of the z axis and this is nothing but our theta lambda. So in other words, theta lambda is minimum of z plus lambda y, where y has to be less than or equal to 0 and we got this point and now, clearly this is not a solution, solution is this. So, what we are interested in finding out is that lambda, which gives us the maximum intercept on the vertical axis and.

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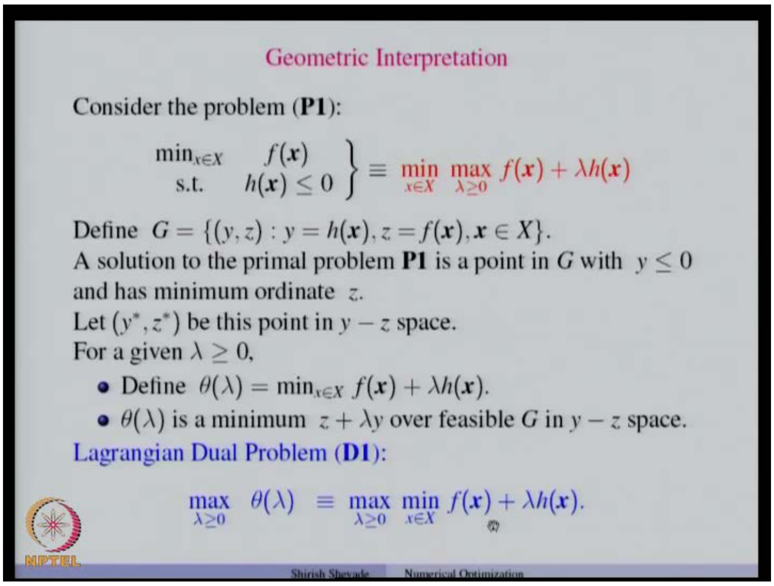


So therefore, what we are interested in solving max of theta lambda, where lambda is a non negative quantity and max of theta lambda is nothing but max of min of the Lagrangian. So now, you will see the dual relationship between the two problems, so this is our primal problem and this is our dual problem. The primal problem is the min max of the Lagrangian, the dual problem is the max min of the Lagrangian.

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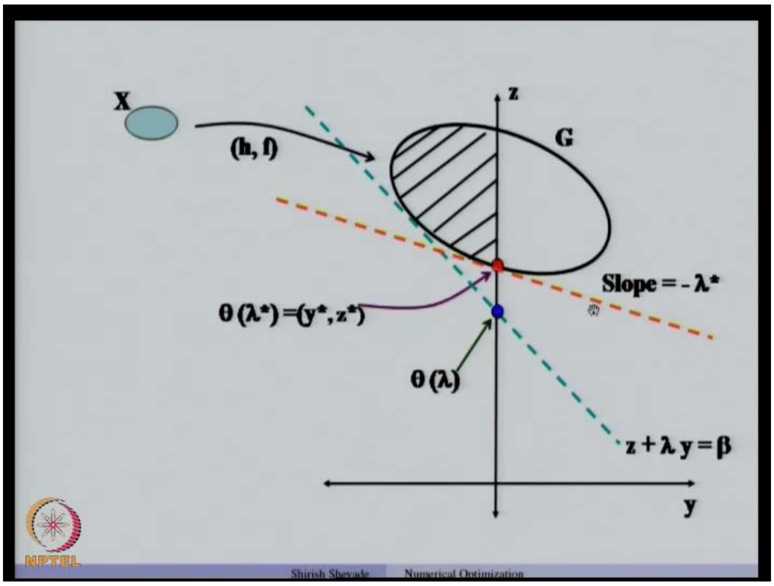

So in other words, what we are interested in is finding out that lambda star or the line with slope minus lambda star, which has the which supports the feasible set, feasible subset of the set G and which has the maximum ordinate. So, with a particular value of slope to be minus lambda, we got this line and this was our theta lambda and with the slope minus lambda star, we got this line which supports this set and the ordinate is maximum and the theta lambda star is nothing but y star, z star. So, theta lambda star is nothing but z star in this case. So in other words, the the primal problem solution is same as the dual problem solution here.

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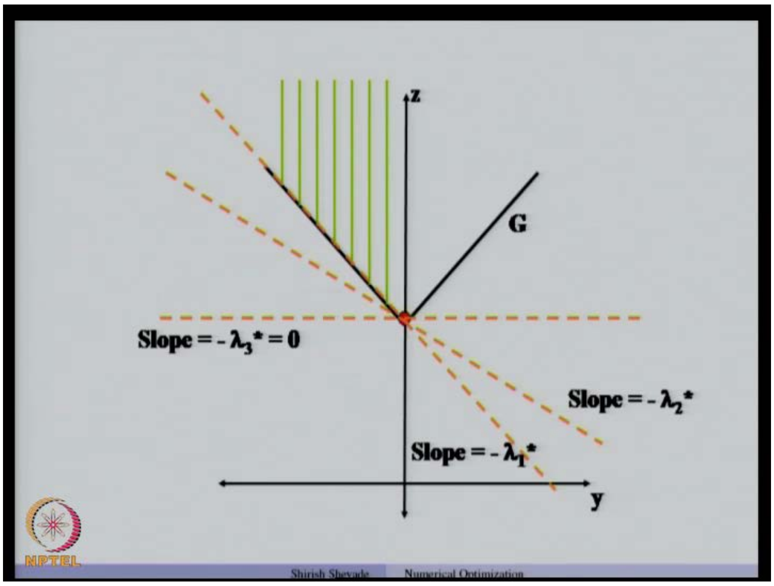


Now, let us see some examples, so the first example that, we are going to see is one here. So, the set G is the set of all points above this function and out of those possible points, the set of feasible points is marked here, so these are the points, where y is less than or equal to 0. And what we are interested in finding out, the minimum the minimum z such that, y less than or equal to 0. And therefore, clearly this is the optimal primal objective function.

Now, here is one possibility, that line with slope minus lambda 1 star supports this set and it gives us the intercept on the vertical axis is same and and therefore, the primal and dual objective function values are the same in this case. Now, the question is that, is this the only possibility, so let us see. So, we got another line, which again supports the given feasible set and the intercept on the vertical axis is same as the optimal primal objective function value.

And in fact we will see that, there exist infinitely many supporting lines with different slopes, which give us the same solution. So, any line passing through this point and having the slope in 0 to minus lambda 1 star is a possible candidate for the supporting line. And note that, here is a line with slope 0, which is also supporting line for the given feasible set and also passes through this optimal primal objective function value. The intercepts of all this lines on the vertical axis are the same.

So, here is an example, where there exist infinitely many possibilities of lambda with support the given feasible set from below. Now, there could be other problems, where such a line line with finite slope is not possible, the line which supports the given set and has a finite slope; so let us see an example.

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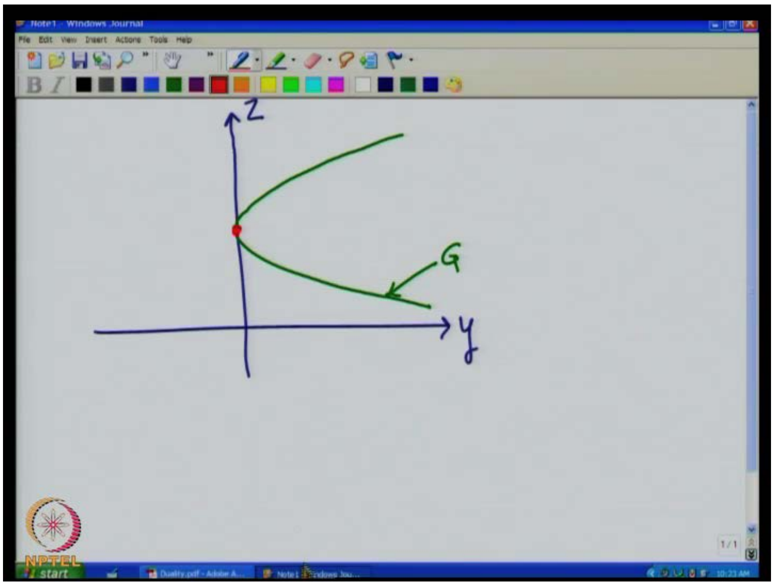


So, suppose we have in the y z space, suppose that the set G, so this is our set G and now, the only feasible point, which is here is this. So, this point is the only feasible point because what we are interested in is the set y less than or equal to 0. So, this is the only feasible point, so that that means at, it is obviously the solution to this problem. Now, you will see that, the only supporting line to this feasible set is the vertical line, which does not have finite slope. So, in this case, we cannot find a line with finite positive slope, which supports the give set although this point is the minimum point. So, here is an example, where one has to finitely many possibilities of the supporting lines, which have the same intercept on the vertical axis as the optimal primal objective functionally. And we saw another example, where there does not exist supporting line with finite positive slope.

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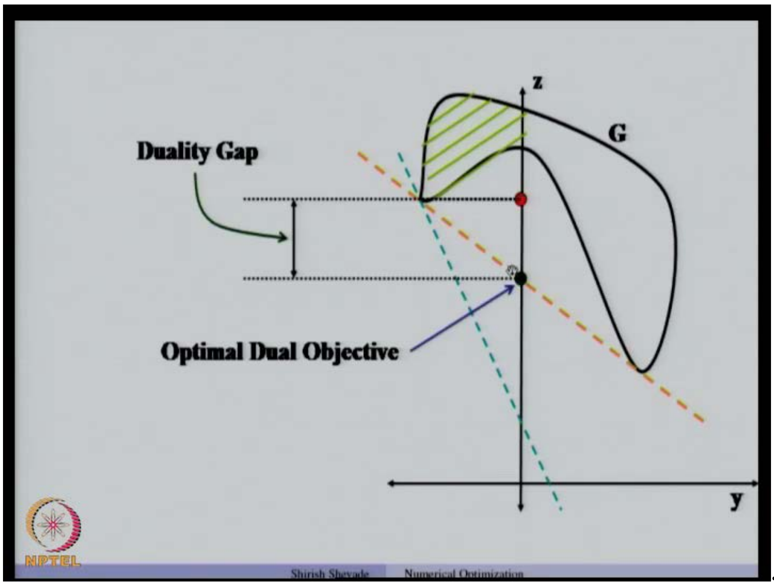


Now, let us consider another example, so we have set G but only that part of the set G, which is feasible is the set, where y is less than or equal to 0. Now in this set, which is feasible the optimal primal objective function value is this and which corresponds to this point; so this corresponds to the optimal primal objective function value. Now, now let us look at the supporting hyper planes to this set feasible set so one hyper plane, which supports this is shown here and for this particular slope, this is the value of theta lambda.

Now, among all those possible hyper planes, which is support the given feasible set, we want to find out the one, which has the maximum value of theta lambda. And that hyper plane turns out to be this hyper plane, which supports the given set feasible set G and which has the maximum intercept on the vertical x axis or z axis. So, this value is our optimal dual objective function value, now remember that, this was the optimal primal objective function value.

And therefore, the two optimal values are not the same or in other words, what is called the duality gap. So, duality gap is basically a difference between the primal and dual objective functions value. Now as, we will see later the dual objective function value is of a feasible point is always less than or equal to the primal objective function value. So so therefore the optimal dual objective function value is always less than or equal to the optimum dual optimal primal objective function value.

And if there is a difference between the optimal primal objective functional value and the optimal dual objective functional value, then we say that duality gap exists. So, if the duality gap exists, there is no incentive in solving the problem because we have not solved the original primal problem completely.

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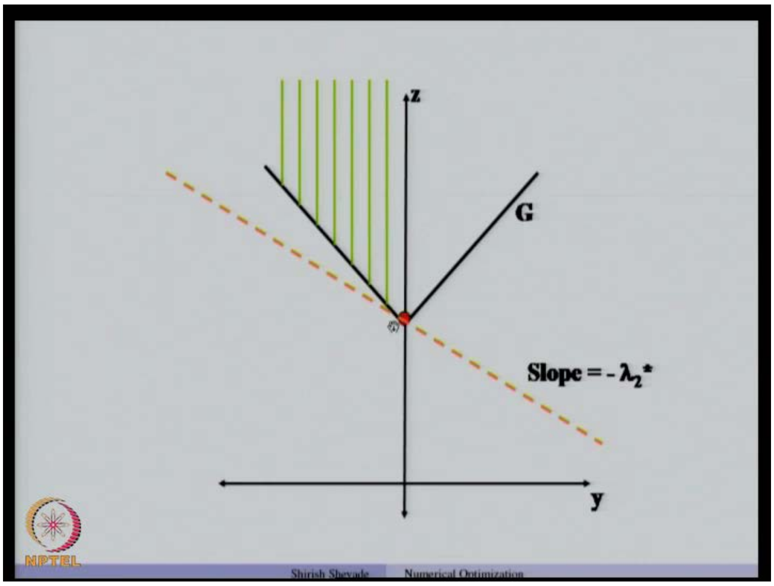


But on the other hand in situations like this, where the primal and the dual objective function values are same or the situation like this, where the slope minus lambda star the primal and the dual optimal objective function values are same. In such cases, it is sometimes simpler to solve the dual problem than, the primal problem. So, what are the conditions under, which the duality gap does not exist and how do we get such points, so that is the topic of our discussion now. Now, before we go into those details, we will first prove that the optimal primal value primal objective function value is always greater than or equal to optimal dual objective function value.

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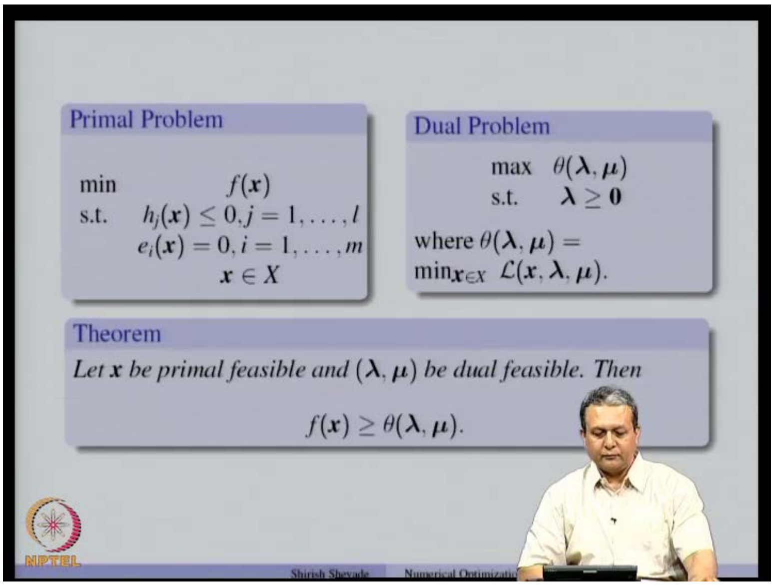


So now, let us look at the general non linear programming problem, where we want to minimize effects subject to the constraint h j (x) less than or equal to 0. There are l inequality constraints and then we have equality constraints of the type e i (x) equal to 0, where i runs from one to m and of course, x belongs to the set x. Now, the dual problem is the maximization of theta (lambda, mu) subject to the constraint that lambda greater than or equal to 0, that theta (lambda, mu) is defined as the minimum of the Lagrangian subject to the constraint that x belongs to the set x.

And then we have our result, which says that, if x is primal feasible and lambda, mu is dual feasible. So, x is primal feasible means that, x has to satisfy all these constraints plus x should belong to x and lambda, mu is dual feasible. So, if you look at the dual problem, there are no restrictions from mu, but there is a restriction on lambda. So, lambda non negative and mu unrestricted such, a point is dual feasible point.

Then the theorem says that the primal objective function value is at least the dual objective function value at evaluated at lambda, mu. Remember that, x is any primal feasible point lambda mu is any dual feasible point. So, the value of the primal feasible objective function is at least the value of the dual objective function. Note that, we are not still talking about the optimal primal and optimal dual objective function value, but that result simply follows from this. So, we will first see this result and then show that the optimal primal value primal objective function value is at least the optimal dual objective function value.

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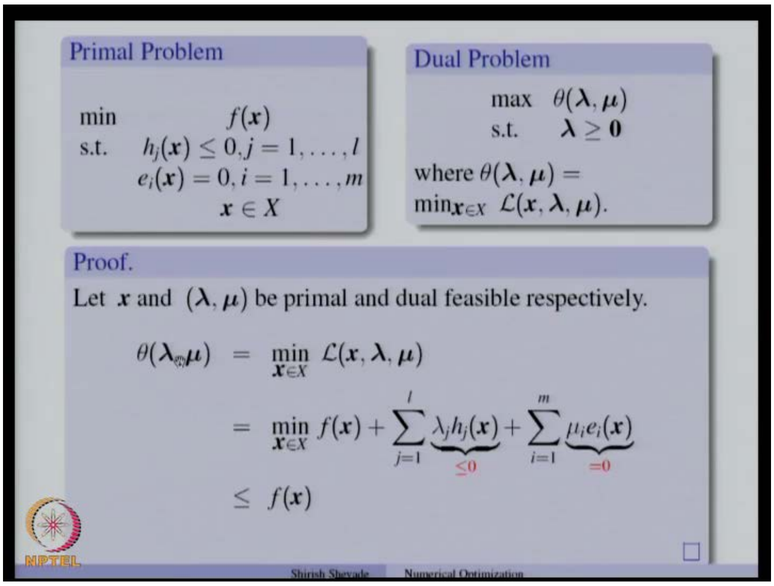


So, these are our primal and dual problems. Let us consider x and lambda, mu to be primal and dual feasible respectively. Now, if you consider the dual problem so the dual objective function is theta (lambda, mu) and what we want to show is that theta (lambda, mu) is less than or equal to f of x. Now theta (lambda, mu) is nothing but minimize of l (x, lambda, mu) subject to x belongs to x.

Now, if we expand this l (x, lambda, mu) as we it is a Lagrangian function. So, that Lagrangian function is same as f(x) plus sigma lambda j x plus sigma mu i e (x). Now since, x is primal feasible, we have h j(x) less than or equal to 0 lambda is dual feasible, so we have lambda's non negative. So, lambda h j(x) for feasible x and lambda is a none positive quantity and again x is primal feasible. So, e I (x) has to be 0 mu i's are any unrestricted in sign, so this quantity is 0, this quantity is non-negative.

And therefore, this minimum will be less than or equal to f of x, because this quantity is non positive and this quantity is 0. So therefore, what we have seen is that theta (lambda, mu) is always less than or equal to f of x or primal objective function value at a feasible x is always greater than or equal to a dual objective function value at any feasible lambda and mu.

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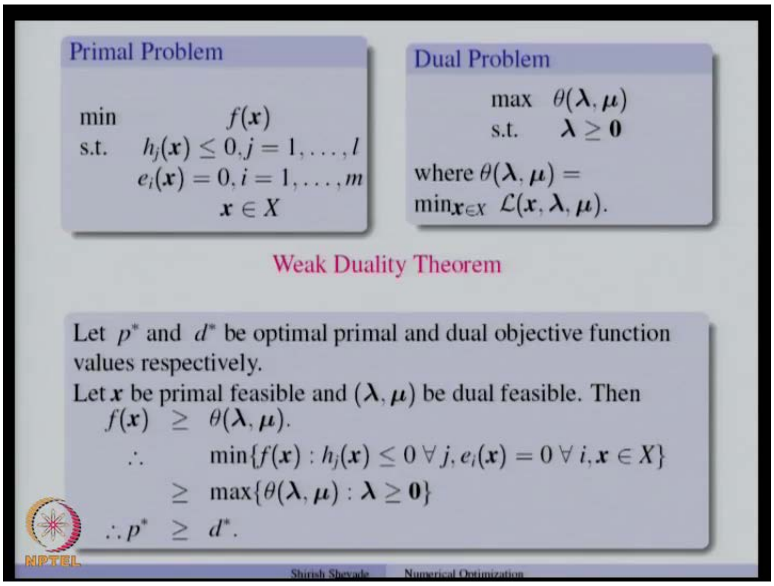


Now, now let us look at the weak duality theorem, which states that, if p star and d star are optimal primal and dual objective function values then p star is greater than or equal to d star. This is a straight forward extension of the previous theorem, that we saw, note that, if x star is solution to this primal problem and lambda star, mu star is dual is a solution to this dual problem. Then p star is nothing but f of x star and d star is nothing but theta of lambda star, mu star.

So, let us assume that, x is primal feasible and lambda u lambda mu to be dual feasible and then by the previous theorem, what we have is f (x) is always greater than or equal to theta (lambda, mu). So, if f (x) is greater than or equal to theta (lambda, mu) for any x lambda mu which are primal and dual feasible respectively, then minimum of f of x will always be greater than or equal to maximum of theta (lambda, mu). And that is, what is clear here that, minimum of f of x subject to the constraint h j (x) less than or equal to 0 and e i (x) is equal to 0 is greater than or equal to the maximum of theta (lambda, mu), lambda greater than or equal to 0.

The reason, why these things are written here is that, our x has to be primal feasible. So that means, it should satisfy all the inequality equality constraints as well as it should belong to the set x. And this is always greater than or equal to max of theta (lambda, mu) and this quantity minimum of f of x subject to these constraints is nothing but e star and max of theta(lambda, mu) is nothing but e star. So, we have e star greater than or equal to p star.

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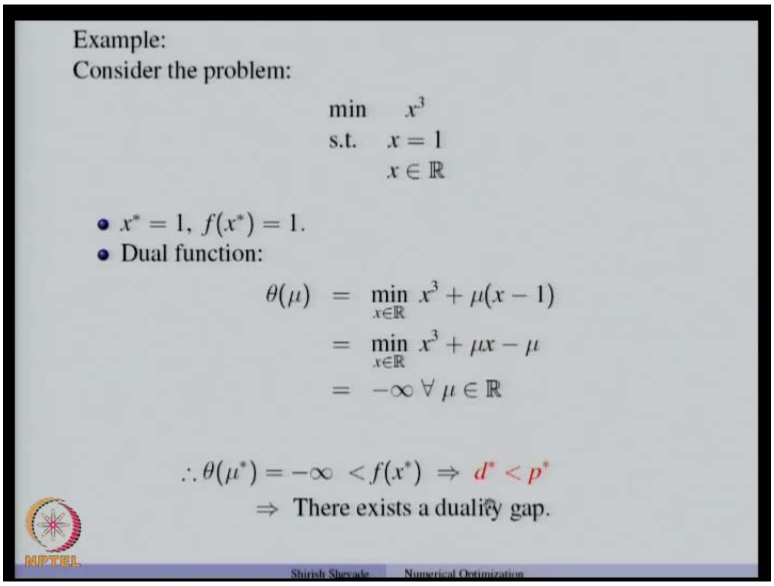


So, let us consider an example, where we want to minimize x cube subject to the constraint that, x equal to one and x belongs to R. So, this R is basically a redundant constraint here, but this example is just to show that, the way we define our constraint set the dual problems depend on that. So, the set x is same as the set R here. Now, the only feasible point is ha x equal to 1 so obviously that is, the solution. So, x star equal to 1 and then the corresponding optimal primal objective function value is also 1, which is nothing but x star cube. Now, the dual function note that, there is a equality constraint so the dual function is theta (mu); theta (mu) is nothing

but, minimum of x belongs to R the Lagrangian function, the Lagrangian function is the objective function x cube plus mu into x minus 1 and this is nothing but minimize x cube plus mu x minus mu. Now, you will see that, the minimum value of this for every possible value of mu minimum value of x cube plus mu x minus mu, for every value of mu is minus infinity, because x belongs to the set R.

So, this value belongs to minus infinity. And therefore, if we look at theta (mu star) theta (mu star) is minus infinity and minus infinity is strictly less than the optimal primal objective function value, which is 1. And therefore, here is the example, where the optimal primal objective function value is one and optimal dual objective function value is minus infinity. So, this is the example, where the duality gap is infinity. So, p star is less than p star and therefore, there exists a duality gap. So, this is the example, where the duality gap exists or d star less than p star.

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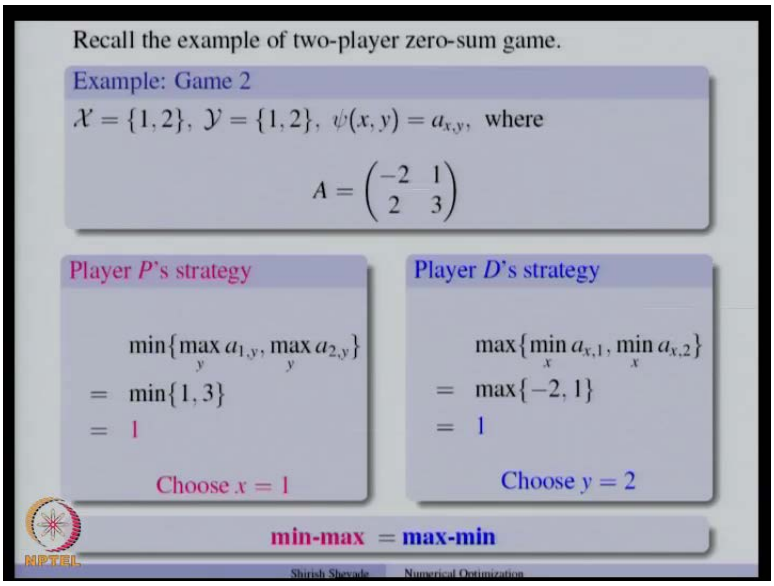


Now, the next question is that under, what conditions the duality gap is 0. Now to understand that, let us recall the example of two player 0 sum game that, we saw earlier. So remember that, in the two player game each player had two strategies one and two and we had a payoff function psi (x, y), which is defined using the matrix A. So, the row corresponds to the player p and the column corresponds to the player d. So, the player p strategy was to solve the min max problem and the min max problem gives the value 1.

So, the player will choose the strategy 1 and player d's strategy was to solve the max min problem and with respect to this payoff matrix the player d the first choose the strategy number two. And since, these two values are same, we say that the min max problem is same as the max min. Now, in our non linear programming terminology, this min max problem is a primal problem and max min problem is a dual problem.

So, the solutions to this primal and the dual problems are the same in this case. So, when we studied the two player 0 sum game, we saw that, if there exists a saddle point, then the game is in equilibrium. And if the game is in equilibrium, then there exists a saddle point. So, in the min max problem or the primal and dual problem context, what is the saddle point that, we want to look at and how do we ensure that such, a saddle point exists. So, we will discuss that now.

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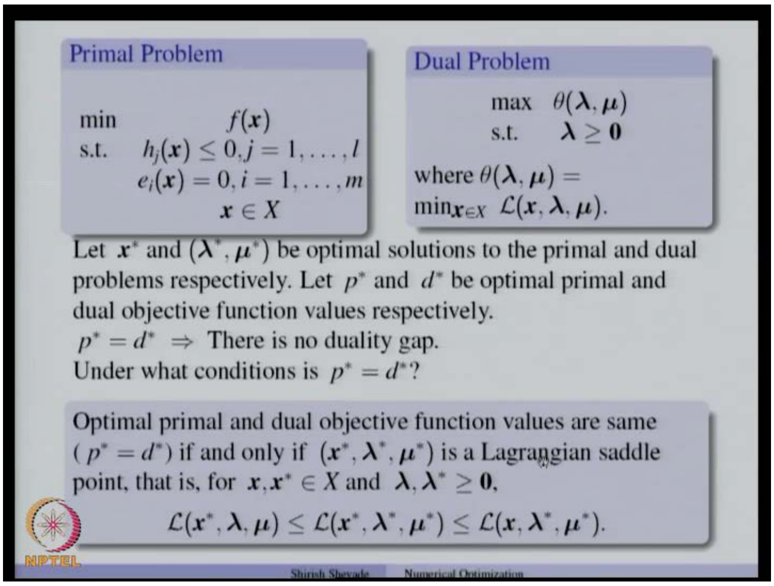


So recall that, this is the primal problem and corresponding dual problem is given here. So, let us assume that, x star is a solution to the primal problem and lambda star, mu star is a solution to the dual problem. And corresponding optimal primal objective function value is p star and optimal dual objective function value is d star, then if p star is equal to d star, there is no duality gap. So, the question is that, when is there a when there is no duality gap. Now to see that, we need the definition of a saddle point

Now, recall that in this primal and dual problem formulations, we use the Lagrangian function as the payoff function. So naturally, we will be interested in looking at the Lagrangian saddle point. Now, it turns out that, under the Lagrangian saddle point conditions, there is no duality gap or if there exists x star, lambda star, mu star. Such that, x star lambda star, mu star is Lagrangian saddle point then there is no duality gap. And the result is true the other way also that, if there is no duality gap, then there exists a Lagrangian saddle point.

So, optimal primal and dual objective function values are same, if and only if x star, lambda star, mu star is a Lagrangian saddle point. That is, for every feasible x and given that x star belongs to x, x also is feasible and lambda and lambda star both non negative. Well, x star, lambda star, mu star is less than or equal to l (x, lambda star, mu star) and l(x star, lambda star, mu star) is greater than is equal to l (x star, lambda mu). So, this very important result, which says that, under Lagrangian saddle point conditions. The duality gap does not exist or p star is equal to d star.

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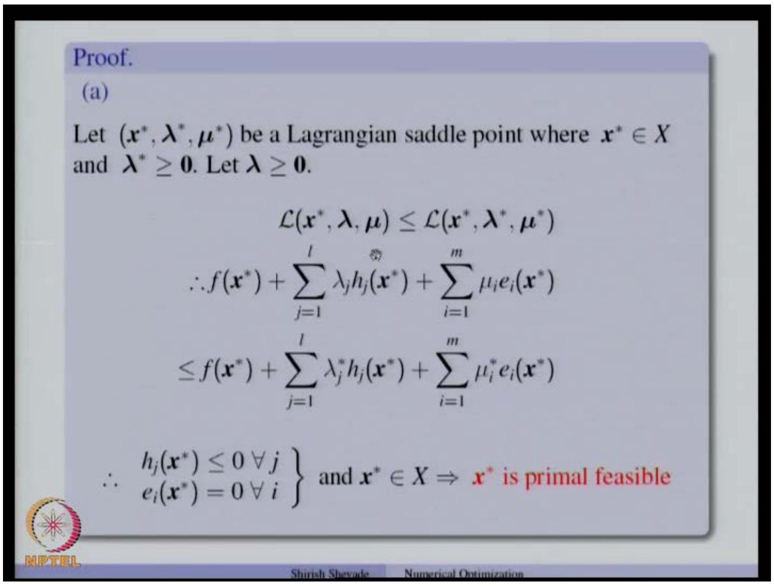


So, let us look at the proof of the problem proof of this result. So, let us assume that, x star, lambda star, mu star is a Lagrangian saddle point. Now under this condition, what we want to show is that, there is no duality gap.

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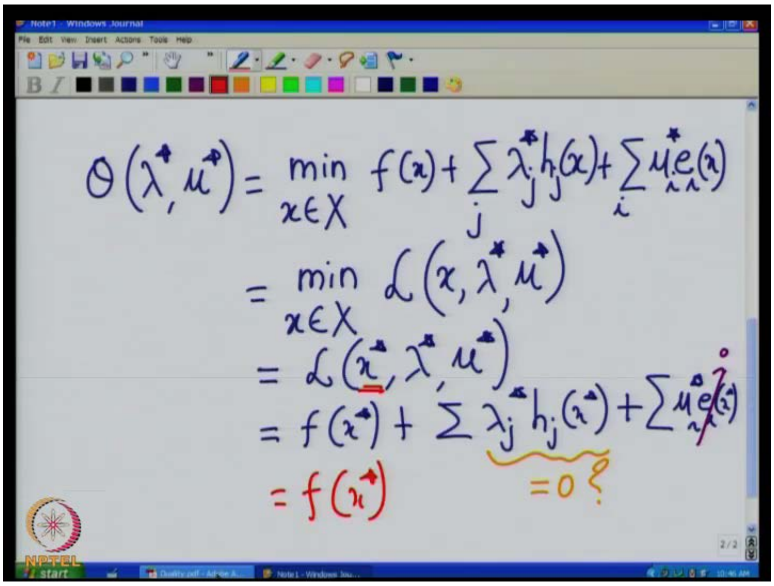

So, if you look at theta (lambda star, mu star) is nothing but minimum of x belongs to x f (x) plus sigma lambda j star h j (x star) plus sigma i mu i star e i (x). So, this is by the definition of theta (lambda star, mu star) that, given lambda star, mu star theta (lambda star, mu star) is defined as minimum over x belongs to x f (x) plus sigma lambda star h j (x) plus sigma mu star e i (x). And this is nothing but minimum of x belongs to x Lagrangian of x lambda star, mu star.

Now since, x star, lambda star, mu star is a saddle point the minimum of l (x, lambda star, mu star) will be same as l (x star, lambda star, mu star) provided x star is a feasible point. So, the first point, we want to prove is that, x star is a feasible point, then using the saddle point conditions, we can say that, minimum of l (x star, lambda star, mu star) or x belongs to x is same l (x star, lambda star, mu star). Now, if you expand this again, what we get is f of x star plus sigma lambda j star h j (x star) plus sigma mu i star e i (x star).

Now, since, if we show that, x star is primal feasible, then obviously e x star will be 0. Now, what we want to show is that, under the saddle point conditions theta (lambda star, mu star) is equal to f of x star. So, what we want is, this quantity is equal to f of x star, which means that, we want this quantity to be 0, so this is the question that, we would like to answer that is, this quantity 0. So, there are two questions that, we would like to answer, the first one is x star primal feasible only then we can write minimum of l (x, lambda star, mu star) as l of x star, lambda star, mu star.

This things x star, lambda star, mu star is a saddle point. So, the first thing is to prove that, x star is primal feasible. Now given that, x star is primal feasible, which means that e of x star is 0, is it possible that lambda j star h j (x star) is 0 for all i for all j. Now, if that is true then we can write this such f of x star. So, the optimal dual objective function value is same as the optimal primal objective function value. And therefore, there is no duality gap, if there exists Lagrangian saddle point.

So, first we show that, x star is indeed a primal feasible point and the second point that, we show is that lambda j star h j (x star) is equal to 0. So, what we have is x star, lambda star, mu star is a Lagrangian saddle point, where x star belongs to x. Remember that, x star just belongs to x and we are not given that h j (x star) is less than or equal to 0 or e i (x star) equal to 0 and that is what we want to prove first before proving that, the duality gap is 0.

And what is given is that, the lambda star is non negative of course, we do not need any sign restrictions on mu star. So, let us assume some lambda, which is non negative. Now since, x star, lambda star, mu star is a Lagrangian saddle point by the definition of Lagrangian saddle point l (x star, lambda, mu) is less than or equal to the Lagrangian of x star, lambda star, mu star or in other words maximum of lambda mu, where lambda's are non negative.

And mu is unrestricted in sign is equal to l (x star, lambda star, mu star). Now, if we expand the Lagrangian on both sides, what we get is f(x star) plus sigma lambda j h j(x star) plus sigma mu i e i (x star). So, here the lambda's and mu's are the variables x star is fixed. And the right hand side, what we have is that, x star, lambda star, mu star is all fixed. So, f (x star) plus sigma lambda j star h j (x star) plus sigma mu i star e i (x star).

Now, the first quantity gets cancelled remember that, lambda's are non negative. So, for a fixed lambda j star let us assume that, these quantities are not there for the time being let us just look at those second quantities on both l h s and r h s. So, you will see that since, lambda is greater than or equal to 0, it is possible to take lambda j to infinity. And if this quantity is a finite quantity, then we have talking about a case, where this infinite quantity is less than the finite quantity, which is certainly not possible. So, which means that lambda j star h j(x star) has to be 0. Now, similar argument can be I am sorry, which means that h j (x star) has to be less than or equal to 0, because if h j (x star) is greater than 0, then by taking lambda j to very high quantity this inequality will not be satisfied.

So, h j (x star) has to be less than or equal to 0. Now, similar argument can be given in terms of mu i e i (x star) mu i's are unrestricted in sign, so we can again take mu i's to any large quantity and will not will not satisfy this inequality. And therefore, e i (x star) has to be 0, so we have two cases, where by taking lambda j to be very large quantity there will be this inequality will not be satisfied. Similarly, by taking mu to be sufficiently large again, this inequality will not be satisfied.

And since, lambda's j's are non negative, we have to make sure that h j (x star) is always less than or equal to 0 and since, mu i's are unrestricted in sign e i (x star) has to be 0. So, we have h j (x star) less than or equal to 0 and e i (x star) equal to 0 and this along with the given condition that, x star belongs to x means that, x star is primal feasible. So, so we use the first part of the Lagrangian saddle point condition to show that by taking lambda’s to be very large this inequality cannot be satisfied. So, the only way to satisfied satisfy this inequality even, when the lambda are very large is, when h j (x star) is less than or equal to 0.

And since, mu i’s are unrestricted in sign, the only way this inequality will be satisfied is where by forcing e i (x star) to be 0 and this along with the given condition that, x star belong to x makes this x star to be primal feasible. So, this is our first step to show that, the duality gap does not exist. Now, the second step is to show that, this quantity is indeed 0.

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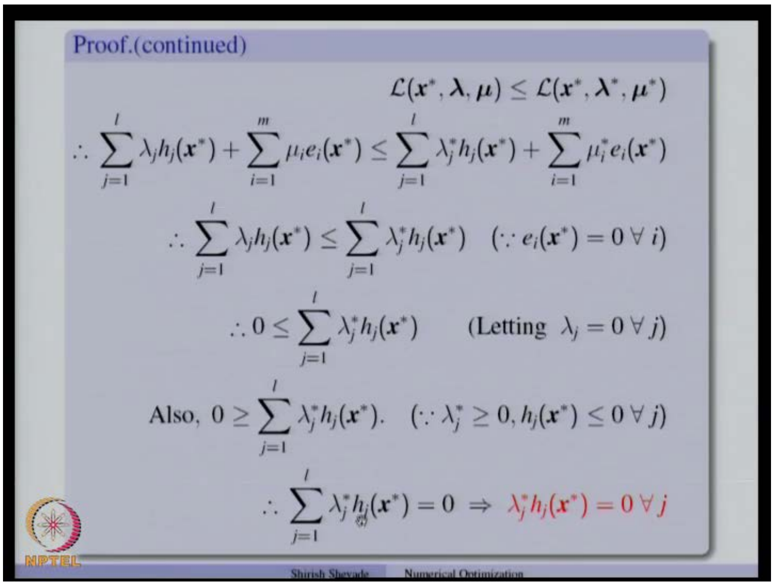


So now, let us use the other part of the. So, let us use the saddle point condition, which is l (x star, lambda, mu) less than or equal to l (x star, lambda star, mu star) and which means that, by taking out the common f (x star) from both sides the inequality is like this. And since, we have shown that e i (x star) equal to 0. So, which means that, sigma lambda j h j (x star) is less than or equal to sigma lambda j star h j (x star), because we have already shown that, x star is no primal feasible. Now remember that, this lambda is non negative. So, if you substitute lambda to be 0 then what we get we get 0 less than or equal to sigma lambda j star h j (x star), but if we look at this the right hand side of the inequality, lambda j star is non-negative h j (x star) is less than or equal to 0, because (x star) is primal feasible.

So, we have this quantity, which is a negative quantity or non positive quantity, because lambda j star equal to 0 and h j (x star) less than or equal to 0. So, sum of all non positive quantities will be non positive. So, we have sigma lambda j star j (x star) is less than or equal to 0 and from this two inequalities since, they are satisfied at all feasible lambda's and x star, we can say that, lambda j star h j (x star) sigma lambda j star h j (x star) is 0 and which again from this we can say that, lambda h j (x star) equal to 0. Because, each quantity is a negative quantity is a non positive quantity and the only way that, the sum is 0, is when in the individual quantities. Quantities in this sum are 0 and therefore, lambda j star h j (x star) equal to 0 for all j.

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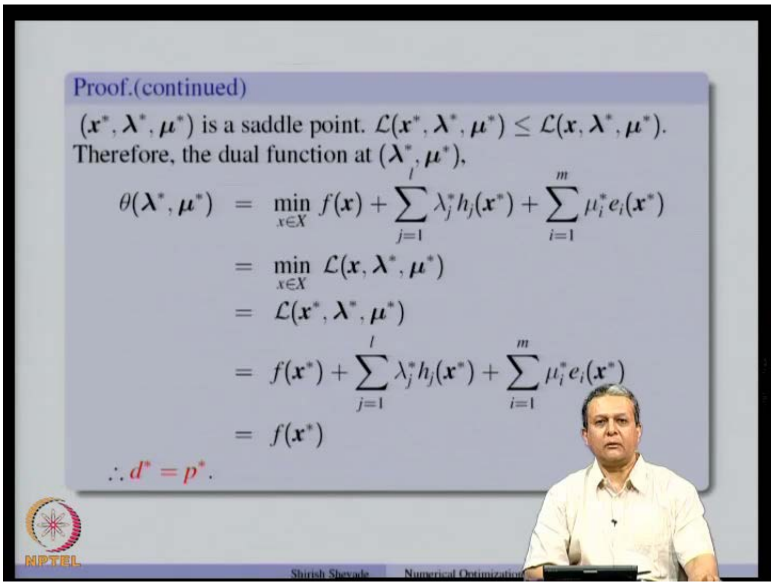


So now, we can show that, the duality gap does not exist under the Lagrangian saddle point condition. So, the first thing that, we showed was that x star is a primal feasible that is, x star less than or equal to 0 e i (x star) equal to 0. For all I, we are given that x star belongs to the set x and we also showed that the complimentary slackness condition, which we studied, when we studied k k t conditions, that also holds. That is, lambda j star h j (x star) equal to 0.

So, we have Lagrangian saddle point and now we use this condition of Lagrangian saddle point. So, the dual function, which is theta (lambda star, mu star), which is nothing but minimum of f of x plus lambda j star h j (x star) plus sigma mu i star e i (x star), remember that, this Lagrangian saddle point condition was not used earlier. So, we use that, now and this function is nothing but the Lagrangian evaluated at x lambda, star mu star.

Now, if we look at this condition that the minimum of l (x star, lambda star, mu star), where x belongs to x occurs at x star or this is nothing but l (x star, lambda star, mu star). Now as, we did earlier, we will expand this Lagrangian function, which is nothing but f (x star) plus sigma lambda j h j (x star) plus sigma mu i star e i (x star). Now, x star is primal feasible, so e i (x star) is equal to 0, so this last term vanishes, we also showed that, the complimentary slackness condition holds. So therefore, lambda j star h j (x star) is 0 and therefore, what we are left with is only f of x star. And so theta (lambda star, mu star) is nothing but f of x star and therefore, d star which is same as theta (lambda star, mu star) is equal to p star that means, that the duality gap does not exist, if x star, lambda star, mu star is a Lagrangian saddle point.

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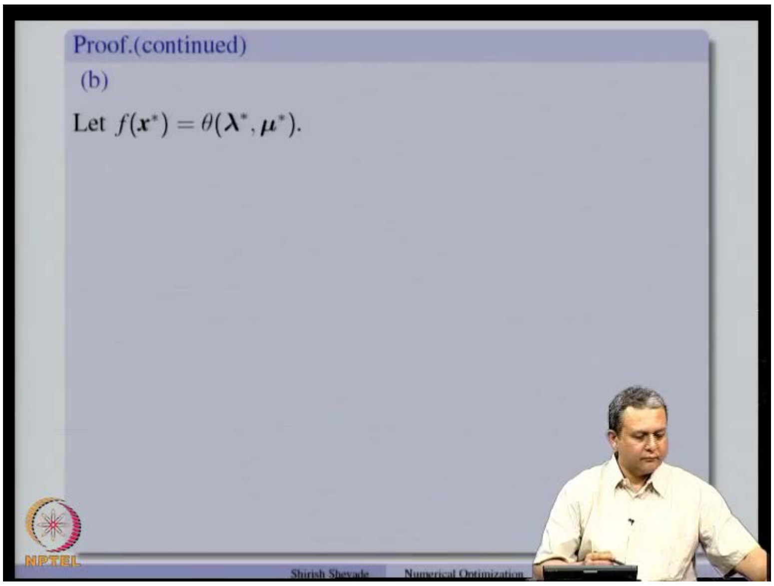


Now, let us look at the other part of the proof, so the other proof that, we want to show is that, if there does not exist a duality gap then there exist a Lagrangian saddle point or in other words.

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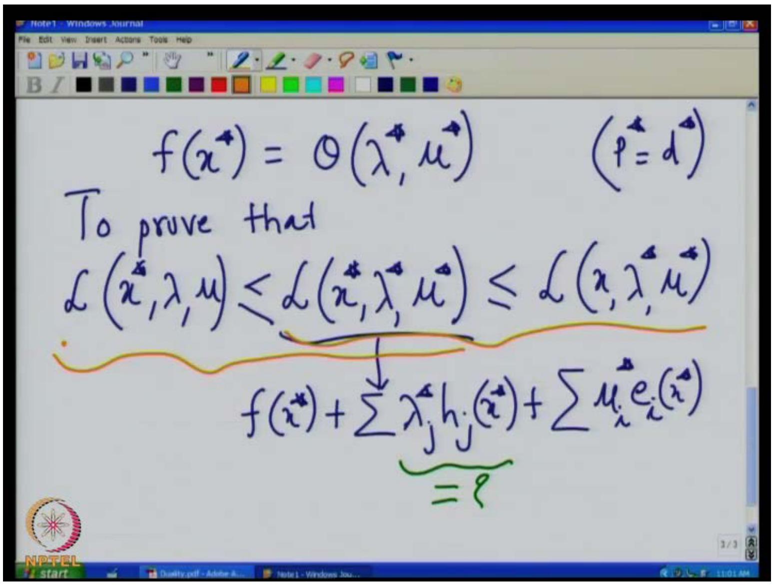


So, what we are given is f (x star) is equal to theta (lambda star, mu star), which means that, p star is equal to d star and we want to prove that, l (x star, lambda star, mu star) is less than or equal l (x, lambda star, mu star) and this quantity is greater than or equal to l (x star, lambda mu). Now, if you look at this quantity, this quantity is nothing but f (x star) plus sigma lambda j star h j (x star) plus sigma mu i star e i (x star). And if we show that, if this quantity equal to 0, that we do not know as of now and x star is primal feasible, which is true because x star is a solution to this problem. So, this quantity is 0. So, what we get is f of x star and then what we want to show is that f of x star will be same as f of x star, which is same as theta (lambda star, mu star).

We will use the that condition to show, that the Lagrangian saddle point condition is satisfied or in other words, what is will show this part first and then show that the other part also holds by using the fact that, f of x star is equal to theta of lambda star, mu star. So, in that process, what we want to show is that lambda j star h j (x star) has to be 0 and that, proof is similar to what we solve just now and since, x star is a solution this quantity is 0. So, this fact together with this fact will be used to show these inequalities, so we will do that in the next class.

Thank you.