Graph Theory

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Lecture No. #07

Dominating set, path cover

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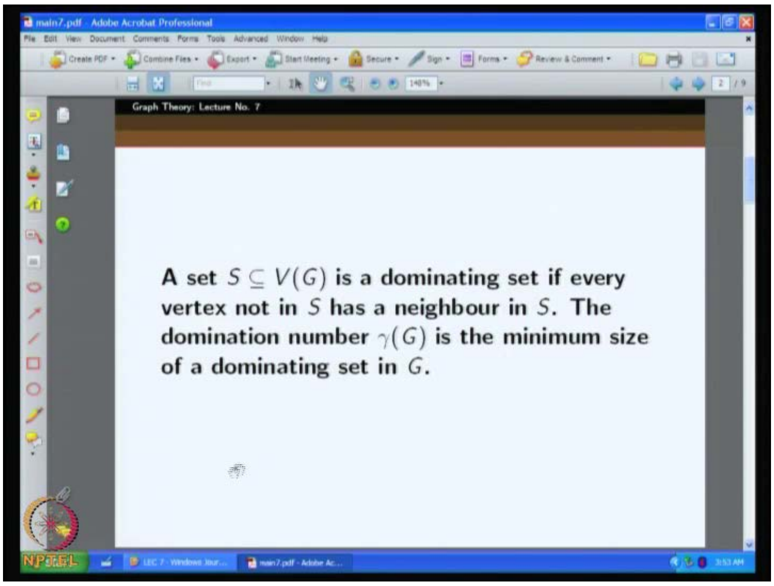

So, welcome to the seventh lecture of graph theory. So, we today we will look at a notion called dominating sets. So, in the last few classes we had discussed about vertex cover, edge cover, etcetera. Suppose, so we start from the vertex cover now. So, remember that vertex cover was a collection of vertices subset of vertices, such that all the edges are covered by those vertices; that means, at least one end point of each edge belongs to that set S.

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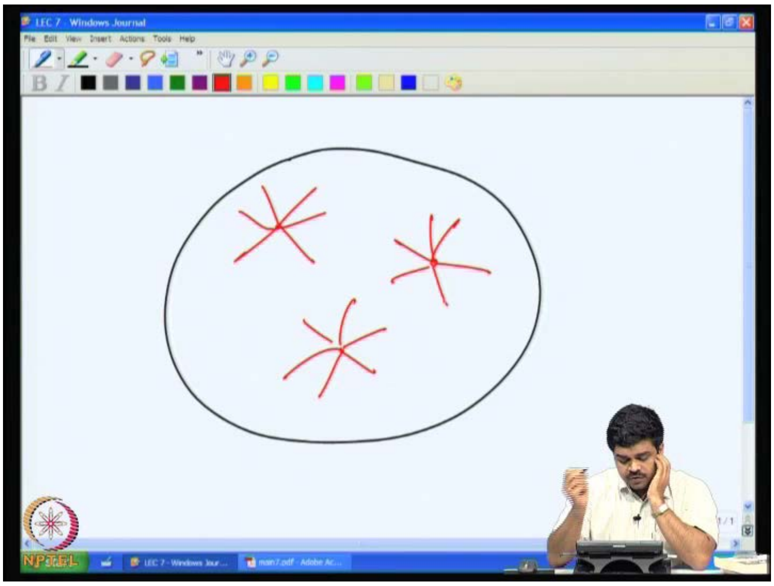

So, it is equivalent to as we once mentioned, that if we have a graph it is like finding out the minimum number of stars such that you get each edge in one of the stars right, you can you can see it is like covering with the stars like minimum number of stars. Now, suppose I want to cover the number of vertices of the graph with the minimum number of stars not necessarily the edges, for instance let us look at the example.

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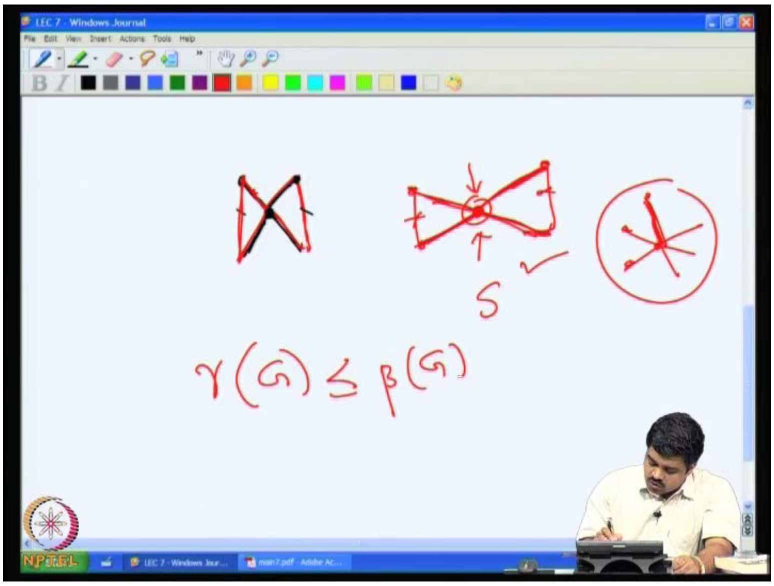

So, for instance if I am talking about this graph, now you know what is the vertex cover of that if I take this vertex, I cover this edge, this edge, this edge and this edge, but of case this edge and this edge is not covered. So, I will have to take this and this edge. So, therefore, it is not possible to get a vertex cover by selecting only this vertex, but on the other hand if I was on interested to cover only the vertices for instance in this graph I am not bothered about the edges, but what I want as a collection S such that every vertex is an neighbor of it right. So, so here this is enough because, this is an neighbor of this, is an neighbor of this, is an neighbor of this, in other words if I was looking for a collection of stars star means a structure like this. So, such that all the vertices become part of at least one star.

Then we see we need only one star here this covers all this thing because, this edges I am not bothered. So, this parameter is called the dominating set and any dominating set which as the minimum cardinality is called minimum dominating set and let us go to the formal definition of that so a set S subset of V of G is a dominating set if every vertex not in S has a neighbor in S, the domination number gamma of G is the minimum size of a dominating set in G.

Now if you look at this pictures we can see that, in all cases the dominating set need not be equal to the vertex cover of there are several cases you have already seen, in some cases they can be equal, but there is no guarantee right now it looks like in most of these cases we drew the dominating number gamma of G since to be less than equal to beta of G, but it is always true of case it is not always true for instance you can you can consider a graph which consist of only isolated vertices.

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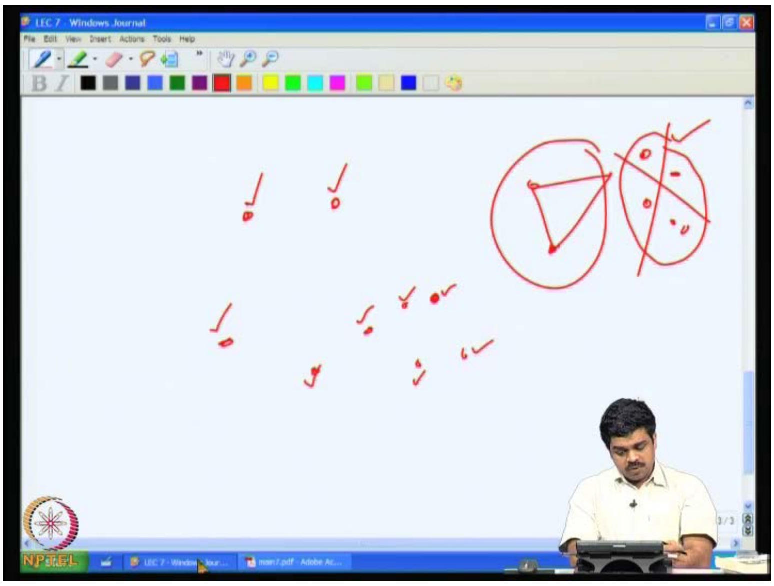

So, if you want to cover it with stars. So, you have to take this, this, this all of them separately right. So, while as far as the edges are concerned there are no edges here. So, there is nothing to cover for instance if you have here considering a graph with an isolated vertex.

So, with some isolated vertex here so, for each of this isolated you may have to take them separately because, nothing else can cover them in as for as the dominating set is concerned, but when we are looking for a vertex cover because, there are no edges between them they are irrelevant. So, we would not even have to worry about them. So, refer in those cases the gamma of G the domination number can be bigger than the vertex cover, the minimum vertex cover cardinality so, but suppose there are no isolated vertices in the graph.

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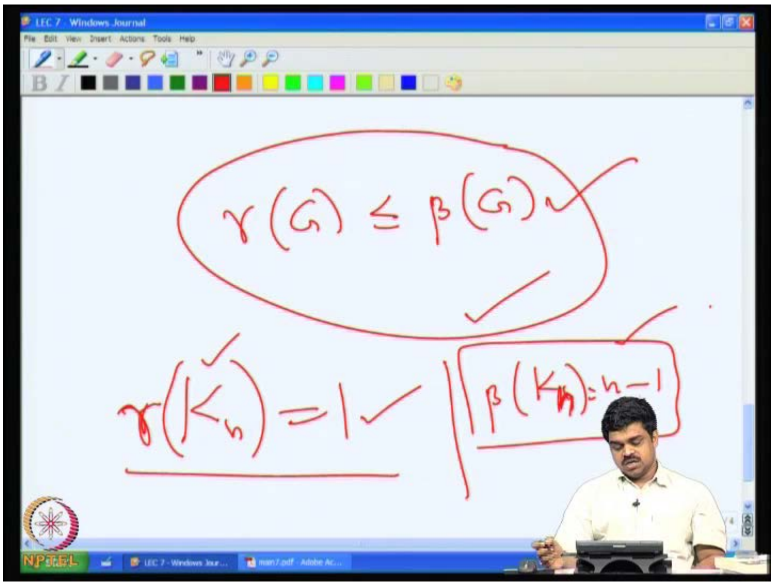

Then you can easily see that the this in equality holds why because, if you take a vertex cover that is indeed a dominating set also it covers all the edges and since every vertex has at least one edge incident on it because we are assuming that there are no isolated vertex either it is inside the dominating inside the vertex cover. So, it is dominated already by the vertex cover or it is outside and it is edge the other end point of the edge is in there in the vertex cover right.

Therefore, the vertex cover dominates all the vertices so, therefore, vertex cover is indeed a dominating set provided there are no isolated. So, therefore, this inequality is always correct, but but does not mean that dominating number domination number has to be equal to the cardinality the minimum vertex cover or very near to that, sometimes it can be very small.

So, compare to the cardinality of the minimum vertex cover, the extreme case happens when we consider the complete graph K n. So, the gamma of K n is just one as you can see because, if you have just pick up one vertex all other vertices are its neighbors. So, it dominates all of them. So, it needs only one vertex in the dominating set while the vertex cover number beta of K n yes we know is n minus 1 because, even if you drop two vertices from the vertex cover. So, what will happen is there will be an edge between them and nobody will be no will be there in the vertex cover to cover those things.

So, therefore, in the case of complete graph that gap can be very big. So, gamma can be much smaller than beta that what is we have seen, then next thing is so, now let us look at some interesting observations about dominating set, the first think is if you have n vertices in the graph and the minimum degree of it is K then we will always be able to get a dominating set of cardinality at most n into 1 plus log k plus 1 by k plus 1 in this what we are going to prove now, but before that so, can we say that there is some some minimum value for the dominating set say in so, some lower bound for the domination number.

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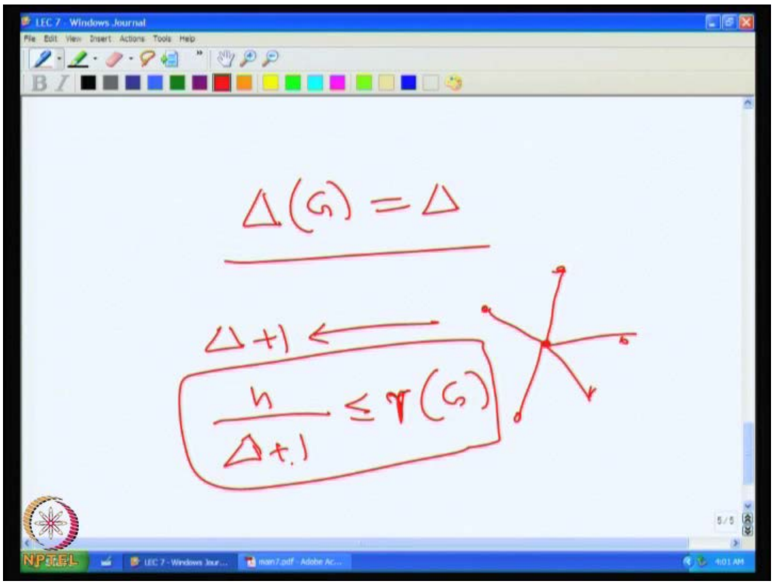

So, suppose the graph is say say a maximum degree K graph, we can consider a maximum degree so, delta of G equal to so, y K let say delta then you can say that any vertex if you pick up and add in the dominating set it will dominate only delta neighbor and itself together delta plus 1 vertices per vertex so, but then there are n vertices to be covered right. So, total n by delta plus 1 will be required at least so, gamma has to be at least this much right.

So, this an easy lower bound for dominating number in terms of this now, if it is a K regular graphs this becomes n by k plus 1. So, now, come to the case of minimum degree k. So, if case the with respect minimum degree you can get same much biggest minimum degree can be of much smaller compare to the that degree of most of the vertices, but you can get an upper bound interestingly like what we are going to do is this so, we will show that, so here is a strategy greedy strategy. So, where we try to take the vertex pitch dominates the maximum number of all vertices which are not yet dominated this is the strategy.

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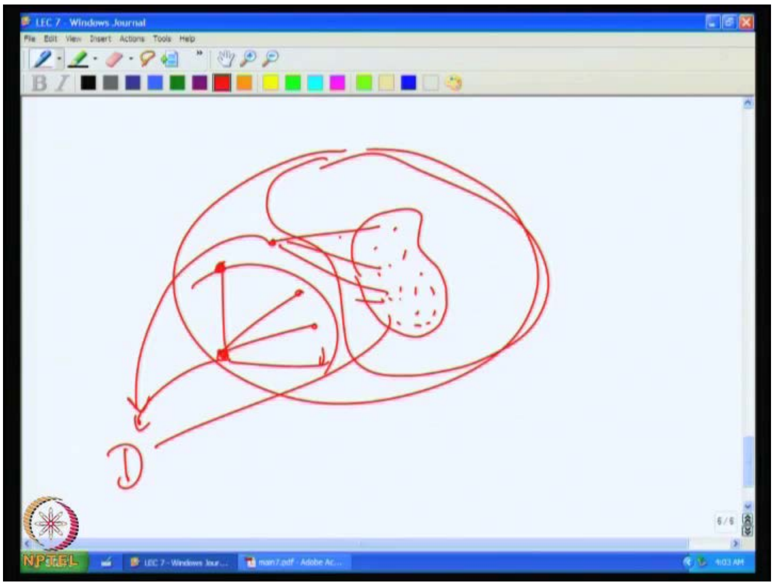

In other words, we can suppose this is the graph you can see which vertex as the maximum number as of now and pick this and then you see this vertex we add added in the dominating set. So this D, I will add it D and then all this vertices already dominated, now you look at the remaining vertices now from the graph we will see the biggest I mean which vertex can dominate the biggest number of vertices from the vertices are not yet dominated and then we can add it the at that vertex to the set and then in the remaining graph we will do the same trick like that right. So, what we do? So, how fast how many how many times we will have to repeat this procedure is what we are going to consider now.

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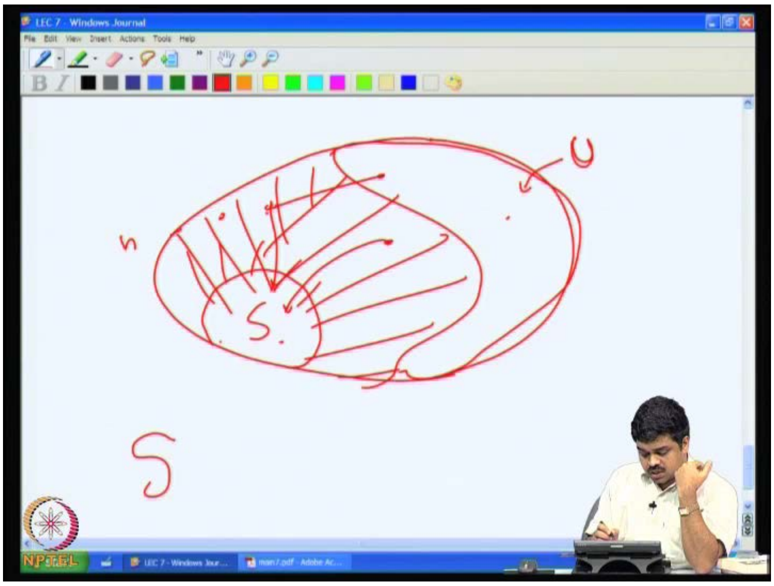

So, this is the way we are going to approach it now suppose you have n vertex graph n vertex graph and you already got some set S. So, you picked some vertices and added to S. So, by our procedure say and then this is that set S here right now this s dominates say this many vertices let us say S dominates all this vertices; that means so, all these vertices are dominated by S. So, by some means for every vertex here you have a neighbour in S already. So, I am not bothered about these vertices now. So, what I am bothered about is this the remaining vertices so, let us say the remaining vertex set is u the undominated vertices as of now with the respect to this S.

My plan is to look for a vertex here which will get me maximum from this group, but then what if all of them are very bad; that means, what if say every one dominates only one or something like that. So, which which will not help me much right every time you will add one more vertex, but then you will have to keep on doing this for may be cardinality of few steps.

So, because initial number of vertices and you may end up with too much so, but what we claim is that in this collection outside S you will always find one vertex such that so, such that it dominates at least U into k plus 1 by n, number of undominated vertices out of this U into k plus 1 by n factor the U vertices which are undominated will be dominated by at least one vertex. So, that vertex we can add to s.

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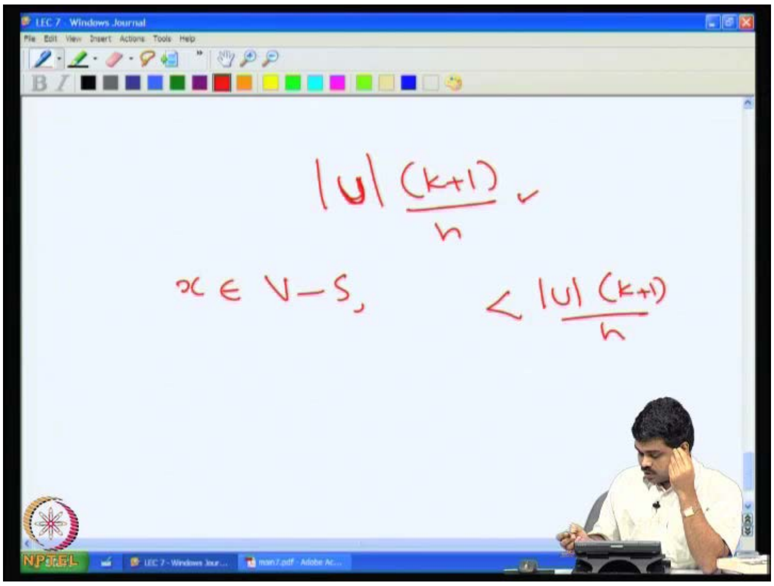

So, which will be reasonable portion of the undominated vertices how do I show that suppose this is not true; that means, every vertex x outside S is such that, so the number of vertices it dominates from U if less than cardinality of U into k plus 1 by n strictly less than this in that case we will show a contradiction. So, I will to explain for the for the ease of explanation I will do it like this.

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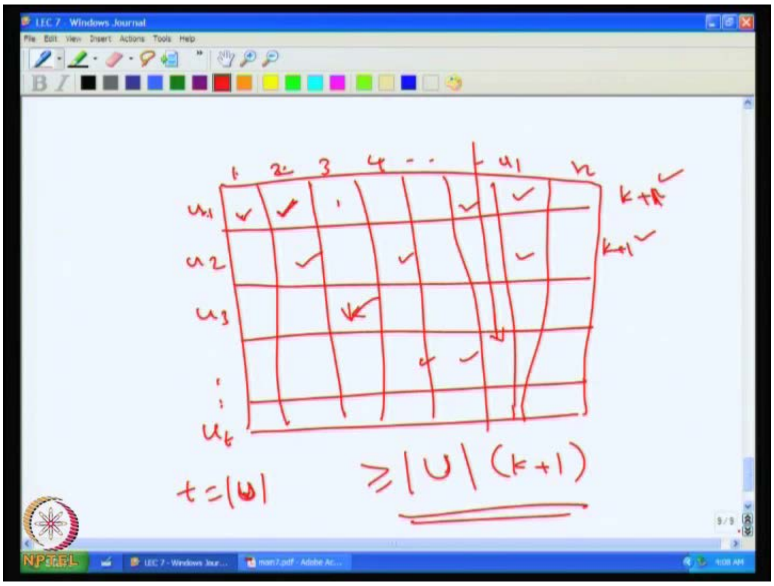

So, let us make a table. So, I will write down the number of vertices 1 2 3 4 up to n here and then help of columns here for each of them. So, this is for the sake of explaining and then so, for each vertex from U, I will say u 1, u 2, u 3 and this is U, t say t is equal to the cardinality of U. Now we, now what we going to do is to put at pick if u1 can dominate that vertex here, function even domain you can if you want dominates 2 I will put a tick here, even dominates 1 I will put a tick here so, even may not be dominating 3. So, I will not put. So, like that I will put ticks how many ticks will come in a row you know that minimum degrees k. So, a tick will come corresponding to the on the on each of these squares in the column corresponding to its neighbors and also for itself for instance u 1 may be here itself it will come because u 1 dominates itself. So, k 1 ticks definitely at least may be more than that, here also K 1 ticks. So, like that so, k 1 similarly in each row will have k 1 click click ticks.

So, if I count the total number of ticks you see that there are at least U into k plus 1 ticks in it. now if you look at column wise, you see that if none of them for instance if you take any vertex from S you would not say any tick here because, you know none of the u 1, u 2, u 3 etcetera can dominate the vertices from S because, there is no connection between them right. While if you consider a vertex here, so if you consider vertex here, so this vertex because, how many ticks can be there in this things in a in a column function for instance in this column how many ticks can be there. So, of case we have a assumption we said that the there is each vertex x outside S is such that it dominates only less than U times k by k plus 1 by n vertices from U..

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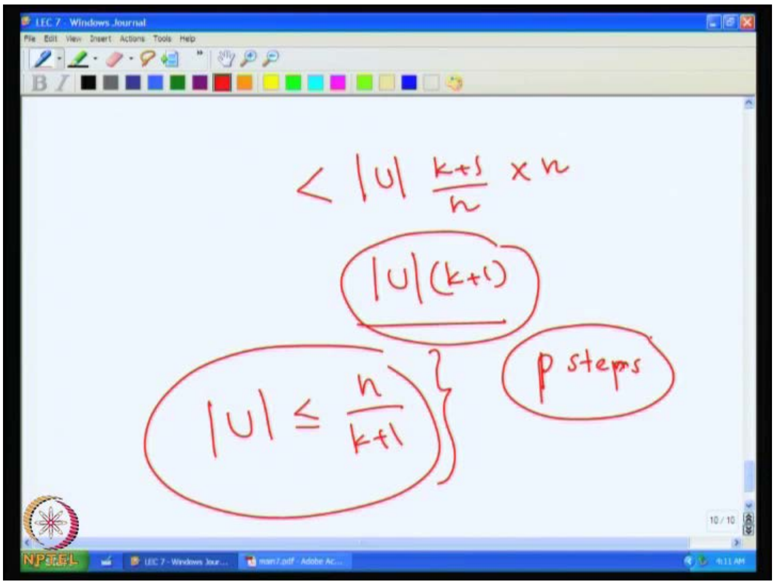

So, the number of ticks we see in that column will be less than cardinality of U into k plus 1 by n right strictly less than that so, this means that if you consider all the columns total number of ticks will be we can multiply by the number of columns right n columns are there so, will only we see is strictly less than U into k plus 1 ticks, but we know that the number of tickles more than equal to U into k plus 1 which is contradiction.

So, what do we infer we do have a column for there are at least U times k plus 1 by n number of ticks. On other word, there is a vertex outside S such that it can dominate at least k plus 1 vertices from U. So, it can be inside U or not can be inside U or not does not matter. So, we pick up that vertex and add to our set S.

So, if I start from an empty set and keep doing this thing we will gradually reduce the set of undominated vertices, we will do it until the cardinality of the undominated vertices becomes less than equal to say n by k plus 1 there we stop and then we add all the vertices in the remaining set into the dominating set because, they them self dominate each each vertex in it dominate itself. So, that the total suppose if we reach this situation in t steps, suppose if I reach this situation in say k steps sorry sorry p steps. So, I will say p steps then what can I say?

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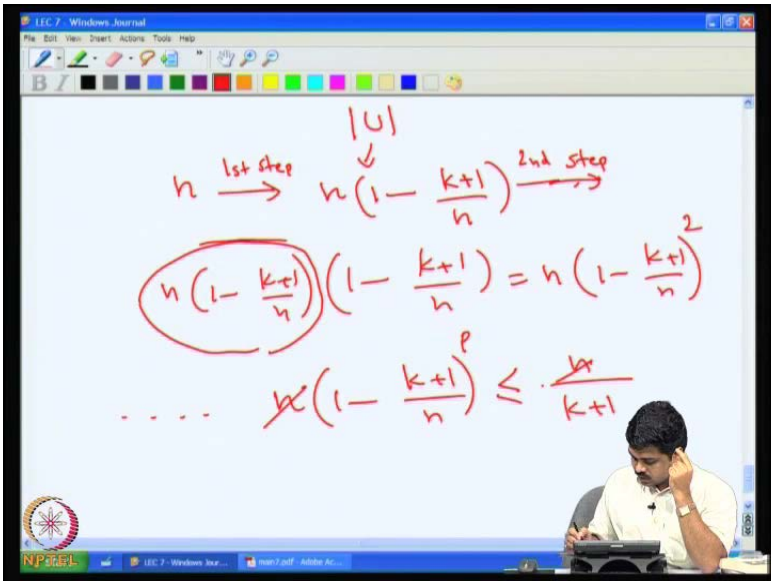

So, in so, then I start from n and after the first step how many vertices will be there there in this U? U will become n into 1 minus k plus 1 by n right after the second step so, this is the first step, the after the second step after the second step what we see is? We will our U will become n into 1 minus k plus 1 by n into another 1 minus k plus 1 by n while because, this is the total number of outside we are taking a way this fraction of that so, into thing this will become n into 1 minus k plus 1 by n to the power 2. So, after p stages we will see n into 1 minus k plus 1 by n to the power p undominated vertices still undominated vertices. So, suppose I want this to be less than equal to n by k plus 1 then how big should be the p should be how how how how much p should be? how big should be p?.

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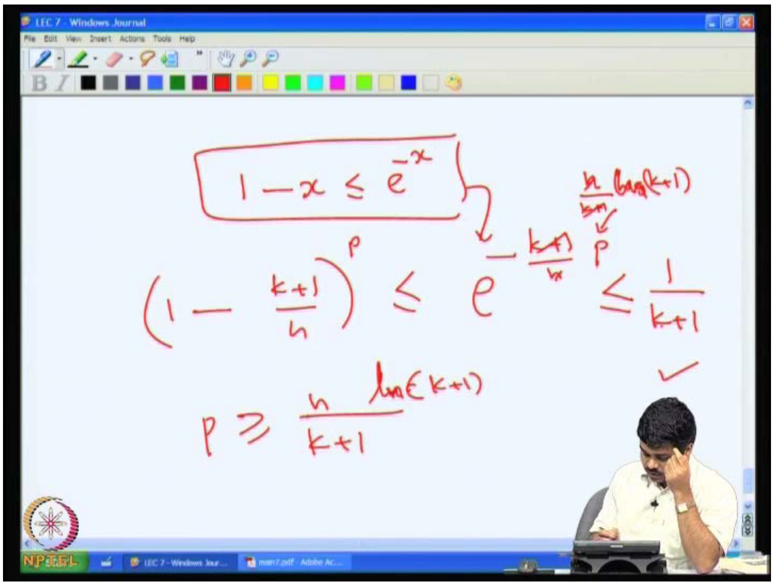

So, we can set these things so, we can this is any so, for instance I can always 1 minus x is less than equal to e raise to minus x. So, well known inequality so, what I want is 1 minus k plus 1 by n to the power p to be less than something. So, I know that this is less than so, I using this is an equality I can I can I can write this is less than e to the power minus k plus 1 by n into p say if this itself is less than equal to k plus 1 by n then of case sorry less than equal to not k plus 1 by n, 1 by k plus 1. So, we dropped and of case cancelled and from both sides. So, then we see that so, we can add the rest of the things as such right. So, what will the value of p be? So, you can take so, if think of substituting p with n into n by k plus 1 into log k plus 1.

So, what will happen this will this n 1 you can cancel this and this k plus 1 will cancel this, log k plus 1 minus will become 1 by k plus 1 less than equal to so, this will be true. So, if you take p to be greater than or equal to n by k plus 1 into log sorry this log I should put this log because this log respect natural logarithm k plus 1, then we see that this is this will become less than thing.

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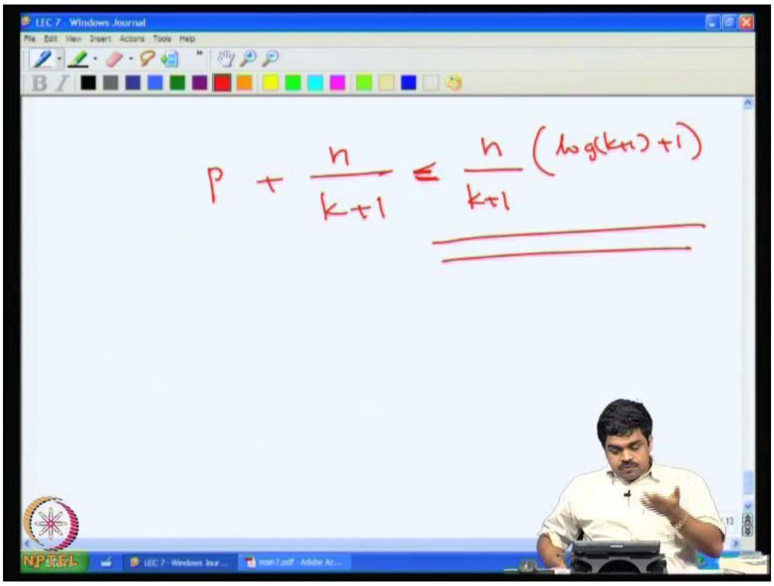

So, now along with this p vertices we also have to add the remaining n by k plus 1 vertices; that means, I need p plus the final size of the dominating set is this much that is equal to n well less than equal to I can say I can take n by k plus 1 into log of k plus 1 plus 1.

So, this is what we get a dominating set like this algorithm. So, which produced a dominating set of cardinality this much so, therefore, this statement of the theorem. So, this is what the statement of the theorem. So, this sized dominate set will come if the minimum degree of graph is this much.

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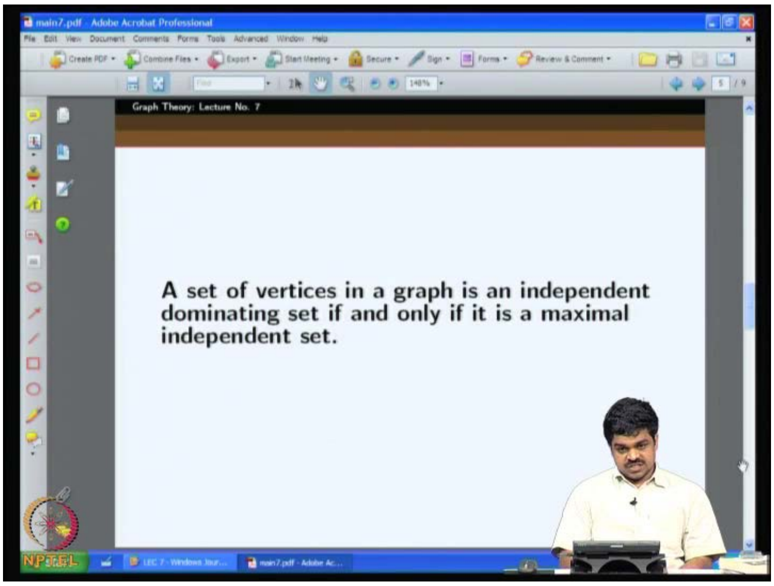

Now, so we can talk of various kinds of dominating sets variations of this dominating set problem by adding some additional constraint about the dominating set that we get for instance, we can ask that is a possible to get dominating set which is connected of case we can get it, but only think is, that if you look for the minimum cardinality the cardinality of the connected dominating set. So, of case the graph has to be connected then right. So will be definitely more than equal to the dominating set because, the connected dominating set itself is a dominating set so, the condition for connected dominating set is just say that it is not only a dominating set induce the sub graph on the dominating set is connected.

So, another kind of dominating set is independent dominating set which is so, when we require that the dominating induce the sub graph on the dominating set is an independent set then it is called an independent dominating set. Now, there is a something called total dominating set where we require that there are no isolated vertices in the dominating set. fine Let us look at this independent dominating set and see an interesting statement about it. So, the independent dominating set I mention that independent dominating set is actually a dominating set whose cardinality is sorry whose induce sub graph is independent, now I can say that it is essentially a maximal independent set what is it mean? So, maximal independent set means it is an independent set, but not any independent set will be a dominating set.

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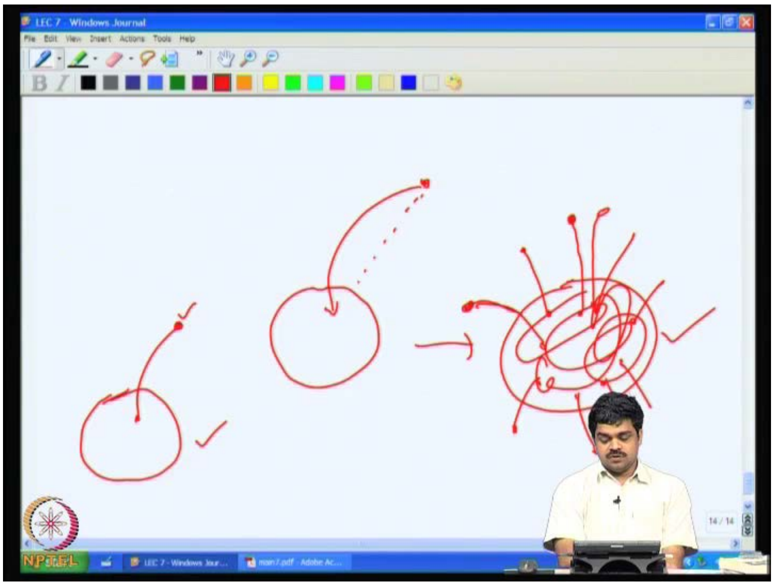

So, you can for instance take an independent set and then suppose you can add one more vertex to it add; that means, if there is no edge between this set and this outside vertex then you can always add it and you will get a bigger independents set, you keep adding it and finally, you will reach a situation that you cannot add any more vertices to it and make it bigger, from any more outside vertices it vertices to it and make it bigger why is it so? Because any vertex outside if you take they will have at least one connection into this dependent set we try to pull into the independent there will be conflict. So, this is this kind of independent sets are called maximal independent set.

So, it is not necessary that a maximal independent set is a maximum independent set. So, the maximal only means that you cannot given this set independent set you cannot add one more to it and make it bigger. So, there is no other independent set such that that the super set of this, but maximum means cardinality wise it should be the biggest. So, you can always talk about other notions like minimal vertex cover. So, vertex cover we have discussed about the minimum cardinality vertex cover, but suppose if a vertex cover is such that if none of its proper sub sets are vertex covers of the graph then it we call there it is a minimal vertex cover. So, it need not be cardinality wise minimum. So, there are these words minimal, maximal which we usually use in graph theory. So, we have already used in the context of the proof of dots theorem, this word edge maximal graph like. So, now, we claim that so this maximal independent sets are dominating sets why because, you see any other vertex outside so, it should have edge end to that it is so, this independent set is dominating the outside all the outside vertices. So, therefore, it is a dominating set.

Now, again if the dominating set is an independent set. So, it is indeed a maximal independent set why because, you take the dominating set any outside vertex should have an edge into that because, it is a dominating set. So, therefore, you cannot move any outside vertex into it because, then a conflict will come. So, it means that the maximal independent sets are indeed the the the dominating sets that have that give us an independent set an induce of graph, independent dominating sets. Now we will look at an interesting statement about independent dominating sets.

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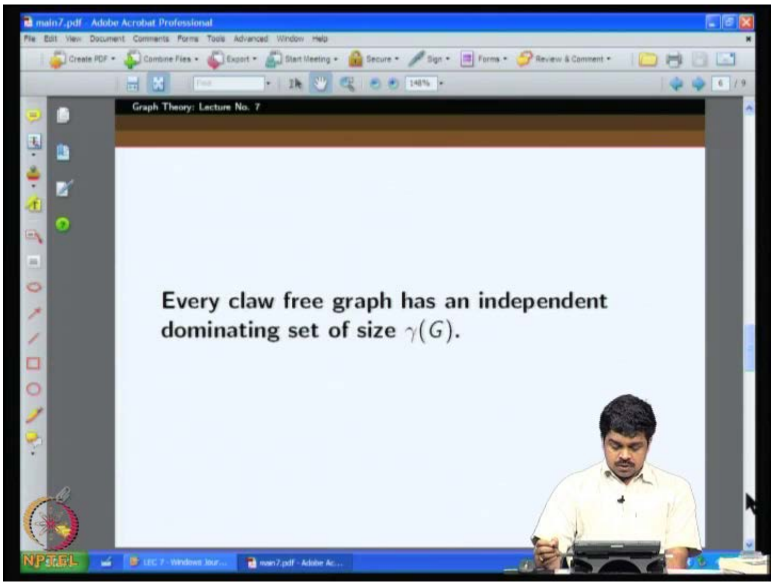

So, it says every claw free graph has an independent dominating set of size gamma of G that mean gamma of G remember it is the cardinality of the minimal dominating set without any constraint. So, of case if you are asking for independent dominating sets even if you take the minimum it has to be of cardinality at least has much as gamma of G there is no guarantee that in a general case, in the general case you may get an independent dominating set whose cardinality is gamma of g, but if you take it class of graph is called claw free graph, then you will always get the independent set. An independent set which is also a dominating set such that it is cardinality is indeed equal to gamma of G that is the cardinality of the minimum dominating set. So, why is it possible before that I have to explain what is a claw free graph?

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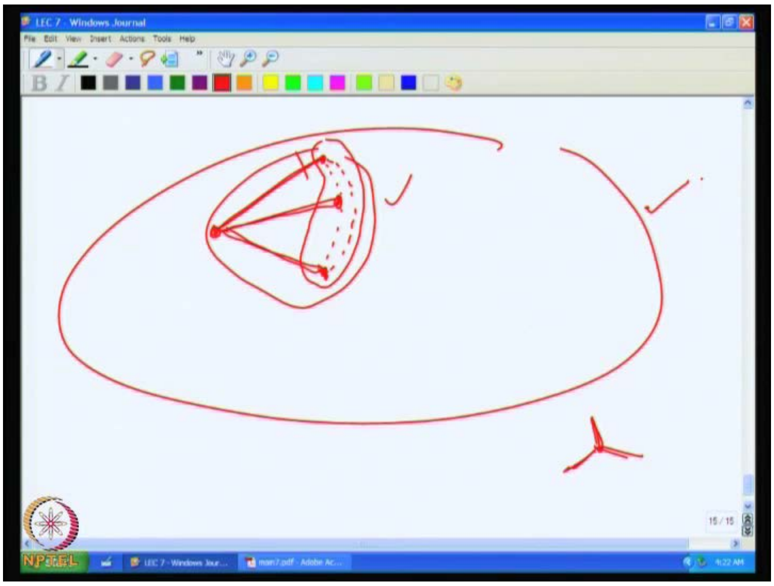

So, first for that you should understand what a claw is? suppose in a graph you see a vertex like this and you see at least three edges going out of it, suppose three edges going out of it such that, here this edges missing, this edges missing and this edges also missing and this is called a claw, there can be many other things in the graph I am just taking this vertex and three of its neighbours and if it is. So, happens that this three of its neighbours are independent that mean they do not have any edge between them, then this is then entire star this three star right. So, this set up is called this structure is called a claw and if the graph does not have any claw in it then it is called a claw free graphs then it is called a claw free graph.

So, we claim now that the if you give me a claw free graph then the cardinality of this minimum independent dominating set will be equal to gamma of G the cardinality of the minimum dominating itself set itself or another words, we will able to find a minimum dominating set which gives me an independent set in a sub graph, with the cardinality gamma of G. So, how do I prove this? So, now, you because, it is real to see that any in any independent dominating set should have cardinality greater than or equal to gamma of G. So, we just have to show that. So, here is an independent dominating set whose cardinality is less than equal to gamma of G.

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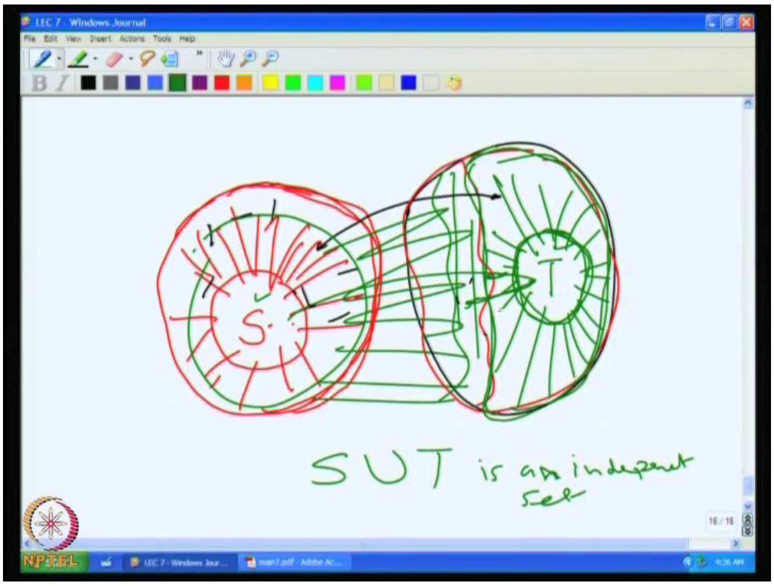

So to do this thing, just first consider a dominating set minimum dominating set let us say this is the minimum dominating set. So, what does it mean that this will the remaining remaining part, I will draw like this is the remaining part. So, I am instead of so, it is essentially means that any vertex here will have a edge back to this any vertex in this side will be connected to this. Now, there is no guaranty that this is an independent set there can be some edges inside this, now what we do is we will look for the look for some maximum maximal independent set here some maximal independent set here.

So, let say this is the maximal independent set so, this we called S. So, what is the meaning of this because, in this part in this induce sub graph this S is the maximal independent set this is a dominating set of this, of case this dominates all the vertices of this, but of case this would not dominate all the vertices here because, if it dominate all the vertices here also then s itself is a dominating set and then be smaller than this one. So which will be a contradiction? To the assumption, that this is the minimum dominating set. So, what a unless this S itself is independent set S itself is equal to this which means we are whatever we are looking for we already got we only want that to demonstrate an independent dominating set. So, now, let us say so, we should assume that so, here only this S can definitely dominant all the vertices in this part, but it will be able to dominate only a little bit from this side. So, let us say say it dominates is able to dominate this much. So, this is the remaining part which is not dominated by S. So, what we do now is to pick up a maximum maximal independent set in this part in this induced sub graph in this induce sub graph we will pick up a maximum dominating set let this be called T.

Now, one think we can immediately observe is S union T is independent is an independent set is an independent set set. Why is it so? Because, this is an independent set, this is an independent set there is no edge between this and this, as you see in that case we would have put this thing here in this group right. So, there is it is S and T together is an independent set is T also. Now the next thing what we can observe immediately is that together S and T dominate all the vertices because, S dominates all this vertices and all these vertices while T because, it is a maximal independent set in this part it dominates all the vertices in this part right maximal independent set dominates here. So, this entire graph is dominated by S union T.

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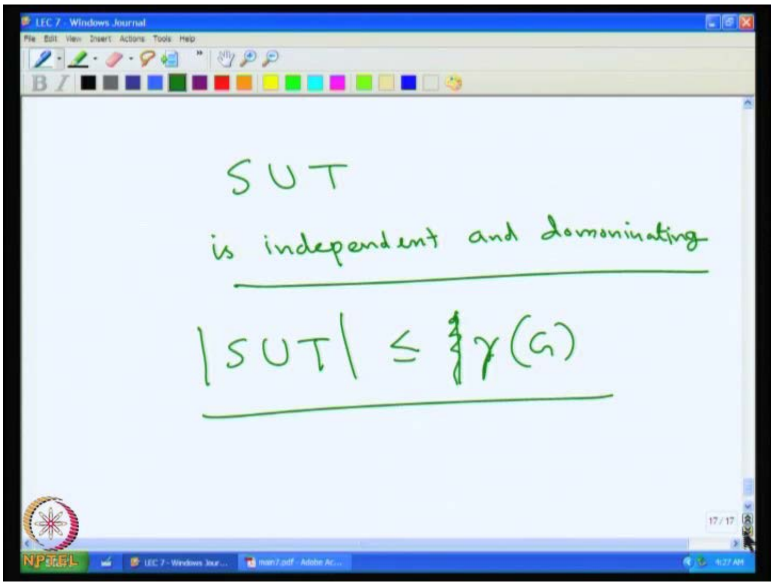

So, what we have now shown is S union T is independent and dominating. So, it is indeed a independent dominating set. So, the only thing to show now this S union T independent dominating set we have shown is less than equal to the cardinality of the so, less than equal to gamma of G right.

This is all we need to show, but we go back the gamma of G is essentially G the cardinality of this set right let me redraw it here the next page.

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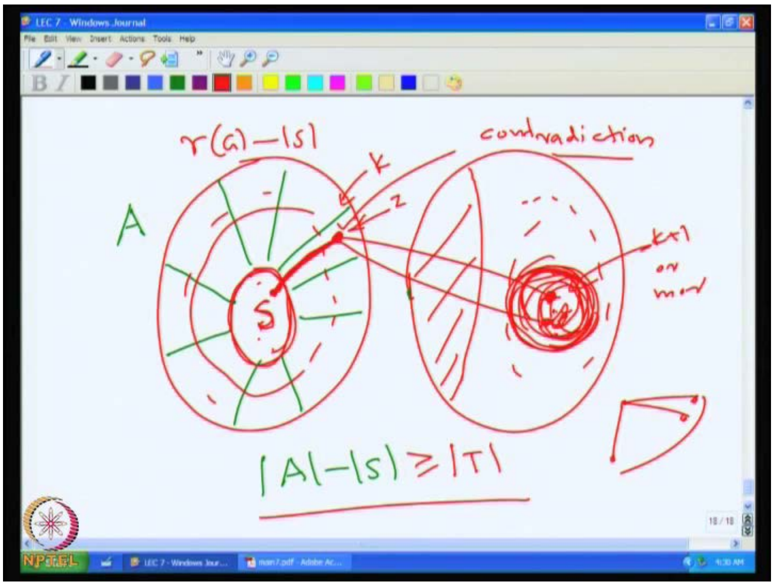

So, we have this is the gamma of g. So, we have the s here. So, now, we are interested in what is the gamma of G minus the cardinality of S that is essentially the cardinality of this right this one, this set. So, this set now you can say for instance the minimum dominating set is A. So, this is the cardinality of a minus s is what we are interested the A minus S. now on the other hand, our T our T was so, this much was already dominated by S right. So, this was already dominated by S. So, here t was a maximal independent set T and this dominates all the vertices in this part in this part. So, if it so happen we have to show that A minus S cardinality is greater than or equal to cardinality of T, in that cases plus T will be definitely less than S plus A minus S, that is A. So, it will be less than equal to gamma of G because, cardinality of a is indeed the gamma of G, we need to show that this cardinality the cardinality of this set is less than equal to the cardinality of this portion.

How do we do that, see suppose this is smaller, suppose there are K vertices here and there are say k plus 1 or more vertices here k plus 1 or more then you know that there is at least one vertex here because, this set has to dominate this entire thing because, S none of the vertices in S has any connection to this portion. So, this k plus 1 vertices k plus 1 no more what is this has to be dominated by this K what is s. So, there should be at least one vertex here which has two neighbours into this right two different neighbours because, all of them if all these K vertices only one neighbour in this. So, then how can they dominate k plus 1 vertices (( )) they should be principle, they should be at least two neighbours for at least one vertex here. So, let this be this vertex be called set. Set as two neighbours here and because, maximal independent set so, they should be vertex which is in S which is a neighbour of this they should be an edge like this into S otherwise I would have pushed a set into S that S not a maximal dominating set.

So, this set has one neighbour in this, one neighbour here and one neighbor here and you know that this together with this is an independent set. So, it is it so it so, happens that we have a claw here these set S three neighbours which are independent right. So, it is a claw so, but then originally with all that the graph is claw free. So, it is a contradiction therefore, this set will give a contradiction. So, we infer that in a claw free graph, we should get an independent dominating set here S union T such that its cardinalities, equal to gamma of G this is the reference to this is so; we will wind up the dominating set with this. And now, so the next topic we are going to consider is an enough the kind of the cover called path covers, let us say this is K here is the simple statement in a directed so, suppose if we take an undirected graph and then suppose I make this statement that I can find a path cover in it whose cardinality is less than equal to the maximum independent set. So, what do I mean by mean by that. So, so you can so, first we have to say what is the path cover?

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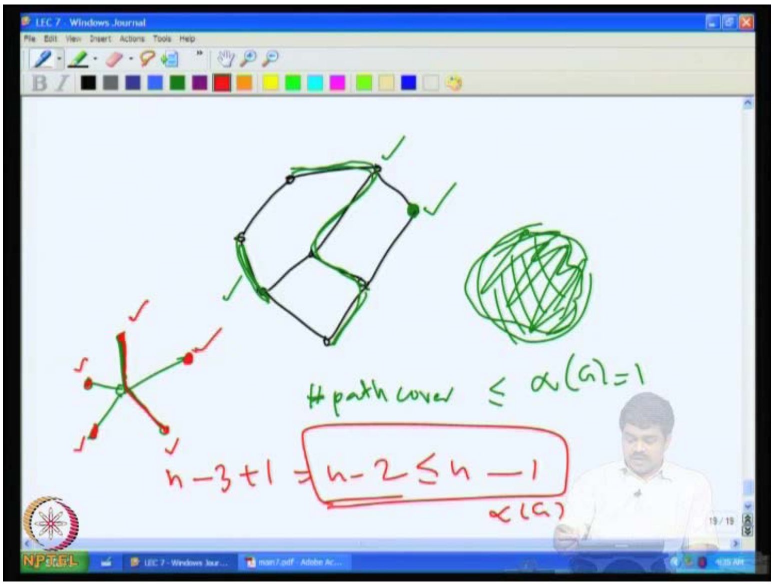

So, you can say that it is a collection of paths such that it covers all the vertices for instance if you take this graph, now you can say that this is the path cover. So, this is is this path and this path say this single return path, they cover all the vertices right. So, I take this path and I take this path and I take this path.

So, it is a set of paths such that they disjoint there would not share anywhere any vertices and together the union of the vertex sets will give you the vertex set of graph now look at this statement. So, am interested to get the minimum number of the minimum cardinality of a path cover path cover and we say that this minimum cardinality of a path cover is less than equal to the maximum independent set size of the graph. So, we can check it in create some examples and do that for instance if you look at a complete graph. So, this is one case where the independent set maximum independent set is quite small. So, the alpha of G here is equal to 1. So, do we get a path cover of this with cardinality of path cover with number of paths in it right. So, number of paths in it just 1 because, 0 is not possible. So, of course we can get a path Hamiltonian path ,if we have a Hamiltonian path then definitely we know that the path cover is the cardinality of the path cover is 1 because that just 1 path is enough so, that is enough right

So, for instance if we take a star, in this star you see you can once you take this edge because this is a biggest path you can take right on two right not more than this. So, you cannot come back. So, now, three vertices is gone all the other vertices you have to take S this trivial paths 1 2 3 4 5. So, in a star s and so, for instance there are n minus 1 hence than what you get is n minus 3 plus 1 is equal to n minus 2 is the number of paths, in the best path cover in the minimum path cover. Now, we can see so this, the independent set but n minus 1. So, this is still less than equal to the biggest independent set alpha of G. So, by from some observation we see that this is always correct, but this statement is very trivial proof in the undirected case. So, for instance so, what I have to show is that if you give me the minimum path cover, then I can show a independent set of cardinality at least as much as the number of path in the path cover.

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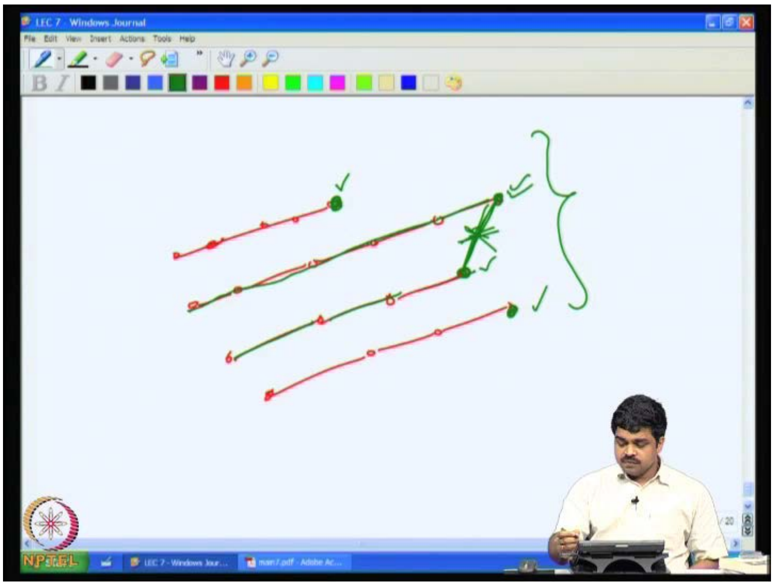

So, you can consider this path cover. So, so you take one path cover. So, may be this can be the path cover or the graph. Now you claim that this is minimum ,now I will show you an independent set from this thing such that I will I will be able to place one vertex in each of the path and say that collection of vertices I have picked up by placing one vertices in each other path is an independent set.

So, therefore, the cardinality of this will be less than equal to then particular biggest independent set cardinality. So, this is the way to pick up, you take the n points of this thing now you can see that it is; obviously, 1 2. So, this is an independent set why because, suppose if there is an edge like this suppose then we would have taken a longer path like a longer path like this right and then what will happen the minimum path curve will be even small, right this is the argument the assumption that what we got was the minimum path curve is wrong. So, therefore, these edges this kinds of edges cannot be there. So, by just picking the n vertices of the path cover we will get an independent set. So, in an directed graph this statement seems to be very easy, but more a non trivial statement can be made; that means, if we consider a sorry So, we are talking about undirected graph here after now so, when we consider a directed graph and then also this statement is true, but in a directed graph what is the path cover? So, the paths now are directed paths the paths now are directed paths.

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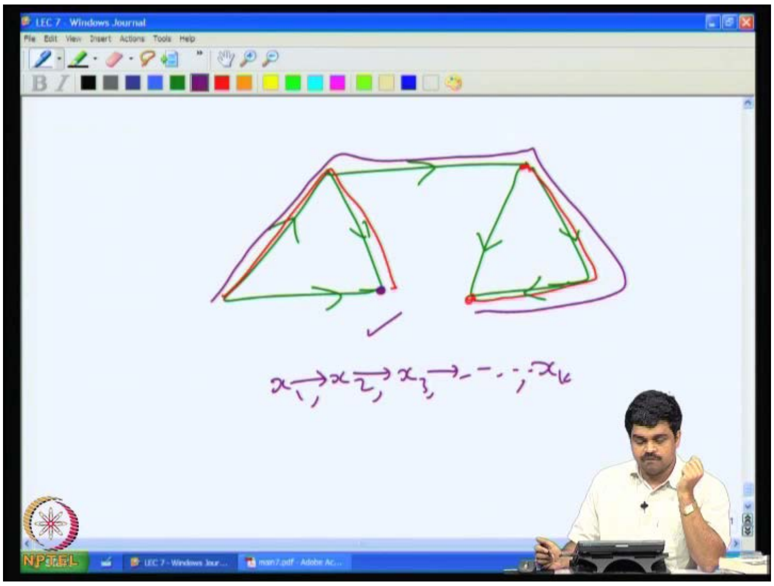

So, for instance I can consider this graph say so, here I will mark some path cover. So, this is a path cover here it is a direct path right here also this is this is a path cover. So, all the vertices are covered by two disjoint paths here or I could have marked another path cover like I could have taken this path I could have taken this path and then this single vertex path. So, this another path cover so, here it is the path cover concept is same as in the undirected case, but just that this time we have to take directed paths. So, if you say that x 1, x 2, x 3, x K are the paths then they should be an arrow from is the path is a path then it should an arrow from x 1, x 2 like this. Now, we are again interested in the minimum cardinality of the path cover. So, again the question is it possible that the minimum cardinality of a path cover is still less than equal to alpha of G see from if you understand that when you consider an undirected graph. So, there are several possible ways of orienting its edges. So, some method of orientation, some way of orientation oriented the edges some some when we give the edges some direction it is possible that the path cover size may increase drastically.

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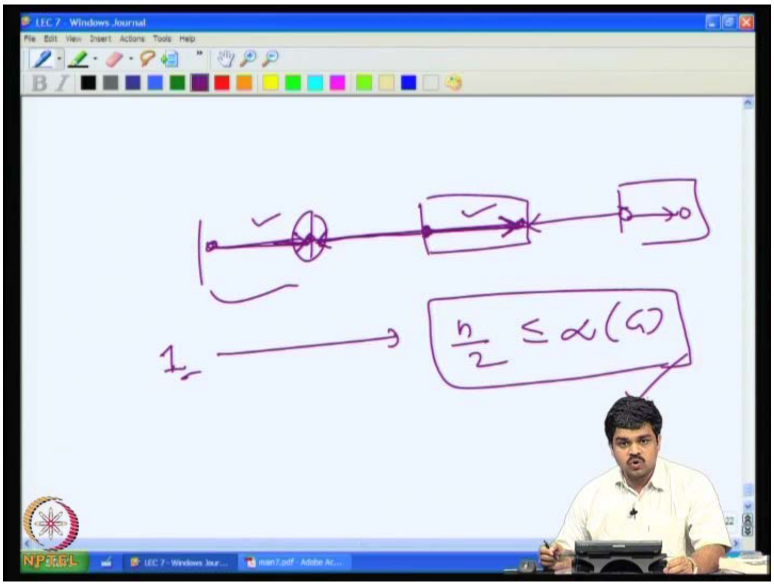

Let me see so, for instance you look at this simple graph this is the path, in this the undirected graph the path cover is just 1 because this is long path, but suppose I orient the edges of this graph and make it a directed graph by doing this, now what is the path cover? for instance if I try to go from here to here you get stuck because, the direction here is the wrong way in whichever I go I take only one edge here; that means, one path covers only two vertices here. So, the best think I may be able do is to do take this and then take this, right take this like that. So, may be n by 2 is the best thing that you can get so, but you can see that in the undirected cases case the minimum path cover cardinality was 1 and it went to n by 2 depending on even or odd right. So, if it is even number you can make it n by 2 exactly otherwise I guess you will get one more, but this is big increase, but even then you can see that the independent sets say still greater than or equal to this the minimum cardinality of the path cover or in other words the minimum cardinality of the path cover even after directing like this remind less than equal to alpha of G. So, the question is it possible that in the directed case also this statement is true. So, we can before getting in to that we can just check for if your some physically we can check in the bipartite (( )) case.

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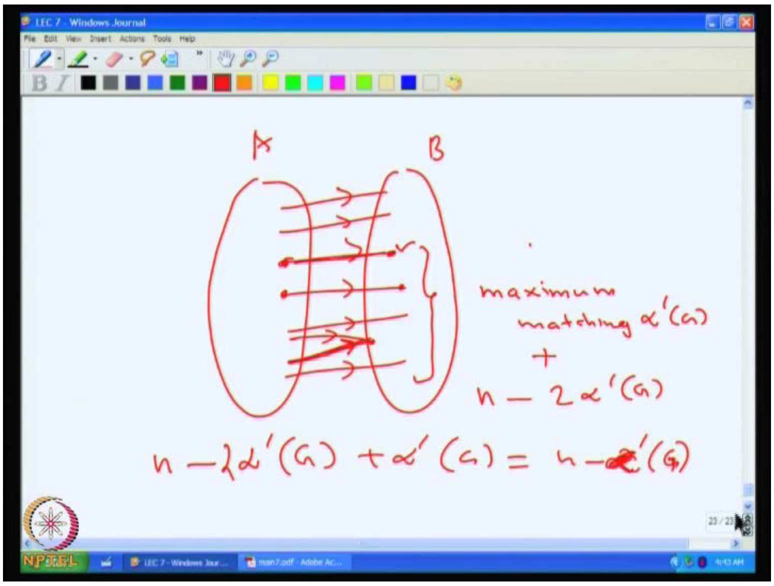

So, one special case so, this the bipartite graph. So, this is A side and this is B side suppose I orient all the edges from A side to B side. So, edges are always going to the A side to B side. Now, I ask what is the cardinality of the minimum path cover, of case if you want to minimize the path cover the best strategy would be to take longest paths possible then here any path can be almost 1. The length of the any path length of the any path can be 1 or 2. 1 means it is a single the single vertex path, 2 means it is an edge. So, you cannot once you can go like this you cannot go further because, it is a bipartite graph. So, you have to go back, but then always the edges are oriented from A to B. So, I guess if you want to minimize the number of paths in the path cover, the best strategy would be to take as many number of two lengths paths, two vertex path as possible and that two vertex path is an edge and they have to be disjoint therefore, we are looking for the maximum matching matching that is alpha dash of G plus the remaining right the remaining vertex each of them will need 1. So, how much is remaining n minus 2 times alpha dash of G. So, what is the cardinality of the minimum path cover here? That is of case n in to alpha dash of 2 alpha dash of G plus alpha dash of G is essentially n minus alpha dash of G n minus alpha dash of G this is the cardinality, but then is it less than equal to alpha dash alpha of G. So, in independent sets this is alpha dash of G that biggest matching.

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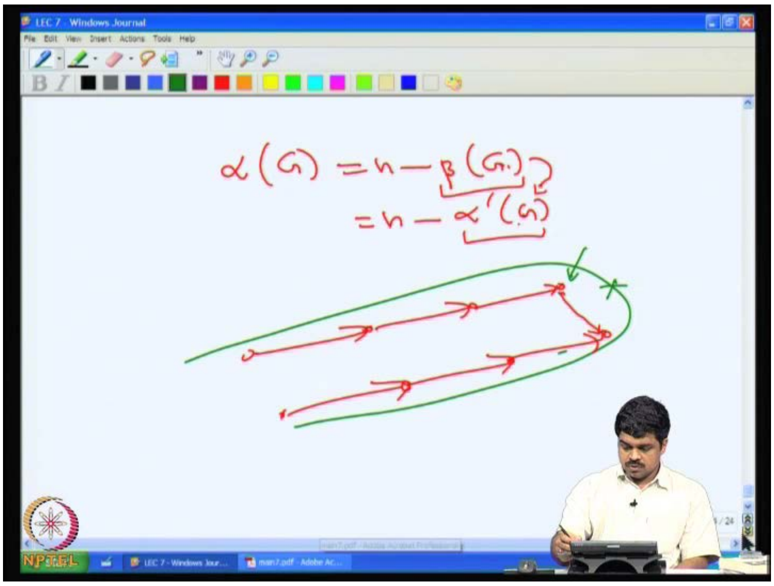

So, but this alpha of G is I know that it is equal to n minus beta of G the vertex cover and this is actually n minus alpha dash of G also because this by konig's theorem this and this are same the biggest matching and the smallest vertex cover has a same cardinality and it is seen earlier this quantity is exactly the cardinality of the path cover therefore, we end up in the bipartite case K is it just touches the upper bound we looking for; that means, it is equal to alpha of G it is equal to alpha of G in when when when we orient all the edges from A side to B side. So, at so, now it looks like an interesting question probably, there are some graphs whose whose minimum path cover cardinality can be more than alpha of G, but in the next theorem we will see that it is it is not possible. So, always the minimum path cover cardinality in a directed graph is at most the independence set not only that little more than that, we have seen that in undirected case we could make and independent set from the paths in the minimum path cover by picking up the n vertices of each path, the terminal vertices that is what we call usually the last vertices of the path is the terminal vertex. The picking of the terminal vertex of each path we call, we proved that that is going to form in an independent set.

So, now we say that, but in the directed case that is no guarantee that if you pick up the last vertices of each path a going to form in independence set because, see you can look at this example. So, for instance if I see a directed path. So, this is the directed path and this is another directed path in the path cover, now you can have an arrow here right. So, have an arrow here, we cannot say that I will in the earlier case what what we told is. So, I would have taken this longer path, but then now then this is in the wrong direction because the arrow is in the wrong direction. So, if is in both ways we cannot we cannot go like this and that is not a directed path. So, if that argument does not work here in this case. So, it is so, by picking of the last vertices which we called that terminal vertices of each path we cannot get independent set is it.

But our theorem say that need not by the last vertex, but you can always pickup the V p one vertex from each path set that if you put them together you need get an independent set there is set V p element of p of vertices set, that this V p is coming is vertex from p right set each path will contributed vertex and together they will make an independent set together they will make an independent set this is vertex set. Now to prove this thing we have we need some concepts, the first concept is we're interested in minimum path cover, but let us say if suppose somebody gave as the path cover.

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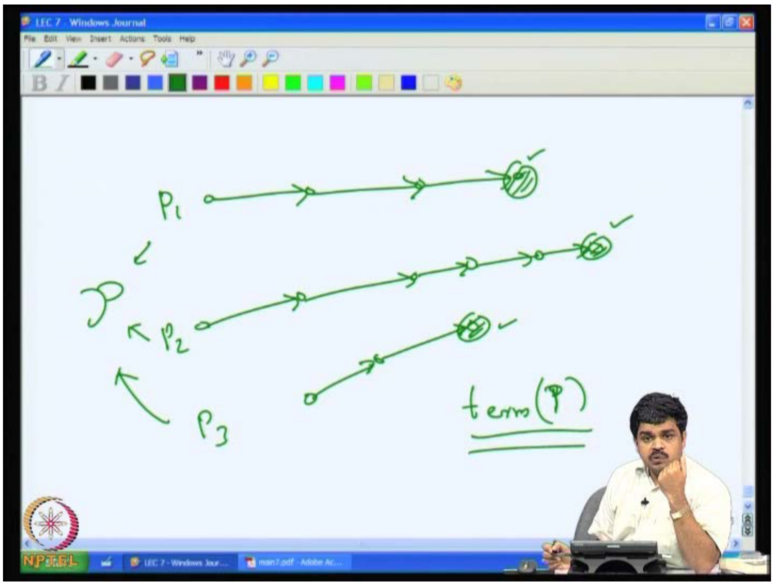

So, this is the path cover flow that given graph there are two graphs here. So, here directions are there. So, these vertices as I told these are the terminal vertices this set. So, suppose this p is the path cover and this is p 1, p 2, p 3 etcetera the paths in it, this all the path in it. This setup path and these vertices will say this vertices, this set is called term of p that is the terminal vertices of p; that means, pick up the n vertices of each path from p and then that set is called term of p.

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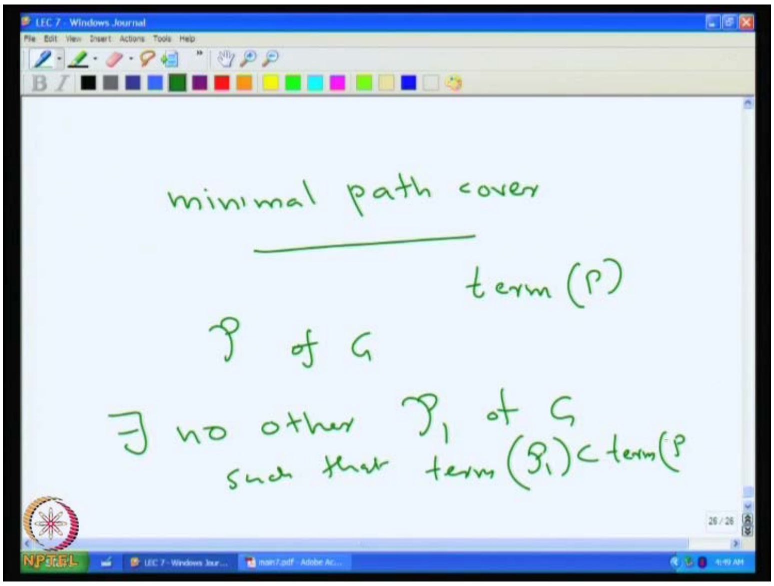

And now, we define concept of minimal path cover. So, this is the minimal path cover is not like so, for instance I give a path cover I would not say that minimal path cover if if no proper sub set of it is a path cover of the graph that that is not a vertices because, any proper sub set will not even be a path cover for any any path cover that will be a true because, give any path cover if you remove one path from that the remaining thing is not going to cover the entire all the vertices in the graph therefore, that is not the way. So, we will define with respect to this set term of p.

So, the minimum path cover p is a minimal path cover of this directed graph G if there is exists no other path cover say p1 of G, such that term of p1 is a proper subset of term of p; that means, we cannot get another path cover in the graph such that, it is setup terminal vertices is a proper sub set of the terminal vertices of this path cover. So, can be to possible there is a possibility that this path cover p which we call minimal is not a minimum path cover it may not be cardinality wise minimum, but just that we have make sure that there is no other path cover with their vertices n vertices of the paths is a proper sub setup of the n vertices of this path and this set, but on the other hand we clearly say that any minimum path cover has to be a minimal path cover also because, if there is a any other path cover with the term terminal set a proper sub set of this, that path cover is going to be a smaller cardinality because, the cardinality of the terminal vertex set is equal to the cardinality of the path cover itself.

So, before it is not possible, because if our 1 is minimum path cover. So, minimum path cover some minimal path cover, but not other way, there can be minimal path covers which are not minimum. Now, how proof will say that you give me a minimal path cover then then we will be able to find an independent set, such that there is representative from each path in the from that, so, we from from each path of the path cover, is what we will we will give proof of this in the next class.