Graph Theory
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Lecture No. # 06
More on Matchings

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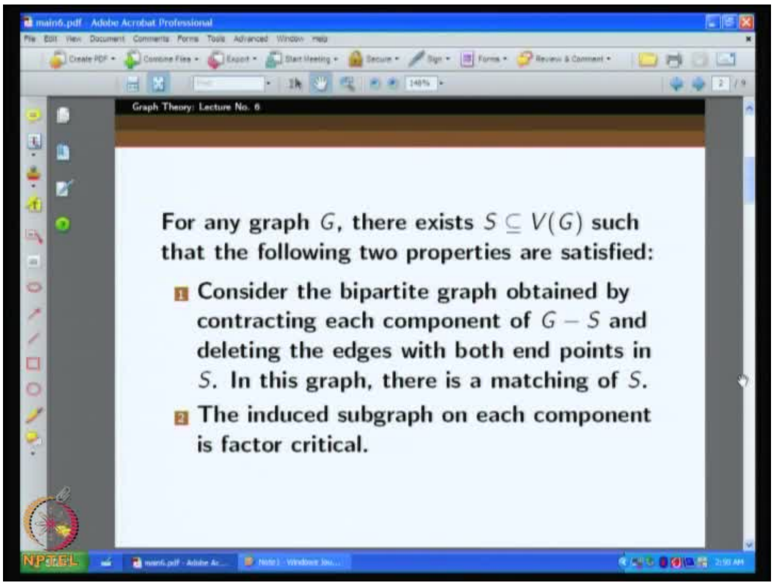

Welcome to the sixth lecture of graph theory statement, that in every graph G, there exists subset of vertices S, which satisfies 2 properties; which are the 2 properties, let us look at that. The 1st property, so the 2nd property is easier, so, which means, if we consider the components, which is obtained by removing that subset from the graph, so for, if the subset is S, if we consider G minus S and its components, each of the component will give us a graph.

The induced subgraph on that component will be a factor critical graph. That means, it is of odd order and if we remove any vertex from that, the rest will have a perfect matching, irrespective of which vertex you remove.

The other condition says, if you, contract each component into 1 vertex and say, if there are some adjacent side, the set S, I just remove those edges, then we will get, take bipartite graph between S and the vertices, which is obtained by contracting the components. In this bipartite graph we can get a matching, which pairs all the vertices of S, a matching of S. That means, S is matchable in this bipartite graph. S will each, we can find a matching, so that each vertex of S gets a partner, with respect to that matching.

So, we show that if this statement is true, then Tutte's theorem will follow from this. So, this, if this statement is true, we can always say, that a perfect matching will exist if and only if the number of components is equal to the cardinality of S, and more than that, we can immediately infer the statements of the statement of the Tutte's theorem from this thing. That means, a graph has a perfect matching if and only if the number of odd components is less than equal to cardinality of S. So, this is what we show. Now, the thing we were trying to prove in the last class was that, this kind of a set exists in every graph. How do, how do, how do we do that?

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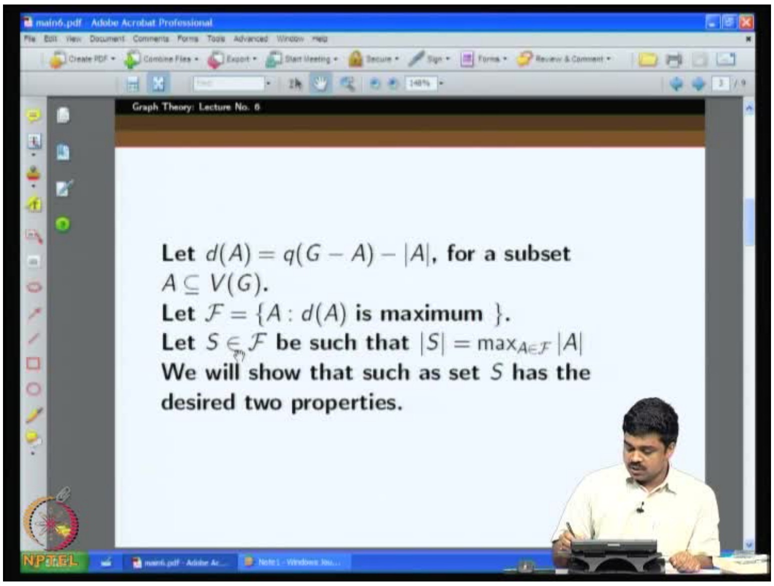

So, we 1st considered this d of a given A subset A. Let us, d of A is equal to q of G minus A minus A and this is defined for every subset A of the vertex set. And now, we consider this set F, where this F is the set of sets and then, we collected all those sets, which will, which will have a maximum value of d of A, the biggest. It is not that there is a unique set, subset A, so that d of A is maximum, there can be many. So, collect all of them, among them, we pick up the biggest set, that means, biggest cardinality set among F from the sets in F, we pick up the biggest set, that we call S.

So, this S has 2 properties: one is, if we consider d of S, that is bigger than the d of A for any other set A and also, among the sets, which have d of A maximum, this S is a biggest set, the biggest cardinality set. So, we will show that such a set S has the desired 2 properties.

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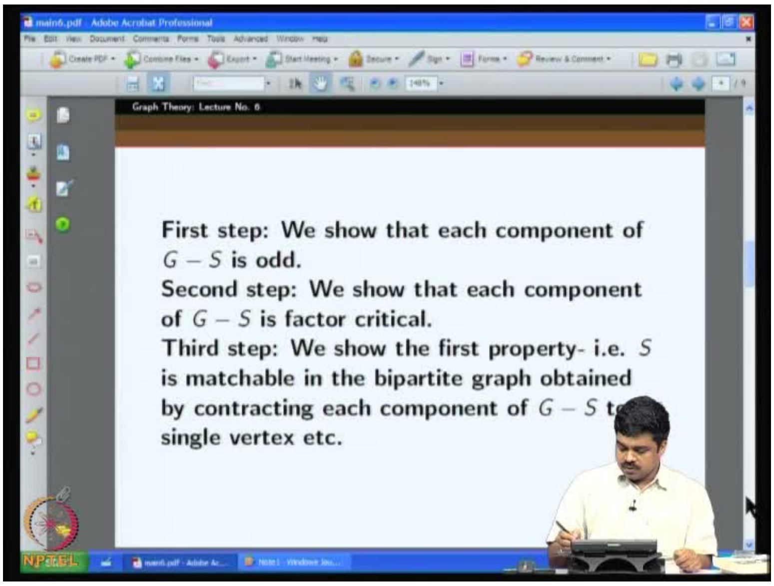

So, what were the steps we followed? The 1st step was to show, that each component is odd. So, we showed, that if any component is not odd, you can move 1 vertex from the, that means, if it is an even component, then I can move 1 vertex from the, that even component to the set S and we will get a new set S union x, and then this set will be a bigger set from F, which will be a contradiction; bigger set than x; that is what we did.

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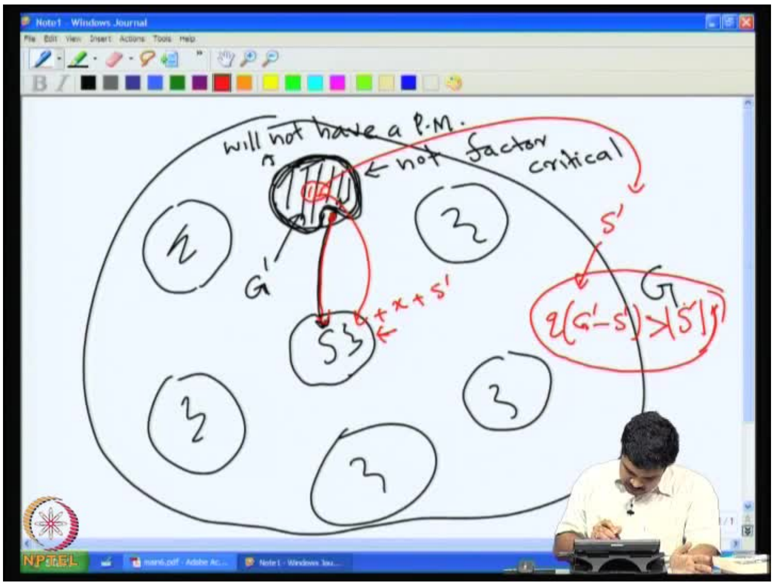

The 2nd thing was, now this is known to be, every component is known to be odd. We need a little more than that, we want every component to be factor critical to show, that we considered, I think I have to continue from here, so select same. This is the set S and so now, we have all components odd; these are all odd components, there is no even component at all; that we have proved already.

So, now, let us consider this component. Suppose, this is not factor critical, not factor critical, then what we can do is we can find out 1 vertex in x; because it is not factor critical, we can find out some vertex x in it and when we remove that vertex from this set, so we are going to move it to S. So, the remaining graph, so I will mark it here, so this remaining graph will not have a perfect matching; this will not have a perfect matching, see, I am writing P M for perfect matching.

So, if it does not have a perfect matching, what I can do is, I can, I can apply the Tutt's statement, sorry, this induction statement to, the, this graph. Let me call this graph as G dash. Now, G dash is smaller graph, G is the entire big graph, this is the entire graph G. Now, this graph is called G dash, see, remember this is just, at least smaller graph than the original graph. Even if this is, this, this, this, this, this, all of them are empty and this also is empty, even then, so we have removed 1 vertex from this thing, before this is, this graph is strictly smaller than the original. So, therefore, we can apply the induction hypothesis on that and we know, that their exists subset S, we, there exist a subset S in it, which satisfy the 2 properties, and as we, discussed, discussed yesterday, so in the last class, this subset will have the property, that it is a bad set.

So, such a subset is always a bad set, that means, so if you say we can always locate a subset S dash on that, let me call S dash, so such that what is the property of the S dash, it will be a bad set with 4 G dash. That means, q of G dash minus S dash will be strictly greater than the cardinality of S, this is the statement of the bad set. So, such a S dash, this is S dash, so cardinality of S dash.

So, this is the property of the, this bad set. Now, we will say, so we can, we want to construct a contradiction. So, you remember, so our plan was to move this x to this and then, now this also I will move to this S dash, also I will move to this so that this will become x plus S dash this.

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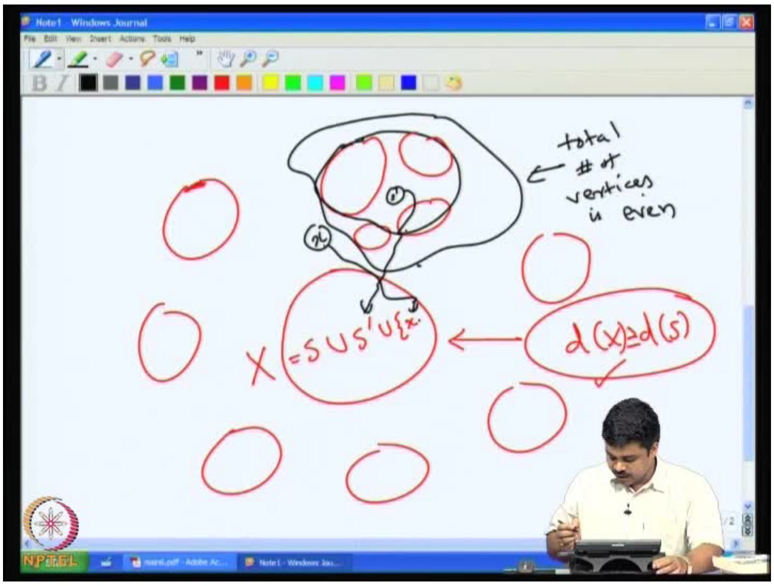

So, finally, I will consider a new set here; this is S union S dash union x. I want to show, that with respect to this subset, if I look at the components, so I will get a contradiction to the original assumption, that S was the biggest subset among F; this subset in S F, so we will contradict this, how?

So, anyway, this is the bigger set, so if I show, that this set is in F, that means, it is a d of, so if I call this as some set X, so if I can show, that d of X equal to d of S or greater than or equal to d of S, then I will be contradicting, I will be proving, that X also belongs to F, either S is not in F or X is also in F, that will, that is what I will prove, so how do I do that? So, you look at the component you are considering; this was our component, so from which we have removed X, and, and S dash, so definitely, it is, so it is only the components. So, for instance, this S dash was like this here, S dash was here, so this we moved to this, X is also moved, so the total number of vertices here is, total number of vertices is, here in this set, in this collection is odd, why? Because total number of vertices after removing X is odd, sorry, is even, because that is why, we already proved, that this entire thing was odd. So, we removed X from that and moved into this. So, the remaining vertices including S dash and its components will be, odd, will be even.

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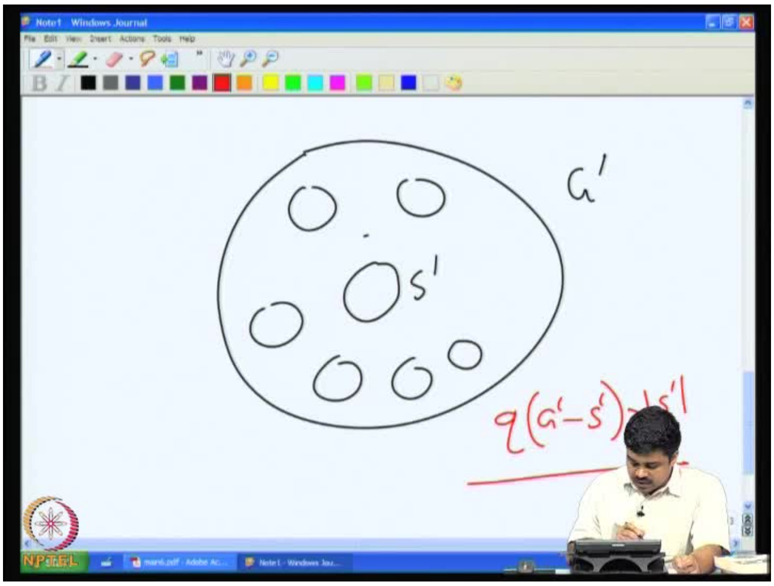

So, now, let me, let me consider, so that, I am considering this graph G dash alone. So, G dash that is we have, S dash and its components. So, see, that whatever it may have, even components and odd components or according to our statement, all the components can be odd, so it is a bad set. Anyway, whatever number of odd components is more than this, so what we know is, q of, q of G dash minus S dash is strictly greater than cardinality of S dash, is what we know.

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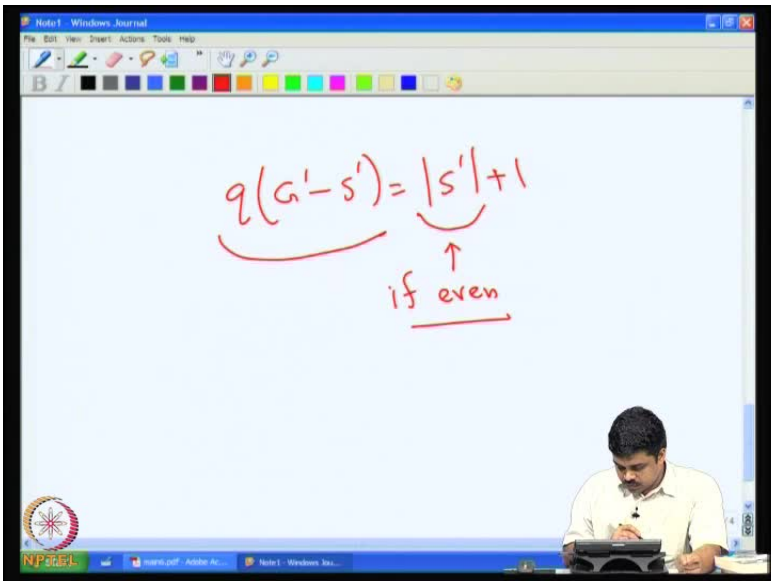

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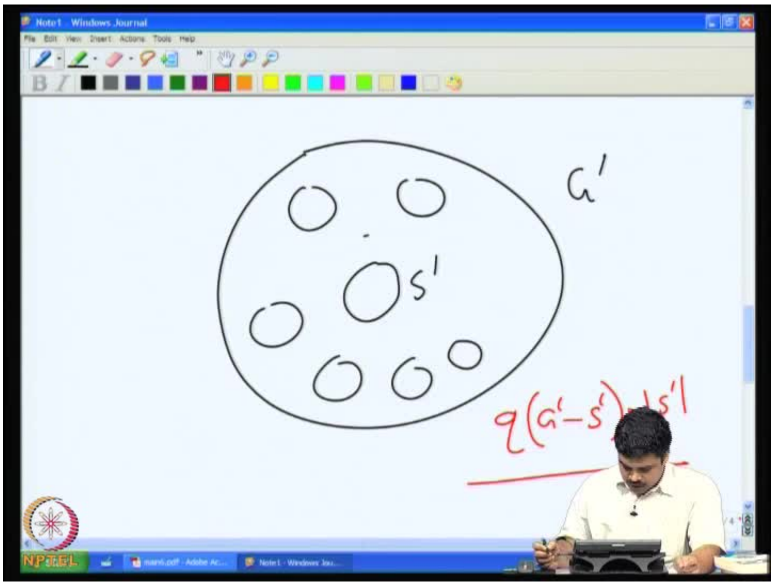

But then, when I say, strictly greater, is it possible, that it is equal to S dash plus 1? Is it possible, that q of G dash minus S dash is equal to S dash plus 1? That is just 1 more, so we claim, that it is not possible, why? Because suppose S dash is, suppose S dash is an even number, suppose, if even, then this has to be odd. So, the number of odd components is odd, so what happens is, yeah, here if S dash happens to be an even number and the number of components here, the odd components, yes, is odd, then therefore, what happens is, the total number of vertices in the odd components is odd plus S dash as an even number of vertices, the total happens to be even here; so, just a minute.

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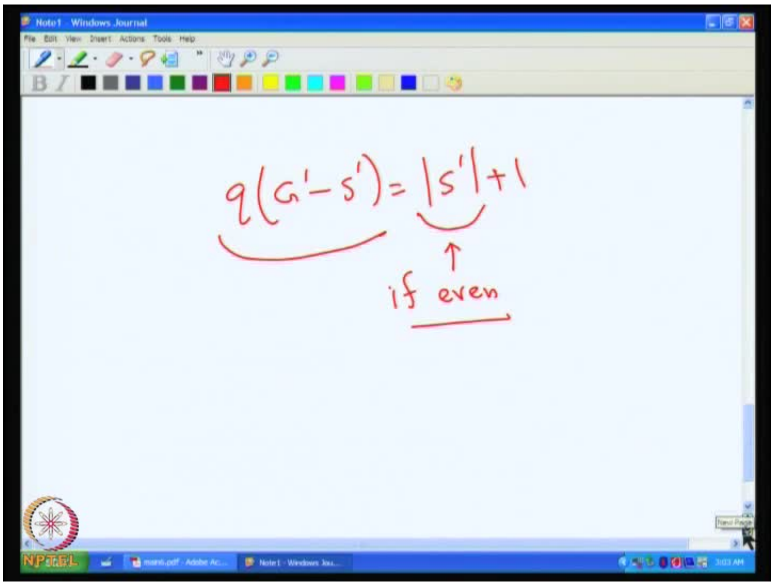

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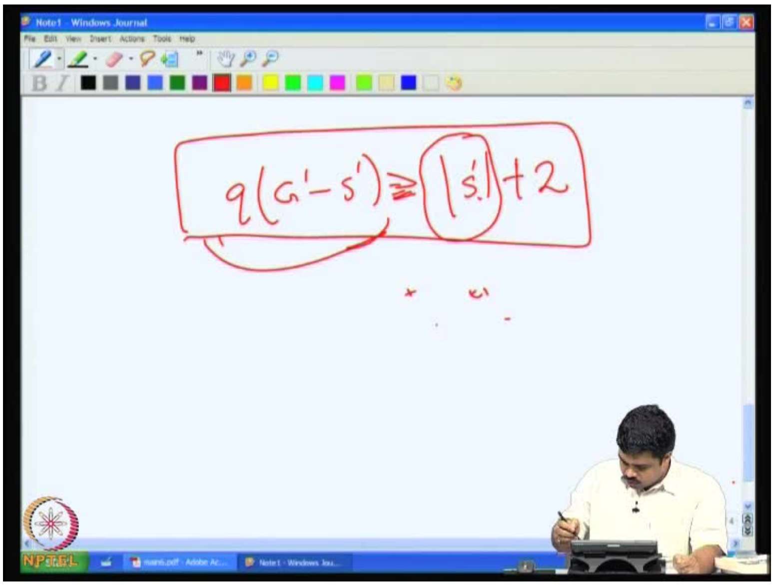

So, we are telling, that can we, can we show, that the number of odd components of S dash has to be, at least strictly greater than equal to 2. This is true because if we consider the parity of S dash is equal to, that means, if S dash is an even number, then this has to be an odd number, sorry, this also has to be an odd number because together they have to make an even number.

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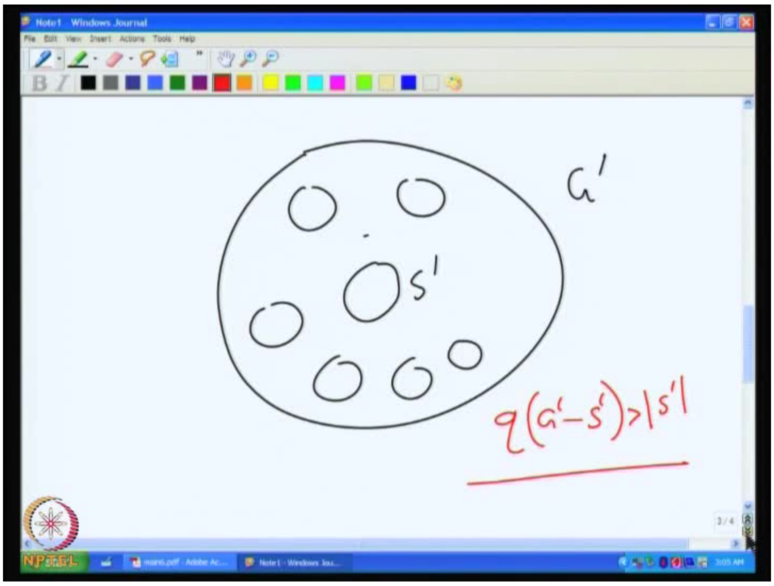

So, similarly, so if this is an odd number, then this has to be an odd number. Therefore, it has to be at least 2 more, so that is why the difference is to be at least 2. Now, this is just a question of counting this, so the total number has to be even; so, the total number has to be even. So, because initially it was an odd number, we removed a vertex x from that, therefore, it became an odd number. Now, you can see that the parity of the number of odd components and the cardinality of S has to be same. So, if S dash is even, then q of G dash S dash also has to be even; if S dash is odd, then q of G dash minus S dash also has to be odd, sorry, also has to be even.

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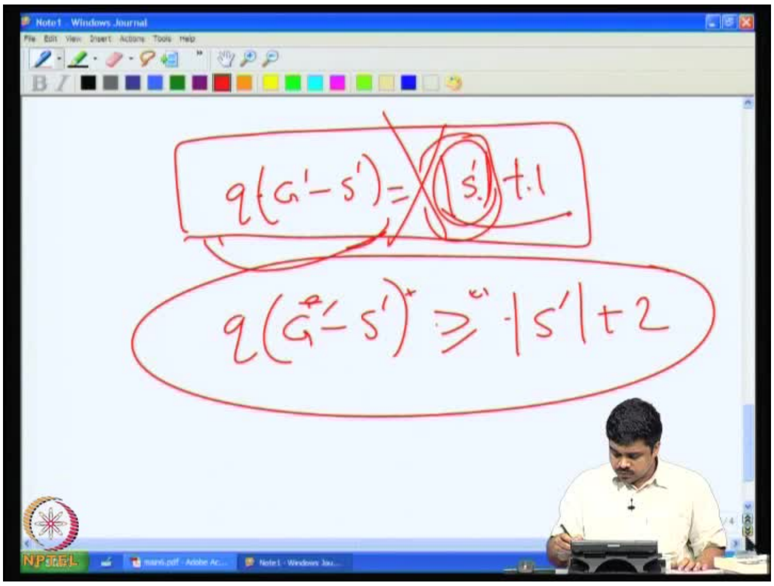

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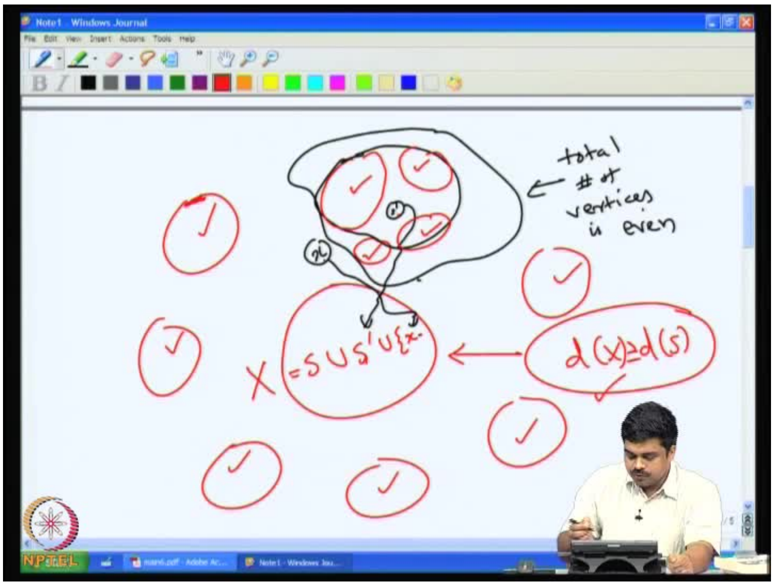

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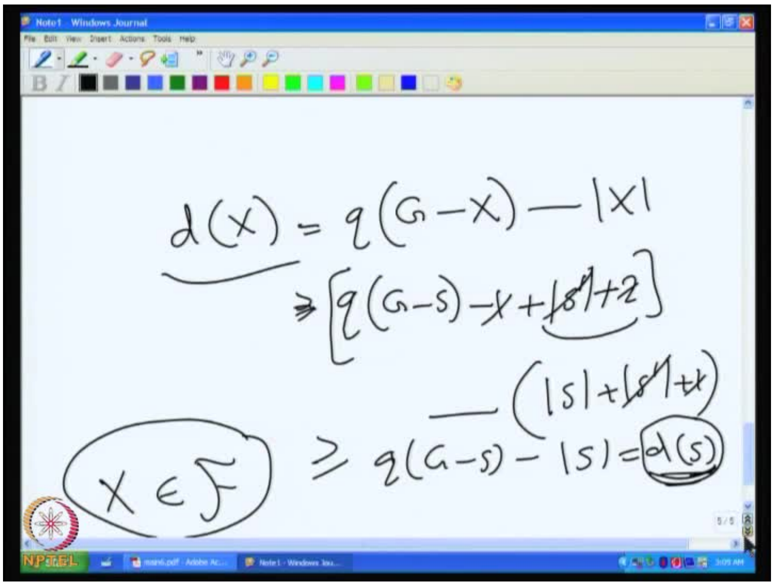

So, therefore, they cannot be differing in, they cannot be differing in just 1 value, so for instance, not possible for them to differ in exactly 1, 1, so therefore there has to be like this, so this, this is wrong, so that means, because in that case, if this is odd, this will be even and then, therefore, we will get this thing greater than equal to S dash plus 2, this is what we get. And why is it useful? Because, so as you see, going back to our original picture, so when I moved X, S dash and S here to form a new set, the number of odd components, so all these things remain as such while new odd components are created for this thing.

How many new odd components are created? So, here, by removing this thing, the number of odd component is 2 more than the cardinality of S while this is only increased by S dash, this is increased by S dash plus 2. So, here we have added 1 more vertex here, so therefore, so if you compute this thing d of say, suppose if you compute d of X, this is essentially the number of odd components of X minus cardinality of S.

So, now, this we know is essentially q of G minus S minus 1 because 1 odd component we destroyed and then we added the number of increase in the odd component is essentially S dash plus 2. We have added this many new and now on the other hand, we have excess increase from S by S dash and 1 more, so this S dash gets cancelled and then here, A minus 1, here A minus 1, this cancels with this, so we will get, that this is, anyway, this is greater than or equal to, this anyway greater than or equal to q of G minus S minus S. So, this is d of S, so what we get is d of X is, at least d of X is at least d of S, so, but of case, it cannot be strictly greater because d of S is already taken to be in F. That means, this was the maximum value, so this has to be equal to this. So, X is, which we, we infer, that X also is an element of F and then why do not we select X instead of S, that is the argument. We can, we can, we could have taken X instead of S, this so because X is the bigger set than S, our intention our plan or what we had assumed S is the biggest set among F. But we see a bigger set here; it is a contradiction, so it has to be factor critical.

Now, to repeat the, the main point here, so what we have done is to consider a component. We already know, that the cardinality of the component is odd and then we removed a vertex from this odd component so that the total number of vertices in the component in that, the remaining number became an even number, say, let us call as graph G dash. And now, this G dash is a smaller graph, we can apply induction hypothesis; therefore it should have a bad set. Bad set means, the number of components, odd components is strictly greater than the cardinality of that bad set, we call that bad set S dash.

And by considering the parity, because the total number of vertices has to be even, we can immediately infer, that the number of vertices in S dash and the, number of, number of odd components of it should have the same parity, that is, if one is odd, the other also has to be odd; if one is even, the other also has to be even. So, therefore, it cannot be the number of odd components is more than the cardinality of S dash, but it cannot be just 1 more than, because if it is just 1 more, then their parity will be different, so it has to be at least 2 more; this is very crucial because we need at least 2 more.

So, what we are doing now is to make a new set X by adding the removed vertex X into S and also, this bad set S dash into S. So, we have increased the set, by, from S to X and the number increase, the cardinality of S dash plus 1 is an increase, but then, along with that, the number of odd components also increases because when we removed S dash, so many new odd components got produced. They will all be counted for this new set X. Then, we considered the odd components of G minus X, but we know that there are at least 2 more there; so, for the S, 2 more odd components than S dash. Now, we have destroyed an odd component, added S dash plus 2 odd component, that is the total increase of S dash plus 1 odd components

So, and that balances the increase of S dash plus 1 in the size of the set, so when you consider that d of X, which is d, the number of components, of odd components of G minus X and cardinality of X, so the increase in both the terms is equal. So, therefore, because it is some difference we are considering, it is the same. Therefore, it might have increased, but we know, that it will not go up because d of S is the biggest value possible. So, now, what we see is excess from the set F and we had already assumed, that S is the biggest possible set that I can pick up, therefore it is not possible. It is, it is a contradiction, therefore it is a factor critical graph. Now, let us see, the, the, the last thing we have to show is by considering, that bipartite graph, which is obtained by contracting the components and which is obtained by, say if at all there are some adjacent side, as you delete it, you make a bipartite graph out of that, so there S is matchable is what we wanted to say and so we can easily show these things.

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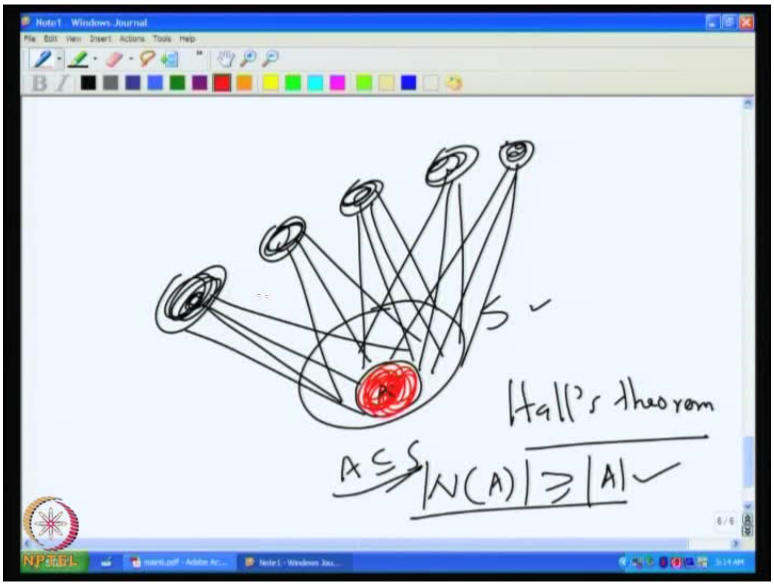

So, we consider S here. So, these are the components, so I am just drawing it, we can assume, that this has got contracted into 1 vertex, so and then, so their some connections will come. So, I have explained what contracting means in the last class, so it will, it will give bipartite graph like this.

Now, suppose, this I want to show, that there is a matching of S, we can use Hall's theorem; what was Hall's theorem? Hall's theorem says that, Hall's theorem says that, any subset, say A of S, if I take, if I take A subset of S and if I consider N of A, if the cardinality is greater than or equal to A for any subset A, then we say, that the Hall's condition is satisfied and then, if the Hall's condition is satisfied, then there will be a matching of S; this was what Hall's theorem says. set

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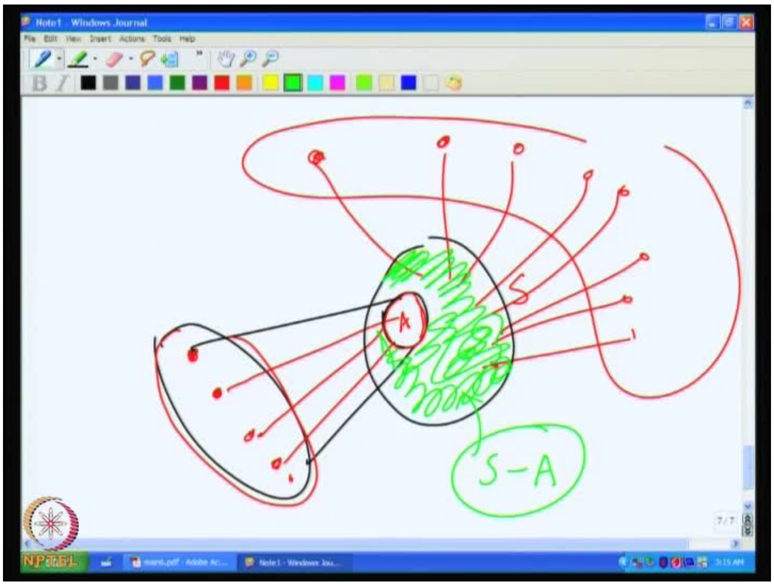

So, now consider a subset A here, so now suppose the Hall's condition is not valid here, so what does it mean? If I consider the neighborhood of it, so it will be like this, this is some set A. So, if we consider the neighborhood of it, so will see some, these are the components; some, these are the components, we will see some components. So, it can be like this, but the number of things, you see, in this thing is strictly less than this, this, let me say, this is A, now this total was S and there are other components here, of case these are the other components. So, these are, but you see, any none of them will be connected to it, so they are all connected into this only.

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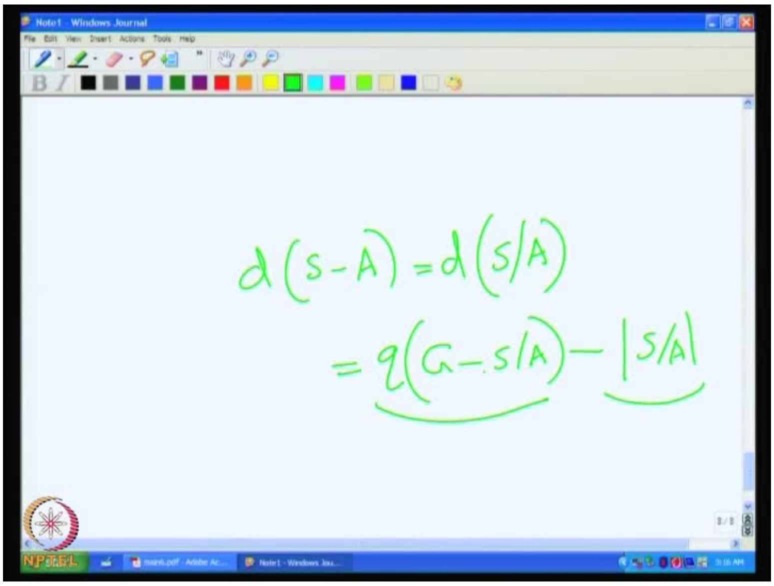

Now, the question is what if I consider, say this set, this set, which is essentially the S minus A; this set is S minus A. now, I could have considered this set S minus A, so what happens is, so I can consider the d of S minus A, what will be the situation? So, of case d of S minus A, so let us say d of, I can use this notation, that will be easier, S bar A is equal to is q of G minus S bar A and then minus cardinality of S minus A; so, cardinality of S bar A. So, now, you see, there is a reduction here in the number of components, odd components, there is a reduction here also.

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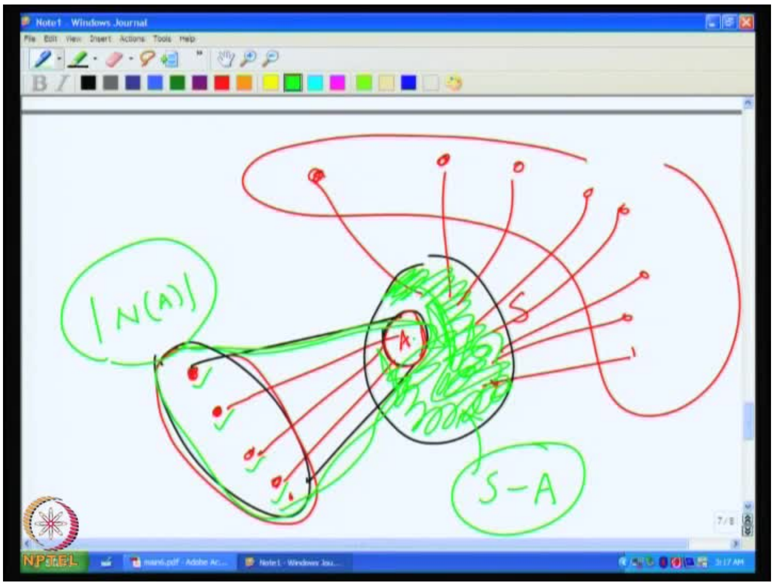

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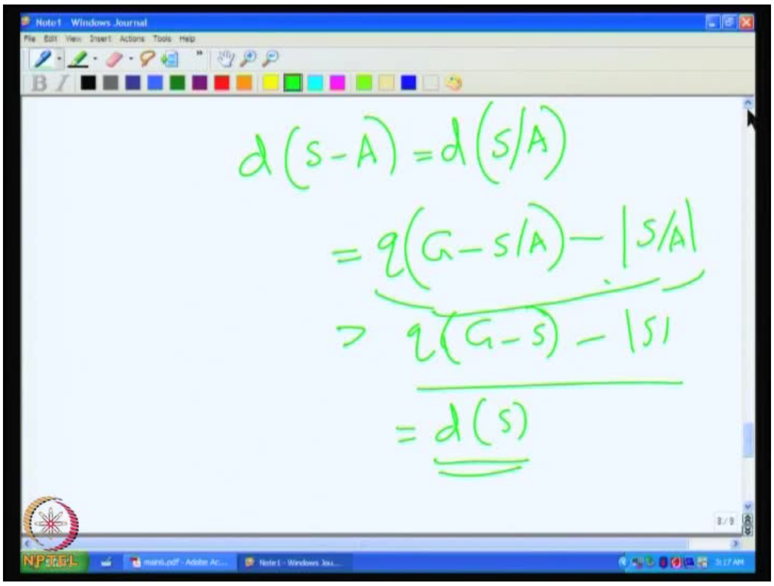

Why is there a reduction here? Because, so there may be a reduction here, so if at all there is a reduction, the only the components, which are the neighbors of S bar A will be reduced and here the reduction is, sorry, that neighbors of A will be reduced, for instance. So, here, these components will disappear from I count because they may not be connected to this, so they may not be, for instance, all of these thing may become a new component, and they may not be even, may not be odd. It may become an odd component, so we have to just look at this portion; we have to just look at this portion. So, how is a reduction? Reduction is this, the N of A. How many, the cardinality of N of A will be the reduction while, while the 2nd term, the reduction is cardinality of A, but you know, the cardinality of A is a bigger term, therefore you are reducing more here. So, the overall, this will be bigger, this will increase; this will be bigger than q of G minus S minus S.

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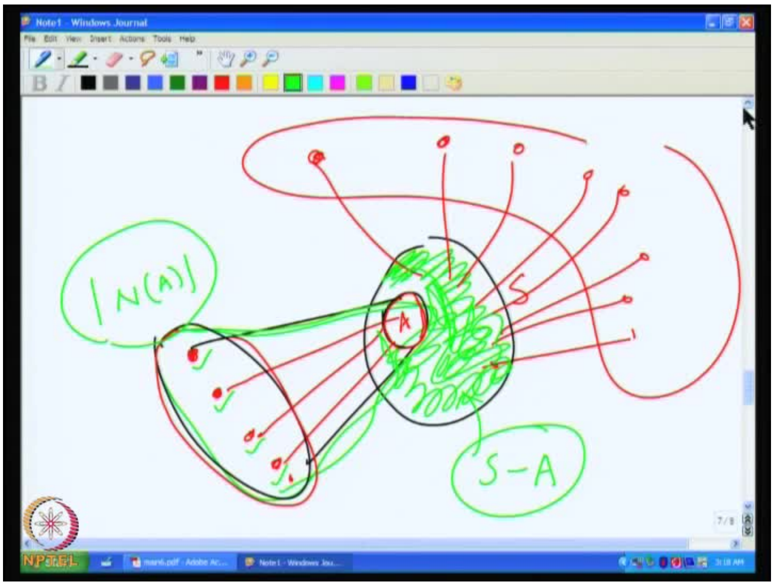

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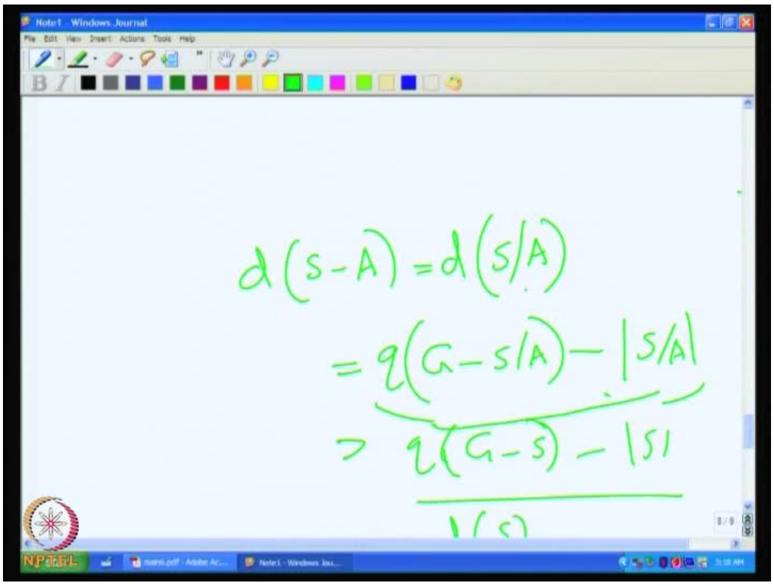

So, this is what, so not bigger than, will be strictly bigger. So, therefore, this will be a contradiction, you see. So, what happens is this quantity is the, this quantity is the, this quantity is the d of S and we know, that this is supposed to be the biggest, but you know, when we consider S bar A, it is d of S bar A. This became the, so this became bigger than that, so it is going to be contradiction. So, therefore, we can say, that the Hall's condition is always valid for this thing.

So, this portion, to summarize what we did is, we considered set S and to show, that there exists a matching of S, we wanted to show, that the Hall's condition is valid for this bipartite graph from the S side, so we can take any subset A and then, look at the neighborhood of it, the neighborhood of it is, suppose the Hall's condition is wrong, some for, some set A, some subset A, the neighborhood has to be smaller. So, now, what we do is, instead of S, we considered S, the set S minus S bar, so we say, that the number of odd components may reduce by N of A while the set's sizes reduce by A. So, the total, the expression q of G minus S minus cardinality of S, this expression for this new set will give a bigger value. So, therefore, it is a, it is a contradiction, that is why, the Hall's condition cannot be false for this graph and because the Hall's condition is true we get a matching of S, that is what we see.

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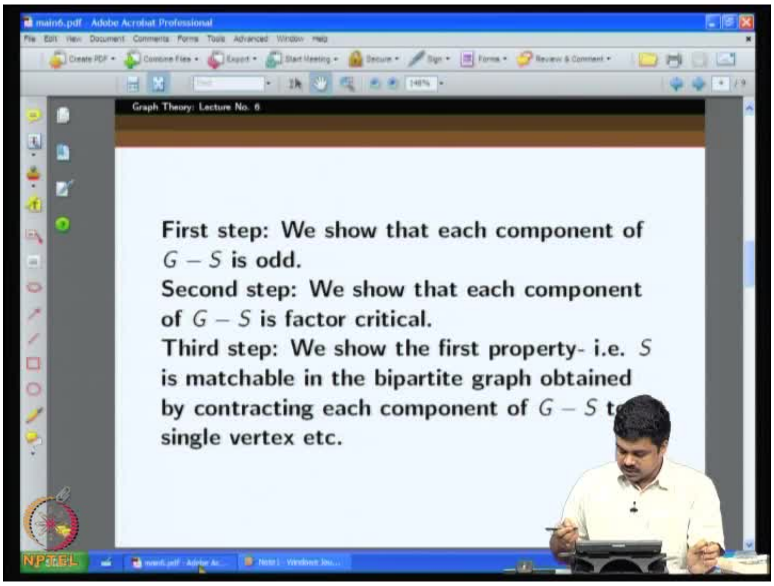

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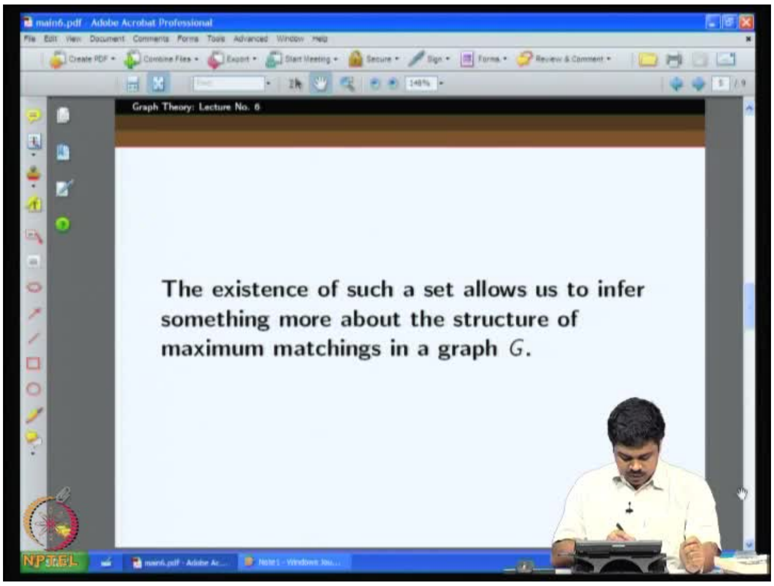

So, finally, what we have shown is this 3 properties, sorry, 2 properties is true for the specially selected set S, so that is, each component is factor critical and also, that the bipartite graph, which we, which we produced out of this has a matching of S.

So, now, not only that, if such a set S X is, then we can tell little more about the structure of maximum matching's in G. So, what can we tell? So, let us, let us look at the thing, what can we tell about the maximum matching's on this?

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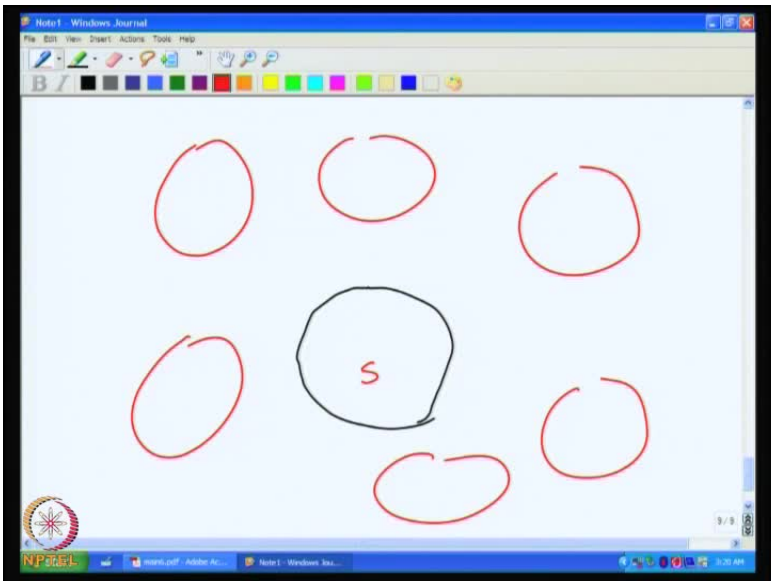

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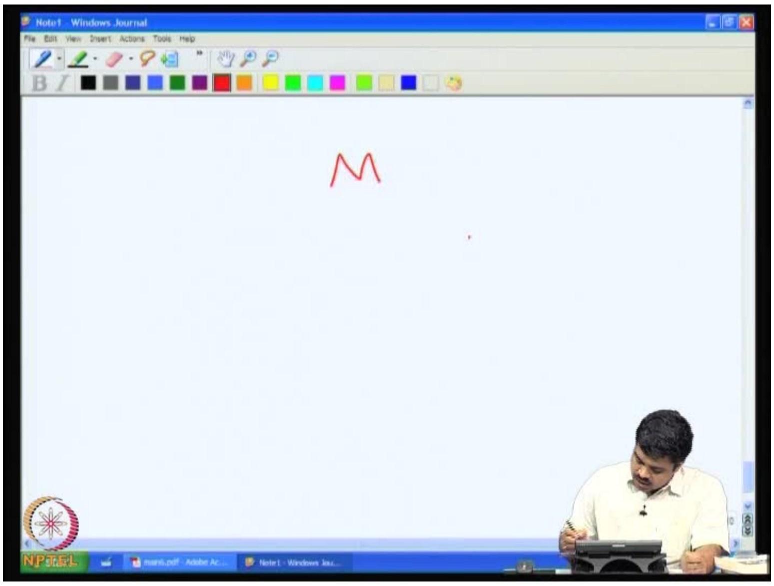

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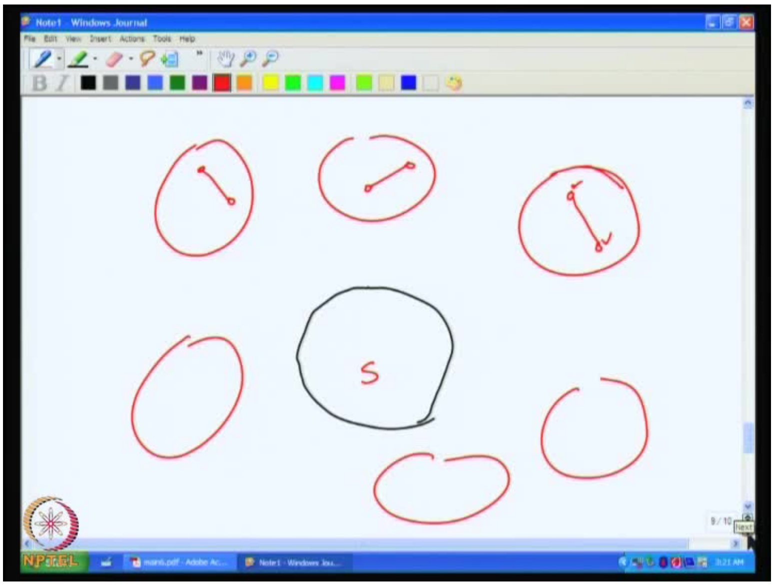

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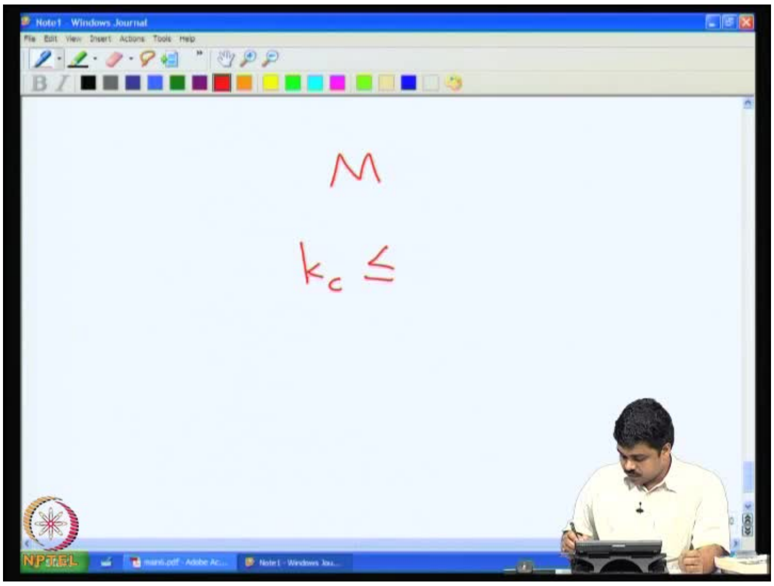

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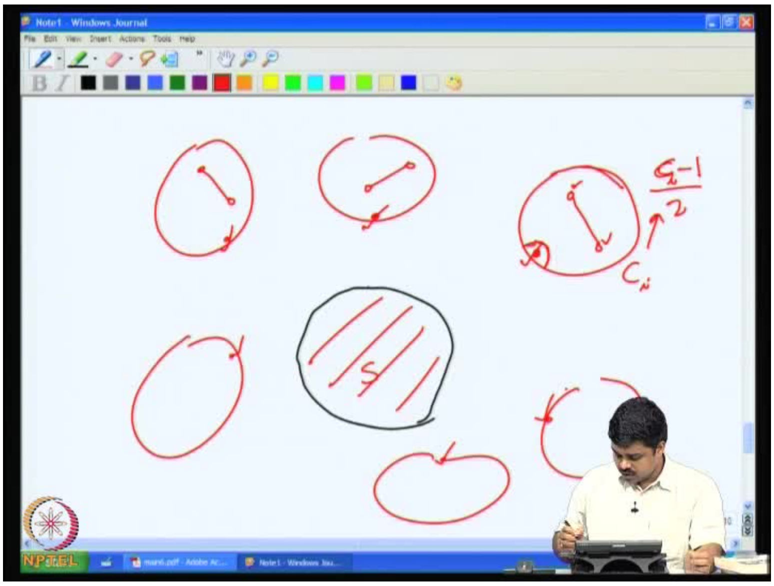

So, if you look at this set, suppose this is a set S, which has this 2 properties and so I will say, this is the S and these are all its components, these are all, remember these are factor critical components of G minus S, this is S. Now, you can say, you give me any matching M, so for any M I can put 2 kinds of edges. See, some edges are like this, since for instance, they can have both the end points in components, it can be like this or it can be like this within, see both the end points can be inside a component. So, this, let us say, they are called k c. The number of such matching edges from M is k c. Now, we see how many such edges can be there, maximum? So, this k c has to be less than equal to the total number of vertices in the, total number of vertices in the component.

So, here, we see, that the maximum possible is because as you can, as you can see, there is in each of this factor critical component, so you can, you may, you will always get because this is odd, you will always get 1 vertex, which does not have a pair here, is not possible to get the entire component matched. So, essentially, the maximum matching you can get is only, see if this is C I vertices in this, in the Ith component, so the C I minus 2 is the maximum matching I can get.

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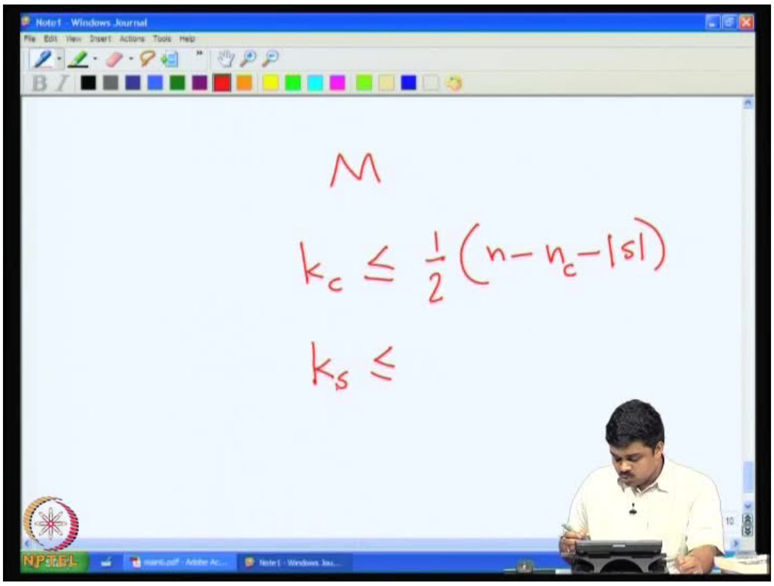

So, other than the vertices in S, you also have to give up 1 vertex from each component. So, therefore, the biggest number of matching edges, such that both its end points are in some component is only half of the total number of vertices. Let us say, n minus the total number of components, I will use this notation for that call so c plus or may be, I can, I can use later for this, the total number of components can be taken as say n c and of case minus cardinality of S, this is the biggest such number of edges we can get from this thing, k c for given a matching.

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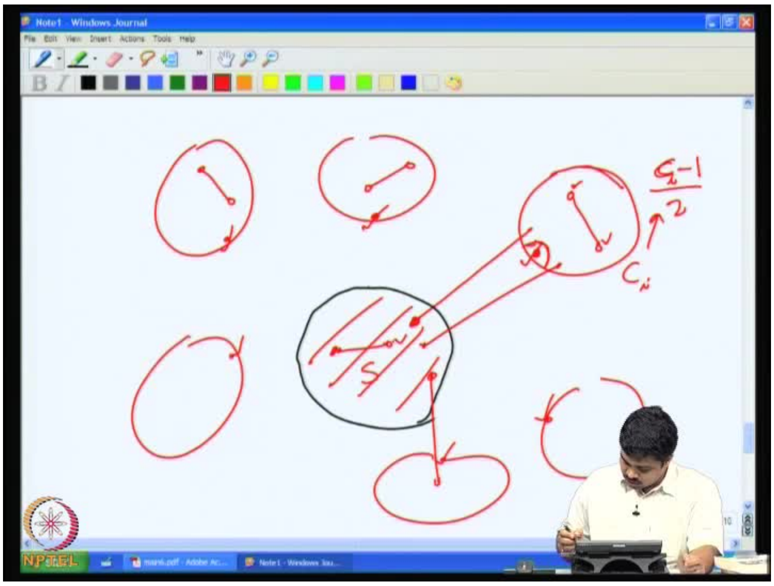

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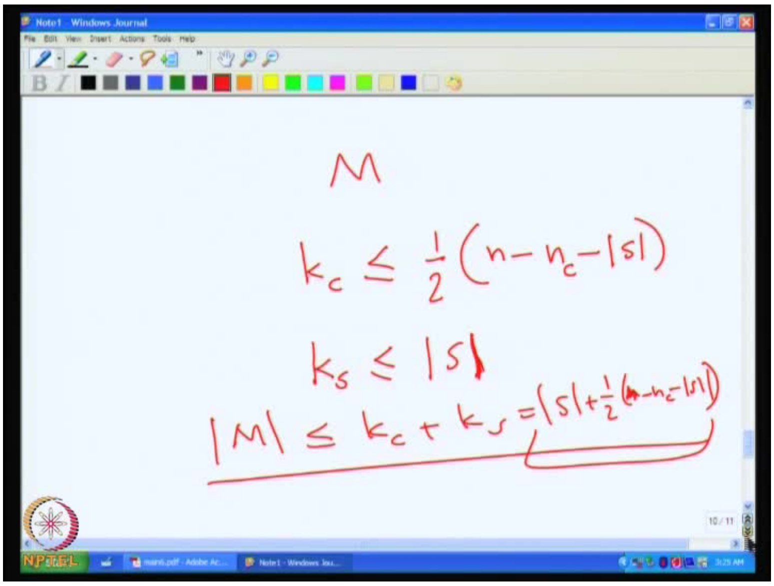

Similarly, we can have another type of edges called, say I will use the notation to, notation k S to denote, that which is having at least one end point from S, which is having at least one S, one end point from S. So, essentially, you see, so this is, so it can be an edge like this, so it can be an edge like this or it can be an edge like this, so it can be an edge like this, both end points here. So, if you are considering such kind of edges, you can have only this many because there are this many vertices on S. So, even if each matching such edge is taking 1 vertex from S, so we, you will have to show, that, you will see, that k S is less than equal to S. So, therefore, the cardinality of any matching has to be less than equal to k c plus k S. So, that is, cardinality of S plus half into n minus n c minus cardinality of S, this is what we see. So, this will, this will be correct.

So, now, this is n. Now, we can see, that not only that this is an upper bound, there are some matching's, which achieve this bound. So, any matching M should have cardinality at most this only, that is, the cardinality of S plus half of n minus n c minus cardinality of S.

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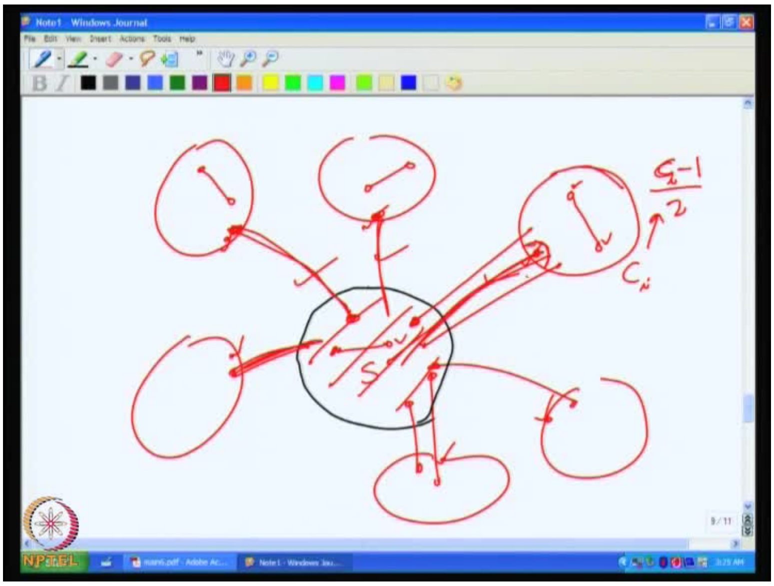

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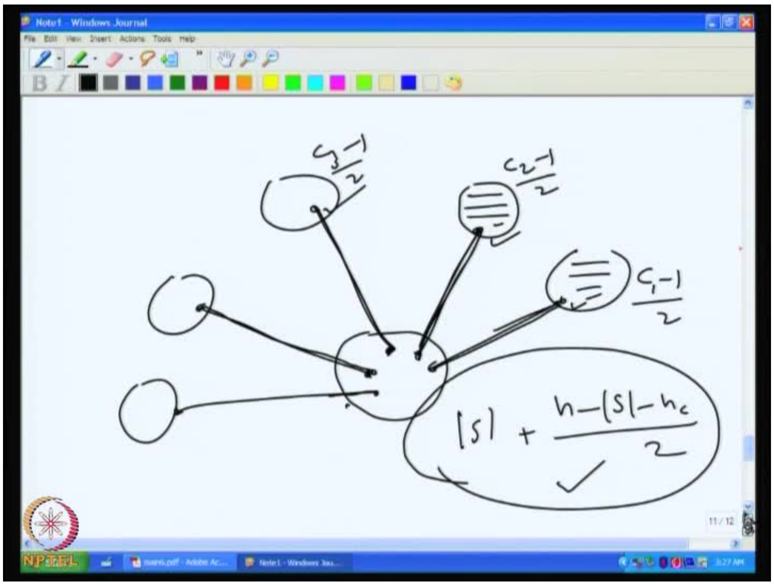

But if we consider this kind of a matching for instance, let us go back to our picture where, yeah here it is clear, that because this is a factor critical graph, you can remove the 1st and also, you can 1st get a matching of this versus this, by, is a 2nd property, sorry, 1st property. You can get each vertex of S matched to some vertex in the components, so one matching edge going from one vertex of S and to a vertex of S and then, so we will, we will able to match it off like that, or if I want to draw here in other one, so here this is the S and then, we know by this 1st property, if you had contracted all these things, you will get a matching from here to here, that is what this 2nd property says.

So, therefore, you can find matching edges going from each vertices of S 2, the components like this and so once I have removed 1 vertex this, 1 vertex here, so each of them are factor critical, therefore I should be able to pair them up completely. So, here, we get C 1 minus 1 by 2, C 2 minus 1 by 2, C 3 minus 1 by 2; so, total of, when you sum up, you will get n minus S minus n C by 2. See, yes, because this much is not there in them, this, because 1 is being lost for each of them, so n minus n C plus we have S of them.

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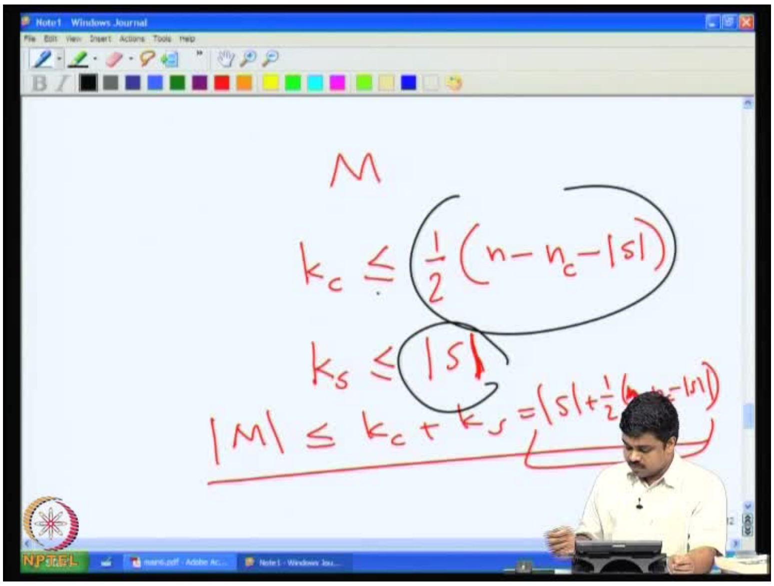

So, here is a matching, which can be formed with this cardinality. So, that means, the cardinality of the maximum matching has to be this much, so we repeat, that their exist a matching of this cardinality and any matching has to be of cardinality less than equal to this. So, that means that S is the maximum matching and not only that, any maximum matching should be of this type, why? Because you can see, that if it is a maximum matching, it should have this much size, this plus this (( )) together, but if this was k S was strictly, for that particular matching k S was strictly less than S, so to achieve this value, this has to be the other term, has to be bigger than this, which is not possible. So, therefore, this has to be a strict inequality here, sorry, cannot be a strict inequality here, that means, it has to be in equality here, this also has to be in equality.

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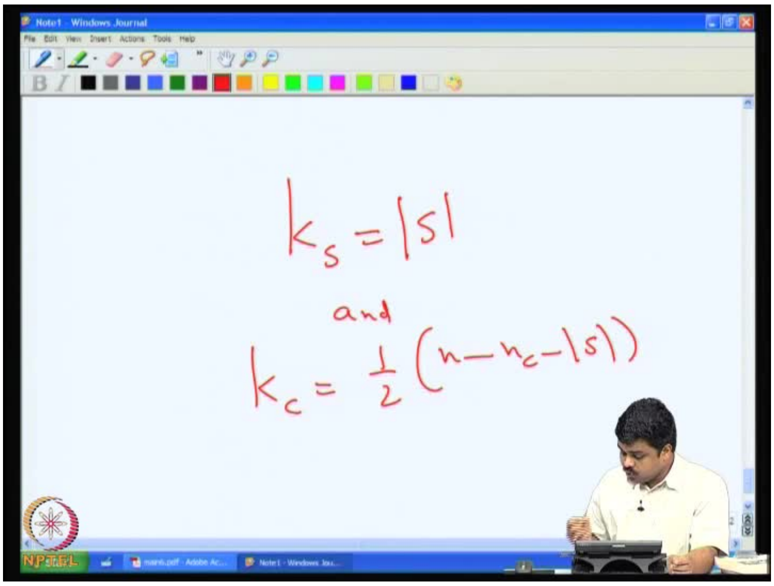

So, that means, in any maximum matching we should have k S, in any maximum matching we should have k S equal to cardinality of S and k C equal to half of n minus n C minus cardinality of S; this much is important.

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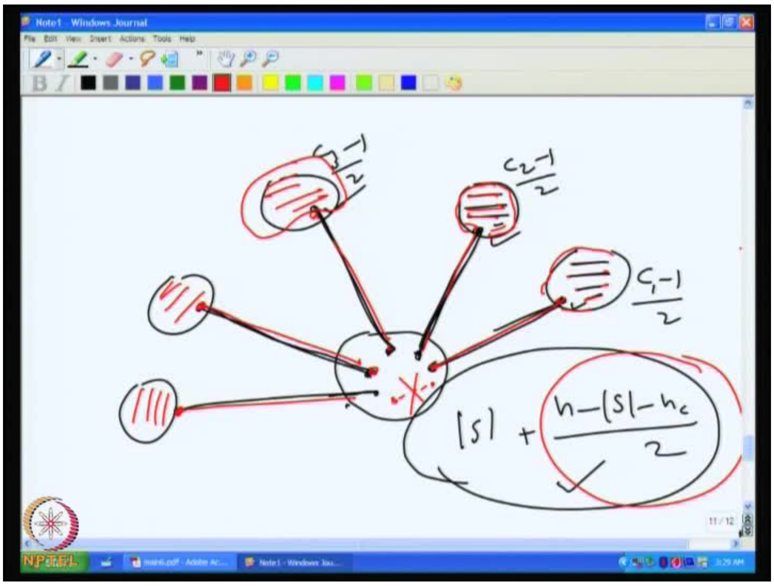

Now, it tells us, that it is not possible to do it arbitraly, for instance if you want to get this, this much, this, this value for k C, then the only way is to match as much as possible in this thing. That means, you can only leave out 1 vertex from this thing, so or otherwise we will go below this thing. So, similarly, you have to take all except 1 from this thing, so you should somehow match. So, similarly, here also, I should match all except 1 from this thing and then, every odd component here, every component will leave out 1 vertex, they should get their partners from S, so that it has to be like this, should be a perfect matching between this and this. There would not be any edge of this sort, that will never happen,

So, the S vertices will always get mapped to the components and the remaining things will have to be completely from the odd component, and the component, and each component will give you a matching, which matches everything except 1 vertex. So, this is the only way the maximum matchings can exist, so that is what it says.

So, now, again, this finally, let me summarize the last part. It says, by proving, that their excess set S with the 2 properties, not only that we could infer Tutte's theorem, we could also infer some structural information about the any maximum matching in G.

So, that means, any maximum matching with respect to this, such a set S has to behave in a certain way, so it should match each of the component as much as possible. That means, it can only leave out one 1 vertex, you will have to leave out because it is an odd component, that means, there is everything cannot be matched. So, 1 vertex only it can leave out and that left out vertex should get its partner from the S, this is the only way the maximum matching can appear. If you violate this, then you will not get a maximum matching. So, that is what it says about the structural perfect matching.

And now, coming back to, so to give a, this to remember recall all the things, that we were doing. So, we already covered several concepts like one, sorry, sorry, I have one more theorem to finish here.

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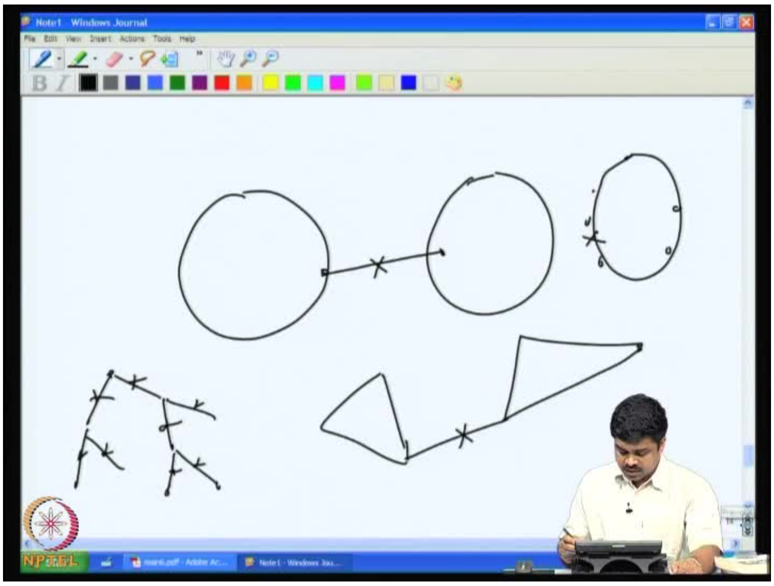

So, here is the theorem from Peterson, which says, every bridgeless cubic graph has a perfect matching. So, what does it mean? So, you consider a graph, a bridge means something like this. For instance, if you, if you remove an edge, if the graph gets disconnected, then it is called a bridge. So, for instance, this is the bridge, for instance here, this is a bridge, here this is a bridge, this is a bridge or in a tree, this is a tree, all are bridges, this is a bridge, this is a bridge, this is a bridge, all are bridges.

So, in this graph, in a cycle graph there are no bridges. So, you can cut any edge, the graph will not get disconnected, so the, that is what a bridge is. So, it should give 2 components, one of the n vertices go on one side and other should go on the other side. Suppose, the graph does not have any bridge and then you can always find a perfect matching in it. So, this is an application of the Tutte's theorem.

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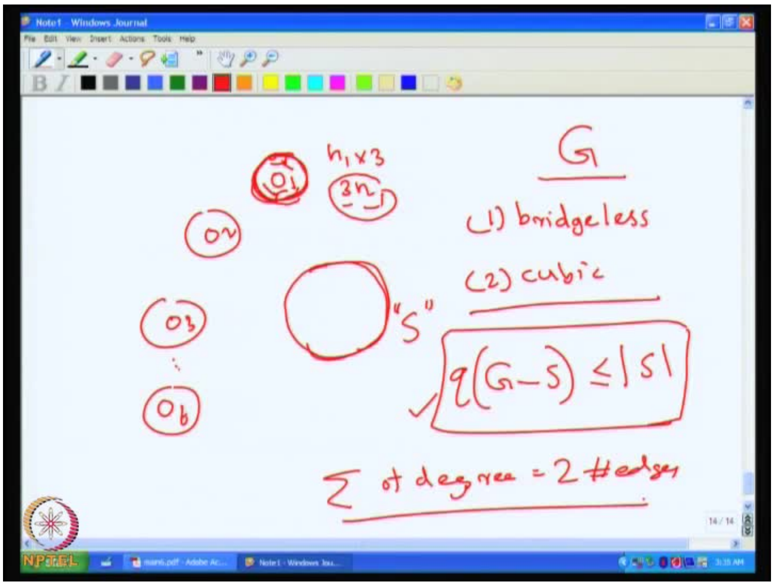

So, we will just quickly go through the, as, take that is proof of this. So, suppose we consider a bridgeless cubic graph G. So, now, to show, that this is 2 properties are there, so bridgeless 2 is cubic, cubic means it is a 3 regular graph, so that means, each vertex has degree 3.

Now, you can consider any subset S. I will show, that your q of G minus S is less than equal to cardinality of S. If you show this thing for every subset, then the Tutte's condition is true, therefore you will have a perfect matching. This is an application of Tutte's theorem, so I considered this subset S and now suppose you consider the odd components of it, these are odd components, so they can be even components that discard them for the time being, this is the odd component 1, o 1, o 2, o 3 like that, so this is some o t.

Now, you see, so this being odd, so and each degree, each vertex has degree 3, so each vertex has degree 3 here, so what is the sum of degrees here? So, here, suppose your n 1 vertices here, n 1 into 3 is the sum of the vertices 3, n 3, n 1, but then 3 is an odd number, n 1 is also an odd number, this has to be an odd number.

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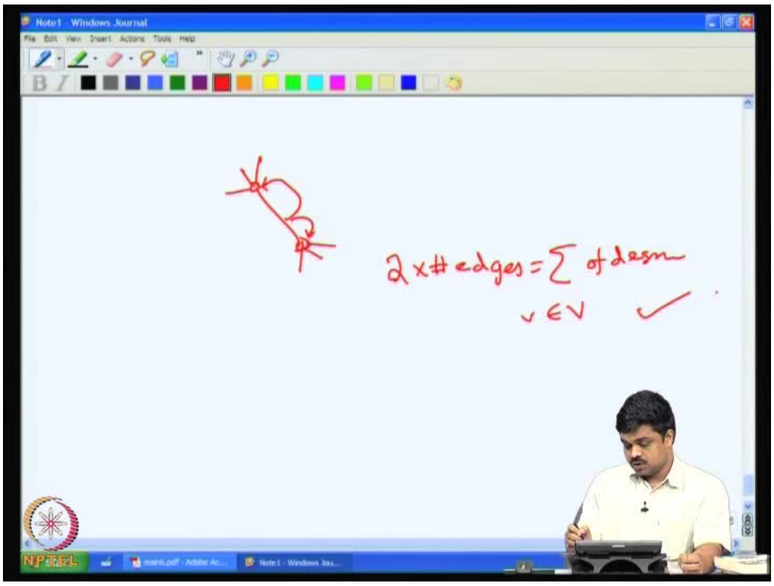

So, what is the sum of degree is for the induced sub graph here? It is always an even number because the sum of degrees, because sum of degrees is equal to 2 times number of edges in the graph because, why it is so? This is because, if you consider any edge, it will contribute 1 to the degree of this and another one to the degree of this, so each edge is contributing 2 to the sum of the degrees. So, 2 times, so actually 2 times the number of edges is essentially the sum of degrees or select all the vertices if you consider.

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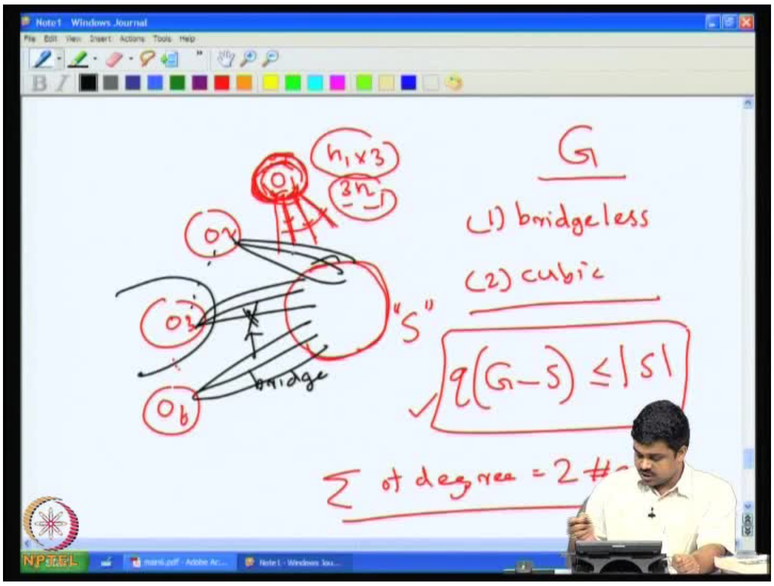

So, it so happens, that the, when you sum up the degrees you will get 2 times the number of edges. Therefore, it has to be an even number you see here, but this is an odd number, what does it mean? Some edges has to go out of this, every edge cannot be inside this. So, how many edges has to go out, because the total is sum of degrees is odd and here we have, inside this we can get only an even number, so we should get an odd number of edges going out of it because for each outgoing edge, we will count 1 here, not 2. So, therefore, an odd number of edges have to go out.

The question is, which are the possibilities? So, can it be that just 1 edge goes out? For instance, out of this, is it possible, that I had just 1 edge going out. You see, this, as for as the oddness and evenness is fine because odd, even plus 1 is an odd number, that does not give it any contradiction, but then, then this will become a bridge, why? Because if you remove this thing, this component will get separated from the rest of the graph, so it is not possible to have just one edge going out of it.

So, the next number is 3. It means, at least 3 edges has to go out of each component and out of means, it should go into S because it cannot, no edge can go from component to component, so each component will sent 3 edges.

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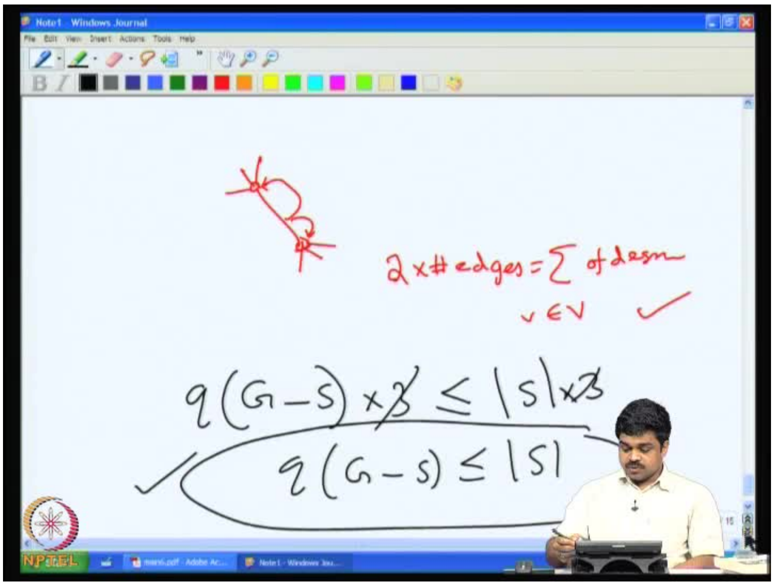

So, therefore, we get q into G minus S into 3 edges are going into S, but then S should accept all these things, but each vertex of S can take maximum 3 edges because anyway, it is a 3 regular graph. So, we need cardinality of S into 3 to be bigger than this quantity because this many edges are coming only, this many edges can be taken by S. So, cutting this, we get q of G minus S is less than equal to cardinality of S and this is the Tutte's condition and this is true for every set by our argument. So, the Tutte's condition is valid, so there is a perfect matching in the graph, this is what, so we are, so we can, we can prove using Tutte's theorem for cubic bridgeless graphs.

Now, the ideas about matching, that are sufficient and necessary condition for matching is this much and now, we will again recall the things we were doing.

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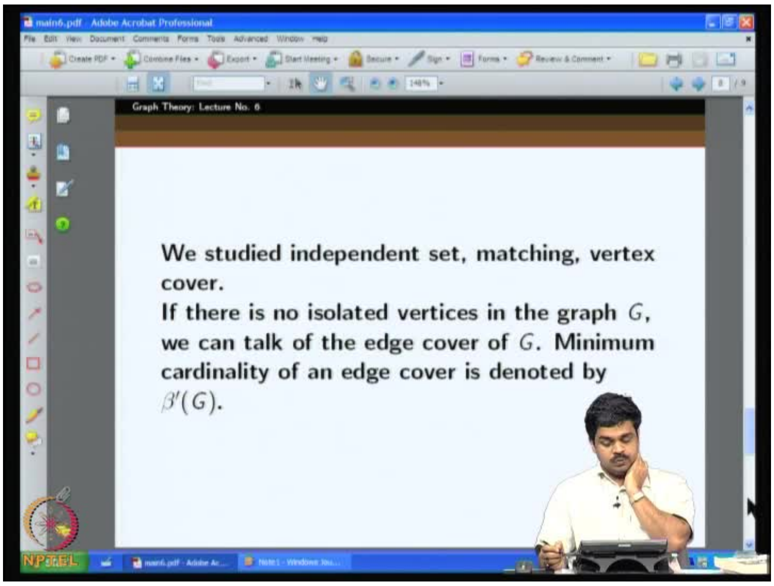

So, we studied independent set matching and vertex cover, these are the main 3 concepts we went through. So, what do we mean by that? So, independent set was a collection of vertices, subset of vertices, such that there are, if we consider induce subgraph on those vertices, there is no edge in them while this is matching is somehow an equivalent notion with respect to the need, some similar notion with respect to the edges. For instance, a matching is a collection of independent edges in the sense, that if we consider 2 edges in the, in a matching, there is no, they are not sharing any vertices, they are independent.

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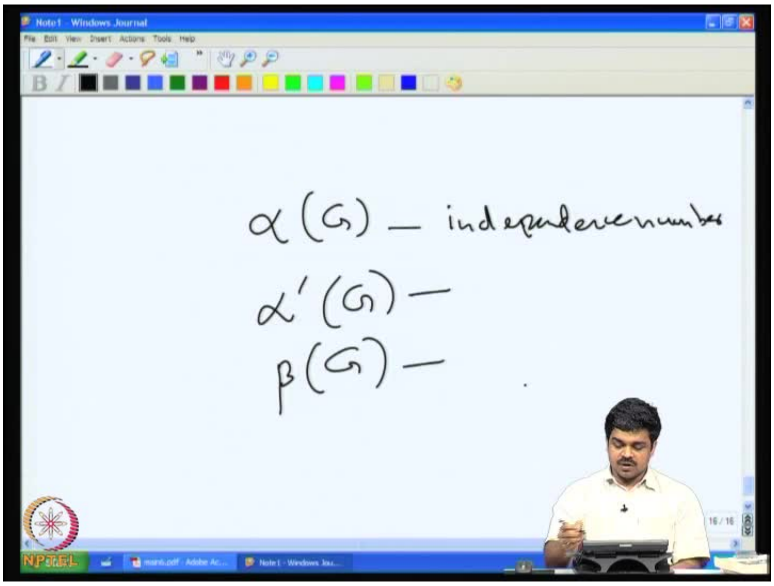

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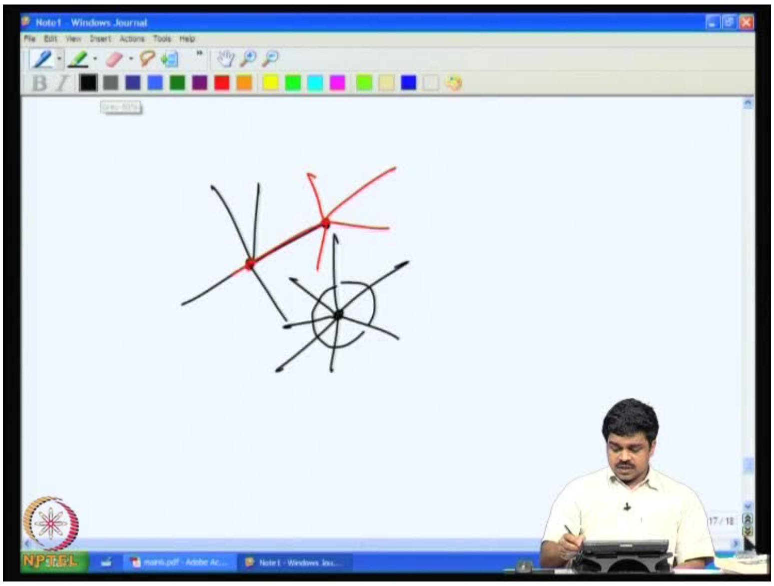

So, with respect to the independent set, we have a parameter called alpha of G and this is the independence number, independence number, stability number and similarly, with respect to the matching, we had defined alpha dash of G, namely the biggest cardinality of the biggest matching, which can reach of 10 by 2 when it is a perfect matching. And the other notion we studied was vertex cover, so which was called beta of G and this beta of G is the number of vertices required, the minimum number of vertices required to cover all the edges of the graph. So, essentially, we can say, that it is the number of stars required to cover all the vertices or the edges of the graph, because if we take a vertex cover, what we see is, with respect to each vertex in the vertex cover, it is covering the edges, which are incident is, this is the star graph.

So, how many star graphs can cover, minimum how many star graphs are required so that it can cover all the edges of the graph? It does not mean, that for instance, you can take, it is possible that you have selected this, you can always select these also, that is not a problem, so how many star graphs are required, had to cover, this is the question?

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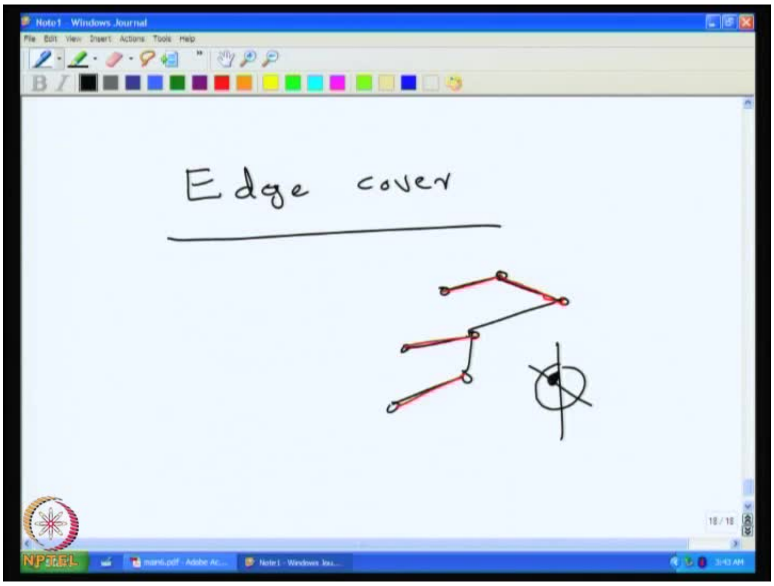

So, now, see, so it is a natural question to ask, so after this vertex cover, is there a notion of an edge cover? So, let us see, what is an edge cover? Edge cover means, like the vertex cover I am interested in covering all the vertices using edges, I want to select a few edges, such that all the vertices are covered. So, for instance, if this is the, if this is the graph, you can select, say this edge, this edge, this edge and now this vertex is not covered, you may have to select one more, so this is an edge cover. You can see if the graph has a perfect matching, so that is an edge cover the perfect matching, covers all the vertices, matches all, I mean, a perfect matching touches all the vertices, therefore it is an edge cover.

So, now, you can see, that, so that much is necessary also because n by 2 edges are there, in that case in the edge cover. So, of case, you cannot do better than that because any edge can cover only 2 vertices. So, if you want to cover an n vertex graph, you need n by 2 edges, so our question is, again to minimize the number of edges, so before that, let us ask, is it valid parameter for every, every graph? Is it possible, that every graph has an edge cover? So, of case, it is not possible for some graphs. For instance, if the graph is an isolated vertex, so this vertex cannot be covered using edges because there is no edge on it. So, therefore, we, from now on we can, whenever we talk of edge cover, we can assume, that there are no isolated vertices in the graph. So, in that sense, an edge cover will exist whenever there is no isolated, vertex, vertices.

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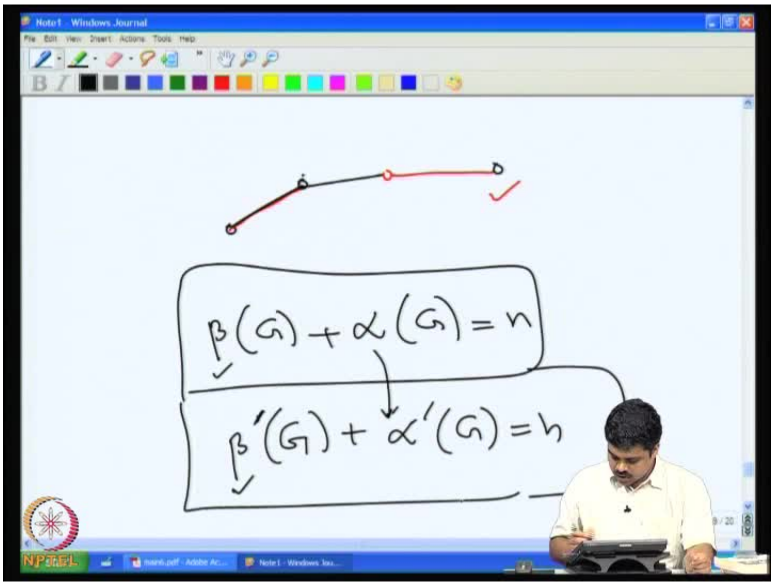

Now, we can, we can talk about the minimum edge cover also. I will not give too many examples here because… So, anyway, it is very much like the vertex cover, like, so now you can, so for instance, you can see, that in this case, so this is an edge, this is an edge cover, the perfect matching and this is an edge cover. So, now, let us say, there is an interesting statement, that we studied about the connecting vertex cover and the biggest independence set, so what was the, that we studied, that beta of G plus alpha of G is equal to the number of vertices in the graph n. Similarly, we do have a relation between the edge cover and minimum edge cover, so minimum edge cover, so can be, can be denoted by beta of, beta dash of G because beta of G is the vertex cover. Let us call it beta dash of G plus alpha dash of G, this is the biggest matching, so see this matching. Remember, you have the edge corresponding notion in with respect to the edges, so that is why I am, we are using alpha dash. So, beta dash of G plus alpha dash also can be shown to be equal to n, so how can I show that?

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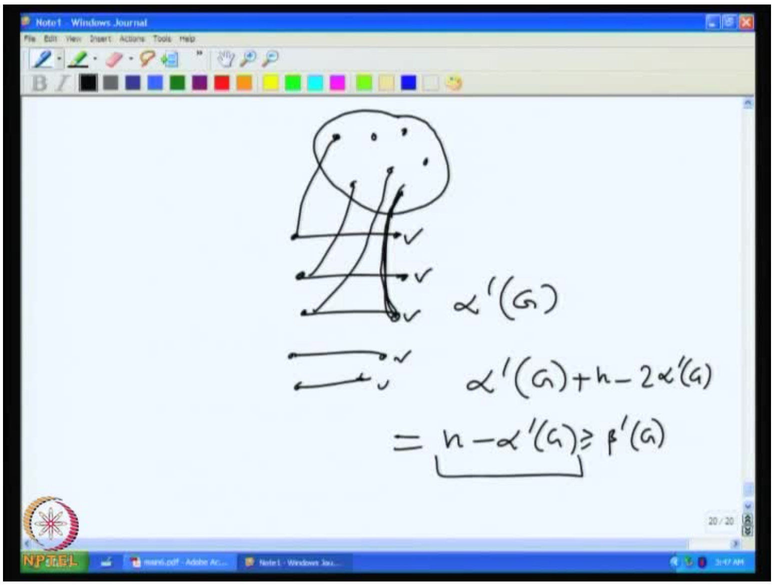

So, this is easy. So, one side at least a little bit, so for instance you can see, that suppose you get an edge cover, sorry, biggest matching in the graph. Now, this is alpha dash of G matching, then you can show, that from this thing you can get an independent…

By selecting these edges we have covered these many vertices. There are some vertices, which are not yet covered, so they, because there are no isolated vertices, they should have some edges connecting to this, see why? Because this is an independent set. Now, there cannot be any edge among them because this is already the biggest matching possible. If there was some edge we could have got a little more, then some edges will come from here to here, so to match each of them, I can pick up, pick up an edge connecting it to one of these things.

So, what do we get? So, we get alpha dash of G plus, what is left here, n minus 2 times alpha dash of G because 2 times alpha dash of G vertices are over, so this is equal to n minus alpha dash of G. So, we see, that this parameter is, this value is definitely an upper bound for our beta dash of G. So, this is, this only shows, that n is greater than or equal to beta dash of G plus alpha dash of G. So, what have I now shown? I have taken a biggest matching and showed, that using that biggest matching, how can I get an edge cover of the graph, so that its cardinality is n minus alpha dash of G. So, of case, our edge cover can be less than that, so we write this equation.

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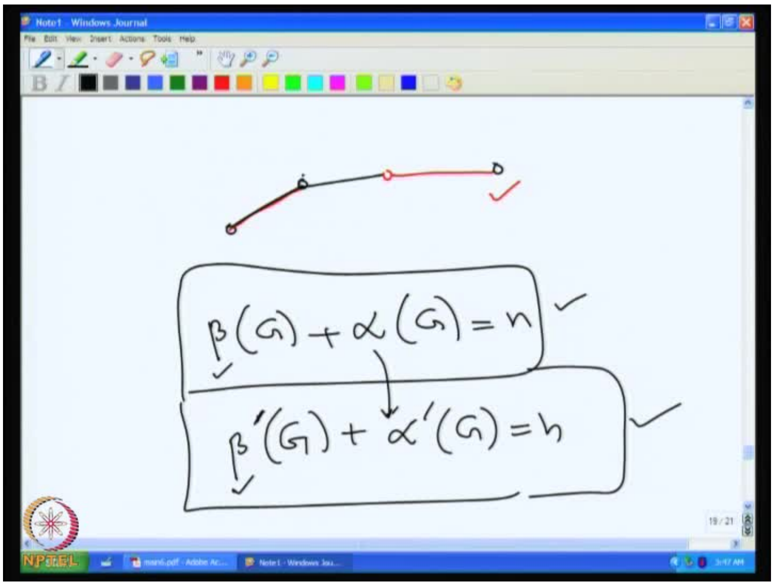

Now, we have to show the other side; that means, you take an edge cover and create a matching from that. So, we can take an edge cover and look at the subgraph formed by the edge cover and then, you take the biggest matching from that and then, you try to prove, that now the other side of the inequalities is also true. So, this is an easy exercise, I leave it to you. So, it so happens, that so we get, we like this inequality, like this inequality we can prove this inequality also. So, these 4 parameters are inter-related, somehow by this in this way and then, in the next lecture we will look at some related, very related concepts and again, some other covering problems. So, see you in the next lecture.