Graph Theory

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Lecture No. # 05

More on Tutte's theorem

Welcome to the fifth lecture of graph theory. In the last class, we talked about a necessary and sufficient condition for the existence of a perfect matching in a general graph not bipartite graph.

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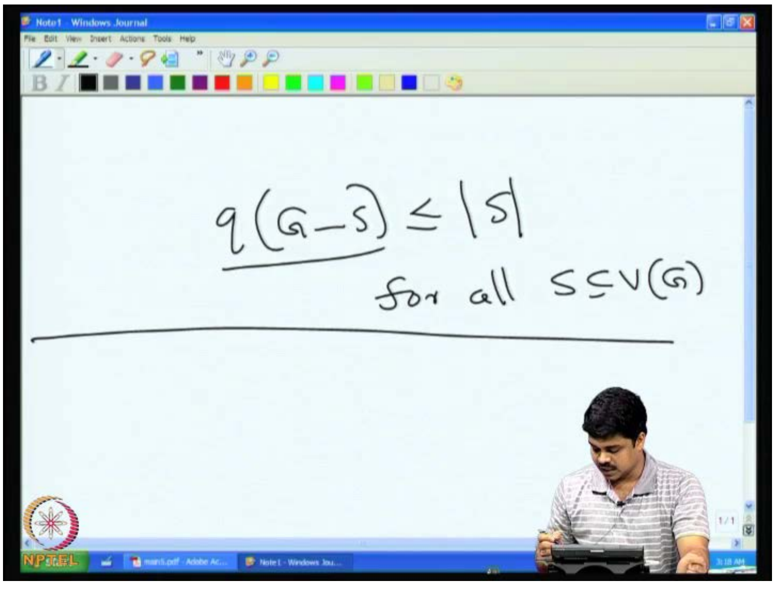

So, this condition was the tutee's condition which was express like this, q of G minus S has to be less than equal to cardinality of S for all subset of V of G. Because, if this was greater q of G S greater than S, then they will be this is the number of odd components, they will be at least one odd components in which one of the vertices will not get a partner from S.

So, therefore, we want have a perfect matching, and other way was more serious or it required a more sophisticated proof, that is to proof if there is no perfect matching in the graph then there is always a bad set; that means, there is a set S which violates this condition, that means there is a set S such that q of G minus S is greater than S, this proof we did in the last class. So, the major steps were to consider a graph G without a perfect matching. Then our intension was to show that there is a perfect matching in it, instead of showing it directly.

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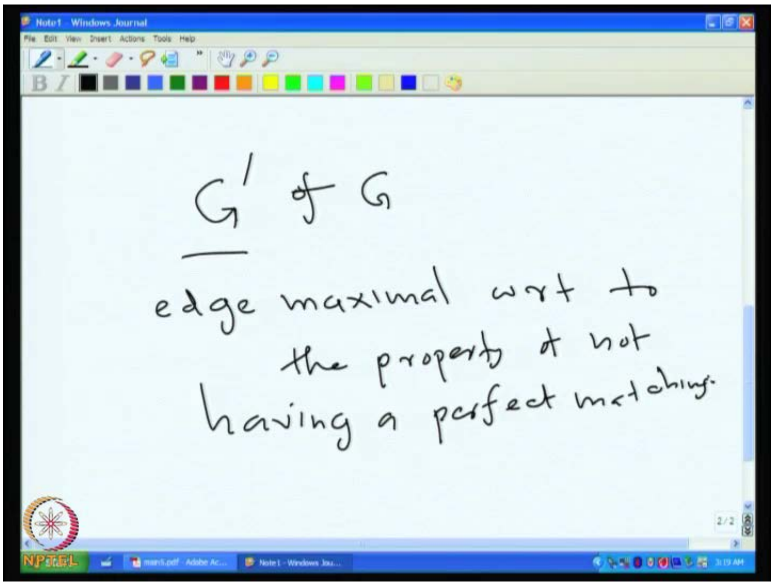

We first considered, it is considered a super graph G dash of G and this G dash was called edge maximal super graph of G with respect to the property of not having a perfect matching. In other words we added edges into G until it became edge maximum and then we argue that if you show a bad set in G dash that is enough the same bad set will be a bad set in G that was the first observation In, after that we observe that a bad set instead of looking for a bad set in G dash the edge maximal super graph of G.

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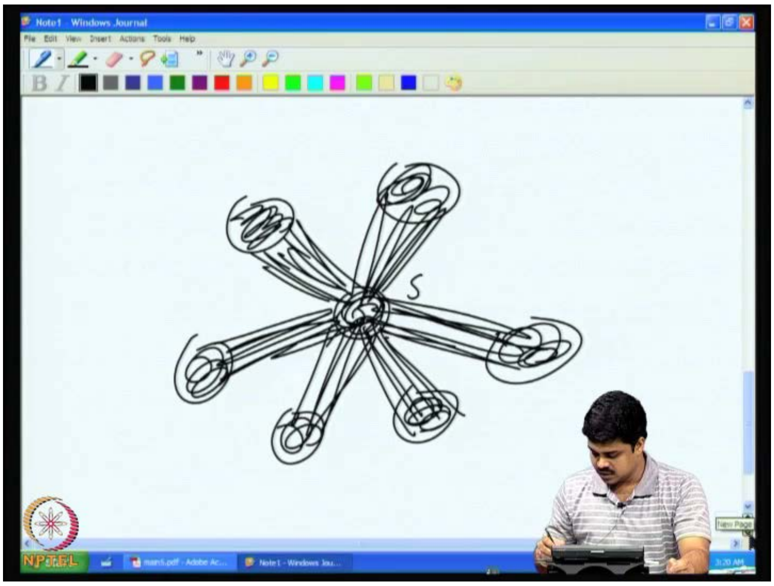

We can as well look for a special kind of set which was like this some s such that, when a consider the components of G minus S they are all complete graphs they are all complete graph and the connection between S and each component is complete like this and more than that each within S also it is complete, if you can find out this kind of a set S that will be a bad set this was the second argument, then after that we only had to search for such a set in the edge maximal graph G dash.

So, what did we do we did the most natural thing namely we collected all the universal vertices; that means, the vertices which are adjacent to all other vertices and called it s.

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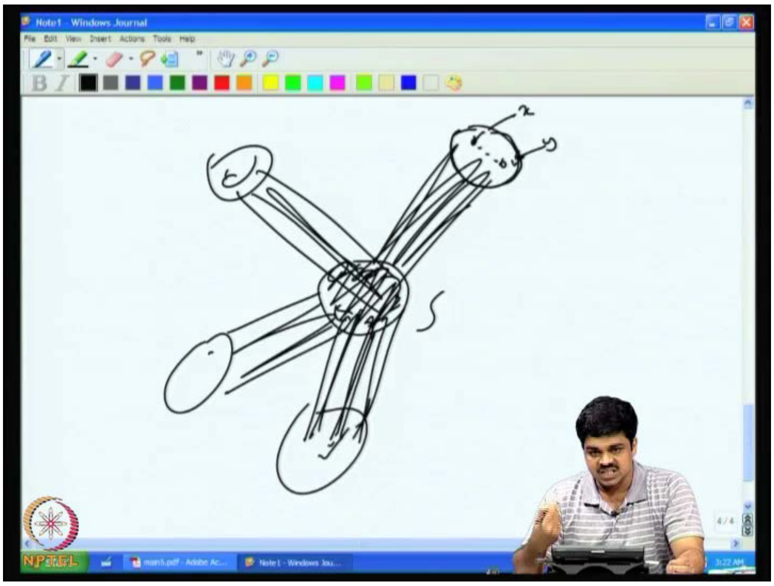

So, naturally this S was, then clique complete graph, because all the vertices inside that are universal vertices. And then, if you take any component; the connection between a component and this thing is complete, because they are these the things in S are now all universal vertices; that means, they have to connect everybody else in particular they should connect all the vertices here, all the vertices here, all the vertices here, all the vertices here right.

So, the only thing we had to verify was that if you considered any two vertices inside a component say some x and y then they will be an edge between them. So, we proved it is by contradiction; that means, we assume that suppose they are to vertices x and y with in a component such that there is no edge between them.

And then we came up with a with a contradiction by showing that in that case I can show a perfect matching in this is an edge maximal graph without a perfect matching then if you show that there is a perfect matching in it will be a contradiction. How did we show that?

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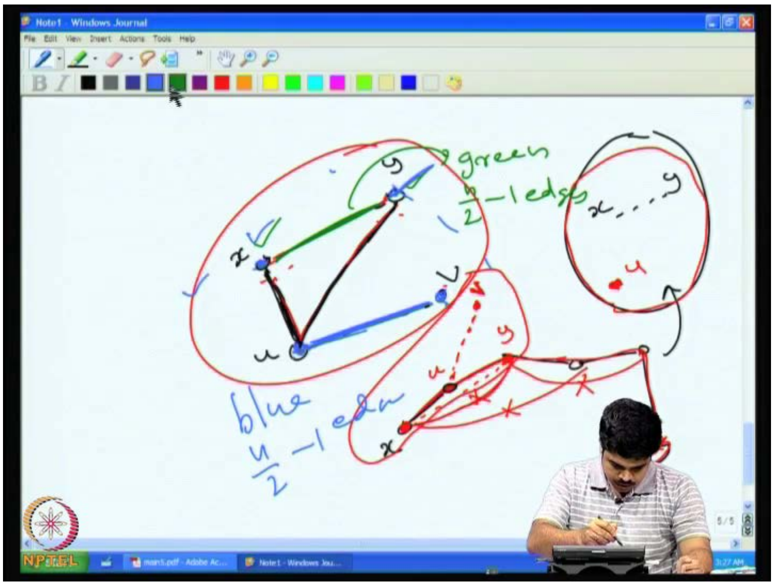

So, the first we somehow we proved that it possible to get two pairs of non adjacent vertices if there is one pair of non adjacent vertex vertices in in the component in a component right this was x y selected then, we can also find another pair u v in the graph not necessarily in the component now, such that they are also non adjacent not only that this u is connected to x as well as y in the graph, this kind of a structure is we can we can get this is what we showed right.

So, this was showed because, this x y is there. So, we considered the shortest path between x and y in this component. In this component we consider short this should be a path from x to y because this is a connected component and then if there is a path, there is always a shortest path the good thing about shortest path is that there are no edges of this sort or this sort or this sort, this kind of edges will not be there because, then it found a shortest path you can always travel like this. Right. So, this would be a shorter path. So, therefore, that edges is not present. So, therefore, you can see if you take x as this and i as this. So, though initially this this were the two this thing I rename this as y.

So, x as this and y as this and then it is very clear that they are going to be non adjacent, this edge is not present and again the other non adjacent pair u v was constructed by considering this as u, the good thing of considering this as u is that there is this two edges connecting this u to this x and u to y, S we wanted here S we wanted here right.

So, and the final thing was just to get that vertex v which not edges into u how was this guaranteed because, u is in this somewhere here in this component right and therefore, it is not in S that is u is not universal vertex, that means if u was the universal vertex then it would have been selected to be to go to be added in S we collected all the universal vertices and put in S therefore, any vertex you are getting from a component will not be universal vertex.

If it is not a universal vertex there should be another vertex in the graph to which it is not connected let let us call it as v. So, that is u v is not it. So, now this structure, these four vertices together has this decide structure. So, u is connected to both x and y, this x y is non adjacent, u v is non adjacent now look so, the ah idea was to considered the perfect matching that sorry then here perfect matching say the almost perfect matching which which just misses x and y; that means, so, it it was presented like if you add this edge to the graph. So, for instance if you add this edge to the graph you will get a green matching of G with exactly n by 2 minus 1 edges in it such that x and y are the only unmatched vertices in it.

Similarly, if you add the the the this edge; that means, u v then you will get a say blue perfect matching sorry blue matching with n by 2 minus 1 edges in it which leaves only u and v unmatched. These two matching's we considered to create an augmenting path for instance we we we were we were we got an augmenting path starting from x and reaching y. So, what do we do, we started from x via a blue edge right blue is there and then we followed a green edge and then we followed blue edge like that it travelled and it could only if it ends in y, it is fine we already got an augmenting path.

If it ah if it is an augmenting path for the green matching if it ends in sorry I just got confused 1 minute. So, so we we we started we try to get an augmenting path starting from V instead of and try to augment the path. So, I will take out to the next page.

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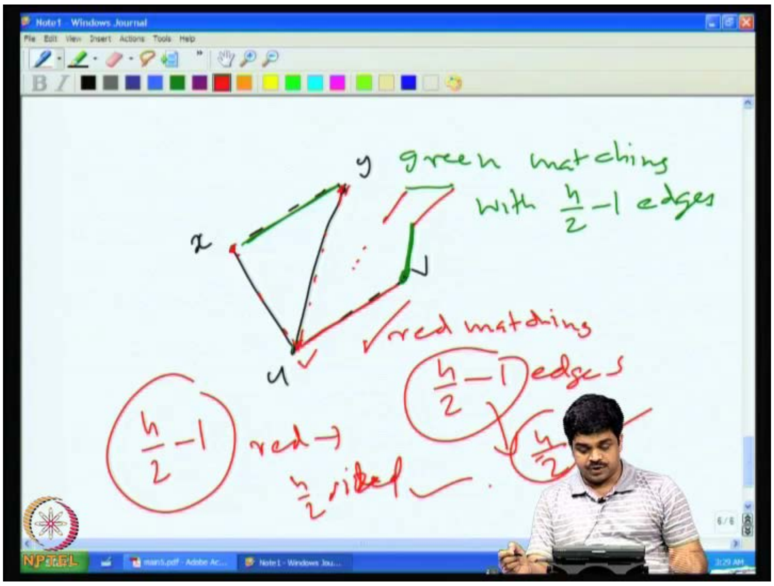

I am repeating, so this is the construction x y and u v these are the two edges. So, I because here I considered say let say I consider this if I put this edge I will get a red matching with n by 2 edges, n by 2 minus 1 edges. And then if I put this green if this edge are all be getting green matching with n by 2 minus 1 edges. So, I see that with respect to the red matching this v and u are the only vertices which are unmatched, but then there is a green edge going out of this.

So, then we can follow the green red, green, red like that. So, it will keep going and then where can it end? If it ends in u well and good because, it will be an augmenting path of the red matching. So, this n by 2 will become n by n by 2 minus 1 will become n by 2 and we will get a perfect matching it is a contradiction.

On the other hand if you reach y or x then you can follow this this edge and come here or this edge and come here depending on whether it is whether it reaches here or here. And then that will be an augmenting path of the red matching once again so, this will also allow us to the augment the n by 2 minus 1 sized red matching to an n by 2 sized sized red matching. So, which will be a perfect matching and thus a contradiction.

Because our by according to our assumption the edge maximal graph does not have a perfect matching this is the, this is idea then now to repeat what we have done is to prove an necessary and sufficient condition for the existence of a perfect matching in a general graph and, but this theorem can be proved in a slightly different way which will be which will be giving more insight and to prove that, we will prove the following statement. So, I will.

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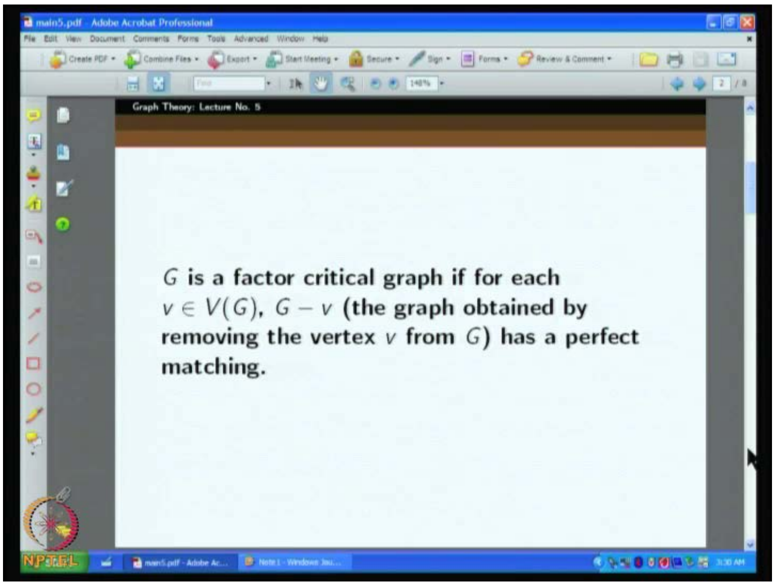

So, we need some definitions. So, first consider a graph G, suppose if you remove a vertex from G and it. So, happens that there is a perfect matching in it and now you can take another vertex from g and removes a vertex from that even then if you get a perfect matching and for if you remove any vertex from G, if you end up getting a perfect matching then such a graph G is called a factor critical graph; that means, for new vertex V G minus V should have a perfect matching.

Such a graph is called a factor critical graph, it is it is clear that if the graph is factor critical then the cardinality of it has to be odd because, you remove a vertex and you get a perfect matching if you if there is a perfect matching it has to be even because, you have remove a vertex already the original graph has to have a number vertices odd right.

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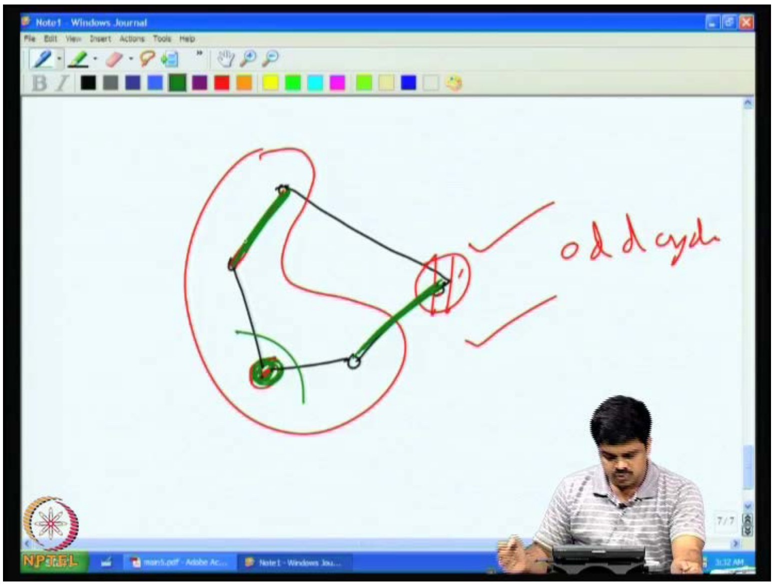

Now, let us look at some examples of factor critical graphs. So, for instance if you can consider this one say this is an odd cycle 1 2 3 4 5, this is an odd cycle. So, odd cycles are factor critical because, you can try removing this vertex say remove this vertex what will happen. So, you end up getting this there is a perfect matching here 1 2 right. So, odd cycles are factor critical.

So, not that I removed one vertex you should try removing another vertex, for instance this vertex if you remove then also there will be a so, I will do with green now suppose if I remove this one then what will happen is there is there is a perfect matching like this in the remaining graph.

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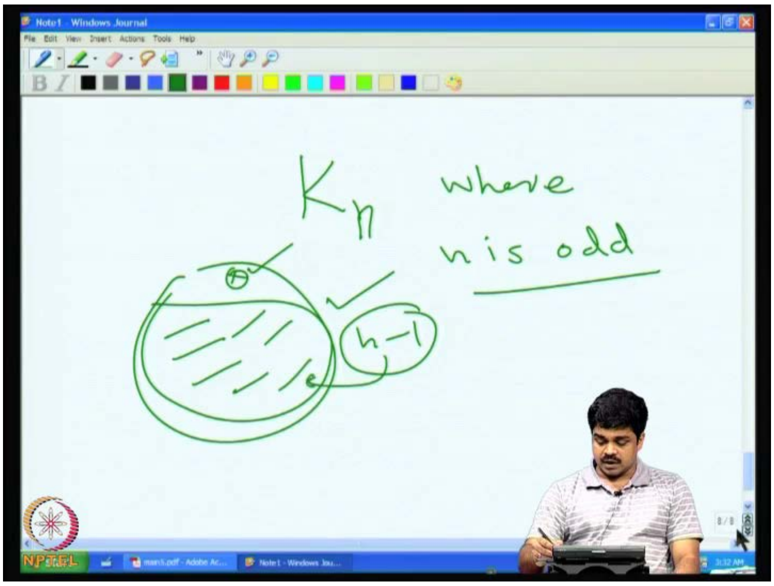

So, if you remove any vertex you see that in the remaining graph there will be a perfect matching, a similar example is a complete graph for instance you consider a K n where n is odd. Now, what will happen? I you can remove any vertex from a complete graph and then see any vertex it does not matter which vertex is?

And then the remaining graph has a perfect matching, because this n minus 1 now the n minus 1 vertices are there it is an even number you can pair up the vertices because, it is a complete graph. So, there will be a perfect matching in k n m, so K n minus 1. So, therefore, K n is also factor critical graph n is odd.

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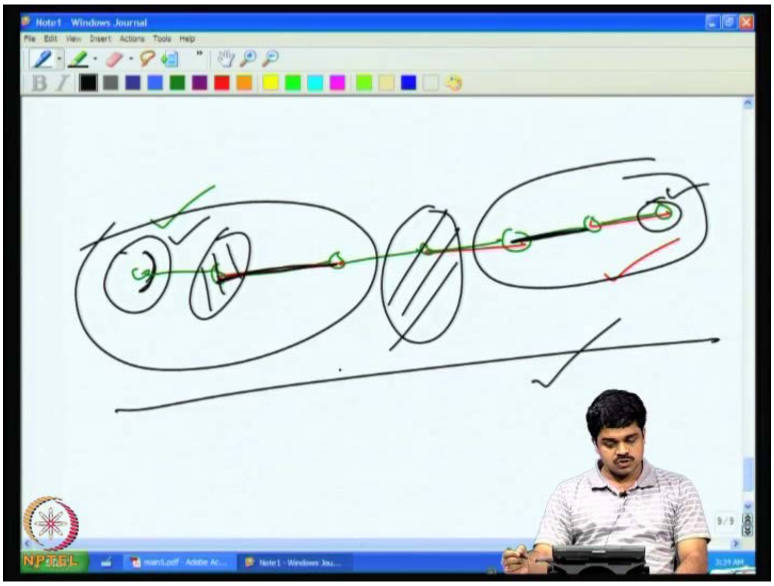

So, let us take another example what about so, a path so, of course it has to be odd or otherwise there is no point there is factor critical. So, here is a 7 node path 1,2,3,4,5,6,7, seven node path, now what about this is it a factor critical graph? So, for instance if I remove this vertex then of course you will get a perfect matching here here is a perfect matching. So, is this a factor critical graph? No because, suppose instead of removing this instead of removing this I removed say this vertex. Suppose I remove the this vertex suppose I remove this vertex then what will happen what I see is this on one side this graph and other side this graph. So, so you do not get. So, you can probably match this one. So, then one vertex will not getting a partner here.

If I match this thing and then this will not getting a partner here you do not see any perfect matching in this graph right in these two, three node graph. So, therefore, this is notified. So, you can try any other vertex what in sense if you had removed this vertex then also you will not get a perfect matching because, this k will not get any partner. So, that way this is an example of a graph just not factor critical all, but number of vertices all, but not factor critical.

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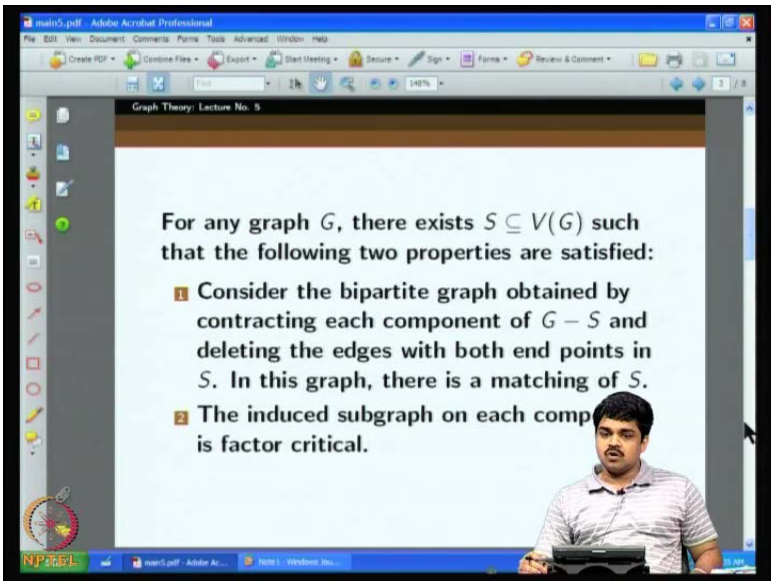

So, this is the definition of the factor critical graph. Now let us see so, the statement says give me any graph G then I will show you some subset S in it such that the following two properties are satisfied again the first condition is somewhat complicated, but the other condition I will explain where is second condition is much easier to explain; that means, this S will be such that if you remove S from G the components of G minus S each component will be factor critical naturally if or it to be have factor critical they whole have to be odd right each component has to be odd not only that they will be of odd cardinality, but also they will be factor critical so, but the first properties says, so suppose let me let me explain what the first property says the first property says.

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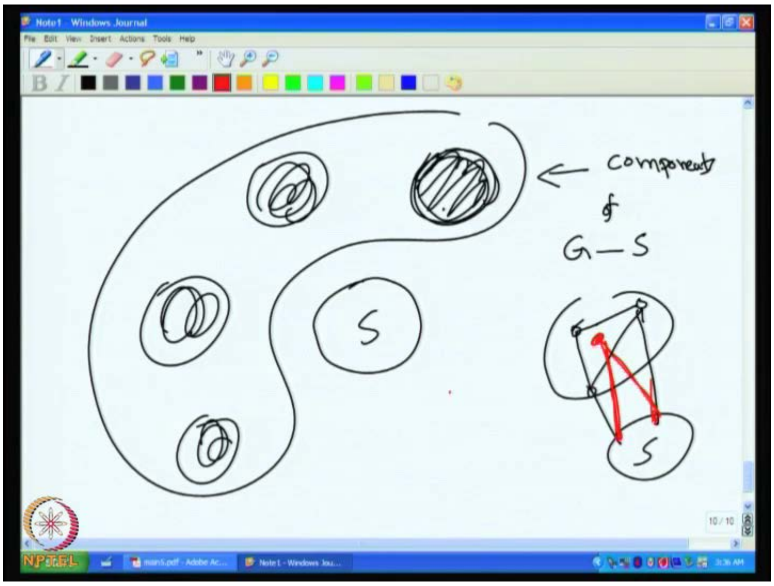

So, that I will be able to show a set S such that if you consider this let this be the components of G minus S these are the components of G minus S. Now, see think of a bipartite graph which is obtained by contracting this is the entire set in to one vertex, this contracting this into one vertex, contracting this into one vertex. So, you will what do what do you mean by contracting it into one vertex for instance if you if you get this kind of a structure.

So, if this is s so for instance this component was like this and there was this connection. So, what you do is you replace this with a just one vertex say you can replace it with one vertex and the connection will be like this so because, this is connected to this. So, this vertex also will be connected this, connected to this, this connected to this.

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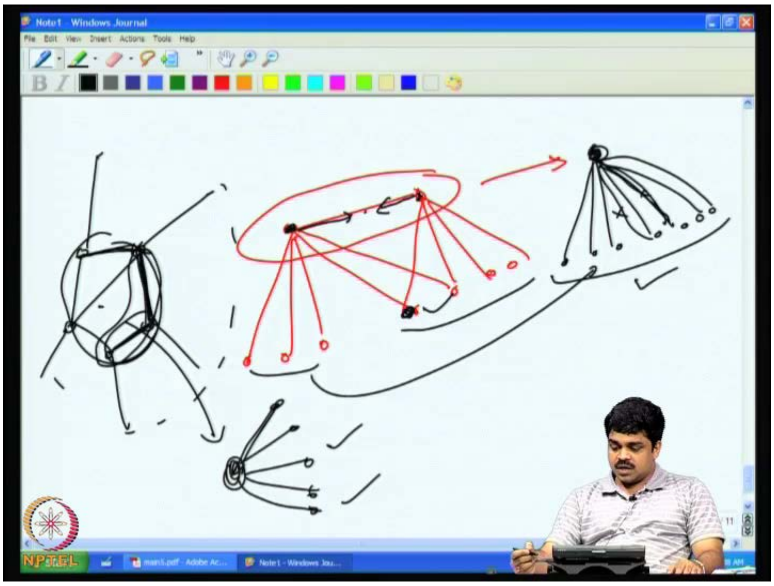

So, in general the contraction operation means if you have an edge like this and then these are the neighbors of it. So, these are the neighbors of this vertex, these are the neighbors of this vertex say some common neighbors also and then if decide to contract this then what will happen is I will I will merge these two vertices into one; that means, I will get a new vertex and all these vertices 1,2,3. So, all these vertices this total seven vertices will be its neighbors.

So, may say that there are some double edges here because, originally this and this both are connected to this vertex. So, therefore, there should be two edges, but then you can always discard it if you are not interested in multi edges. Sometimes when you are interested in multi edges you may have to keep it, but if you are only talking about simple graphs you can. So, this is the contraction operation.

So, when I say you contract a for instance we can always contract a connected induce sub graph into one vertex by one by one contracting it is edges for instance contract this edge first and then you will together with this, new vertex you will contract this and then together with this newly formed vertex will contract this.

So, finally, this entire thing will convert to one vertex and all the neighbors of neighbors it has outside it; that means, any vertex outside this induced graph with at least one edge going into the induced graph will be its neighbors. Multi edges, multiple edges will be removed if we are not since we are not interested in multi graphs as (( )). So, this is the contraction operation and then now I am coming back to the, this thing.

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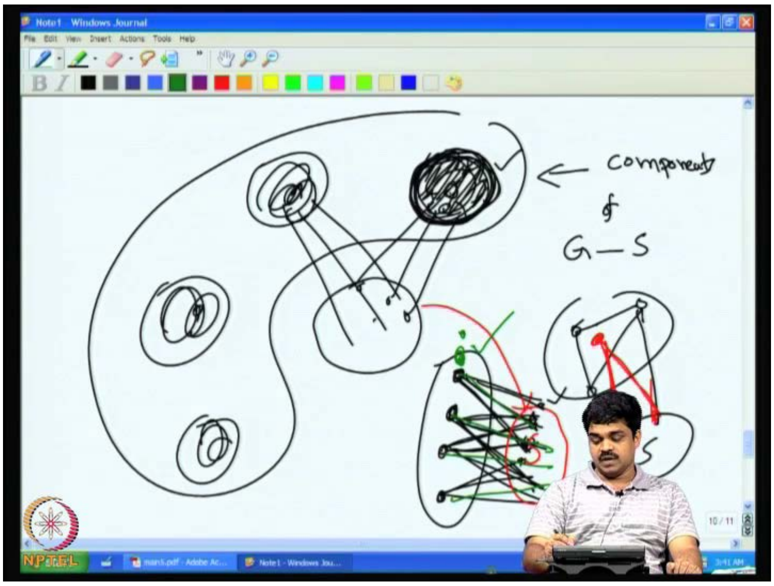

So, here I am asking you because, this is the components and they connected if you look at the induced sub graph on this components you will see a connected graph and I contract all the edges here in to form one vertex. So, I can form, so, this is contract and I keep it here and then this is contracted into a one 1 vertex I keep it here, this is also a contracted and then this is and within S there can be some edges for the time being what I do is I will remove this edges so; that means, I will I will I will I will empty those edges and then then make it empty graph.

And sorry empty, within this it will be empty. So, so here we will we will have S like this is S, this is S and then now the connection between this and this will be as before like for instance all the vertices to which this component is connected we will keep the edges to those corresponding what is s here?

And then the all the vertices to which this is connected we will keep the connections here so, that in so we will get a bipartite graph. So, what I have done is each component is shrunk into one contracted into one vertex. And inside as if you had some edges we just discarded them, removed them and we retained only the connections across. So, if some edges coming from this component to a vertex here and that edge will be now coming from this contracted vertex in to that vertex.

So, this graph is a bipartite graph now when we form this graph from the given original graph and this subset S then the statements says you will have always a matching of s; that means, you will be able to get a matching such that all the vertices of it is matched. Something like this you will able to find a partner for each vertex of S here they can be extra vertices on this side, that every vertex on S side will get a partner such a matching will be available again we will look back.

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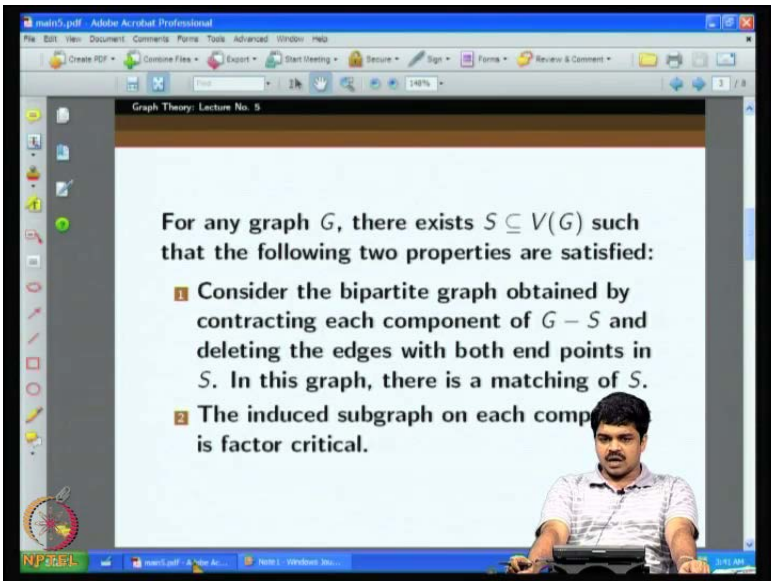

So the, these are the two conditions. So, give me any graph G then you can always get some subset S of V of G such that two conditions are satisfied, one is if you create the bipartite graph from by shrinking the or contracting the components of G minus S into one vertex and may be deleting the edges inside the S.

Then you will always get a matching of S; that means, you will get a matching such that each vertex of S is S has a partner there can be extra vertices on the G minus S the other side right. And and if it second condition is if you look at the induced sub graph on each components it is going to be a factor critical graph is going to be factor critical graph it goes without saying that the number of vertices in each component now has to be odd.

So, we earlier we had considered even components also, but now we say that if we consider is this special set S you will always get each component odd and factor critical right apart from that matching property.

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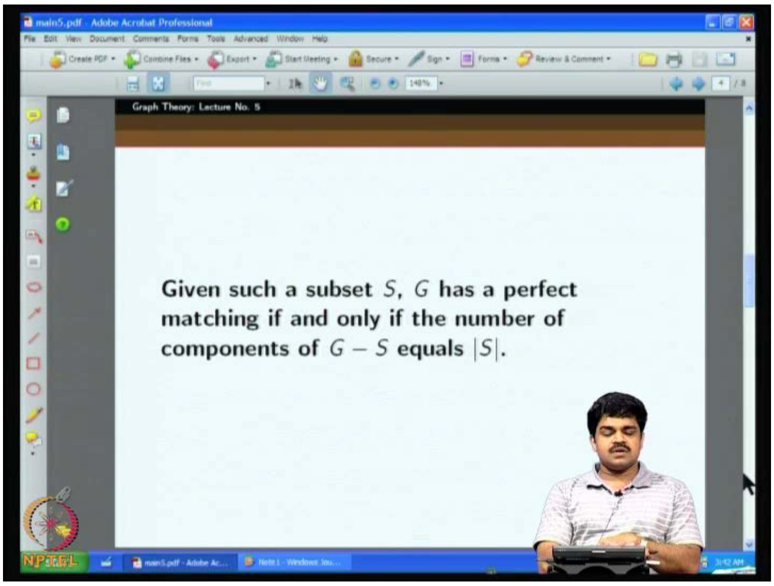

So, such a set excess and then says not only that it excess. So, the test for with respect to such a set S testing for the existence of a perfect matching would be straight forward what will we do? So, we just have to check whether the number of the components of G minus S is equal to the cardinality of S or not.

If the number of components of G minus S is equal to the cardinality of S then the there will be a perfect matching on the other hand if the number of components of G minus S is not equal to the cardinality of S.

Then we are not going to have a perfect matching in other words the existence of a perfect matching is equivalent to whether testing, whether that cardinality of S is equal to the number of components of G minus S.

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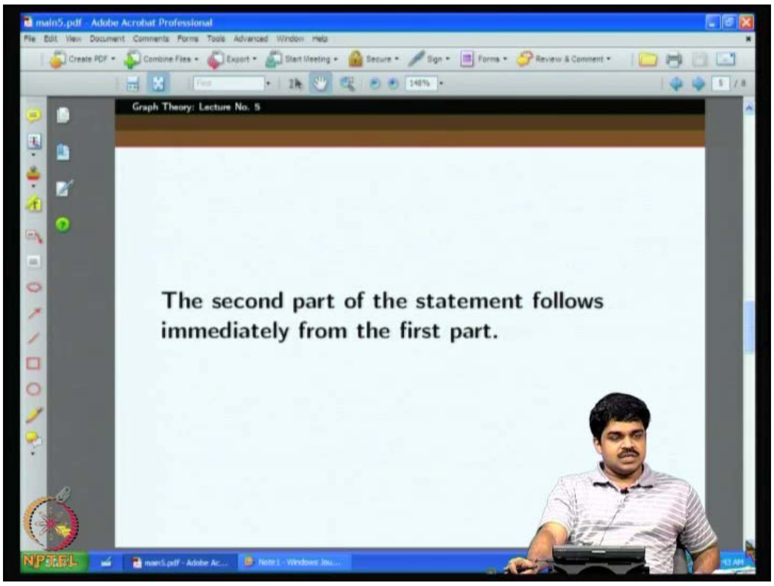

See the first thing we should look at the second part; that means, checking for perfect matching is equal to checking whether the cardinality of the number S is equal to the number of components of G minus S is very easy consequence of the conditions that the set S itself satisfies.

For instance if you had got a set S which satisfies the two conditions listed here and then it is easy to verify that the second condition is true for instance second statement is true; that means, about the perfect matching. Why is it? So, we know that if there is some matching of S in the bipartite graph. So, definitely the cardinality of S has to be less than the less than equal to number of components right.

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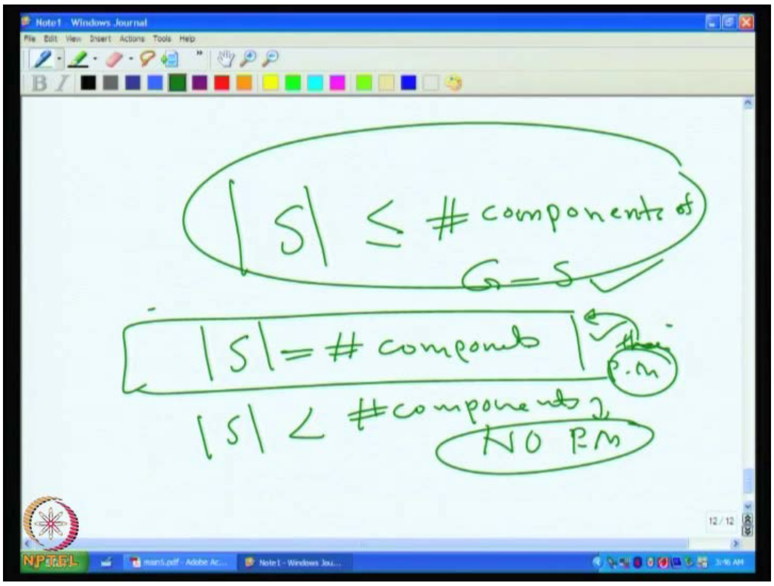

So, the we can what we can get from that is the cardinality of cardinality of S has to be always less than equal to the number of components of G minus S is (( )). Why because the because we we know that in the bipartite graph we form by contracting each component of G minus S into one vertex and taking S on the other side, the S is match able; that means, S gets a partner for each vertex of s get a partner from the other side. So, there should be at least as many vertices as there are in S on the other side, right so, the number of components of G minus S to be greater than equal to S.

Now, this means if S equal to, it can be either it can be equal to the number of components or it can be strictly less than the number of components right. So, these are the only conditions if such a set is given if the set satisfies this property right. So, these are the only two condition now says this if this is true, then there is a perfect matching there is a perfect matching if this is not true ,then there is no perfect matching this what it says right if this is true there is no perfect matching, if this is true this is there is a perfect matching.

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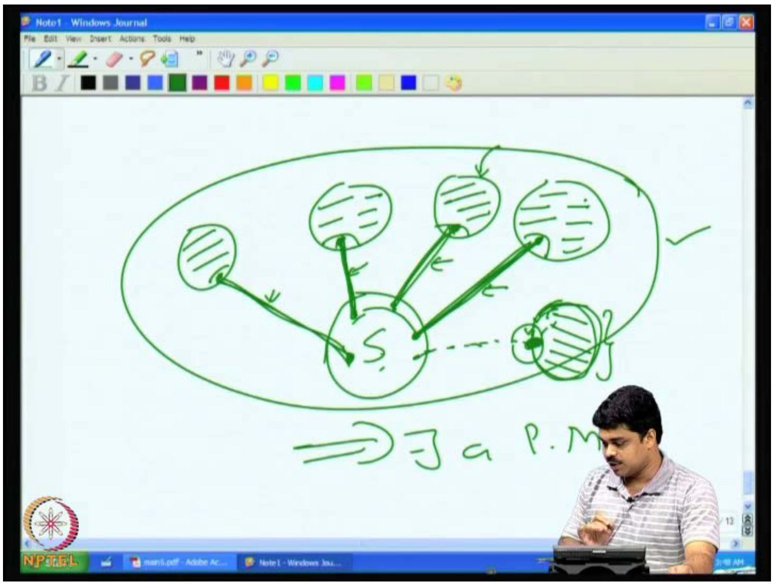

So, that if this is true, that there is a perfect matching is easy to see because, suppose if S is this and the number of components are equal to the cardinality of S what we can do is we can we know that also S you get a matching of S like this somehow every vertex of S get a partner on this side.

So, there will be one vertex which it is connected here that so, that is why I getting a partner right. So, so what I do is I take this matching edges first this matching edges and then. So, once I remove this vertex from the component because, it is factor critical you will be able to get a perfect matching inside this component similarly here also one side remove this thing because, this component itself is factor critical you will get a perfect matching here right you will get a perfect matching here like this.

So, together you can match every vertex. So, this there is nothing inside every vertex is got partners from outside. So, that together it gives me a perfect matching. So, F is a p.m. So, not this was possible because, we had a number of components are exactly equal to S suppose there was one more component then what will happen there is an odd component here which does not get a partner here because, all the vertices of S is finished.

Because one given to this, one given to this, one given to this, one given to this. Now there is one here which is not match able to S. So, it has to get its partner within this, but then this is an odd component cannot be perfectly matched. So, even if you after managing to get the biggest matching here you will still have one vertex left it will get will not have a partner. So, therefore, you will not have a perfect matching.

So, you can see that if it is so, happens that the number of components is strictly greater than the cardinality of S you cannot have a perfect matching. So, if you on the other hand if you have the cardinality of S equal to number of component (( )) as you see can get a perfect matching. So, therefore, that is a very simple consequence of the structure of that set S if you can get a set S like that in any graph G then this statement will be true; that means, the cardinality of S equal to the number of components will be the condition for the existence of a perfect matching.

So, now the only thing we have to really prove is a that such a set S is but before considering the proof that see why this proves the tutee's theorem again because, so this looks like another condition for the existence of perfect matching. That means it seems to say that there is there exist some kind of a set and then you take that set you just consider the cardinality of it and compare it with the number of components in the corresponding G minus s.

So, if they are equal number there is a perfect matching otherwise there is no perfect matching why is it equal to saying that there is always a perfect matching if there is no bad certain in a graph that was what we did in the last class right that was what tutee's theorem said.

So, for instance if there is a bad set then there is a no perfect matching is a trivial thing. So, you do not need a any proof for that. So, we needed a proof for the other side; that means, if there is no perfect matching there is a bad set. So, now we claim that the bad set is this set S. So, if this set which satisfies these two conditions will be very much suitable to be taken as a bad set why because you know that, so, that this because if there is no perfect matching in the graph then the cardinality of this set has to be less than the number of components.

And as we know each component with respect to this set S is an odd component its factor critical itself, but it the it is an odd component therefore, the number of odd component is strictly greater than the cardinality of S therefore, it's a bad set. So, from the statement the statement here; that means, if such a set S of x is satisfying these two conditions then we can infer that the tutee's theorem is true; that means, if there is not perfect matching then we should be able to get a bad set and that bad set happens to be this particular set.

So, this seems to be a stronger statement. So, in other words in the tutee's theorem we just told that if there is no perfect matching there is a bad set now we are saying what kind of a set it is or in other word we are giving more information about the structure of that set.

How do we yeah. So, now, information such right now sorry. So, now, we want to prove this statement. So, the how do we prove that such a bad set excess to prove this thing?

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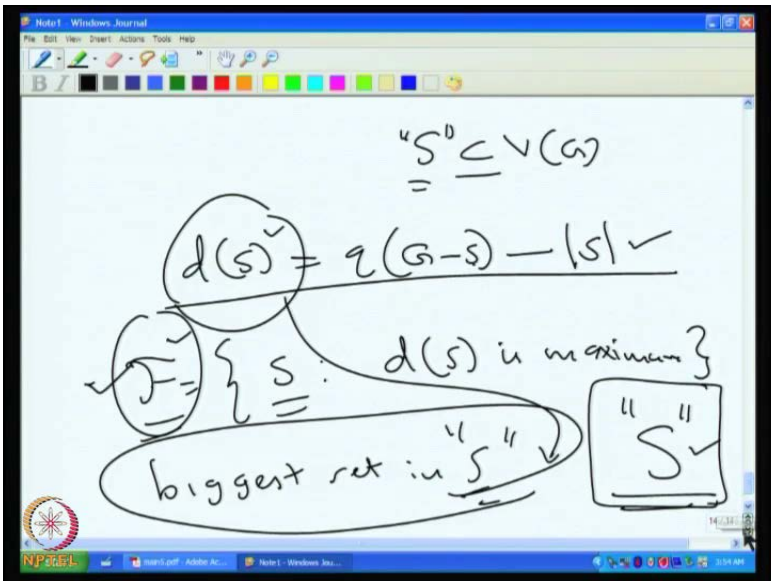

So, we do the following thing we first consider this difference for instance you take any set S state some set S. Now, let us say d of S is equal to q of G minus S minus cardinality of S this is essentially the excess. So, in case there is no perfect matching we know that q of G minus S for some set q of G minus S is going to be greater than S. So, there may be some positive excess here.

But it is it can be negative for instance q of G minus S may be less than the cardinality of S it can be 0 they can be equal right if there equal. So, we define for any S a parameter called d of s which is essentially the number of odd components of G minus S minus cardinality of S.

So, now we collect that F be the set such that d of S is maximum possible what do you mean by that so, for each S you get a value of d of S and you can see which has S the biggest d of S. But then it need not be a unique set there can be several sets with the maximum possible value of d of S among all the sets. So, let us say we collect those sets in f right that that set is called the set of set is called F.

Now, among all the sets in F you can pick up the biggest set in S set in F. So, this also need not be a unique set, but we can so, we can take one biggest set from F, first F is the set of S is subsets of V of G such that d of S is maximum and then we picked up one particular set such that it is it is cardinality is maximum among the among this set enough not among all the possible set in that case I will be seeking the entire set V of G not that.

So, among the sets which have come in f I will pick up the biggest set. So, let us called this so, this will be called a S from any now now on because, this I am going to prove that this particular set is that the kind of thing I am looking for this will have the special structure I am looking for; that means, the two properties will be satisfied by this set right. This is what I am going to do right.

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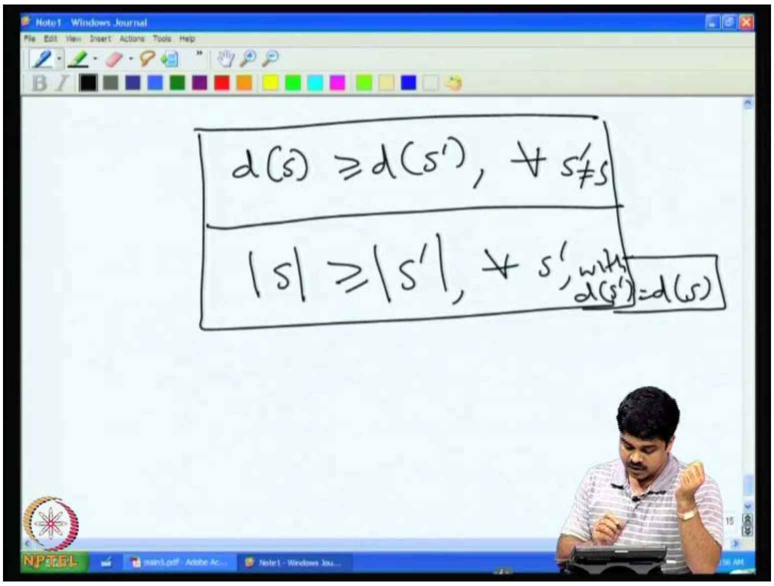

So, so I will I will for calling purposes this set S has this property. So, d of S is greater than or equal to d of S dash for any S. So, for if you consider any other S dash d of S is going to be greater than or equal to S dash, this is 1 property right for all not equal to S. Now, this was one this is one property and also among all the sets which have the same d of S this will be F the highest cardinality; that means, we have S is greater cardinality of S greater than equal to cardinality of S for all S dash with d of S dash equal to d of S. This is the two properties we have for this S so, we will we will we will come back to this later we will we will we will try to use that the set S is the kind of once we wanted looking at the looking at this property trying to contradict these two properties.

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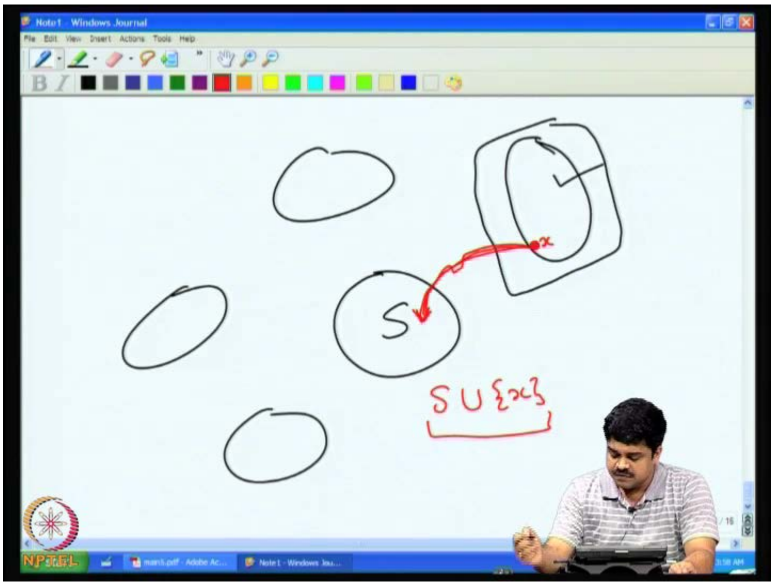

Now, what I am going to do is suppose so, I consider this set S and then I consider the components which it gives suppose these are the components the first thing because, what I want to show is that each of this component is factor critical to begin with then I will show the matching three properties I have to sorry two properties I have to show.

So, how do you will show that this is factor critical so, before showing that factor critical let me show that this the component cardinality the number of vertices in the component is odd; that means, what if it is an even right. So, for that let let us pick up one vertex. So, let so, let us mark a vertex here right now Consider moving these two here it is possible right. So, it let me call it x. So, in other words I am asking you to consider the set S union x.

Then the question is of case S union x is a bigger set. So, is it possible that when I remove S union x from the graph instead of S the number of odd components in the resulting graph is more than the number of odd components sorry number of odd components and the resulting graph is more than the number of odd components in this graph.

For instance, if it increases by one what will happen? So, that is exactly what we are going to tell so, I am in interested in this parameter what is q of G minus S say minus x? So, what is S and x I am so, or otherwise I can write it as S union x is removed from S right.

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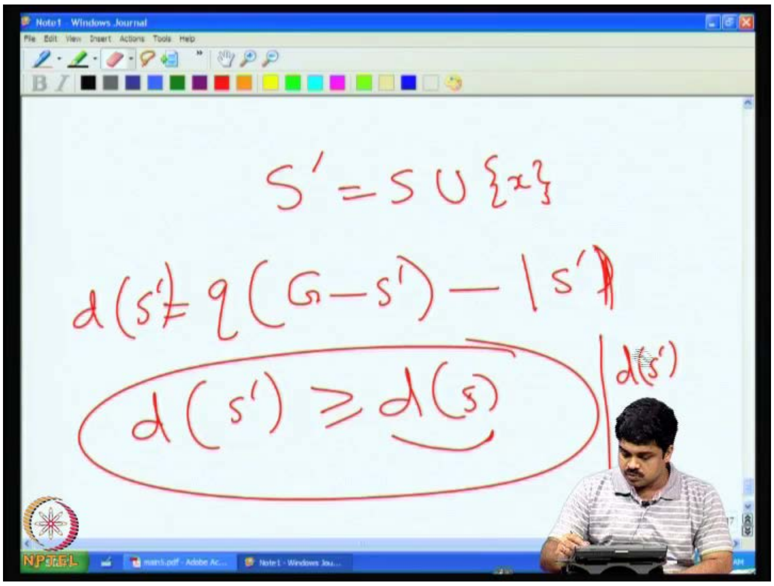

So, let me call it so, for convenience so, convenient I can call this S dash equal to S union x. So, know I am interested in this parameter d of S dash. So, sorry G minus S dash minus cardinality of S dash this is what I am interested. So, the question is can d of S dash be greater than or equal to d of S can it be bigger or equal. So, if it is bigger then we already have a contradiction because, in that case S would not have been in F right we would not have called d of S is than maximum possible d of S. So, S be a set with maximum value of d of S so, on the other hand, so, so which means that is so, the only possibility is d of S dash and the only possibility is d of d of S dash equal to d of S because if d of S is greater.

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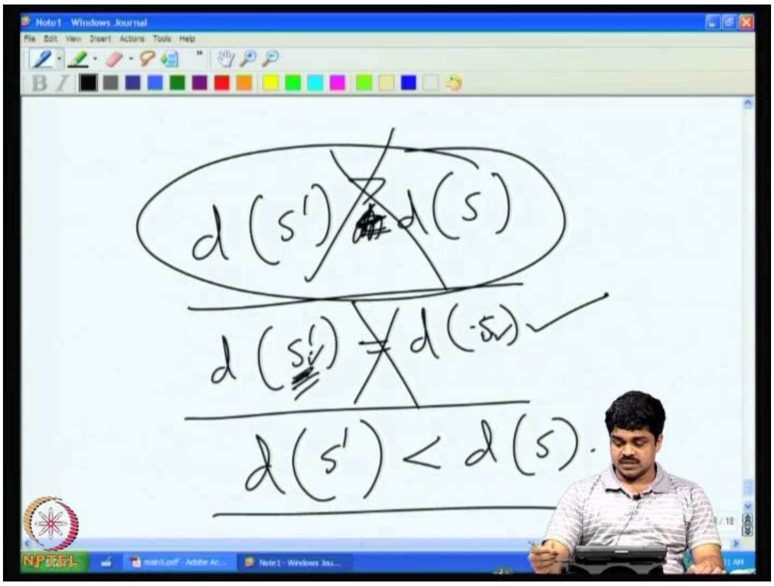

If d of S is greater then what will happen was that the number of we we we have selected S to F thinking that S is such that d of S is maximum now you are re saying that d of S greater than the d of S that is not possible. So, you can have this or less than equal to this that is possible of case these are the only two possibilities right.

Now, what will happen if d of S dash is equal to d of s that is also not at because S dash is a bigger set than S then I would have selected S dash instead of S remember from F the among the sets with the value of d of S maximum I have picked up the biggest set. So, I would have, but then S dash is bigger set than s. So, this would be a contradiction to our selection. So, this is also not possible. So, we need d of S dash to be strictly less than d of S that is the only possibility. right

So, we cannot have this thing because, the S dash bigger sets in d of S and then also this strictly greater is also not possible because in that case S would not be enough at all right. So, now let see so, if d of S dash is less than d of S then I claim that, so, that component has to be odd as we saw this earlier. So, here we considered selecting some vertex x and removing it to S right.

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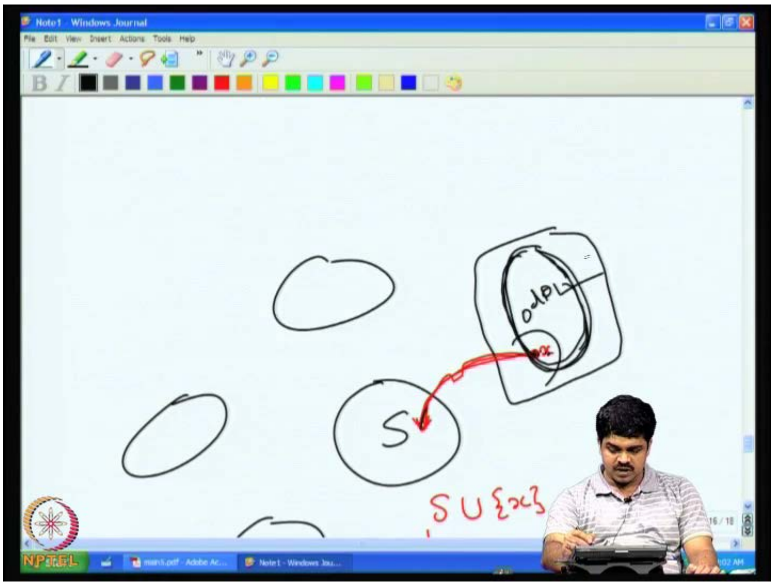

So, if this is an even component then what will happen even I move an x from this thing. So, here this total will become odd this will become odd when I cut this so, it is possible that when I remove this vertex x from here this make cut into several pieces right.

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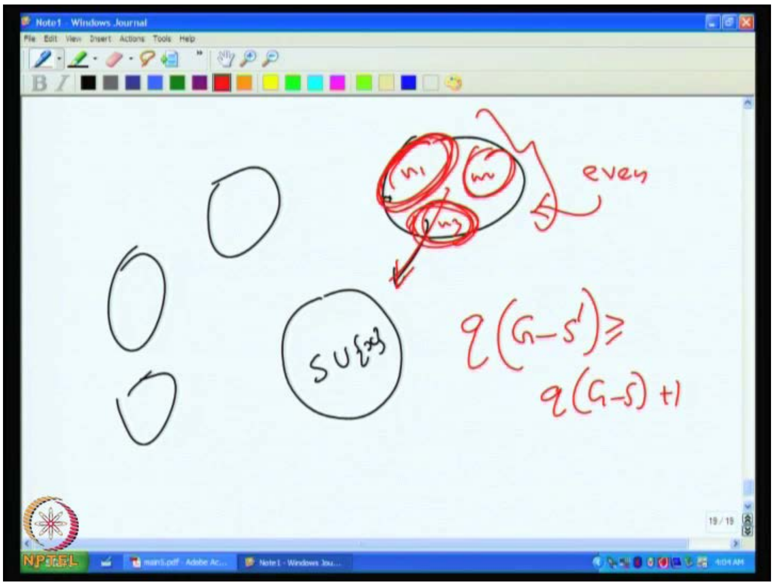

So, for instance it may happen that, so, you took S union x here. So, when I move so, all the other components will remain same, but this particular component from which x was taken. So, may cut into say it may cut into several pieces. But then together the number is odd if several this is n 1, this n 2, this is n 3 etcetera. If all of them add together to an odd number there should be at least one odd number in it right. So, this at least one odd component in the in the piece among this new components formed. So, this new components is formed because, we removed x from this thing if this be component got cut into several or divide into several new components at least one of them has to be odd because the total has to be odd.

So, therefore, what we can say is so, instead of initially there was an even component here either it got converted to one odd component or it got split into several components out of which one is odd. So, one more odd component got created. So, we can clearly say that q of G minus S dash is greater than or equal to q of G minus S plus 1, this much I can say right one more it created.

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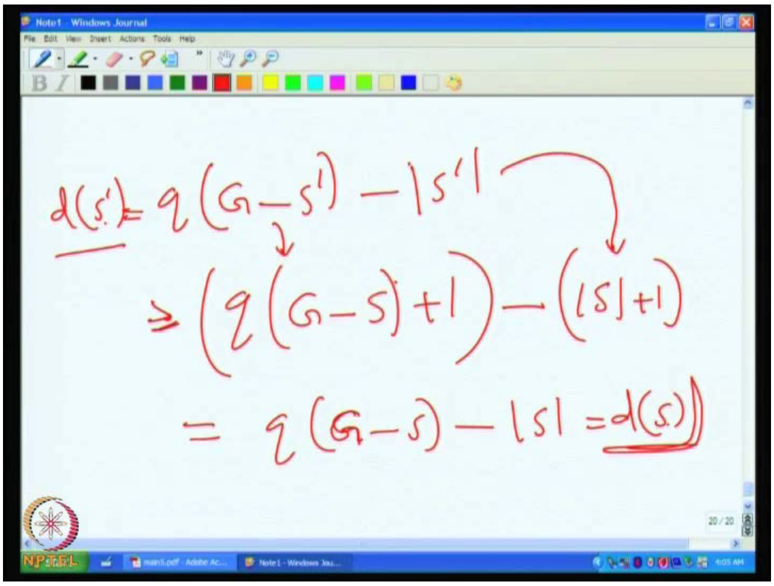

So, what about q of G minus S dash minus cardinality of S dash? So, that is essentially greater than or equal to G of its greater than or equal to q of G minus S plus one because, this part is bigger than this minus S plus 1 this is exactly S plus 1 right because you had an one more vertices this is q of G minus S minus cardinality of S this is d of S.

So, what we get is q of this happens to be d of S dash what we get is d of S dash is greater than or equal to d of S. So, in both cases we have seen contradiction it cannot be strictly greater because, in that case S will not be in F at all it will be a contradiction because we selected S from F.

And if there equal S dash is bigger set than S. So, that will be a contradiction to a selection strategy. So, therefore, that is also not possible. Now, the next thing is to consider next thing is to consider this fact that, so, we have already shown that each component is factor critical sorry is odd odd is odd say now we will show that each component is factor critical.

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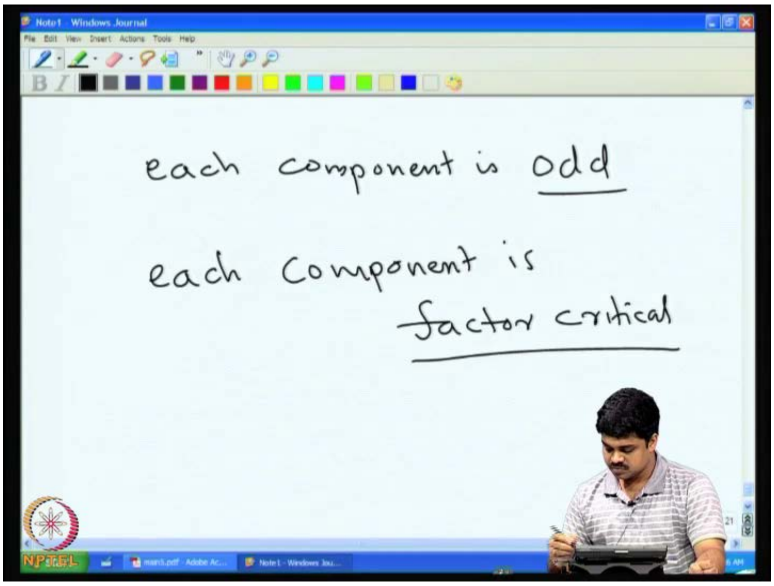

How do you show that it is factor critical? So, the argument is suppose it is like this similar argument suppose it is not factor critical.

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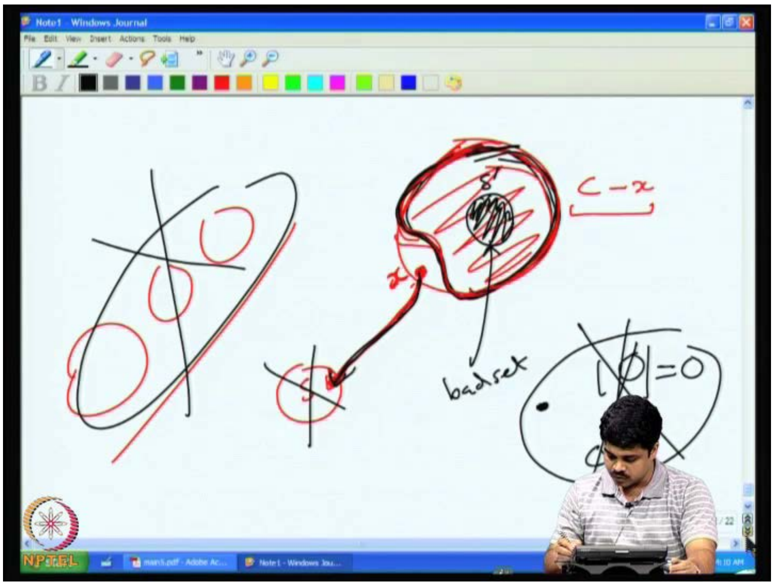

So, then it means that I can lose so, for instance this is my component, so, here is my s. So, there are some other components here I am just discarding them. So, here I am interested in this component. So, if the argument is true for any another component also. So, let us pick up a vertex and move into this right. So, now, this become like this. So, here we have this remaining part. So, I move any x it does not matter I will this is S, I will x is moved in to S first.

Now, what happens is this particular set x so, sorry this is not any x. So, if this is not factor critical there should be some x such that when I remove x from this, so, this becomes; that means, is components c minus x becomes a graph without a perfect matching there is no perfect matching in it right in this graph.

So, have to select that x in such a way that this becomes a graph the remaining becomes a graph without a perfect matching. So, you remove x and move it to S. Now, because there is no perfect matching in it I know that by tutee's theorem; that means, I am not because, I know that this condition is by induction for instance for any smaller graph I can assume that induction is true.

That means I can assume that in any smaller graph here theorem is true. So, therefore, I will say this graph the tutee's theorem is valid. Tutee's theorem is valid means there is there is no perfect matching then there is a bad set I can find the bad set in this graph. So, I should have told about the induction first because we proving it using an induction; that means, first we will prove then we have to prove check it first trivial cases, like when the number of vertices in the graph is just 0, that is easy because you can just take phi as a set if the set S what will happen is it is only 0 cardinality.

So, so, you can there is nothing to prove here and then. So, for a vertex one that is so, for a single vertex graphs also it is very easy to see you can take phi because, so, it is this is the one component and they are all that one component so, one component and just one vertex, it is definitely factor critical because if you remove that then there is an empty graph therefore, it is a factor critical graph and then trivially you can say that empty set is match able to this one vertex.

So, therefore, for small cases it is easy to verify therefore, I assume that all smaller graphs the theorem is true; that means, there exists is a subset S with the above properties right and then as I already mentioned if this statement is true by induction then definitely the tutee’s theorem is also true; that means, there is a bad set if there is no perfect matching in the graph.

So, this is a smaller graph why this is smaller graph. So, what if all the things are empty; that means there is nothing here even this is empty. So, even then this x is there that is x single one vertex is moved to this so, therefore, this is a smaller graph therefore, I can apply induction on that and there is a bad set here. So, is this is the bad set this is the bad set. So, bad set (( )).

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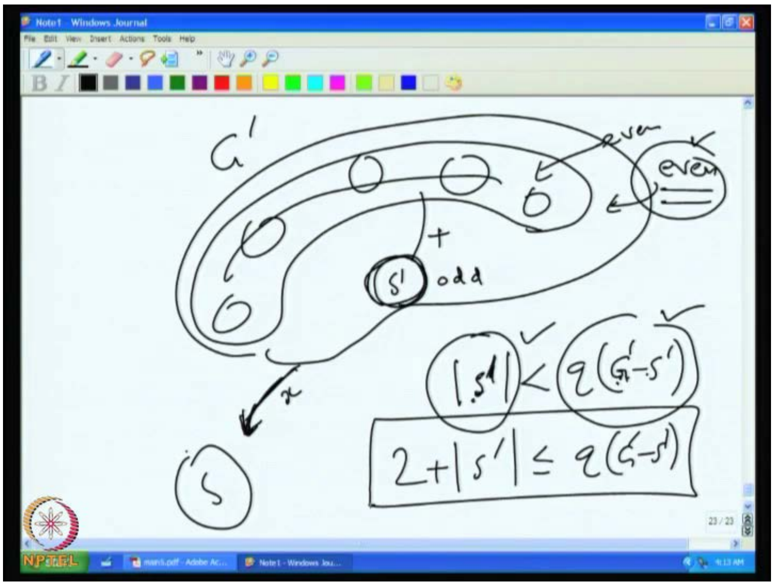

Now, let me call this S dash so, what is the situation for in I will draw that this is S dash this component alone I am drawing drawing this is S dash and then there are several components here. So, this is a situation, now, see remember this is a four component we show this x. So, we had original S here x was moved from here to here. So, this you remember because the originally this is an odd number as we already shown then when I moved x this became an even number even even number in total right.

So, the this is a bad set I know that S dash cardinality has to be the cardinality of S dash has to be strictly less than q of this say this graph this called G dash G dash minus S dash, but can it be just one more. So, if it is just one more then that the parity of this term and this term is different; that means, if it is odd this has to be an even if this is even this has to be odd if it is just one more, but that is not possible because if it consider the total number.

So, if this is odd and the number of odd components is even the total number of vertices here will be even because all the even components will contribute an even number, the odd components will pair up and if there are even number of odd components they will totally contribute an even number.

And even number added with an odd number will give you an odd only an odd, but we know that total is even. So, that will be a contradiction similarly if this was an odd number this S dash was an odd number, then the number of odd components has to be even because, if the number of odd components is odd then the total here will be a an odd number ah and then odd plus odd will be an even number. So, that is a problem. So, therefore, we we know the sorry so that, what I mean is? yeah.

So, the so, the parity of S dash and q of G dash minus s dash has to be different if this is S dash S odd then this has to be even if s dash is even then this has to be odd which essentially means that G q of G minus S dash cannot be just one more than S dash. So, it has to be at least one more than we we get this q of G dash minus s dash this is plus 2 right 2 more should get.

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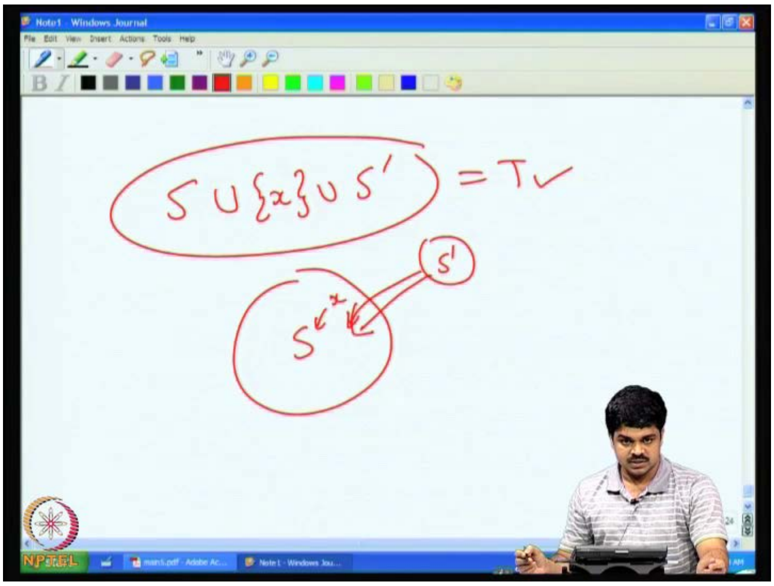

So, this is crucial because, what we are going to do is to consider moving the x. So, this was the original S, now move S state, and also S dash state , so that means, S is moved and also S dash is moved, also that means, I will consider S union x union S dash is the new set, let it be T. Now, I want to analyze the number of components with respect to T. So, this will be done in the next class the proof, we will finish in the next class.