Graph Theory
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Lecture No. # 04
Tutte's Theorem on Existence of a Perfect Matching

In the last class, we discussed about a necessary and sufficient condition for the existence of perfect matching in bipartite graphs. So, that condition was called the hall's condition and it was about it told the hall's condition was if every subset of the a side has neighborhood to cardinality of the neighborhood at least as much as the cardinality of set itself, then the hall's condition was satisfied. And then we told if this hall's condition is satisfied, then the reason bipartite matching in this graph and vice versa. For instance, if there is the bipartite matching in the graph then it is clear that this hall's condition will be satisfied.

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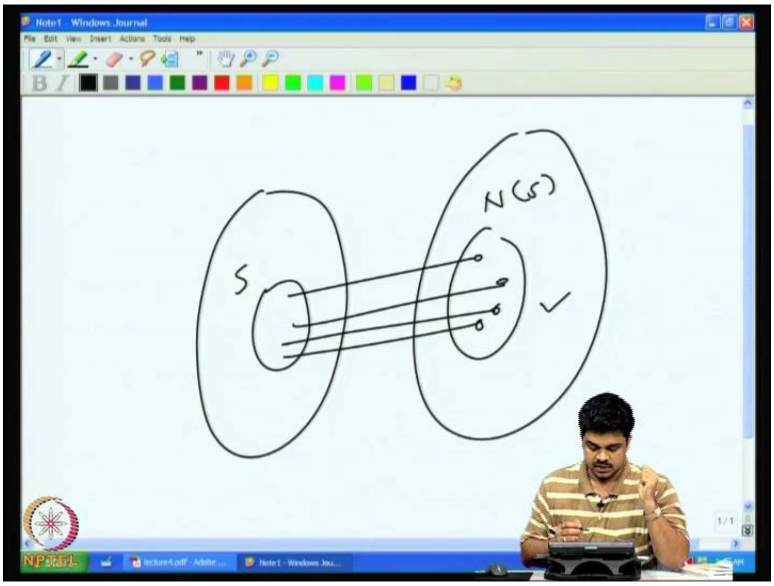

Because you take any subset then this matching will get neighbors for that for instance, if we had this subset this perfect matching will take each vertex of a distinguish neighbors, and together the neighborhood has to be at least as much as the neighborhood of sorry the cardinality of S, therefore, it is an necessary and sufficient conditions. In other words, if a perfect matching is there, in the graph in the graph in the bipartite graph then the hall's conditions will be satisfied. If the perfect matching is not there, if we want satisfied, in other words if it is satisfied there always be a perfect matching.

Now, we will look at general graph this was only about a bipartite graph. So, what about and necessary and sufficient condition for the existence of the perfect matching one factor - perfect matching has been already mentioned. And another word for perfect matching is one factor, we are interested in some necessary and sufficient condition for the existence of a one factor in a general graph general and directed graph.

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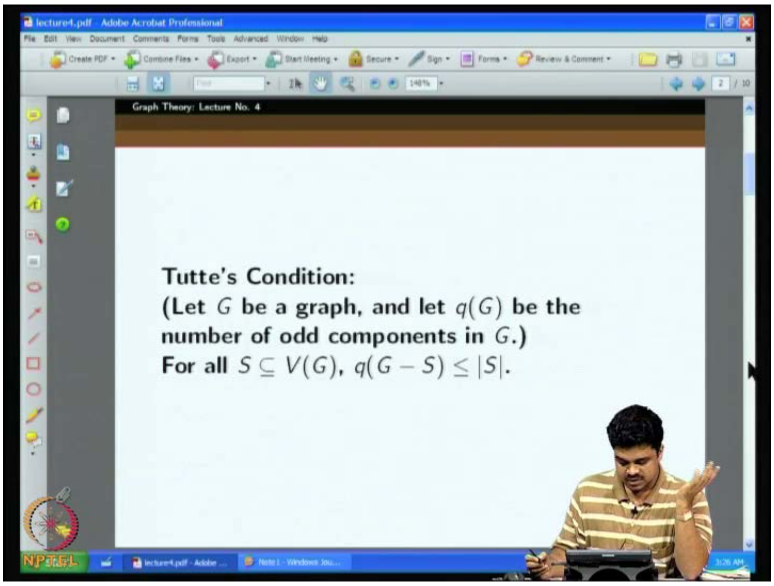

So, this famous condition called Tutte's condition is a necessary and sufficient condition for the atom. What is tutte's condition? Let G be a graph, it is a need not be bipartite graph, now as I told. Now we say q of G is the number of odd components in G, what you mean by odd components?

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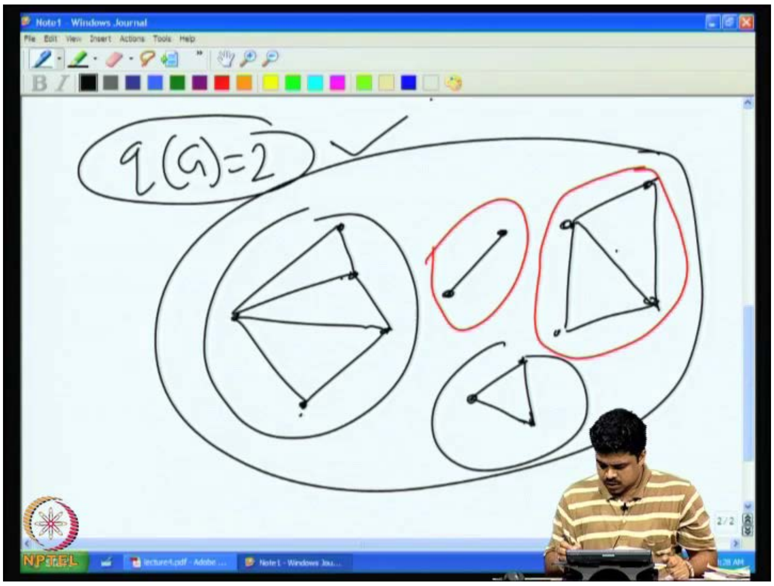

So, let us take a graph. So, the graph can be say you can consider this graph. So, you can consider this graph together; this together is a graph. So, now, this graph consists of 1, 2, 3, 4 connected components; some of the connected components, some connected components have even number of nodes in them; some connected components have odd number of nodes in them; for instance, this is 1, 2, 3, 4, 5. So, it is an odd component; this is an odd component; this one.

Why, because it has odd number of vertices in it? This is an even component. So, let say, I can mark it with another color; this is an even component; this an even component. What about this one? This also an even component, because it has four nodes on it even number of this; on the other hand this is an odd component again.

So, the odd number of odd components in this graph is 2. So, we write q of G is equal to 2. So, this is the notation for that. So, I can do it. So, the q of G equal to 2 is what we write for this graph. So, we do not have any notation for the number of even components. So, that total number of components, because we need only the odd components. So, q of G means the number of odd components of the graph.

So, now we consider some subset S of the vertex set, and consider that we removes that subset S from G in count the number of odd components; this is q of G minus S; G minus S is the graph which remains after the removal of the vertex set S. When you remove the vertex set S, remember you remove all the edges incident on those vertices removed also. So, essentially you will get induced sub graph on G minus S.

So, the number of odd components we considering that and for the Tutte's conditions to be true, we need the number of odd components to be less than equal to the number of vertices removed the cardinality of set we removed. So, we can probably consider an example so, one possibility is we can look at example here.

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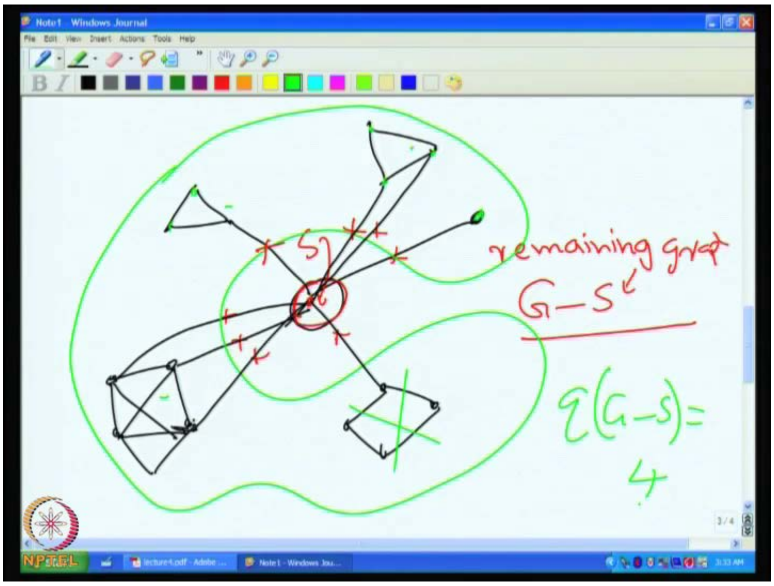

So, here I draw this kind of a graph. One...

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We can consider this graph once comes. So, here let us consider this vertex set say let us say this vertex set Let, me call this singleton set this one point vertex set as S. So, what happens? If I remove this S.

Of course, if when I remove this S all these edges will go away. Now if the remaining graph that is G minus S the remaining graph is G minus S. What will be that? So, I mark it like this say. So, this is the remaining graph. So, let me remove it and place it. So, the remaining graph... So, I can say mark with another color here so, this is the remaining graph.

So, what is coming in inside the green is the remaining graph. So, this graph how many odd components it has? This is one odd component, because there is only one vertex set. This is another odd component, because only three vertex here. This is another one third, and then… Here is this is not an odd sorry this also odd component 1, 2, 3, 4. But this is not a odd component this is not a odd component 1, 2, 3, 4 four odd components we have.

So, we have q of G minus S is equal to 4 here. So, we can take may be slightly different example say was… So, another one…

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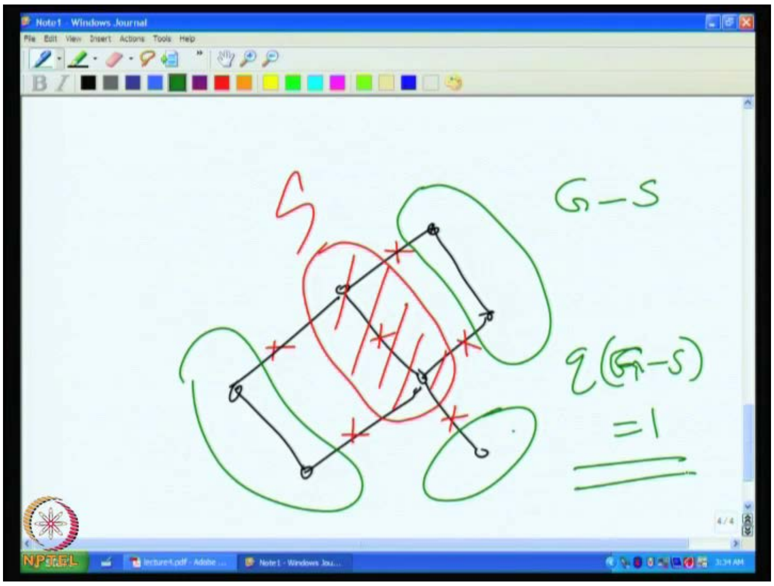

So, let us this graph say, consider this two elements subset S, now this is an S now. Suppose I remove this set so, whatever things will go away? This edge will go away, this edge will go away, this will go away, this will go away and this also will go away.

So, what we get the remaining thing what we get as this one and this one? So, this is the G minus S, and sorry this also. And this is one even component, this is another even component, but this is odd component. So, here q minus G minus S, q of G minus S is in fact, 1. So, this is what the number of odd components means?

Now, tutte's condition says for any subset S of the vertex set of G; when we remove the vertex set S from the graph the number of odd components has to be less than equal to the cardinality of S itself; less than equal to the number of vertices we removed itself. that is what Tutte's condition says.

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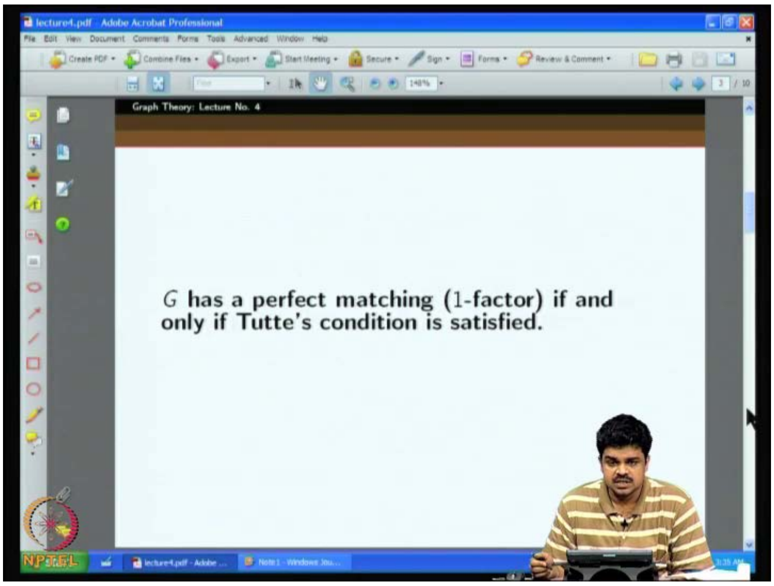

So, it the tutte's theorem says that the tutte's condition is a necessary and sufficient condition for the existence of the perfect matching in G; that means, G will have a perfect matching if and only if tutte's condition is satisfied. Let, us say one side is very easy, for instance if you want to check that the tutte's condition will be satisfied; if a perfect matching is present in G, how will we proceed?

So, this is the way. Let, us say suppose let us consider a graph and we have a perfect matching.

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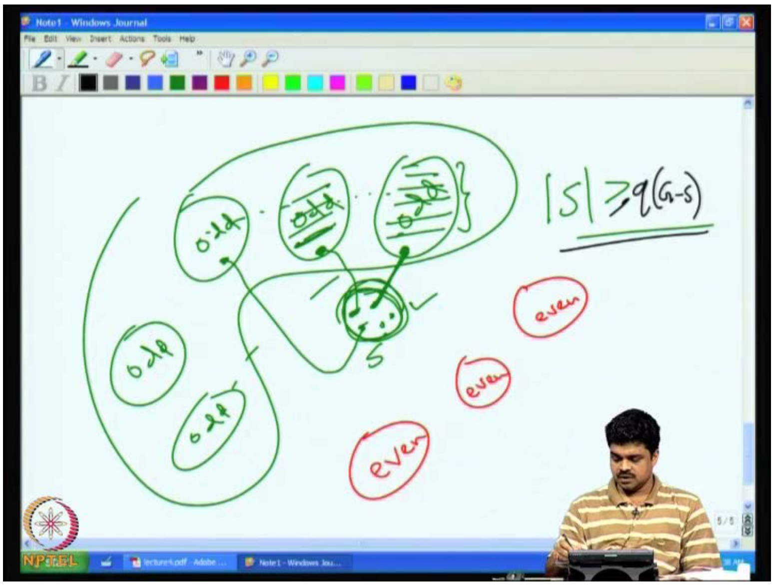

And then you picked up some subset S. So, some subset S there is some vertices I need. And now, this is the odd component of this and these are the odd component of this. They can be some even components also it is possible that there are some even components. This is even, this is even, this is even like that and these are all odd components.

Now, you see if there is a perfect matching here, this perfect matching will be matching things like this, but it is clear that at least one vertex set will remain here unmatched; that means, there is at least one vertex in this component, that test node get a partner with respect to the matching a considering from the same component.

Therefore, where should it partner has to be here somewhere in S right in S. Because there are no edges across here there cannot there is no edge, here there is no edge, and here there is no edge. Therefore, it is partner the bottom of vertex cannot come from this, this, this or any of this, because they are not adjacent. So, it partner has to come from this things, because if it is perfect matching every vertex has to be matched. Now what about this odd component? Same things true after; however and whatever way the vertices are matched here they should be one vertex here which does not get a partner here, because it is a total is odd everything cannot get matched in that.

So, it is partner should come from this side and this cannot be this it should be different. Now, similarly from this also should get partner from this. So, every odd component should get a partner from these things. Clearly they should be sufficient number of vertices in this to provide partners all of them; which means, that cardinality of S has to be greater than or equal to q of G minus S, because there are so many components. Cardinality of S should be greater than q of G minus S. So, let us... So, that is what we know?

So, the one side of vertex is easy; that means, if there is a perfect matching the tutte's condition has to be satisfied for every subset S, the number of odd components of G minus S has to be less than equal to cardinality of S itself. The other side is more difficult; other side means, I want to show that the if there is no perfect matching in a graph it is essentially, because the tutte's condition is violated; that means, the tutte's condition is always violated, whenever there is no perfect matching.

So, if there is no perfect matching we should able to find some subset S. Such that, the number of odd components of G minus S is strictly greater than the cardinality of S, that is what we will need.

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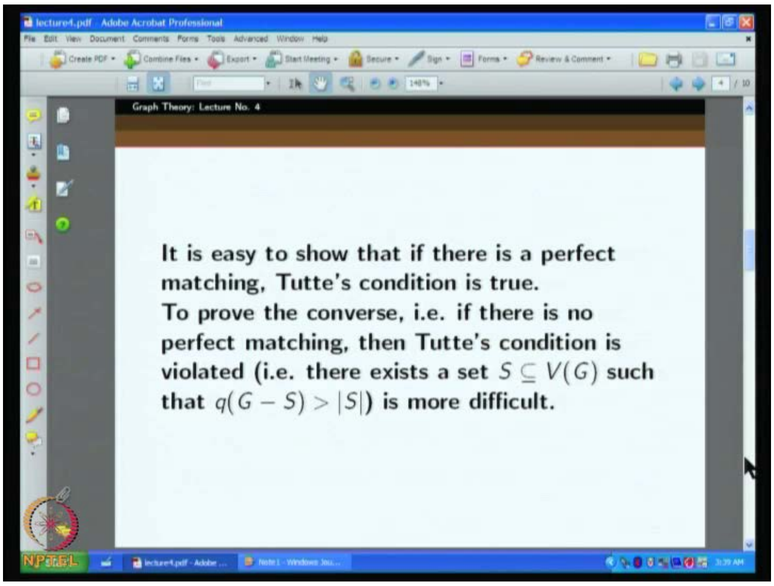

So, such a set is called a bad set; that means, set S for with the tutte's condition set is violated; that means, a set is for which q of G minus S the number of odd components of G minus S is strictly greater than the cardinality of S is called a bad set. We call it bad set.

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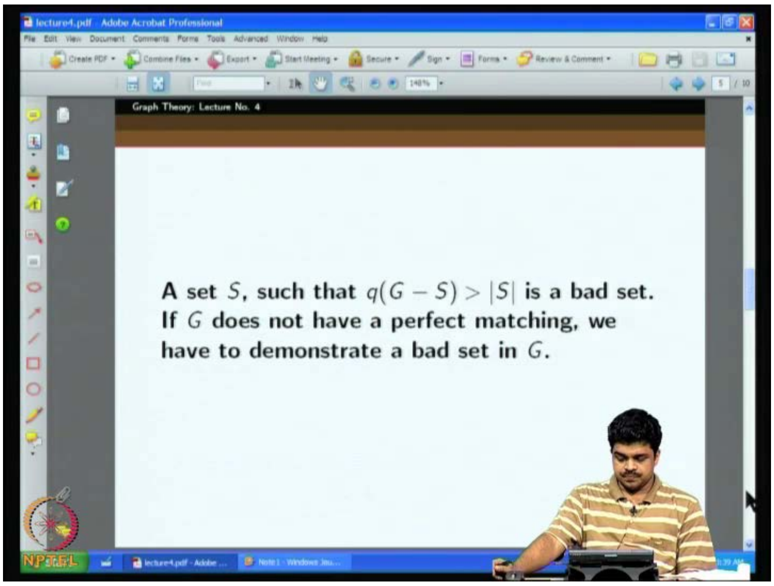

And essentially we are saying that if there is no perfect matching in the graph there should be a bad set, or in other words if there is no bad set in the graph there should be a perfect matching. So, how do you show that this is true? Let, show these things we precede like this.

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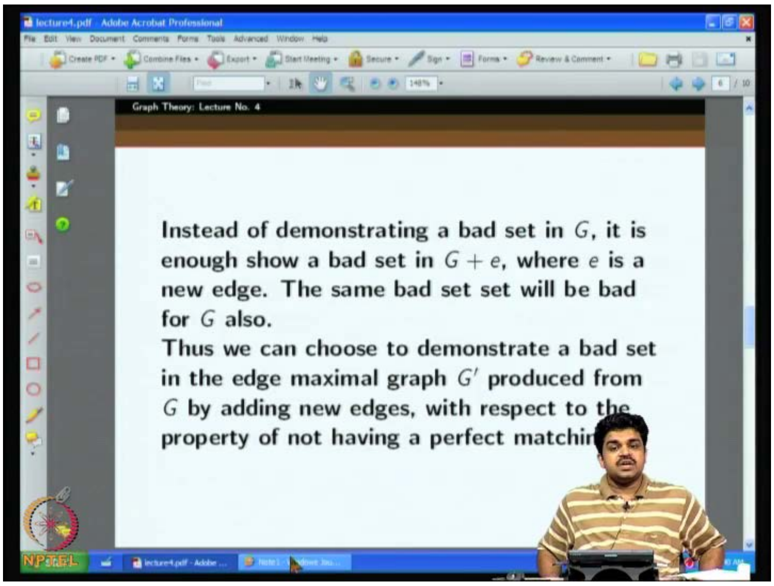

So, our intention is to show that, whenever there is no perfect matching in G then there is a bad set. So, take a graph G, which ever graph you like such that does not have a perfect matching, we have to demonstrated the bad set, show the bad set in these things. To show the bad set so, instead of directly showing it we will use some tricks the first observation is. So, we need not really show it in G itself, we can add some edges for instance if there is some two vertices u and V and G.

Such that, u is not adjacent to V, then V could have added this new edge to G and made it G plus e, and then V could have demonstrated bad set in that. If we somehow managed to get a bad set in G plus e, then that bad set will be bad set in G also. Why is it show?

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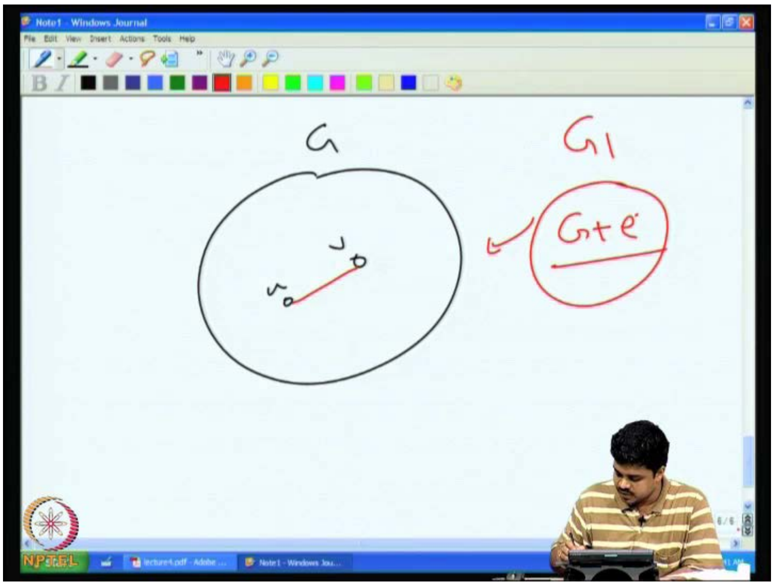

So, let us say, so we have this graph G, what I am saying this if there is any vertex is u v. So, this is new. So, you can try adding this new edge to it. So, we will get this new graph will be set G 1. We can call it so, this is G plus e. right G plus e G. So, I am saying it is enough demonstrate a bad set in G plus e that same bad set will be a bad set for G also.

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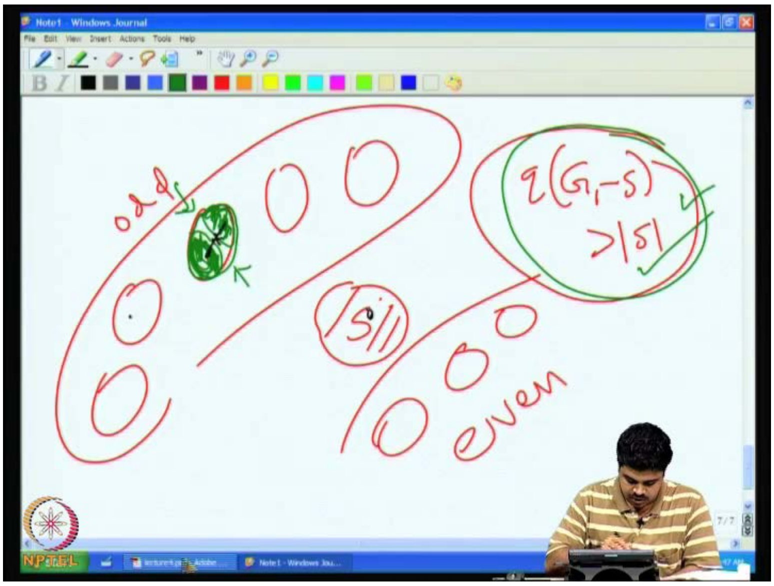

So, we said obvious; suppose say this is a bad set for G plus e, S. So, that means, we have several we look at the odd components of G plus e, these are the odd components. That can be some even components also do not worry about it. This is even, this is odd.

This number of odd components is by definition, because it set is bad set. So, G of q sorry G dash minus sorry G 1 G 1 minus S, G 1 is what we called it? (no audio from 17:51 to 18:00) G 1 minus S is strictly greater than the cardinality of S that is why it is a bad set. Now we consider the corresponding set S in G is it a bad set in G also. So, it depends on where this extra edge e is… So, in G with respect to G, we can thing where was that edge newly added edge occurred.

For instance that the newly added edges like this both end vertexes are inside S. Then the picture is not going to change for G also, because the cardinality of S is not going to change neither the number of odd component is going to change. So, therefore, this inequality will be still true in G. Similarly for instance if the bad set was sorry the edge was like this the edge was like this if the edge was like this. Then again we can see that S the cardinality of S is anyway not going to change, because it is same vertex set only if I putting on edges not going to change anything, or removing on edge it is not going to change the cardinality of S.

Now, it is going is it going to change the number of odd components. So, therefore, even if the edge is from S to some components it is not going to change. The inequality it is going to be the same.

The next situation is supposing the edges something like this sorry not like that. So, because it is in picture is about G dash suppose, because the edge can be now inside this right inside this inside this. Now, now we when we get to the edge to go back to G, this is G dash. What can happen? See if nothing happen, if the this component remains in that as such connected even after removal of this edge. Then it is find again this equality will be satisfied and therefore, as will be a bad set.

But it is possible that can be removing this edge. It may so, happen that this may get cut into two pieces, because of the edge only the got connected. So, now if by chance this was an odd component, both of them happens to be even component. Then we will lose one odd component from the total. And this happens to be just one more than S, then this may be violated. So, then we will be not able to claim that S is bad set in G, but luckily that situation will never occur.

Because it is not possible for both of the things to be even, why? It because both of them are even this together also should be even, number should be even. So, that will not be an odd component in the first place. So far, one of them has to be odd component. So, for the destroyed odd component we will definitely get one or smaller odd component. So, the total number of odd components will be equal or more, it is possible that both of them are odd component here then it may increase. So, this S will still remain as bad component, because this inequality will be satisfied in all the cases.

So, what we have now shown is somehow, if we add an edge to G and ask whether there is a bad set in G plus e, and if the answer happens to be S then the answer to the question whether there is bad set in G is also S. Because the same bad set which works for G plus e is a bad set for G also. But what is a good thing about it? So, you are asking instead of looking for bad set in G you looking for bad set in G plus e does it make the problems many easier.

So, the interesting thing is if you can look at G plus e for bad set, you could have look at G plus say one more edge e plus another edge, for instance. So, let us we could have told.

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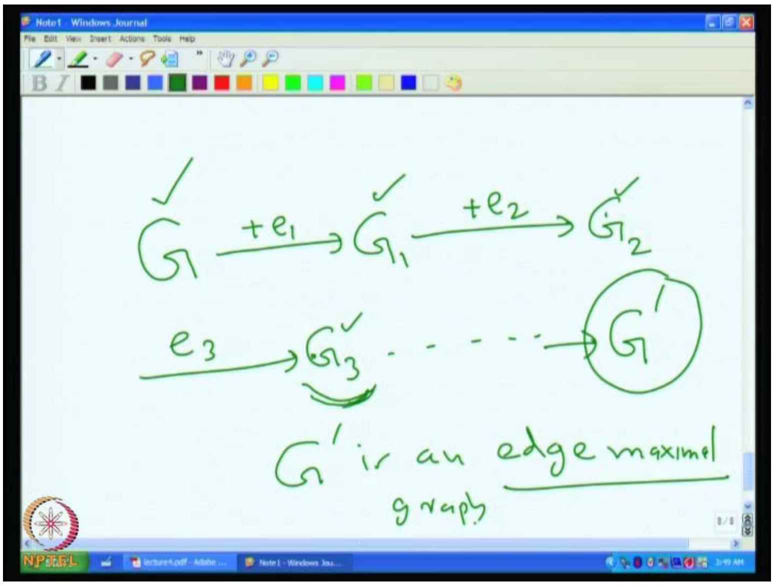

So, I will first start with G, I will add an edge to G and make G 1. I could have add it look at things, but instead of looking at that I could have even added another edge e 2, and then we could have created in new graph with more number of edges G 2.

And definitely I could have added it one more edge e 3 and we could have made it G 3, and then we could have look at these things. Any time if I get bad set, the bad set will be here, the bad set here, the bad set the same bad set will be a bad set here, and it will be a bad set here to be enough to look for the things. And finally, we could have come to a graph which I called G dash such that no more new edges can be added without creating a perfect matching.

So, of course, it is true that this keep on adding a edges, and if somehow get a bad set in any of this graphs which is created the adding more edges. Then the original graph also has. So, our aim is to keep on adding edges till a point that it get's saturated in some sense. So, what do we mean by saturated in some sense? That means, we cannot add any more edge without violating some property. What is the property we are interested in? Because G was graph without a perfect matching that is the property we are interested in. When we adding new edges we will make sure that this new graphs are also without a perfect matching.

So, in other words when we new edges added if you perfect matching comes in the graph, then we will not add the edge we will go we will look for the another edge that can be added. And then G dash such that we do not see any more new edges which can be added to G dash without introducing a perfect matching it. In that sense we say that G dash is an edge maximal, this is the word edge maximal, edge maximal graph edge maximal graph in what sense? Edge maximal graph with respect to the property of not having a perfect matching that is what.

So, we can tell about a graph there is an edge maximal graph, we can do with respect to the several properties any some property, but here the property we are interested in is the absence of a perfect matching. So, G. So, again to repeat, what we are interested in is to prove that? If there is no perfect matching in a graph then there is no there is a bad set in it. Now, instead of directly looking for the bad set in G we will keep on adding edges to G, and look for bad set in the corresponding edge maximal graph G dash.

And this edge maximal with respect to the property and there is no perfect matching, and also it is made from G by adding more and more edges. The good thing is somehow get bad set in G dash this same bad set will be valid for G also, that is what we argued. So, then we can as well look for a bad set in G dash. So, what is good about G dash, why should we go to G dash look for a bad set?

The good thing as about G dash is that it is somewhat some kind of a saturated graph it is edge maximal, in other words if we add any new edge tutte's is not already existing in it. Then the perfect matching will view of all, what can tell about the maximal edges of such a graph? Of course, it is almost there in sense that it is n by 2 minus 1.

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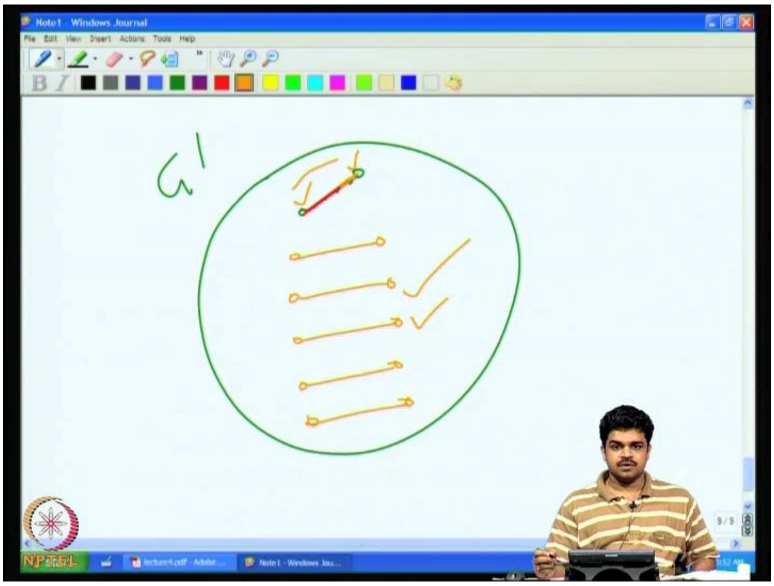

In other words, so if this is the edge maximal G dash this is the edge maximal graph G dash, and you just add an edge suppose say you decided add this edge. Then immediately we can see that there is a perfect matching comes; which means that there was already some matching which was just one less than a perfect matching; that means, every other vertex is match by some things, these are the only two vertices which are unmatched and then this also added. So, there is a perfect matching.

So, this is true for every non adjacent pair, any non adjacent pair will be such that there are the only two unmatched vertex for some matching. So, this is the good property this is going to be useful, that is why we are look at this edge maximal graph? Now how am I going to prove the result for this? Now we are looking for a bad set in the edge maximal graph, but when we are looking for a bad set… What is a bad set? Bad set means, the number of odd components of G minus S, the number of odd components of the graph generated by removing that vertex set S is strictly less than the cardinality of sorry great strictly greater than the cardinality of the number of the set of vertices remove it, if that is a bad set.

But then this is the typical property, how do you find out that such a set in even in the edge maximal graph even with the property? So, what will make this? So, what kind of set it is will be? That is what we first we should first think? For the instance a bad set in the edge maximal graph, how will we look different can we tell something about it? So, therefore, so we let us think about what kind of properties a bad set? If suppose it if it all it exist will have in edge maximal graph?

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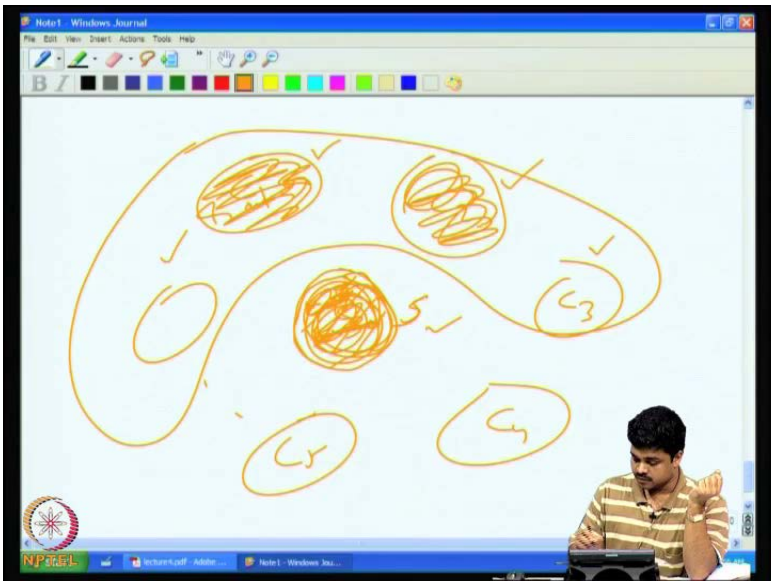

So, suppose this is the bad set, I am not saying that set S is edge maximal. Who is set S is then it should be like this? So, these are the components of the edge maximal so, these are the components of G minus S. So, let me call it C 1, C 2, C 3, C 4 like that.

So, now. So, the question is it possible that there is a pair of vertices in this such that, there is non adjacent u v, there is it possible to have a u and v in this thing such that we do not have any connection between them. So, you can easily see that it is not possible, because if what could happen? This is same set to be a bad set; that means; suppose these are the odd components of this some even components are there, the number of odd components is going to be bigger than this.

Now, suppose if I add this edge in the things, what will happen? Then we know that already after adding this edge there is a perfect matching, if there is a perfect matching you cannot have a bad set. But then this is going to be still the bad set, because the number of odd components is going to remains same by adding this edge and then the cardinality of this also going to remains same, this bad set will remain as a bad set.

So, there is no way of producing a perfect matching in this graph by adding a new edge here within this thing, because any new edge added here will not reduce the number of odd components. So, what does it mean? Because it is an edge maximal graph if there are two vertices inside it which our non adjacent, it should able to add on new edge there and then perfect matching should come. So, we can only infer that… So, this should be a complete graph this should be a complete graph S should be complete graph.

And what about each other components the same argument is true, because if there is a nonadjacent pair we can add that edge; we will get a perfect matching. Then if there is a perfect matching we cannot have a bad set, but this bad set remains that bad set, because the number of odd components has not change, number of vertex set has not changed, so if it is originally a bad set, it has to be a bad set.

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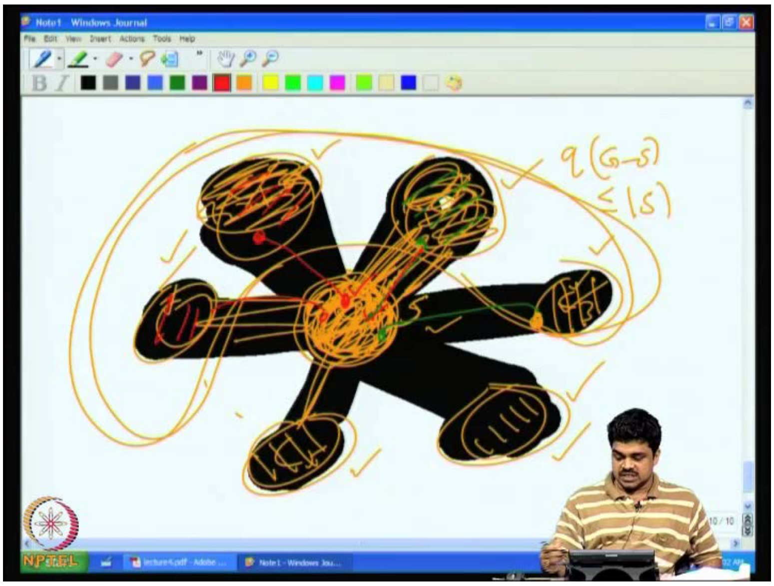

So, all the things have to be this should be a complete graph; this also should be a complete graph. And similarly this is also a complete graph, this is also a complete graph this, this, this similarly this, this all these things are complete graphs, that is what we see.

Now, what about the connection between this say for instance these connections? can I tell anything about them is it possible to have a missing edge here; that means, some edge some vertex here, some vertex here is it possible to have non have them non adjacent. In that case the same argument again tells it is not possible, because if I put this edge again the number of odd component is not going to change, this is the again an odd component this is odd component, this odd component, because it is number have not changed.

And similarly the vertex to the cardinality of vertex set is not going to change. So, this is going to be a bad set again right, this is going to be a bad set again. So, it is not possible. So, what we say is that? It is not possible to have a non adjacent pair here, in other words all the edges across should be present. Because even if one non adjacent pair was there I could have added that the edge argues that a perfect matching came, because of that. And, because the perfect matching is there then my graph cannot have a bad set in particular this particular set cannot be a bad set, but it is not true, it is it remains a bad set.

So, what we can infer is every connection between this, between this, between this, between this etcetera, should be present. So, for instance if I again mark it, what I can see? All the connection between these things, between these things, between these things, between these things should be present entire. A graph should be like this with respect to the bad set we should see something like this.

What I have now told is essentially about the generally… So, if at all there is a bad set, it should like this. It should have this special structure, but I have not told that there is a bad set I have not also told that any suppose… So, I am interested to find the bad set. And the other question is how the bad set should look? Some the bad set will look?

But then if it also true that suppose, I get some set like this look like it is much easier to figure out this whether, this kind of set is present or not? So, suppose if I find such a this kind of a set in the graph, can I be sure that this is going to be a bad set. After I have in prove that, but it is a valid question to ask, because the bad sets have a nice structure like this. can I can also have a reverse question, that suppose we see a set with this special structure that is always going to be a bad set. It is so happens that it is (( )). It is going to be a bad set, why?

Because you know that in the perfect matching sorry in the edge maximal graph we do not have a perfect matching. And suppose I find a set like this and if it is not a bad set, then it should have q of G minus S less than equal to the cardinality of S that means, the number of odd components has to be say let us say these are the number of odd components. I mark the odd components like this. So, this odd component has to be strictly sorry less than equal to the cardinality of S.

Now, I say that this is the contradiction to the assumption that we do not have a perfect matching in the graph. Because we can always match all the vertices of the even components, because there is a complete graph and like every they can be paired of perfectly well, and perfectly. And then here also the odd components can be paired of except for one vertex here say this. This vertex I can saw in a partner from here. Similarly here also the all the vertices can be paired of except that I can one vertex will be left, but it can get a partner from here another vertex of the then this.

Similarly, here also I can pair of all the things, but one vertex will remain that can we get a partner from here so, from each of them. So, I need because the number of odd components here is strictly less than equal to the cardinality of this thing, I will be able to the assign one for each odd component, one vertex from the set at each for each odd component. So, that it can provide partner for the left out vertex of the odd component. So, we get a perfect matching in G.

So, our assumption that this graph edge maximal graph G dash did not have a perfect matching was wrong they did not have a perfect matching was wrong. So, what we see is not only that the bad set has that the special structure; that means, click it is a click and all the components are clicks, and between the component, and it is the we have a complete connection. Not only that, if we get any set with the particular such a special structure, it is has to be a bad set this also easy.

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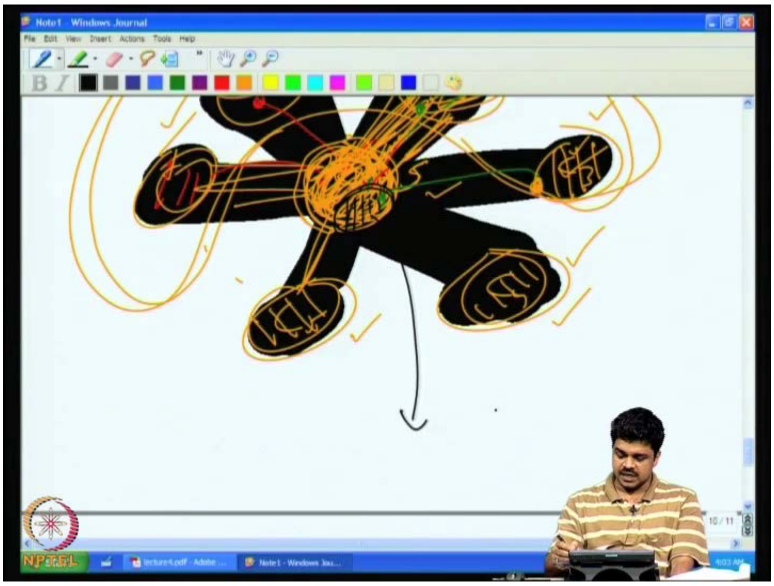

So, then you can see that. So, because one one one point is missed. So, for instance one can see how, what if this cardinality of this S more than the q of means, what if why do you think that the remaining vertices here got can be paired of will be even? If it is even can always pair of that? It is true, because if the number of vertices of the graph is odd, then we know that the set phi can be taken as a bad set.

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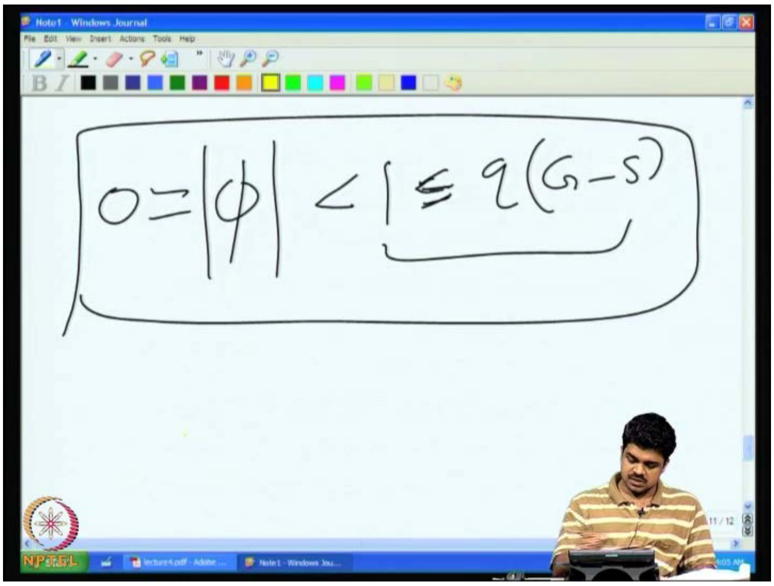

Because if you considering the cardinality of phi and this is always less equal to 0, this is going to be a less than 1, which is the cardinality of the odd component with a odd graph. If it is a connected graph there is one component that is odd. So, but then if it is disconnected the graph, at least one component has to be odd. So, this q of G minus S is going to be greater than equal to 1, and that is greater than the cardinality of phi.

Therefore for the odd graph is the the statement is trivially true, because it is always take phi as the empty set as the bad set. So, we can assume that in a even graph so, therefore, we can see that after pairing of the vertices by some matching we have removed already, so, we have already considered even number of vertices. So, whatever is left out has to be even number, because it is total is even. So, the remaining even number of vertices also can be paired off. That is what? That is why? That can be completely paired off.

So, this we have already argued that, any bad set has to look like that particular special structure and also if any set has that any special structure then it has to be bad set. So, you can see that there is interesting thing here that means, we are initially we have started to find out the bad set in G dash, now we do not look for the bad set which is more difficult to find out, we only have to find out a subset of G dash which will give which has the special structure; that means, it induces the click all the components induces the click and between each a component, and the set we have a complete connection that may which has to find out.

Suppose I want to find out such a set, what will I do? Of course, so because the vertices in S has to connect every other vertices in the graph; can you see that for instance you take a vertex here, you can see it is connected not only it is connected not only to the vertices inside this. So, also it is connected to all the vertices in the component, this component, this component, this component, this component every vertex. In other word such vertices is usually called universal vertices.

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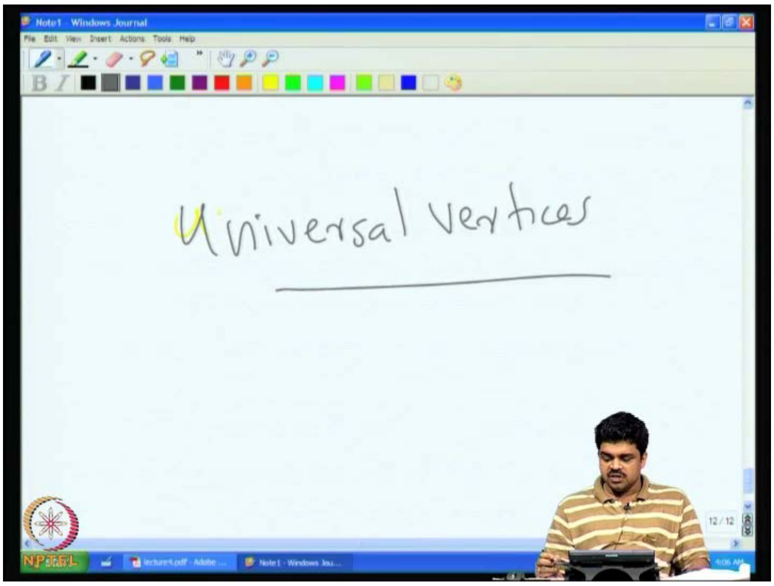

So, this is the word universal, such vertices are called universal vertices. A vertex which is adjacent to every other vertices in the graph is a universal vertex.

And you can see that if such a set X is all the vertices in that set is going to be a universal vertex.

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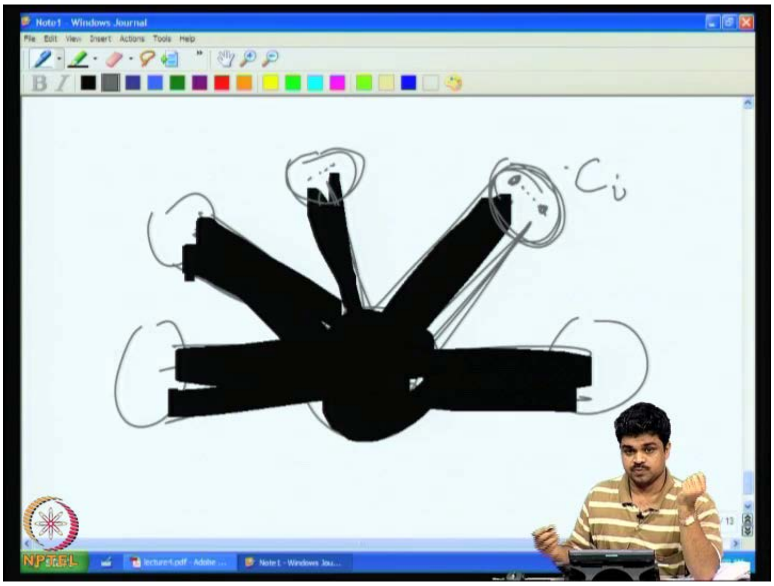

So, it is very natural to collect all the universal vertices in the graph. If you want to get such a set; that is the most natural approach, and if once we collect all the universal vertices then we will get this S, this a set of universal vertices. So, naturally, because it is universal vertices is complete connection will be there.

Now, we look at components of which results by removing these things and naturally, because there are universal vertices all these connection will be there. So, thus that particular structure will be there already. So, all these connections will be there.

Now what is missing? So, you have a already got whatever we are looking for its like this, we will have all these connection, here already got this much we have this connection. So, we have this complete connection inside also. So, this much we got.

What is missing is this probably there is there are two vertices here, which are non adjacent. In other words probably this thing inside this component will may not have a complete connection; probably within things will not have the complete connection, something like that. So, we are almost got it, but still there is a problem this is does not have the kind of structures we are looking for.

So, you may ask, what happens if I did in find any universal vertices in a graph? In that case has to be a empty set. So, it is trivially true this much as happened. So, the components of the graph it is will be the thing, but then they want any vertices in the graph, which is connected to everything else.

So, let us say if there is a contradiction that is if there is a situation that some components say. Let us say this C i. This component has two vertices such that the non adjacent. We going to produce a counter example, sorry contradiction based on that. Why how we are going do that so, contradiction to what? If you remember this was an edge maximal graph, but it did not have a perfect matching edge maximal without a perfect matching.

So, we will show that if there is an edge missing in this component, we will have a perfect matching; this is what? The contradiction here, it may look at a little surprising, and when we are saying that when there is an edge missing here we are going to show that there is a perfect matching; which will be a contradiction, because there is no perfect matching.

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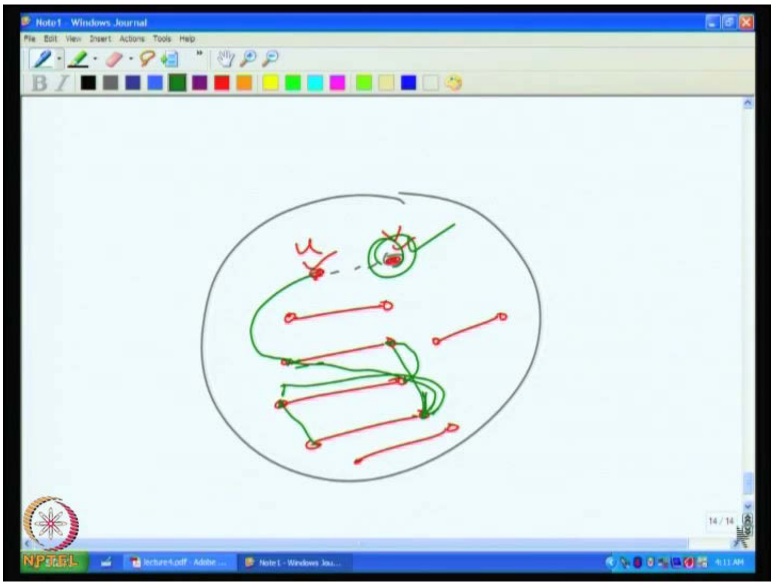

However we going to do this thing? So, we will now we do not need the picture, let us look at this graph. So, if we have a missing edge. So, what is the possible at it? So, see one thing is see you know that, if you add these edges, you will get a perfect matching that is the property of the edge maximality. So, therefore, there is already got a matching of this graph, which matches of every other vertex of the graph.

So, except these two vertices they let us call u and v. So, these are the only two vertices which are unmatched with respect to that matching. So, is it possible to augment this matching? Augment means, can you somehow increase the cardinality of the matching, giving if I matching increase even by one we are done, because it is already n by 2 minus 1, it will become n by 2; that means, it will be a perfect matching.

So, you can make an attempt for instance we can start for one as we studied in the bipartite guide. One method is to find out an augmenting path. Augmenting path means, you start with unmatched vertex follow a unmatched edge and then matching edge, then unmatched edge and then matching edge.

And then finally, if you manage to each the other unmatched vertices then it is the augmenting path. What we can do this? You can replace the edges which appear in that path from the matching by the edges which appear in that path from the outside matching. So, you soft the matching edges with the unmatched edges.

So, is it definitely you can go out of instance you start by path from u, you can always go from u to some vertex. So, definitely that vertex, because it is not this this is not adjacent. I can always follow this red edge now. Now from here I can try to find another vertex; may be I reached here.

So, after some time so, what is the guarantee that? So, I will not all my. So, I keep going and then I reached here, and then what is the guarantee that when I look for neighbors of these thing? So, I can continue my path. So, I will not end up with the already seen vertices something you have to try. So, if we try for some time, we will realized at little difficult to show that. If can you continue with path until, you reach the other unmatched vertices.

So, here is a trick a very nice trick. So, which allows as to complete to go forward. So, what is the trick?

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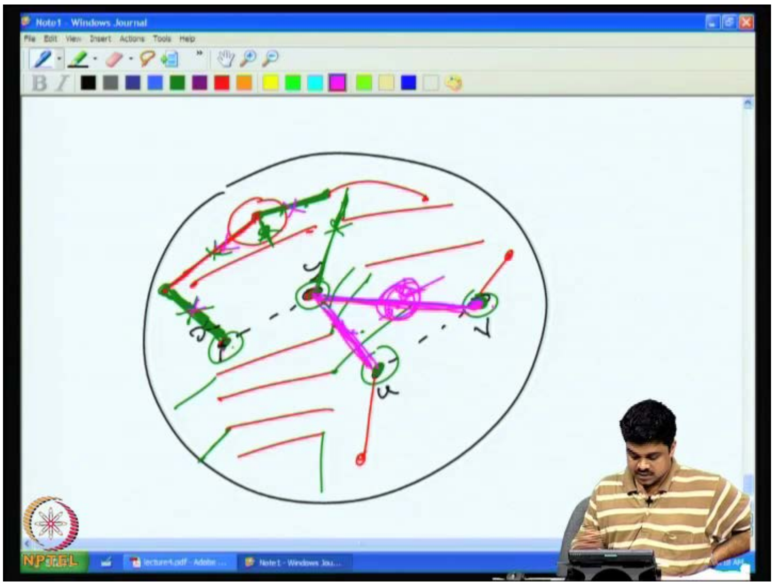

So, the trick is… Suppose you have… This is the graph instead of one non adjacent pair of vertices; you have two non adjacent pairs of it. Let me call x, y and u, v now. So, I may want to start at x. So, let us say, if I put the edge x y then you will get a perfect matching here, I call it right perfect matching. This corresponds to the x y, because it will match of all the vertices, in particular there will be an edge like this also, and that will be also matched. So, everything will be matched by red matching, but only the two vertices are unmatched by the unmatched vertices with respect to the red matching, that is x and y.

Here similarly, if we try to put this u v vertex so, u v edge here then also will get a matching. Let us say it is a green matching, and then green matching will be matching every vertex also; similarly, somehow. So, here also particular will be this matching, some green matching will be there. Every vertex will be matched except u and v here with respect to the red matching only x and y was not matched, with respect to the green matching u and v are matched unmatched.

Now, what is the good thing about this? So, the good thing about it for instances by start travelling from x, and I am trying to reach the vertex y. So, one easy way is to take, because it is only here I see only a green edge, the red edge is not there; because x is unmatched with respect to the red matching. But definitely when I reach here I will see a red edge on that, because that has both red and green there. So, a green is coming in then we have to see red also, when I reach here definitely I will see a green here.

Because this vertex has both green and red by via the red be enter, then through I should be able to exist through green. Similarly here also I will be able to do the same trick by red I can follow. So, I can keep on going, you can see that we will never come back to then already visited for instance; this path is never going to come back then already visited vertex. Because it is only one red edge incident on it and one green edge incident on it, both of them already used. If some via red edge again it has to come or via green or red edge, if again it has to enter it will violate the property that there is a only one green and one red edge adjacent to the vertex.

So, therefore, that will never happen. So, the earlier problem that question, what if it again comes back hits, the same the vertex which is already visited will not happen here, but then are we reaching the destination; that means, a starting from x can we reach y that is not guaranteed, because this can stop at three points. If it is stop set y well and good we got augmenting path.

But, so what will do? Suppose if you reach y then naturally we will some out be entering white with green edge. In that case we will replace all the red edges in path with green edges, we can see we early see that, because it is start with green edges and end with green edges, the number of green edges is one more than the number of red edges. And therefore, the number of edges in the matching will increase, the in other words we are augmenting the red matching by replacing some of the red edges by green edges.

But this will not happened or in other words you can see that in that case, we will we this both x and y you will get matched and all the other vertices also will get matched. But what it fit never reaches y? So, where as it can go? It can go an end in u or v, if it is ending in u or v it is ending with red edge see, because there are no green edges there. So there these augment this not augmenting path. It just in alternating path starting with green then red green, red and it is ending with red. So, we both red edges and green edges in such a path will be equal; by switching the red with green is not going to increase the length of the cardinality of the red matching or with the green matching.

So, the only solution that… So, you can things of sometime whether you can somehow all these issues, but there is it does not look like at there is an immediate solution for that the only these things to have we can think wishfully; what if I have two edges here. Like say I have suppose I have an edge here and then edge here. So, definitely these two edges are not going to be of red color, because this red edge is not going to be incident with y. So, this has to be different, I mean it want be red.

Similarly, this cannot be green edge, because there is no green on this. So, it has to be some other color neither red nor green, so similarly this one. Suppose there are two edges whichever this want be green or red; whatever if at all there are then we do not have this problem, because when we reach this via red edge we can follow via this, and for that time t we can assume that this is the green edge, or in other words we can replace all the red edges in it by this green edges as well as finally, this one.

So, that will be a augmenting path though it is not fully green red, green red. It is a green red green red finally, red and this new edge. This is as could as a green, because you know you are only planning to replaces all the red edges by a different kind of a edges we augmented it, augmented the red matching. So, that can happen.

So, but then there is no guarantee that this kind of a structures make exist. So, how do you make sure that we can pick up two non adjacent vertices from the graph, such that this kind of a structure comes.

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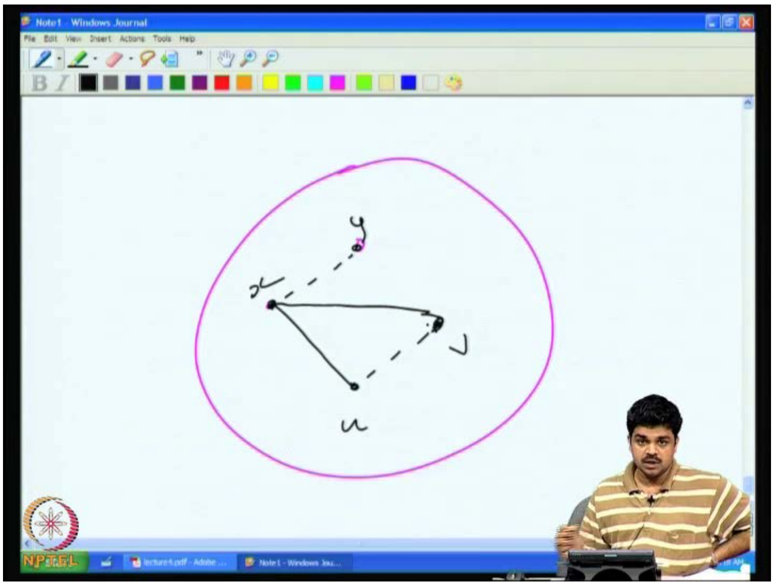

So, this is what we want so, two vertices x and y, u and v. Such that this is non adjacent pair, this is non adjacent pair, but we need this thing. So, this is the trick finally to do this thing. We know that by our earlier picture.

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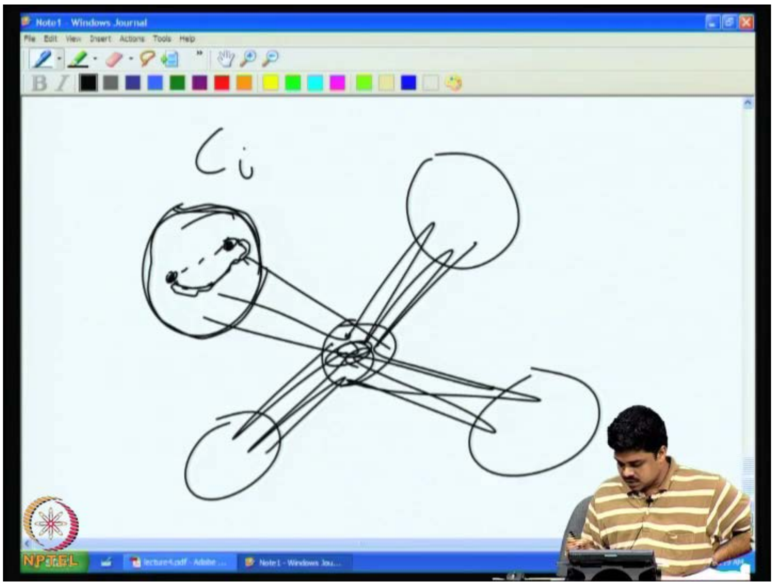

We know that…. So, this was our picture. So, we have all these connections there is some all these inside, but there is at least one component which we call C i. Such that two vertices there are this non adjacent two vertices here. So, I will just take this component, because it is a component and this is two non adjacent vertices on it, there should be a path starting from here to here somehow and reaching here. There should be a path.

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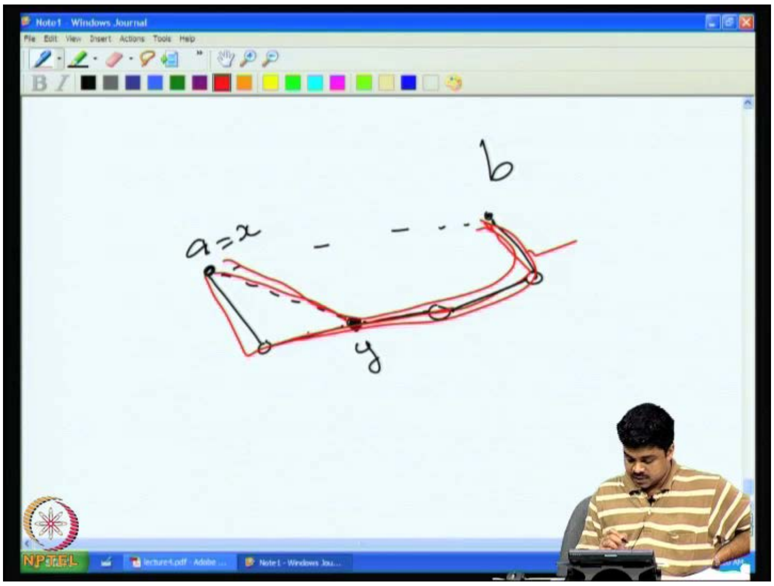

So, we can consider that path, let us say, this is the two vertices. Let, me call it x and y sorry x sorry, let me call it a and b, let me call it a and let me call it a and b, and then this is non adjacent there is a path inside that component which goes from a to b.

So, let us consider the second vertices of this thing this path. So, this will be called x and let us call this y. clearly there is no connection between this and this. See this path; I will take as a shortest path, if it is a shortest path we can clearly see that this edge cannot be there, what? If there this edges there, then we are going to get a shortest path here like this, right then this one then this path going to be short.

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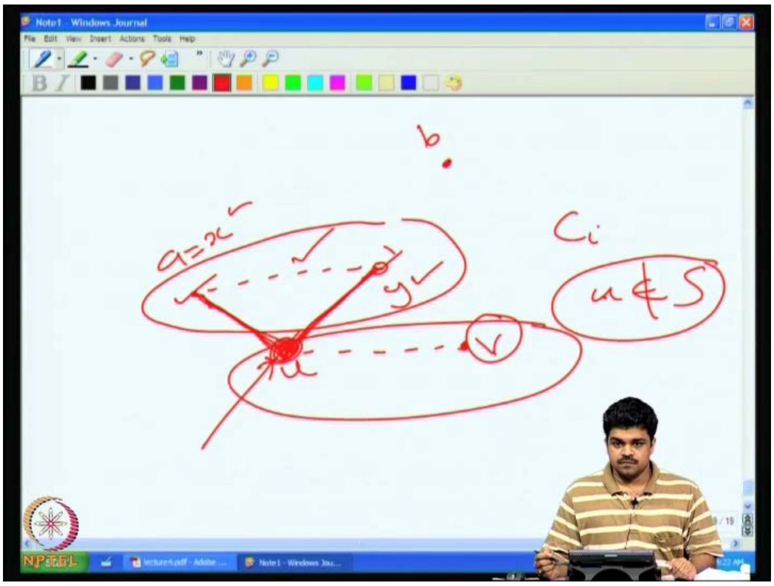

So, therefore, we are not going to have that edge here. Now what we have is an a and b, but if I take the shortest path in it and this third vertex this is a is the first vertex, second vertex and this is the third vertex. And call this is x, this is y and this is non adjacent pair. And both of them are component and it does not matter for us.

Now, we will consider, because this connection, so let us be u. Now can I if I find one other vertex v such that, v is not adjacent to u then we are done, because then we get x y here u v here. And this connection is there; this kind of connection we were looking for. So, two non adjacent vertices x y, u v and then u v is connected to both x and y. But then how am I sure that this will be there, because u is from the component C i. And therefore, it is not in S, u does not belong to S.

So, it is not a universal vertex, because we are collected all the universal vertices in S. So, these should be at least one vertex in the graph which non adjacent to u that will be v, and definitely it is not x or y, because these thing are connected it. So, we get this pair. So, the we can find an augmenting path for this the there matching, which produced based on x y and u v, and then the we will get a bigger matching, that will be a perfect matching that will contradict the fact that there was no perfect matching in the given edge maximal graph, this will be a contradiction. So, these complete the proof, this is what? Guarantee as that if we collect all the universal vertices that will have the kind of special structure, we were looking for and that is going to be a bad set by argument. In the next class will summarize the key points of these proof ones again.