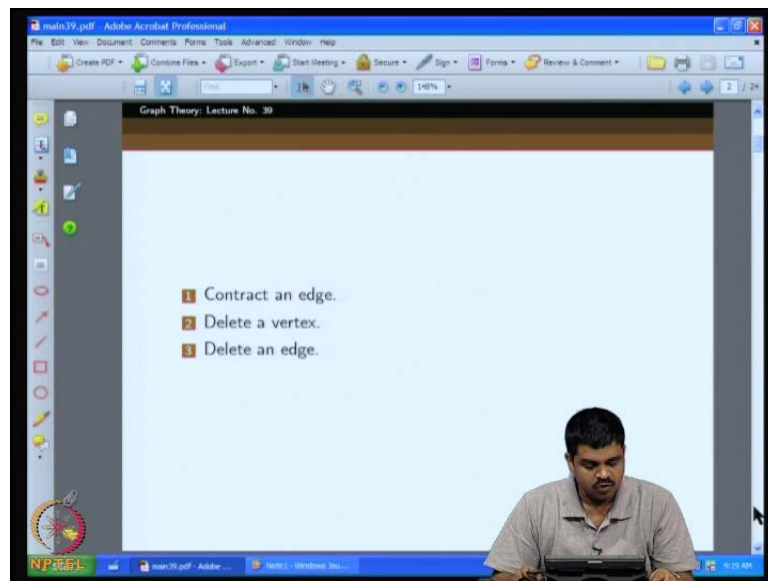


Graph Theory
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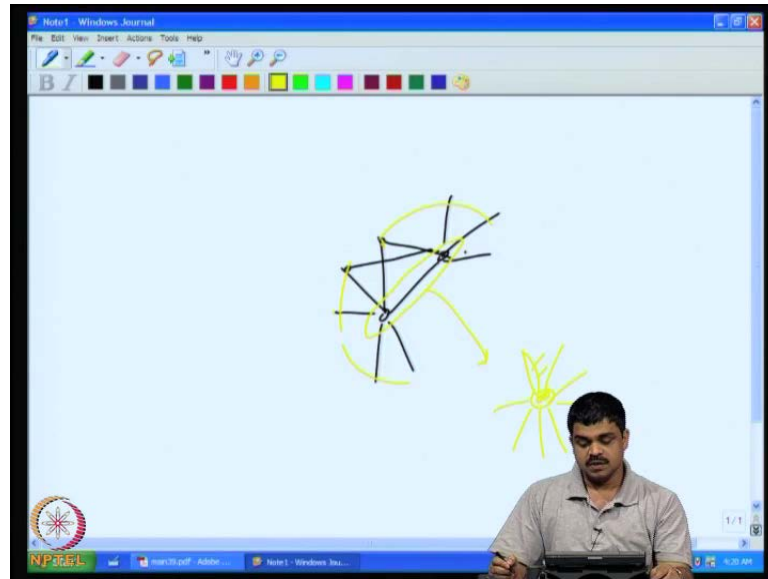
Lecture No. # 39
Graph minors and Hadwiger's conjecture

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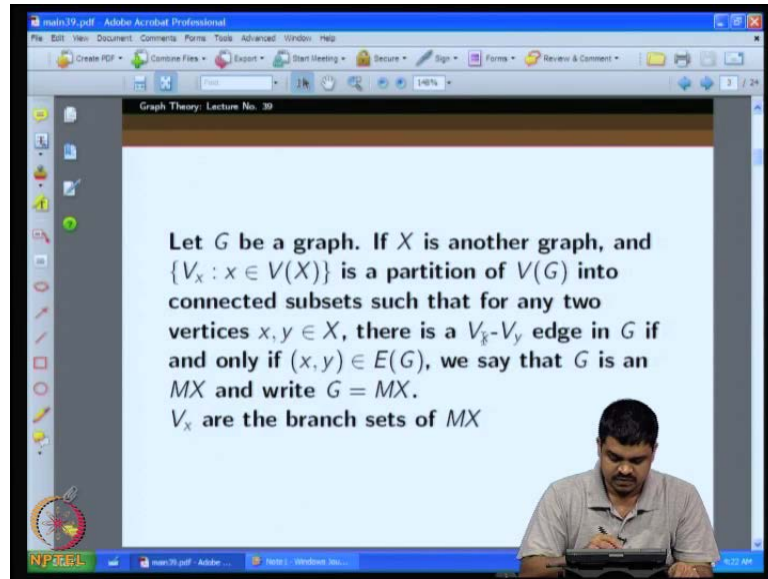
Welcome to the thirty-ninth lecture of graph theory. In this class, we will look at graph minors; of case, we are already familiar with graph minors; I have talked about minors earlier also. But in this class, we will look at it in a little more detail, of case we do not have much time to go in depth, we will just touch some of the initial material of this subject. As the student should be already familiar, these are the three operations involved in taking minors.

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Contract an edge, which means if there is an edge like this, so these are the neighbors of it **say** says some neighbors like this also, then you decide to contract this edge; what will we do? We replaced this with one vertex, new vertex, and all these neighbors will become the neighbors. All these neighbors, all these things will become the neighbors of this new vertex. So, of case, you may see that like, they are two edges here that may become multiple edges, but then we can discard one of them, if multiple edges are not relevant. So, the contraction operation is this; any way, we are already familiar with it, so do not go into again repeating it. So, delete a vertex is the second operation, delete an edge, so you can do it in any order, and in any number of times and whatever graph results is the minor of the original graph, this is what we told.

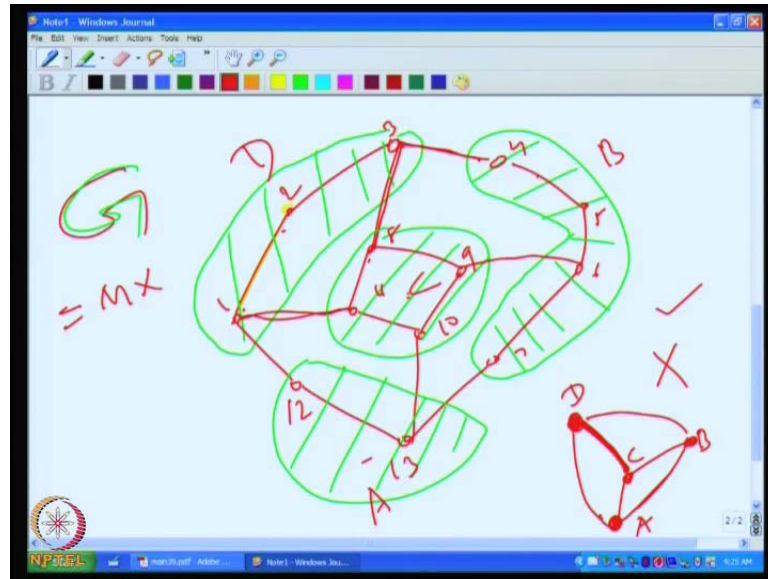
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Let us presently slide frame (()), because so we want to concept called branch set. So, let G be a graph. Now, if X is another graph, and corresponding to each vertex of X , X is the smaller graph, let say compared to G , X is the smaller graph. Corresponding to each vertex of X , we identify a set of vertices of G , so that is what V_X . For each X element of V of X , we identify $V(X)$, namely some subset of the vertex set of G , and get this collection be a partition of the total vertex set of G , V of G in two connected subsets.

The property that we need is each V_X should induce some connected graph in G it is a connected subset such that and another property we need is any two vertices in X in the smaller graph are connected or adjacent you cannot only if there is the corresponding edge between V_X and V_X . So, that $V_X V_Y$ edge in G that is we say that then we will say the G is a M_X and write G equal to M_X , so then the these V_X are called the branch sets of M_X .

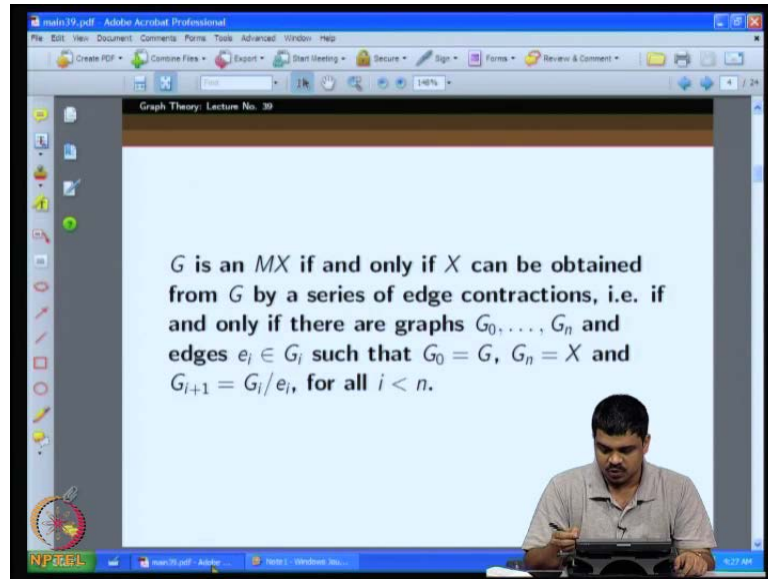
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So, we can take an example and so let us say **so let us say** this simple graph, so we can **right** so this graph so let us identify some collection of connected vertex. So this is one collection, this is another collection and this is another collection and this is say another collection **right**. So, this so this is some vertex and just marking it of so here, so essentially I have partition the vertex set of G so this graph, the red graph is G red graph is G and I have partition the vertex set into say this is 1 2 3, so this and then this is 4 5 6 7 is another group, 8 9 10 11 is another group, 12 13 is another group now you can corresponding to this. So, we can mark a graph like this. So there are 1 2 3 4 vertices so this is M X 1, see this M X you can see this vertex A correspond to this A, B correspond to this B, C correspond to this C and D correspond to this D **right**.

So the point is corresponding to each vertex here we have a subset of vertices of G and then they are connected each of these subset are connected for instance here there is connectivity here there is connectivity, here there is connectivity, here there is connectivity and then now whenever there is an edge became these two, if there is an edge between these two, there is the edge between the corresponding sets here **here** between C and D, C and D C is this, D is this. So, the vertex are edges between that not necessarily one at least one is what we are say **right**, so this is vertex so then we will say G equal to M X, so this and then C of case that are many G which are M X and this G is also M X and which other branch sets here? These are the branch sets these **these these** are the branch sets was X correct.

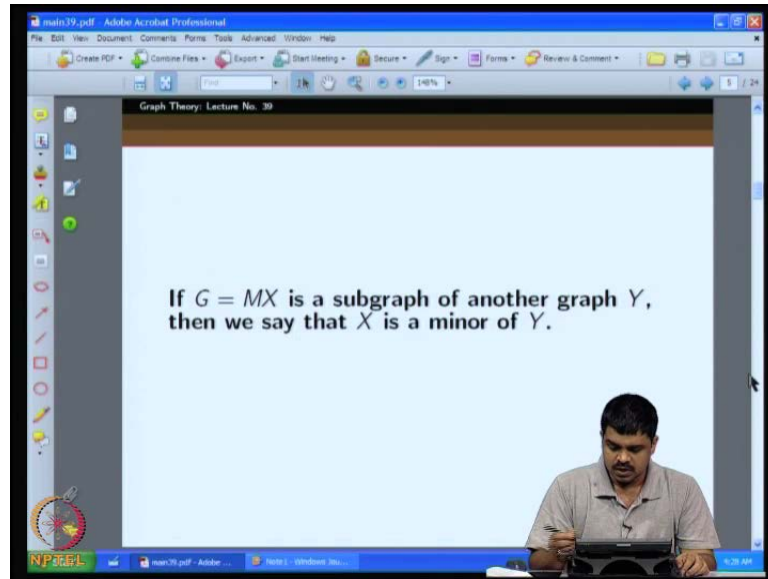
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And then the point is that G, G can be seen as $M X$ only if X can be obtained from G by a series of edge contraction that is that see it you understand for instance, if I contract these edges here this will this can become this vertex vertices, and then if I contract these edges then this will this vertex will come, if I contract these edges edge this vertex will come and contracting this four edges will results in some multiple edges will be there what we can discarded. So, then so by a series of edge contraction I can get this thing and also it is very easy to say that see that, if from G you can if there is a series of edge contraction to get this thing, we can always identify the branch sets there essentially the vertices which code contract into one vertex of X they from the connected set **right**. And also the vertices which code contracted into one set will not be part of the set of vertices it code contracting than other one; therefore, this $M X$ idea and this contraction idea its equivalent **right**, it is **are** same.

Therefore, we can view it the idea of minors this contraction in two different ways: one is thinking of the graph minor **sorry** or **or** the smaller graph as obtained as the result of contraction, the other is we have the identify the branch sets. And then there the part on of adjacent set these between the subsets is captured as a smaller graph that that both way its same its equivalent.

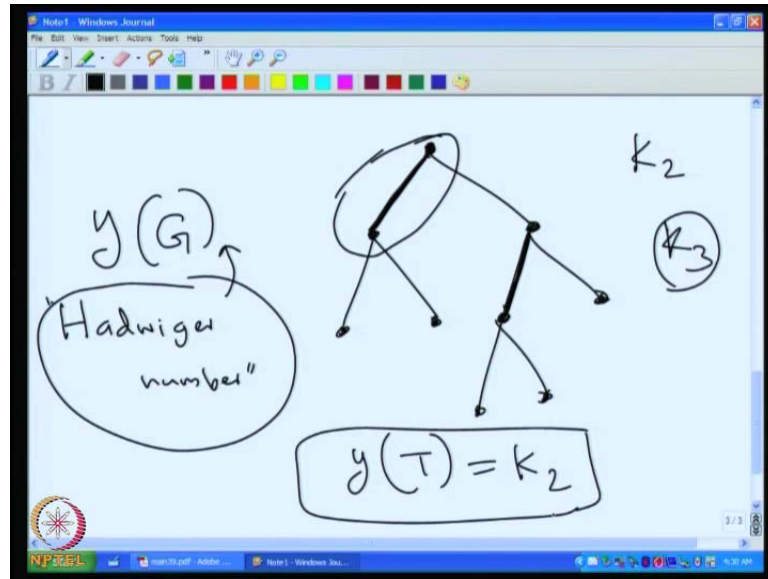
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And of case then we have the notion of the illusion of the vertices illusion of edges to accommodate that we will say G equal to $M X$ is a sub graph of another graph Y , when X is a minor of Y is not that we need to obtain X from Y by a series of contraction along we can also delete some vertices. And edges if you want to accommodate that we will say G equal to $M X$ if G , such a $M X$ is sub graph of some other y and then X is the minor of Y so this what?

So, now let us look at some examples I am already explain what some **some** an example here, so far instance we can ask in general view we will usually ask what is the biggest clique minor that we can obtain from a given graph, why this clique minor is an important, because if a clique minor on some k vertex is available in g then every graph on at most k vertices is available in G **right**. So, because the edge illusion, vertex illusion will allow as to get the smaller graph in its; therefore, the clique minors are usually studied, so and also then see lets that is also easy to understand.

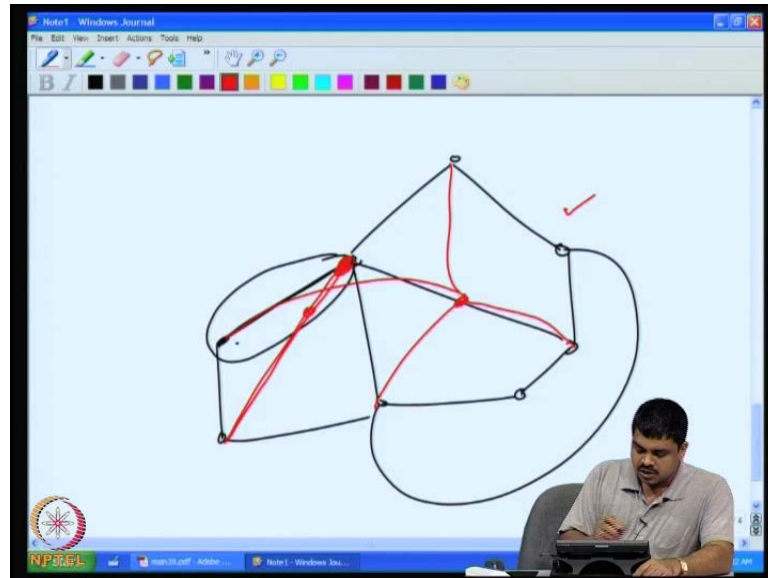
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Therefore let us ask the simple question suppose it is a tree G is a tree what is the biggest clique minor that we can get, so we have probably we have mention it before this biggest clique minor that we can get from a graph G is denoted by η of G , and its sometimes called Hadwiger number. So, which will be clear later, because its associated with the Hadwigers conjectures, so that is why it is called Hadwiger number that we are going to explain a little later. So, let us see from given a tree in a tree, so what will be the Hadwiger number of such a tree, so the question that definitely you will immediately see that you can get k_2 as a minor clique there are some edges.

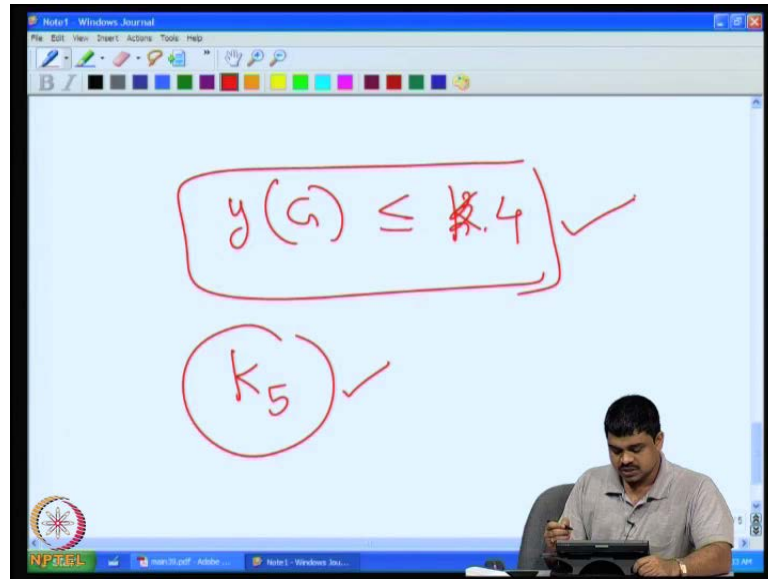
And then so you do not we do not even have to contract we just have to delete other edges, and the vertices just to take this thing, but then can we get a k_3 . So, the crucial thing that we have to see when if you contract any edge here we contract any edge here so we what we get is again a tree; therefore, as we contract the **that** tree property will not be lost it will be tree after contraction it always will be a tree. So therefore, we will never generate a cycle from that therefore k_3 will never come, so k_2 is the biggest clique that we can get from this thing η of T is equal to k_2 for in a tree.

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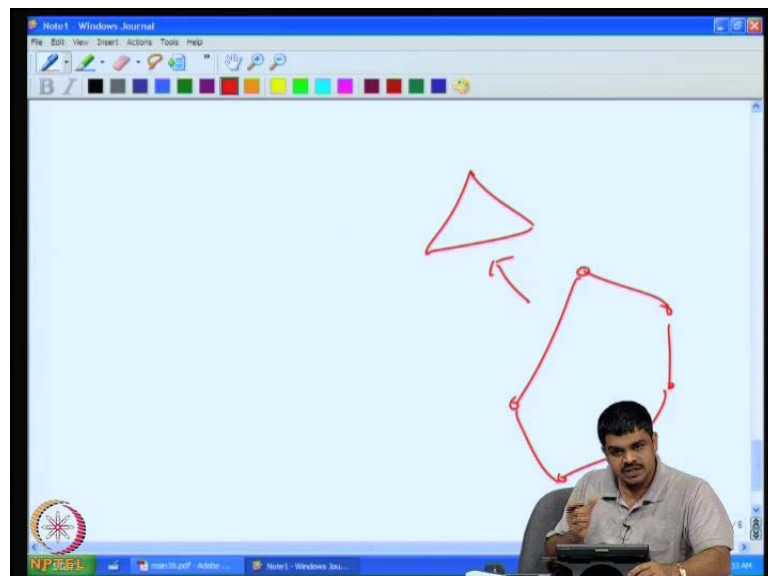
Now, see one interesting observation that we can make is that the similar kind of thing is true for planar graph for instance, if we take one planar graph and what will happen if I contract **contract** an edge here, so far instance this is a planar graph, so I can contract any edges you can see the planarity will be suppose will be maintained for instance, if I contract this then what will happen? So, I will replaced with one vertex here, so not here, so you can I can replaced it here itself, and then this **this** can be drawn like it without **without** destroying the planarity or for instance. If you are try to contract it what will happen, so this so you can place vertex here, so this entire thing will come here, so the planarity will not be lost **right**, so that you can imagine as shrinking the edge slowly and everything get closer. So because we are within in a face, so it is not going to effect the embedding the planar graph in the plane; therefore the planarity is mentioned if you contract edges.

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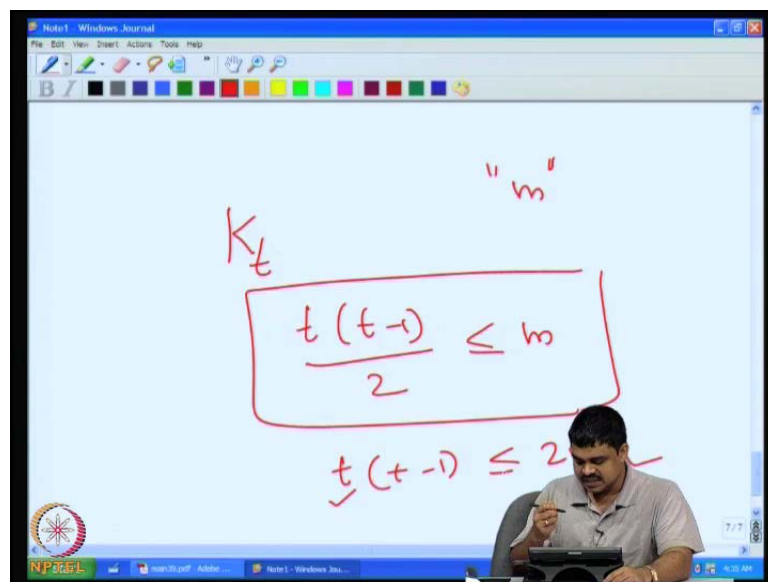
So therefore, the what will be the what can I tell about the Hadwiger number of planar graph, definitely you can say that the for any planar graph χ of G the biggest clique minor has to be less than equal to 4 **right**. K_4 is the biggest you can get why is it so because, if you get a K_5 or more from the planar graph by contracting that means that because the contraction operation retains the planarity property if K_5 is obtain by contracting edges from a planar graph, K_5 also has to be a planar graph, which is not true K_5 or more cannot be a planar graph therefore, the biggest clique minor that you can get from a planar graph is at most K_4 , biggest is K_4 you cannot get K_5 or more.

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So from a cycle what we can get is a triangle, and that is very easy **right** for instance you can you just have to keep contracting, and then you will end up with the triangle, then **right** this are some example of taking minus using contraction. Now, let me ask one question for instance can we see any relation between the biggest clique minor that we can generate from a graph and number of edges in the original graph, so the answer is also clear in fact as we contract the edges of the graph. What we see is then number of edges can never increase we lose some edges, we never increase the number of edges because if you contract what we do not add any edges **right**, we do not any new edges, the old edges only become slightly reposition because two vertices become the same vertex by getting merging together **right**. So the number of edges can only decrease.

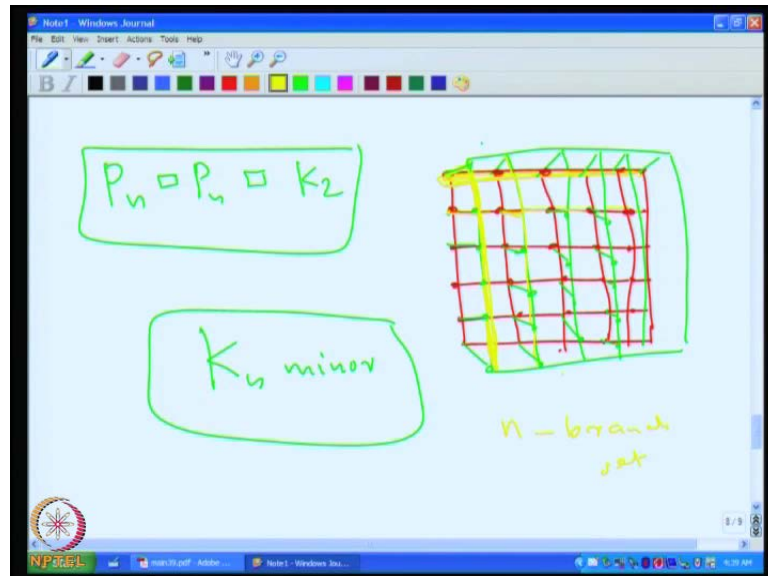
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Suppose, if m was the total number of edges before so then suppose if K_t is the biggest clique minor you can get, then we know that t into t minus 1 by 2 has to be less than equal to m **right**. Because you cannot generate in m any new edge, so from this thing we get t into t minus 1 less than equal to $2m$, and we can get some upper bound $4t$ in terms of m something like $\sqrt{2m}$ will be so little from this thing we can get **right**. So, therefore if the number of edges is very small, so there is the we cannot expect very large minors of case, so it can be much, much smaller than for instance in the tree there can be n minus 1 edges, but so we always get maximum K_2 minor **right**. We can never get planar graph can have at most around three n minus six edges, but still we can get only a K_4 minor from that maximum, so not more than that. So, therefore of case this is just

easy upper bound, and another thing is to look at the idea of branch set to demonstrate how to see the view the minus as branch sets.

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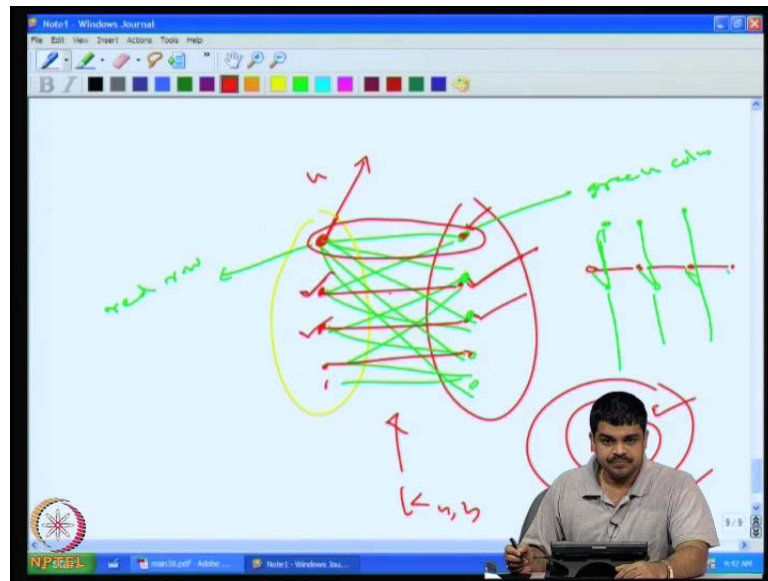


Let me take this example so this is what it is called a grid, so this is an n by n grid. So essentially n by n grid is the Cartesian product of paths, Cartesian product of P_n P_n cross P_n , so then suppose I make a copy of this. So, say something like this **right**, so then so and also connect if this, so that means this is called a double grid. So, essentially formally this is the formally this is the Cartesian product of two paths P_n into P_n into an edge K_2 , this is what now I am interested to show that we can get a K_n minor from it K_n clique minor in it, so n clique minor in it, so that do this thing, so this is the technique. So, we can so I am just marking some branch set, so for instance I will take this and this one branch set; that means this one column from the green grid and one row from the red grid so the first row, and the first column first row of the green grid and the first column of the red grid. So, you see that this **this** row is connected here, this column is connected here, and also because of this edge.

This entire thing forms a path, so here up to here it can come and then now it can jump here and then this is full path this is the connected set. Similarly, we can take the second row of the red grid and the second column of the green grid, and because of this connection the entire this is this two are connected in that that forms a connected. Then we take the third row, and third row of the red grid, third column of the green grid and

that forms a connected set then like that we can get n branch sets there all connected and they are disjoint also it pair vice disjoint. Now, if I contract them what will I get that is the question, now to see this thing, so we can what probably we can **we can** make to it in a smaller step. So one there is, so let us let us define the branch sets slightly differently so it takes correspondingly each red row, we have the branch set its send definitely connected corresponding to each green column we have a branch set.

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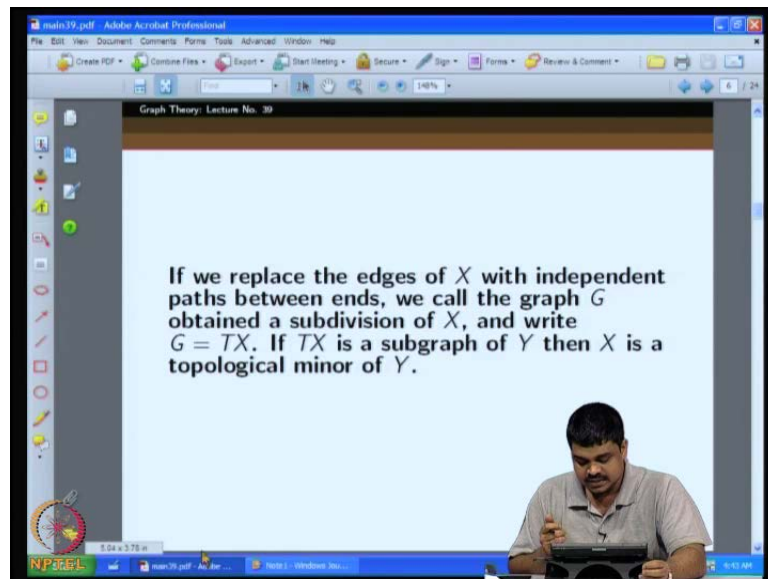
So, that means we will get a graph where is these are all red column **n** of them, and similarly, here we have green, this is red rows corresponding to each correspond to a red row, and this each correspond to a green column. And then of case we know red row intersect with each green column; therefore, intersects means that there is an edge between them. So for instance, if you have a red row like this so the green columns are green like this **right**. So, then we have the connection, so another will be going like this, so you have this connection, so another will be going like this you have this connection.

So therefore, smaller connected so we will get this graph of case there are other connection between them **between them**, but then let us discarded for the **(())**. Now, we can merge dimension before we can merge these two vertices together that means, so originally we are seeing. So, this is what this is complete bipartite graph $k \ n \ n$, and then if we merge this **this this this** together that means is correspond to the first row this the first column, this is the first row here, the first **sorry** second row here, second column

here, is the third row here, and the red and the third column in the green, if we merge it what will we get?

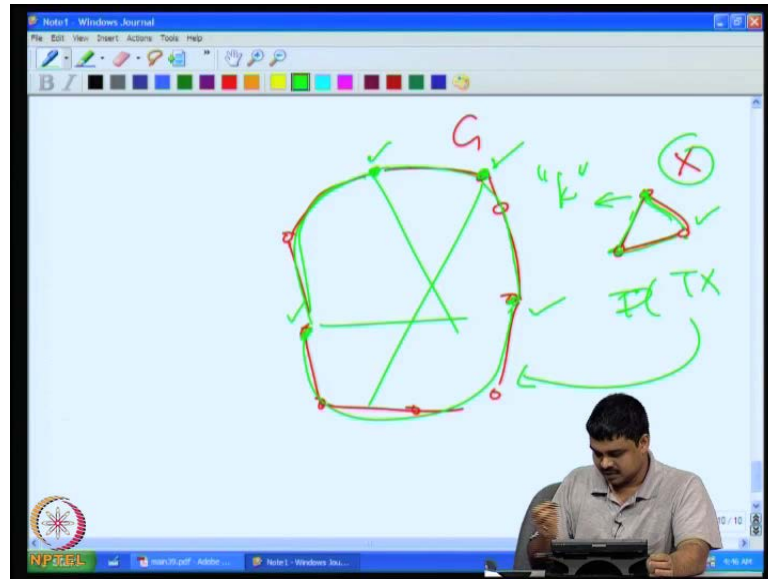
We will get a $k \times n$ **right**, because if the complete bipartite graph there so this is connected all these sides and this is connected all these sides. Now, in the all become 1 1 one of them so they will they will have connections between every pair, so we get a $k \times n$. So, here we see the concept of branch sets more clearly here, we can see what is become $k \times n$ the each vertex of $k \times n$ can be interrupted as a column, and row a red column and a red row and green column, and because the branch set, because they connected and when we contract them we will get this **get this** $k \times n$.

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Now, so this is just we let us try to some of the this thing and another concept you have to look at is the topological minor, suppose you are given a graph X with and then we replace each edge of X it independent path between ends. So, we call the graph G obtained a subdivision of X , we have seen this before also like which has to take an edge and replaced with some paths which are the number of edges in the path is depending on how many times it divide, and this is such a graph is called T of X . $T X$, so if $t s$ is a sub graph of Y then X is a topological minor of Y , **$T X$ is a sub graph of Y then X is a topological minor of Y .**

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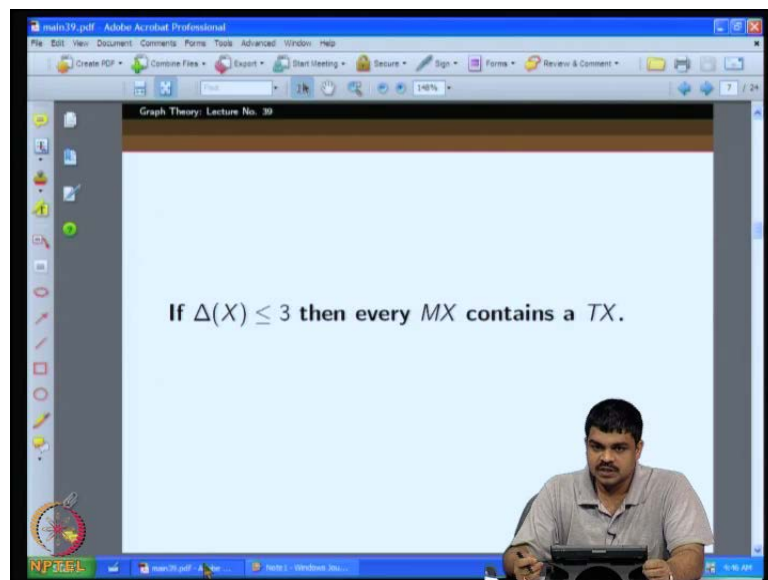
So for example, we can **we can** see that, see for instance, so this is of case this is a cycle and that triangle is a X and this is G. So we can **(())** three vertices to this, so this edge can be seen as subdivided here and this edge can be seen as subdivided like this and this edge can be seen as subdivided like this. Of case we can have other edges here, but because we only want that is T X of big cycle was T X, and then T X should be a subset of this, then we say that X is the minor of these entire graph sorry topological minor of these entire graph.

So of case if X is a topological minor of y x is definitely minor of y minor of y also y because topological minor means we can of case take the some, and we see that definitely we can contract the edges of the path and get X **right**. Therefore, if X is a topological minor of Y then X is definitely a minor of Y also the other way need not to be true for example, we see that these vertices are called the corresponding vertices **right**. We have any corresponding to each we have to fix a vertex here **sorry**, this **this** three vertices we had fixed. These vertices which correspond to the vertices of this thing **right** before subdividing the path, so those are called the branch vertices equivalent to the branch sets.

So these branch vertices, because if the degree here in x is 2. So, two path should be going lot of here for in general if the degree of the branch **branch** vertex is $k \times n$, but we should see k path at least going out of this thing, so the degree of this thing should be at

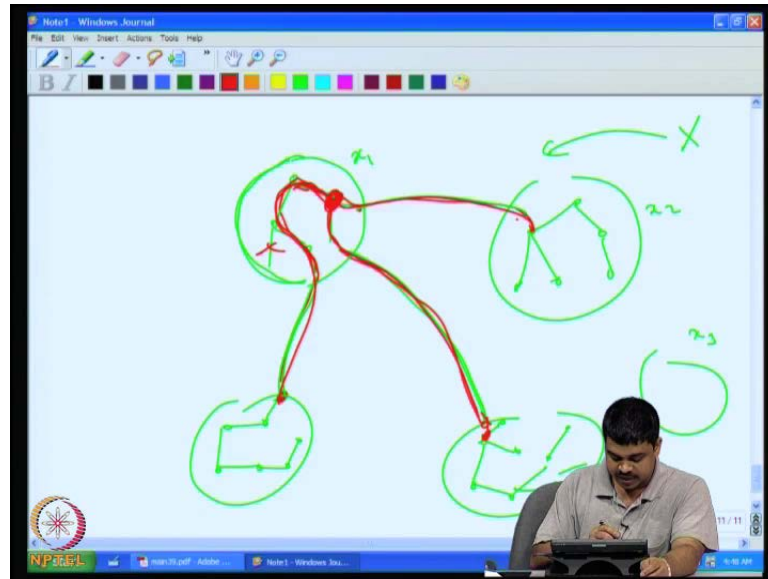
least as much as that. So, it is not possible to get for instance for if G is a k regular graph it not possible to get $k, k + 1$ regular graph as a topological minor of G because, we see this a requirement but on the other hand the minor is different it is possible that we can get the denser graph like this some minor. So, even if for instance we are getting complete graphs big complete graphs, we can get as the minor even if the original graph as path as long as there were number of edges were enough. Then the contraction allowed us to accommodate edges in the same vertices degree can increase by contraction while here subdivision operation cannot increase **increase** the degree **right**; therefore, that is clear.

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But on ((C)) we can see that in some very special situation the reverse is also; that means if we have a minor that means if we can identify branch set corresponding to given X , then we can also have a topological minor. So, its happens when delta the maximum degree of X is at most three, so how do you do this.

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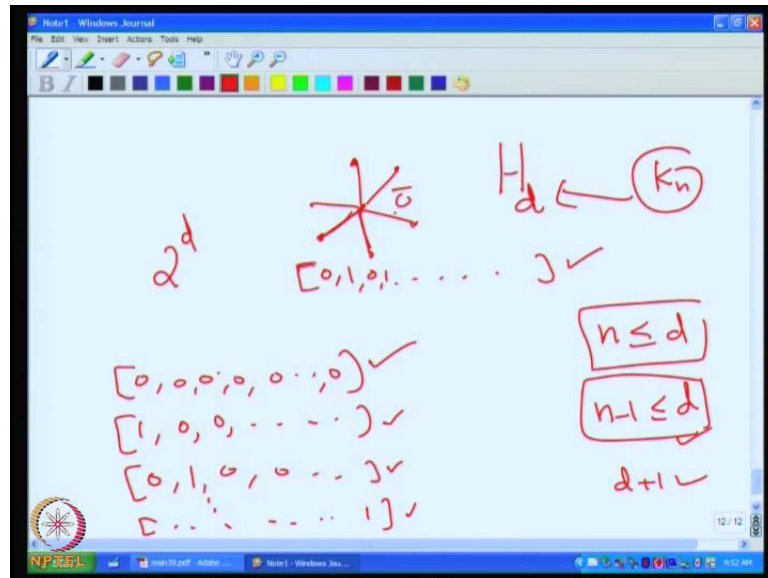
So to see this thing suppose you have we had identify this branch sets, some branch sets for X so these are the vertices of see x_1, x_2, x_3 correspond to these branch sets you know we need connectivity here, we can discard all other edges, just keep a tree here, because you know we can discard other edges, once we retain connectivity it can each branch set **right**. So, let us say this is minimal number of minimal edges we need from G , so that connectivity is there and also suppose, because whenever there is a connection between the two vertices here, edge between the two vertices they should be corresponding edge between this.

For instance, we can see this **this** may have at most three connections outside, so let say we include this vertex take this vertex also and say take this vertex also, and then now look at this tree, this again a tree this thing now, because it is a tree. If we see that between this, and this we have a path unique path, and then this will joint it somewhere somewhere **right**. So, if you take so it is easy to see now that will be one vertex such that you can see paths going out of this, and this **and this right**.

Now, everywhere if you replaced like this then you can see discarding other edges unnecessary edges which not taking path this thing, we can see topological minor we can see it, $T X$ its next contain the $T X$ that is what, but that this happen only even Δ is at most three it is it not happen a when Δ is greater than three. So, now the **the** next, so

before went to the next topic, so we can illustrate the idea of topological minor by taking one example from.

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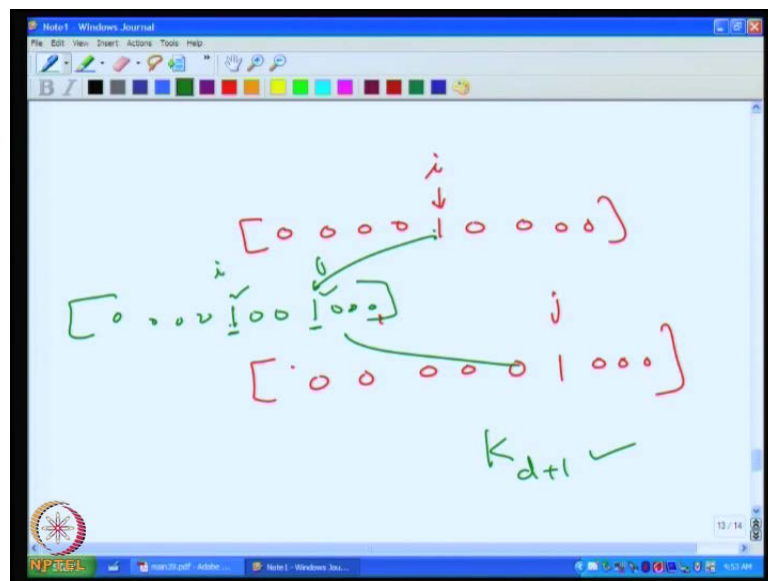


So far instance, which one example for from what we have studied earlier, so far instance look at this graph H_d which hyper cube - d dimensional hyper cube, essentially remember that this is the graph whose vertex set is 2^d vectors, d length vector like the vectors are like 0101 like the 01 vectors. So, the width d components in it, in the 2^d of the other, and when other connected 2 vertices are connected with an edge adjacent with an edge when the hamming distance is 1; that means when they the corresponding vectors differ in exactly one component position.

So we have familiar with hyper cube, so then what is the **the** biggest topological clique minor that we can get in which what is the biggest n such that K_n is the topological minor of H_d , so this question. So, immediately we can say that n has to be less than equal to d **right** sorry $n-1$ has to be less than equal to d , because here H_d the degree of each vertex is d **right**. Now, K_n means they are n vertices **right** so degree is at most d , so the **if** you take K_n then we have a n vertices in it, the degree of each vertices $n-1$ so the $n-1$ cannot be more than d as we have already mention, because from every branch vertex. We should we will see at least $n-1$ path **path** going only d path can go, so we **we** can they only at most $n-1$ path.

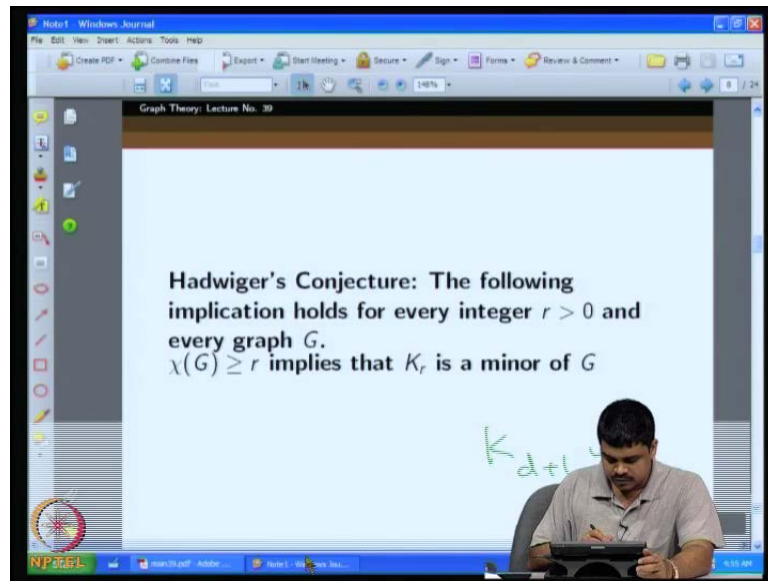
Now, let us also see a topological minor here, so take this vector so far instance one as $0 \dots 0$ this vector, and the other is $1 \dots 0$ this vector, the second is $0 \dots 1 \dots 0 \dots 0$ - this vector and finally, we have the d th vector namely **sorry** d plus 1th **right** d plus 1 th vector is this 1 . So, here this is $0 \dots 0 \dots 1$ this is corresponding to each bit position we just keep 1 all of the 0 so and greater than 1 vectors, and then now I want to show that this vertices formally topological minor. Of case this **this** central vertex, this vertex, $0 \dots 0$ vertex is a connected to all other vertices all other d vertices that we have, because the hamming distance is just 1 for all of them, and then how do show that between any pair of these two vertices.

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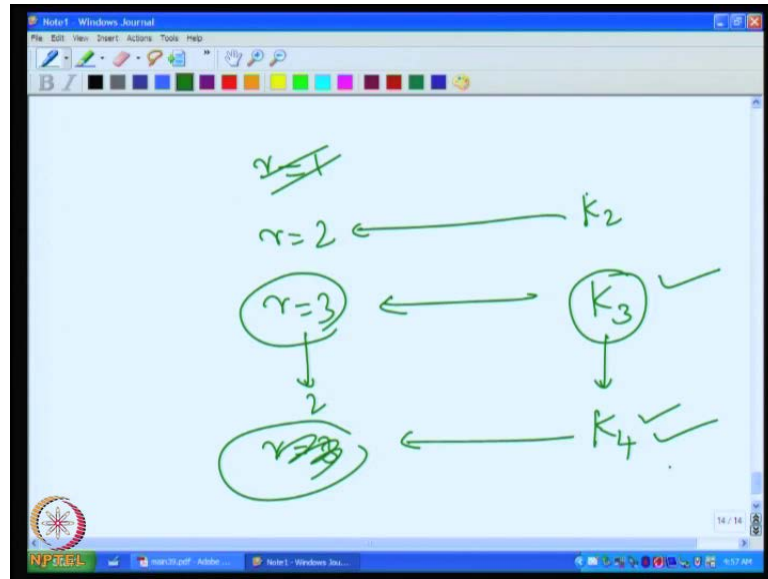
For instance the 1, so let us take one vector a $0 \dots 0 \dots 1$ th position, we have 1 and all others are 0 here, and take another vector where j th position we have all these place, we have a 1 and then all others are 0. So of case so what we can do is we can identify vector here, it is adjacent to both namely it has in the i th position and j th position we have 1, all other position we have 0 **right** this is definitely adjacent to this vector because the hamming distance is just 1. Similarly, this is also adjacent, and then clearly there is an path between this and this, and this intermediate node will not be path for any other i, j , because this ones appear in the i th position. And the j th position for any other pairs two position will be different; therefore, we see that there is a d plus 1 K d plus 1 topological minor here, and then that is the best that is maximum possible you cannot get anything greater than this, so this is clear.

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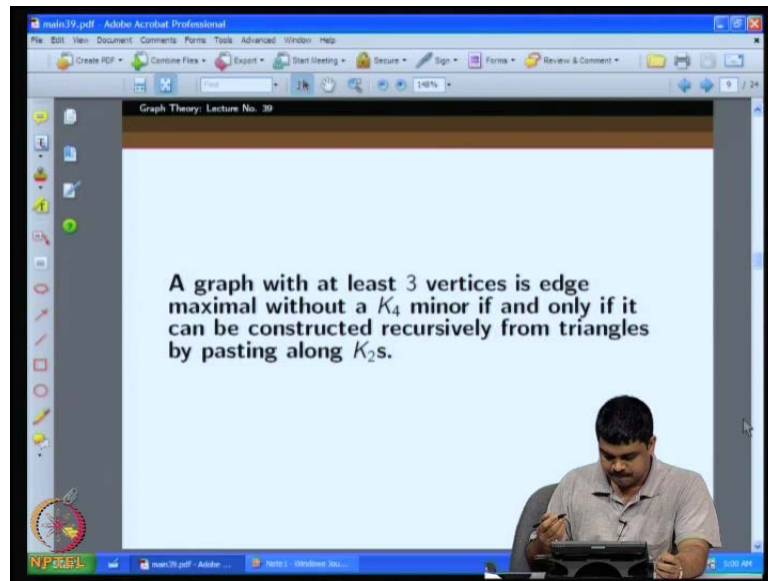
The now we will come to of case this was just introduce this concept, the minus the topological minor we just worked out some examples to understand how this what this topological minors are and what the clique minor **sorry**, usual minor are and we also looks into the clique minor which are of case easier to understand, because between every pair we need an edge every branch set we need a edge. So, this Hadwiger conjecture can be easily understood now, so this is one of the most important conjectures in graph theory, and its try to extend the four color theorem, the this question this **this** conjecture say for every integer r greater than 0 every graph G. If the chromatic number is greater than equal to r then it should have k r minor, the graph should have a K r minor if the chromatic number is at least r. So, the in other words if it does not have a K r minor then we can color vertices of the graph properly using at most r minus 1 colors. So it is why easy to verify for small cases for instance.

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Let us look at χ equal to $1/k$, that means if the chromatic number is 1; that means it is isolated vertices it do have a K_1 minor then χ equal to 2 k 's that means the of case we are saying that if we denote have a K_2 minor, so that means if you denote have a K_2 minor we should be able to color using one color definitely true, because if there is no edge we can color using. So, of case so χ equal to three k 's means, so this is not for so this is χ equal to 3 k 's means if there it denote have a K_3 minor if it denote have a K_3 minor. Then we can color using two colors 2 colors, so if you denote have a K_3 minor we can color using three colors so this is very clear because, if you denote have a K_3 minor, but we do not even have a cycle in the graph for the trees we can colored using two colors. So what about the next if you denote have a K_4 minor, then χ equal to 4 k , K_4 minor then we can color using three colors, so sorry we color using three colors χ equal to 4 k 's right.

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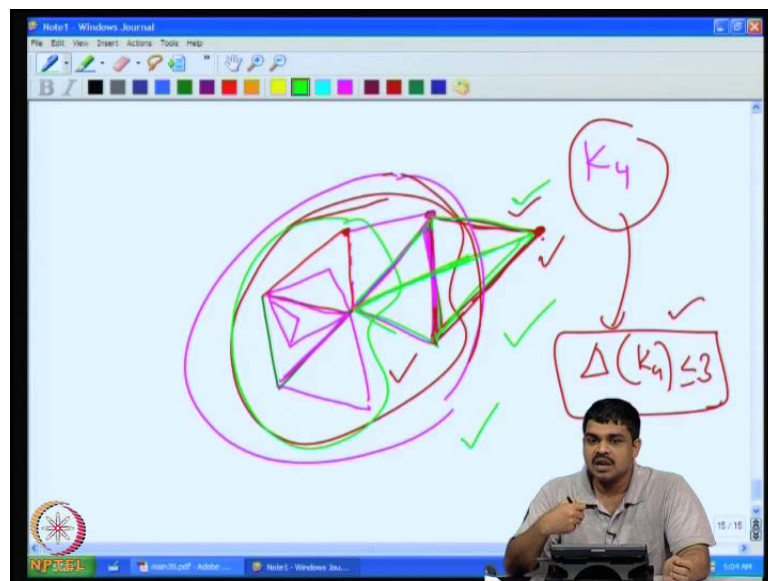


So, this can be so this needs a little more effort, so this **this** we need to understand the structure of the graph without a K_4 minor, then if we denote have a K_4 minor we need to show that it can be color using three colors. So to do this thing we have to understand this structure of the graph without K_4 minor, so this helps here is **(())** a graph with at least three vertices is edge maximal without a K_4 minor if and only if, it can be constructed recursively from triangles by pasting along K_2 s **K_2 s** the K_2 means edges. So, just K_2 s why we will look at it edge maximal graph, so we are interested in coloring graphs without K_4 minor, we want to show that a graph without K_4 minor can be colored using at most three colors. So, instead of generally looking at any graph with K without K_4 minor we are same that we will look at edge maximal graph without k_4 minor with the property, there are no K_4 minor, what do I mean by that, I mean that I keep adding edges in such a way that still in K_4 minor.

So, we come to a situation saturated situation namely any more edge in adding one more edge will introduce a K_4 minor will introduce the k_4 minor in such a case, we will have to **...** So, we will say that it is an edge maximal graph with the property, but there is no K_4 minor. So, it is enough to show that this kind of graph can be colored using three colors **right**, because if this kind of graph can be colored using three colors any graph without a K_4 minor can also be colored using three color, because give me a graph without a K_4 minor than I can always add edges to it. And make it edge maximal with respect to this property of not having K_4 minor, and definitely I have only added edges after adding a

few more edges, I can color the graph with three color the original graph also can be colored using three colors, so that is why we are looking at the edge maximal graph along. Now, so this (()) gives a structural description of that this type of graph edge maximal K_4 minor free graphs, it sense that if they exactly added graph which can be constructed move from triangles by pasting triangles recursively along K_2 s something like this.

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So we can, so we start this is one triangle, of case this is a edge maximal K_2 minor free so we can add a triangle so we can identify this edge we can paste one of the triangle along this edge. So, now this a new graph so we can of case we can another triangle and then say identify an edge may be as this one, this one, in then I can paste one here this triangle for I can paste another one here like identifying this was the edge to paste or we can identify this edge once again. And then paste one triangle here, so I can paste one here, now we can paste one here, so this way so I can paste one here is identify this edge, this **this** kind of graph are called so that graph which are obtain by pasting triangles recursively (()) so, we are we are not along to paste like this **right** because, that is like identify a vertex alone.

So, what identify edge it paste, now why do we thing that this graphs which are edge maximal without K_4 minor, of case they cannot be a K_4 minor here, so if there is a K_4

minor and suppose up to here up to some stage, we do not have a K_4 minor, and then we suppose decided this edge, and then we pasted it triangle here.

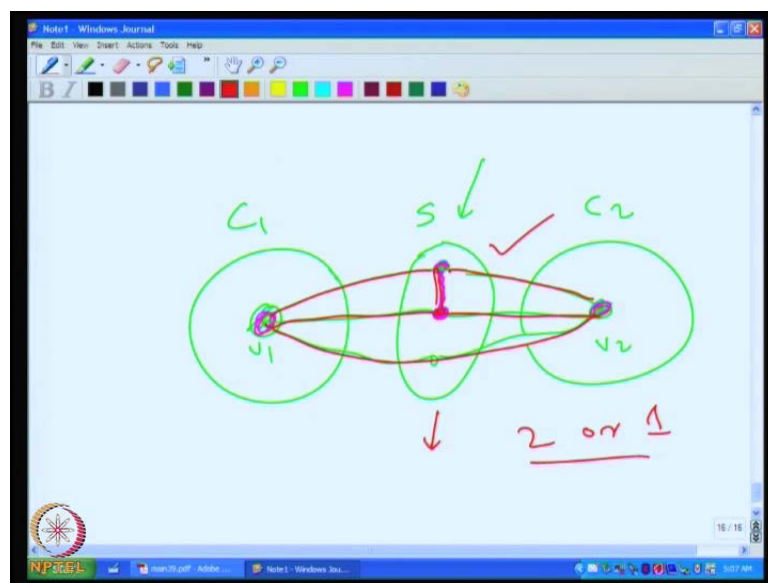
Now $(())$ sudden a K_4 minor appeared, so how will the minor look if k_4 minor appear because, a K_4 minor has maximum degree of K_4 is less than equal to three **right** is three only. In fact therefore, then they should be a K_4 topological minor also **right**, so is it possible to have this is a branch vertex here, if this is a branch vertex the new one, then the other one has to be the other three has to be somewhere here **right** inside 1 2 3 somewhere here in this portion, but then there is only the two edges going out of this thing it is not possible to half this branch vertex. Now it should be all the branch vertices should be here, but then what is the use of having this, we have we can instead of going why are this we could of always use this edge.

And then we will get that K_4 topological minor in the part itself. So therefore, that would be a contradiction that inductively, we assume that earlier they was no K_4 . Suddenly K_4 minor cannot exist in the new graph, and also its edge maximal because, here it is was edge maximal. Now, if this is not edge maximal the edge that we can add without introducing a K_4 it has to be something like this.

Now, clearly you see there is already a triangle here **there is already at triangle here**, and this edge if you contract this entire remaining part will give us the **the** new K_4 **right**, so because they should be some connection between this, and this here **right**. And then also there is a some connection here. So therefore, I so we can we can prove that we can **we can** get a if such a if it is not, if we can add an edge without introducing, so we **we** cannot add any edge without an introducing a K_4 minor. So therefore, this is edge maximal K_4 minor graph definitely that is not the key issue, the issue here is essentially, so these graphs are indeed edge maximal with the property that there are no K_4 minor in it is, but any edge max what we want to say that any edge maximal graph without K_4 minor will have this structure namely the obtainable by pasting triangles on K_2 s **right** recursively. And how do I proves that so **to** to show that so we just need required easy, but I will just start it of but will not do it completely, because we do not how much time now so to see this things, if we can see that suppose we are given an edge maximal graph without a K_4 minor.

Now, can it be a complete graph know if its complete graph it has to be K_3 only, because only three vertices should be there, but if it is a three vertices then definitely it does not have a K_4 minor. So therefore, at least four vertices then it cannot be a complete graph, of case if it is just a triangle, we are through, because it is one triangles like we say it is **it is** pasting of triangles. And what show whatever as we as we required it is **it is** one triangle, but if it is more than a triangle that means four vertices are more, then we cannot have it a complete graph, because then K_4 minor is anyway there, so it is not complete graph; therefore there is a separator.

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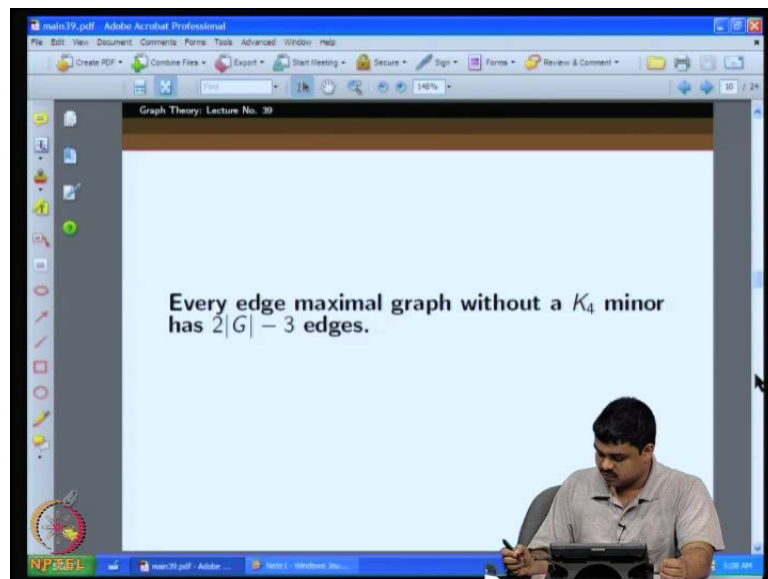


And then this separator cannot be too big, if it is three or more. So let us take two components **right** now some vertex this components, and some vertex in this component say v_1 and v_2 , so they will three paths disjoint path between them **right**. Internally because, of the connectivity, because of the minimum separator we consider so with connectivity is three or more they should be three length path. Now, suppose if I had removed these two vertices still the graph will remain connected, because the minimum separator size is three by removing just two vertices, we cannot disconnect the graph.

So, we can assume that there is one more edge say between one more path between this, and this so between any two of them. So, one pair we will still get some connection let take this thing, now taking this **this** and this vertex **right**, so we already get a K_4 minor and this, this vertex so we have a K_4 minor here. So **right** is the triangle, and so this

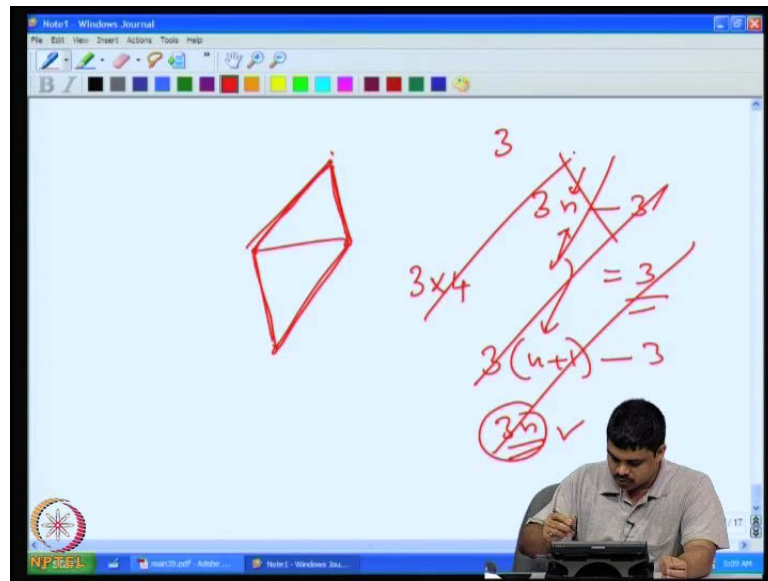
vertex is connected to this, this and this so we have a K_4 topological minor here. So therefore, we can see that that is not possible to have now we can assume that the connectivity of this graph is K_2 or 1, so and then we can analyze again based on the edge maximality and K_4 minor and then show that it is it will **it will** lead to the contradiction, so I will leave it to the student for working that out. So, we will the point is anyway we need not worry too much about the proof of that it, because easy so of case some technical details that is all.

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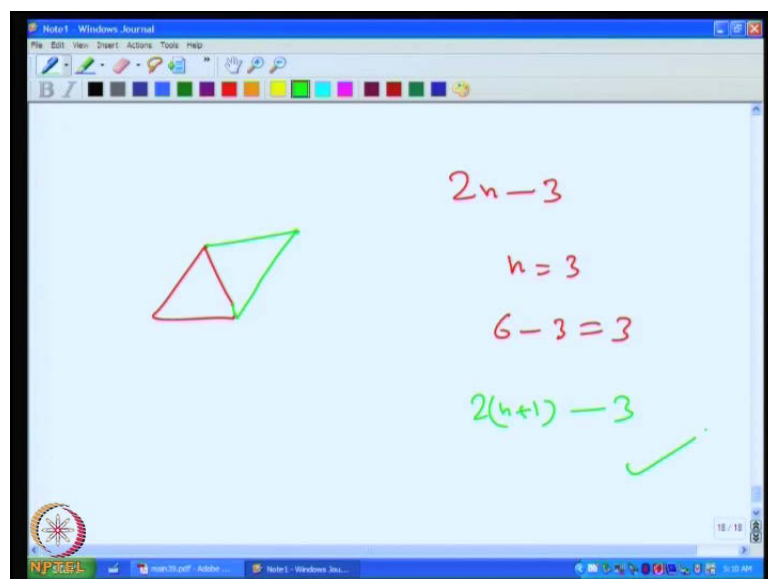
So, we **we** will go to the next statement, what we will what this means is that one of the implication is that see then any edge maximal graph without a K_4 minor has at most 2 into number of vertices minus three edges, why is it? So, this is also a simple induction.

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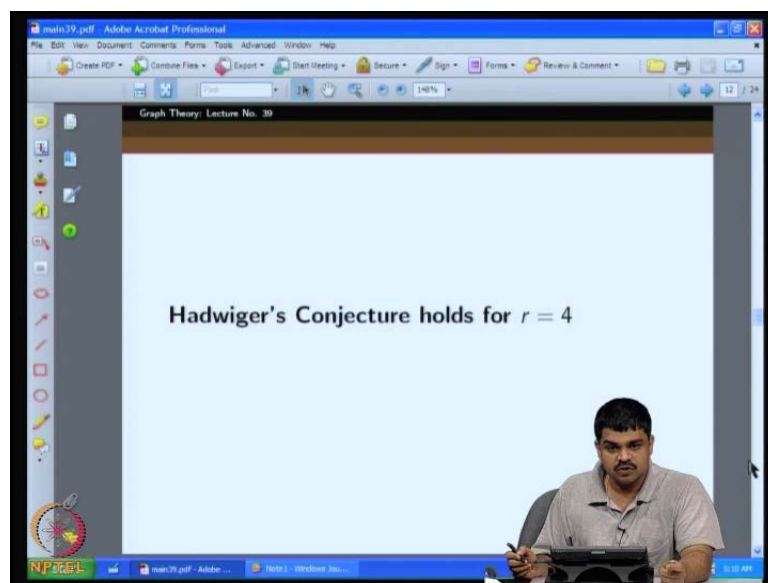
Because you know it start with a triangle, only three edges for three vertices that is three n minus three put n equal to 3, we get three **right**, so that is correct. Now, if paste another triangle, so what do you **you** are introducing in one more vertex, but only two edges **right**, so that is like here we are introducing; of case so 3 into n plus 1 minus 3 equal to three n minus 3 n , 3 n edges is what we are expected have by the statement, of case we are only increasing two edges here, so we are not increasing three. So, **right** for instance the four vertices is what how much we have so four vertices three into four **sorry (())** the number is wrong. So 2 n minus 3 edges what we need, so this is wrong.

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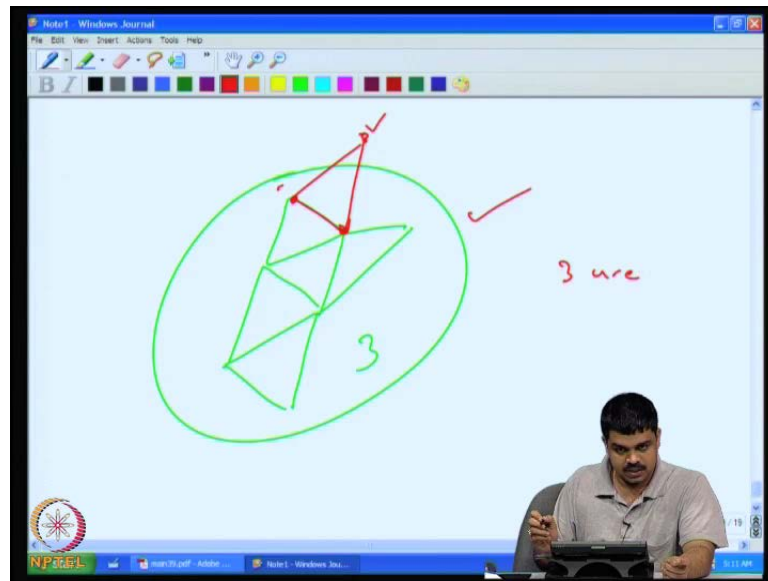
So, the what again we want to show that $2n - 3$ edges **right**, first let us **first let us** let us for the triangle, put n equal to 3 so $2 \times 3 - 3$, so that is $6 - 3$ equal to three that is correct. Now when I am **when I am** pasting a new triangle so we introduce two edges that is becomes $2n - 1$, and **and** this minus 3 that is for **for** every new vertex here only introducing two edges, so this will be true; so this by simple induction it is its verified. Now, so that was just some consequents and immediate consequents if there is no K_4 minor in the graph then the maximum number of edges, you can get in such a graph is $2n - 3$ where n is the number of vertices.

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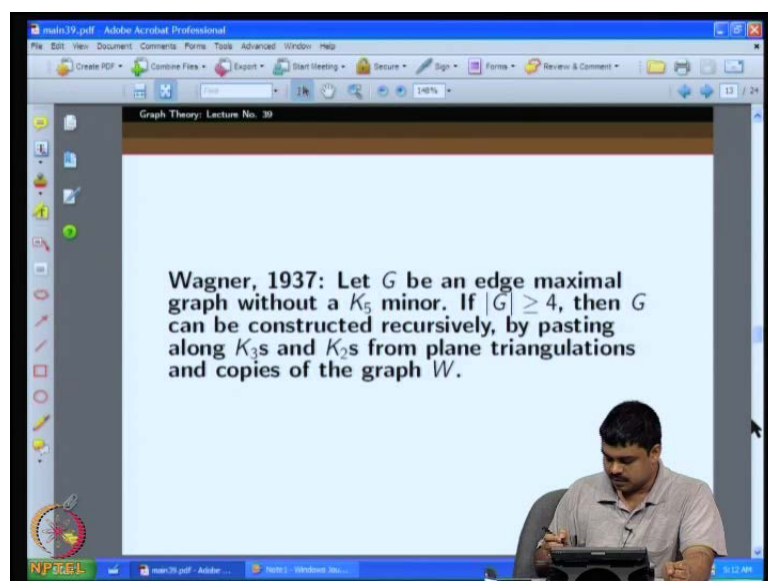
So the point now is Hadwiger conjecture will holds for the r equal to 4 k why it, so because you know the K_4 minor of graph is a K_4 minor of graph I want to show that it can be colored using three colors. So, what we do is we add edges, and make edge maximal, because if the edge maximal **maximal** graph will be colored of case the original graph can be also colored.

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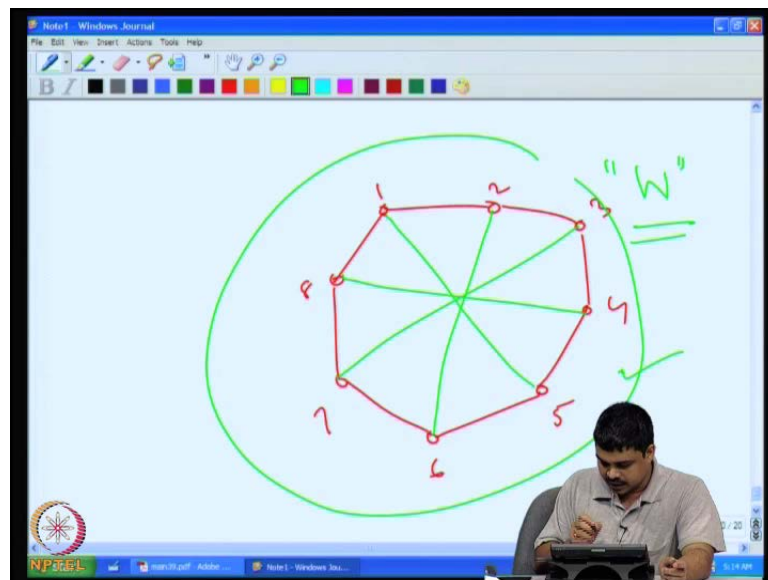
But this edge maximal graph can be constructed by pasting together the triangle, so this is like a just pasting together the triangle now this is colored using three colors, now when you add one more vertex, one more triangle, so definitely so this color, if we gives to the two vertices. And then the third color given to that which very easy to see that always this edge maximal graph can be colored using three colors, so the Hadwiger's conjecture is to for is K_4 ; that means when K_4 minor is not there it can always color using three colors.

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Now then next case is when K_5 minor is not there the sit, we want to show that Hadwiger's conjecture in this case, we want to show that any graph without K_5 minor can be colored using four colors. See of case, so this is requires as structural theorem like this we need to understand this structure of the graph without K_5 minor, it is provided by Wagner in 1937. Again we only on to consider the edge maximal graph without K_5 minor and he told it characterized the graphs without K_5 minor, and edge maximal with respect to the property. So, he says let G be an edge maximal graph without a K_5 minor if G is greater than equal to 4 then G can be constructed recursively by pasting along K_3 triangles and edges k_2 s from plane triangulations and copies of the graph W , what is the graph W ?

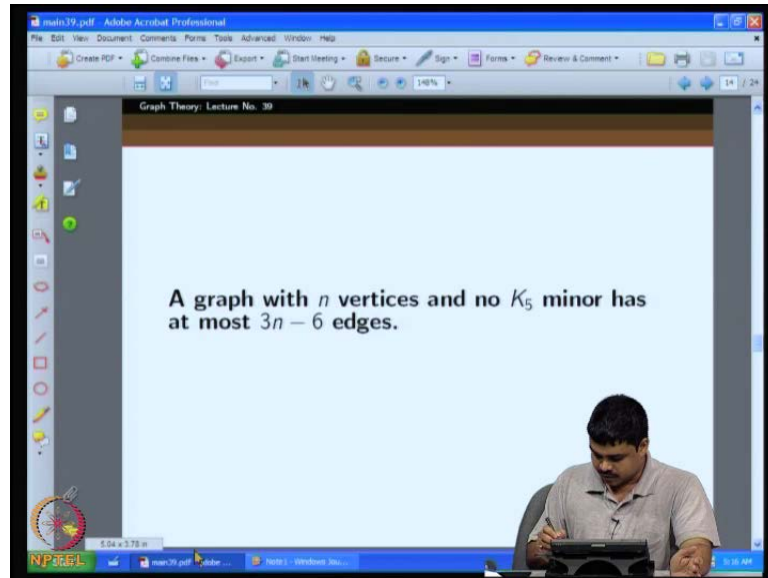
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So, this graph W is a special graph which is eight vertex graph 1 2 3 4, 1 2 3 4, 1 2 3 4 5 6 7 8 so and then there is connection between these two and there is connection between these two, and there is connection between these two, and there is connection between these two, this graph is called W , say eight vertices on it. It can be drawn in several different ways, but let just because we do not have a that much time we will just take this drawing, so this is a special graph so he says take this. This is non planar graph, but this is only non planar graph had to bother about, when we are considering the edge maximal graph this non planar graph. And other than that any edge maximal planar graph that means the plane triangulations, edge maximal planar graphs are essential the plane triangulations, why because when keep **keep** adding when **when** you add an edge.

And if you lose planarity then K_5 yeah, so that is not direct need the proof of that, but because on $K_{3,3}$ the K_5 minor make them, but so we he proves that so in that case including this thing.

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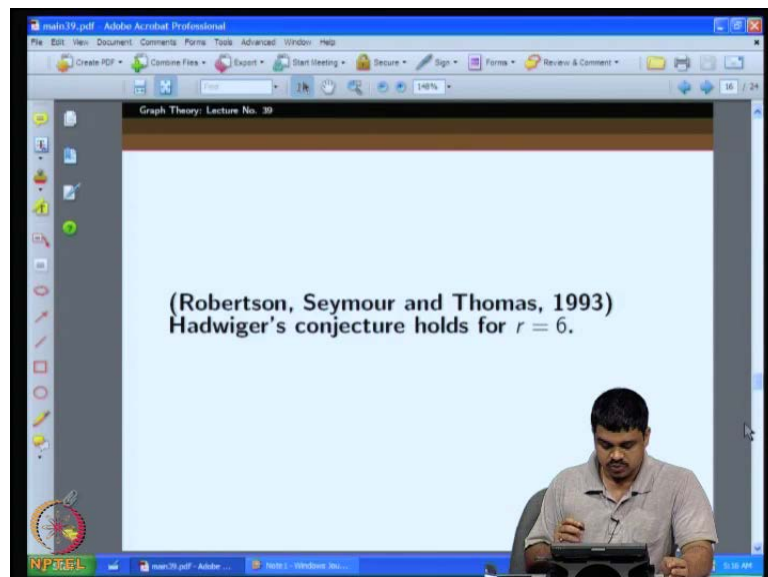
We can make sure that that is only we need bother about only plane triangulations and this particular graph, so these are the edge maximal cases we can what we can do is we can paste them along like in the earlier case, we had pasted along K_2 s that edges now we have to place along paste not only on K_2 , but also on K_3 if necessary, if two kind of pasting are allowed on triangles we can identify a triangle in this graph, and then in the graph. And then and merge them together by putting the vertices of the identifying the vertices that triangle in both the graphs, so are like we did in the earlier case, this is the only way we get edge maximal graph without a K_5 minor.

And from this thing you can immediately conclude two things: One is how many edges will be there a graph with n vertices is no K_5 minor has at most $3n - 6$ edges the same kind of induction, we did here it little more complicated calculations are required, but this is by induction we easily to this thing. And the other thing we conclude is that Hadwiger's conjecture Hadwiger's conjecture holds for r equal to 5 K ; that means if K minor is not there we can always color with four colors, why because each other components. That means we are constructing this graph edge maximal graph without K_5

minor graph will plane triangulations which are obviously colorable using four colors by four color theorem.

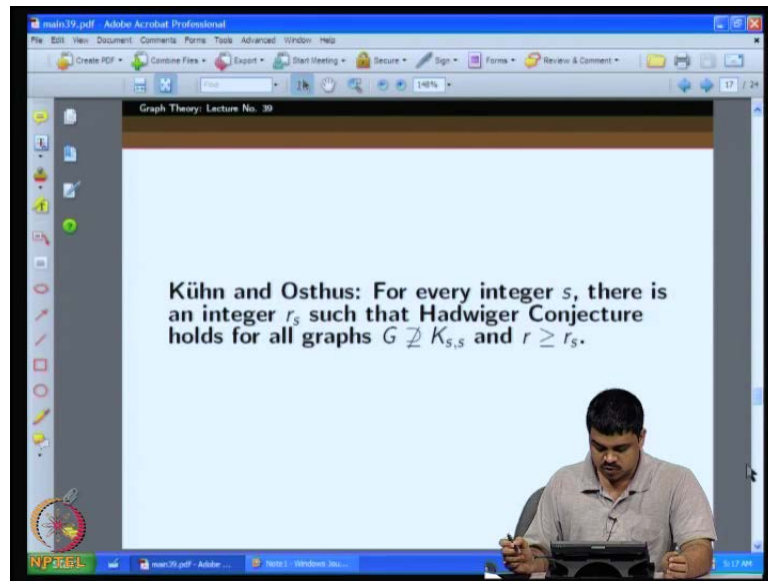
We have already studied that and this particular graph this **this** graph **this graph** definitely can be colored using four colors, and now we are pasting only on the triangles or an edges. So, when we pasting we can rename the color in such a way that the both the both sides are consistent. So, this technique is very easy I leave this student, leave it the student to verify; therefore, in this case also it shown that the Hadwiger's conjecture prove it.

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Now, the next case proved by Robertson Seymour and Thomas in 1993; that means if K_6 is not there you can color it with five colors base by extending the four color theorem. We uses the four color theorem for the purpose of this thing they uses, so but it is considerably more difficult.

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So that is all about Hadwiger's conjecture, so there are some other special cases also like (()) proofs is result. So, we will continue in the next class.