Graph Theory
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Lecture No: # 03
More on Hall's Theorem and Some Applications

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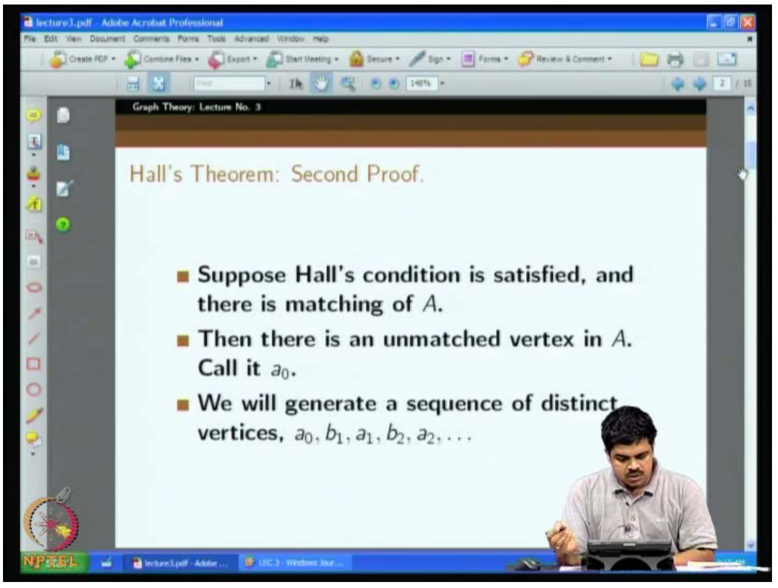

Welcome to the third lecture of graph theory. In the last class, we had discussed Hall's theorem and it is proof. Now, we will see another proof of Hall's theorem; before that let me remained you, what Hall's theorem was… So, it is about a bipartite graph.

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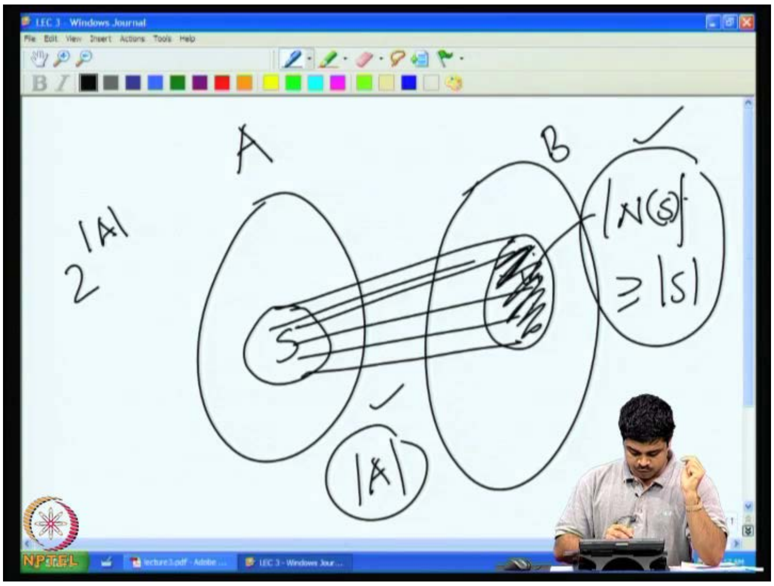

So, let this is a bipartite graph. This set is A, this set is B. Now, Hall's theorem says that if Hall's condition is valid for this bipartite graph, then there will be a matching of a; that means, there will be matching which matches all the vertices of A.

So, what is Hall's condition? Hall's condition says, if you take any subset S of A here, you can take any subset S of A, there are two raise to a possible subsets. Any subset of A if, A if we consider and count the number of neighbors that S has the neighbors means, the vertices of B which is at least one connection to S. If you count this set, and if it is so happens that, the number you get; the numbers of neighbors here is greater than or equal to the cardinality of S. If this is true, then we said the Hall's condition is correct where satisfied this. Hall's theorem says, if Hall's condition is satisfied, then there will be a matching of cardinality A in this thing cardinality A in this sectum, in this bipartite graph.

Now last class, we proved this theorem using a theorem which are which we had already prove - before proved proved before that which was the Konig's theorem. Konig's theorem was about the vertex cover, and the cardinality of the minimum vertex cover, and the cardinality of the maximum matching in the bipartite graph. We use that to deduce Hall's theorem in the last class. Here there is a direct proof. So, we will go through the proof. So, what we are going to do is, to go suppose it is not to go by contradiction; the suppose this theorem is not true; that means, the Hall's condition is valid, but we do not have a matching that matches every vertex of it.

In other words, there excess at least one vertex of A which is unmatched. Let us call it A. So, our plan is to construct a sequence of vertices see a 0, b 1, a 1, b 2, a 2 like that. So, a 0 from the side A which is unmatched; b 1 from the side b which is matched which we will make sure that that it is going to be matched. Then a 1 from the side a, and from b 2 from the other side a 2 from this side.

Like a sequence of vertices which are distinct right; that means, we would not repeat the vertices in this sequence. And show that we can get a never ending sequence which is definitely not possible, because only there are finite numbers of vertices in our graph. Therefore, that will lead to a contradiction. So, how we are going to this things, I will explain.

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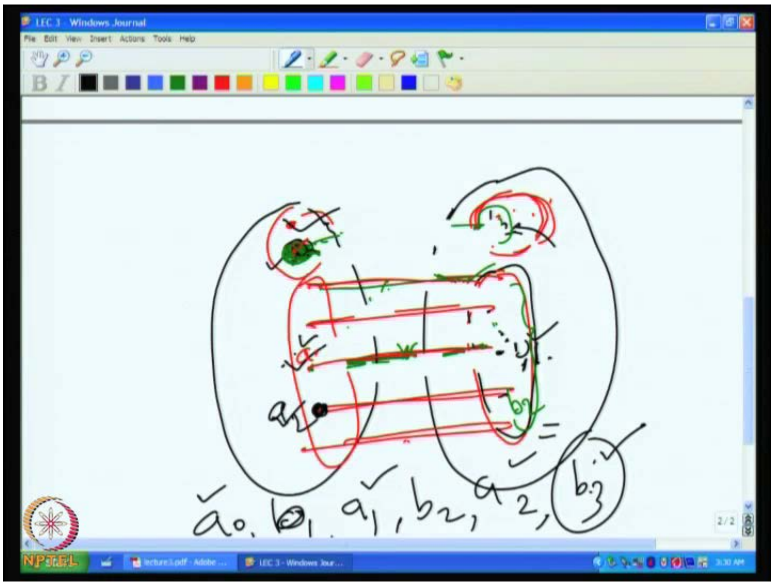

Consider this bipartite graph, where the Hall's condition is satisfied. We considered the maximum matching in this bipartite graph now. So, let us mark the edges of the maximum matching, the biggest matching possible.

So let us say, these are the vertices which got matching from this set; these are the vertices which got matching from this set. Now, because we have assumed that the Hall's theorem is not true for this graph; that means, Hall's condition is satisfied still if there is no matching which matches all the vertices of there, they should be some vertices here right. There can be vertices here also unmatched, these are the unmatched vertices, these are the unmatched vertices on the this side A side.

Now, what we are going to do is is to show a contradiction; we told that, we pick that an unmatched vertex first, let me call this a 0. Now, my plan is to find a vertex which I will call b 1 on the b side; somewhere I here, somewhere here on this side I want find vertex b 1 right.

So, but the essential condition I am looking, when I am seeking looking for b 1, I want make sure that it is a neighbor of a 0. So, they should I am searching for a vertex here which is a neighbor of a 0, and the question first is it possible to pick up b 1 here, such that b 1 is a neighbor of a 0. It is not possible to find anything in this collection which is the unmatched vertices of b, anything in this collection which is which are the unmatched vertices of b, because if there is an edge like this, if there is an edge like this, definitely we could have added it to the exiting matching, and then you would have got a bigger matching. So, that is not possible right.

So, what we can says so, they they would not be any vertex of that is what. Now, so then, is it possible that there is no neighbor of a 1 at all in this b 1; that means, we cannot find a vertex b 1 on this site right; that is also not correct, because since Hall's condition is correct, a this vertex a 0 should have a neighbor somewhere this vertex a 0, should have a neighbor somewhere here on this side. Let us say, yes yes we have told any neighbor of a 0 has to be in the in the matched in this collection only, it is not possible to have in a neighbor of a 0, in the unmatched among the unmatched vertices.

Let us say this is b 1, we pick up one. So, the yes yes you can to remember the Hall's condition says that if you select any set of vertices, it should have at least the number of neighbors it has on the other side should be at least as much as the cardinality of that vertex. So, that of that set we are considering. For instance, if you had consider the singleton set, then will be at least one neighbor; for if there is no neighbor Hall's condition will not be satisfied for that singleton set. This singleton set will violated right.

Now, so let as pick up therefore, this vertex b 1 and write (a 0,b 1) sequence now. As soon as, we got this b 1, because we know this is a matched vertex. So, we can take the partner of it on this side; partner of this on this side. So, let us say this is a 1, let us call this a 1. So, that a 1 will b the next vertex here. Now, we have selected a 0 this one and a 1 on this side, and b 1 on this side right. Now, our idea is to find out a vertex on the b side, such that that vertex is a neighbor of at least a 0 over a 1 one of them, may be both of them; so, a neighbor on the b side which I will call b 2 later, now such that the b 2 is a neighbor of either a 0 or a 1. So, how will I do that.

So, first question is is it possible to have such at least one neighbor for a 0, and a 1 together (( )) which is not b 1. So, if that is not true, b 1 is the only neighbor of a 0 and a 1 together, then Hall's condition will be violated why, because if you consider this set of two elements a 0 and a 1; then according to Hall's condition it should have at least two neighbors on the other side. So, if b 1 is the only neighbor that a 0 and a 1 together has on the other side, then it will definitely valid the Hall's condition. Therefore, there should be one more vertex on the other side which is a neighbors. Let us call it b 2, let us call it b 2 that is call it b 2.

Then, now the issue is where is this b 2; is this b 2 an unmatched vertex here or a matched vertex here; can it be an unmatched vertex here. Is it possible that our b 2 is in this set; that means, can b 2 sit here instead of here. So, is it possible to have b 2 there, instead of here. So, but we see that if b 2 is here, then suppose b 2 we see that b 2 b 2 is connected either a 0 or a 1, as we have already told is not possible to have a connection between b 2 and a 1, a 0, because if say this connection was there, if this if this connection was there; then already we we can make the matching bigger by adding this green edge to the already existing this red edges. So therefore, that is not possible.

So, what we see is we can connect. So, so possibly b 2 is a neighbor of a 1- possibly b 2 is a neighbor of a 1; suppose it is possibly like this. Say, suppose b 2 is a neighbor of a 1, is it possible. So, in that case what we see is see follow this path. So, from a 0 we can reach b 1 and from here, from here we can we can come like this to a 1, and then from here we can reach here right. So, this is definitely an alternating path. In fact, if you remember what we defined last time is called an augmenting path. Why is it an augmenting path - it is starts from an unmatched vertex on the a side, it reaches it follows an unmatched edge and reaches a matched vertex on the b side, comes then travels via a matching edge then travels by an unmatched edge, and reaches the unmatched vertex on the b side.

So, this is an alternating path, because it starts one an unmatched vertex on the a side, and reaches an unmatched vertex on the b side and by travelling, traversing unmatched edge, and then matched edge, unmatched edge, matched edge like that.

So, but then we know that if there is an augmenting path, the matching we have already considered; this red edges that is not a maximum matching. Why, because we could have replaced this matching edge, this matching edge by the this two other edges in the augmenting path. We could have replace things, instead of this we could have replace these thing. So, then these vertices will also get match, this new two vertices also will get matched. So, we will get a bigger matching here using this augmenting path. So, before it is not possible to have b 2 outside. So, b 2... So, the inference is that b 2 is also somewhere in this area; that means, b 2 also should be among the matched vertex of b. So, let us erase the unnecessary things now. So, all this right.

So, so so this is our original matching. Now, what we have seen is, we have a b 2 here on the b side. So, I can write b 2. And b 2 is a matched vertex before as its partner on this a side, say lets it will called a 2; and then that will be added to the next one.

Now, will again ask the same question - can I find b 3 now right. Which what other properties of b 3 is requires, b 3 want we we want b 3 to be a neighbor of either a 0, a 1 or a 2, a 0, a 1 or a 2 right. In such a way that; that means, a neighbor neighbor of this three elements set right, will we get a b 3 yes, because the Hall's condition is true, we have to get one b 3 which other than b 1 and b 2; why because if b 1 and b 2 are the only neighbors of (a 0,a 2,a 1,a 2), then Hall's condition is violated, because there are three elements in this set; there only two elements in the two vertices in the neighborhood of that. So, they should be at least three, they should be one more.

And show let it be b 3, but this neighbor the again the question is can it be unmatched. So, if it is unmatched, as we argued before it it will give us augmenting path why, because depending on b 3 is, if b 3 is connected to a 0, we immediately have a contradiction; if b 3 is connected to a 1, then as before we had a augmenting path like that a 0 to b 1, b 1 to a 1, and a 1 to b 3. If b 3 was connected to a 2, then we have an augmenting path like a 0 to b 1, b 1 to a 1, and a 1 to b 2, and b 2 to a 2, and then a 2 to b 3.

So, that will be a slightly longer augmenting path. Depending on which vertex it is connected; we will get augmenting path, because every vertex we select on the a side; that means, a 0, a 1, a 2 etcetera - will all end in one alternating path. It will be ending in at least one alternating path, and then if it is connect this b 3 is connected to the new vertex on the other side is connected to that vertex, this is alternating path can be extend at an augmenting path.

And once we get an augmenting path we know that, we will get a bigger matching by replacing the red edges in them by the other edges right, as we already discussed. So, that is why we can always get a new b i. So, going back to our slides.

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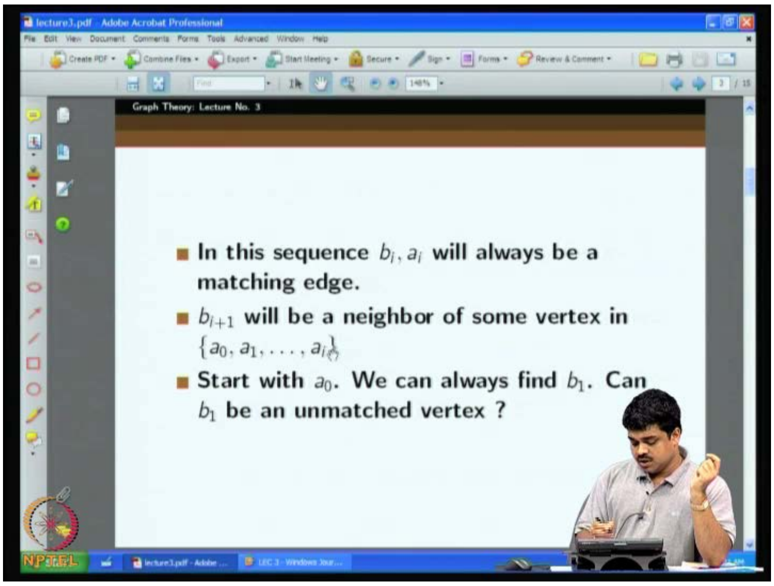

So, we will see that once we get a 0, a 1, up to a i; we can always find out find a b i plus 1; such that b i plus 1 is a neighbor of one of these vertices a 0, a 1, a 2, up to a i; that is always guarantee it, because of the Hall's condition.

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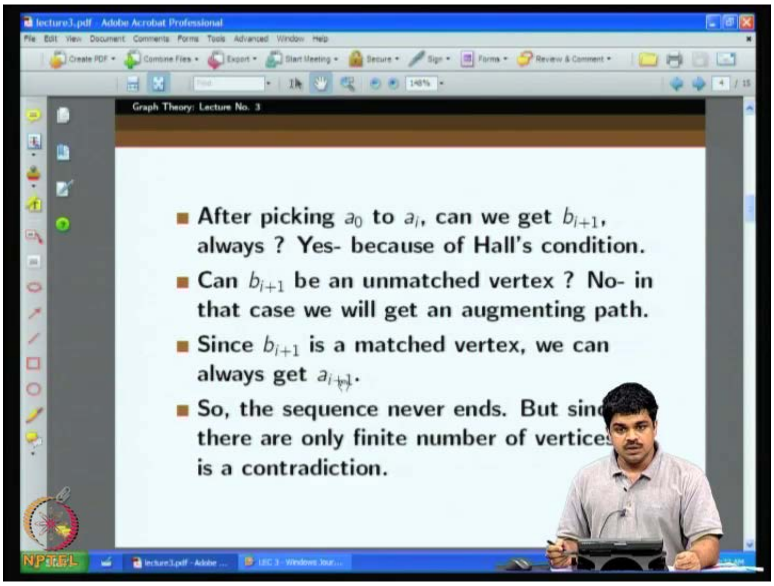

Now, the this b i plus 1 cannot be an unmatched vertex, because if b i plus 1 is an unmatched vertex, then we will end up with an augmenting path; which contradicts the assumption that we have already taken the maximum matching. So, so we can once we get a b i plus 1, it is a matched vertex and then we can always get a i plus 1 by come traversing back through that the corresponding matched matching edge right.

So, the its partner b i plus one's pair partner on the a side will be a i a i plus 1. So, it happens what we see that sequence will keep on going; it will we can we we do not say an entry right, because there is no reasons for it to end, always we will find b i plus 1, always we will find a i plus 1, all are distinct; how is it possible - it is definitely not possible, because we do not have some many vertices on the graph, it will finish off right the matched vertices will finish off that time, at some point of time.

So therefore, at some point of time we should run out of the matched vertices on the b side, and we will not get a new b i plus 1. So, what is the contradiction? It is a the only the the contradiction, the only reason for the contradiction is our assumption that, we could pick up a 0 to start with, because that is the extra vertex right.

So, that a 0 was an unmatched vertex. Though so, it we can (( )) that, we can never pick up an unmatched vertices; that means, all the vertices were matched by the maximum matching; that means, we really had a matching of a in the graph. So, that is the inference. So, this is a another proof of Hall's theorem; that means, if the Hall's condition is satisfied, then there will be a matching of a in the bipartite graph.

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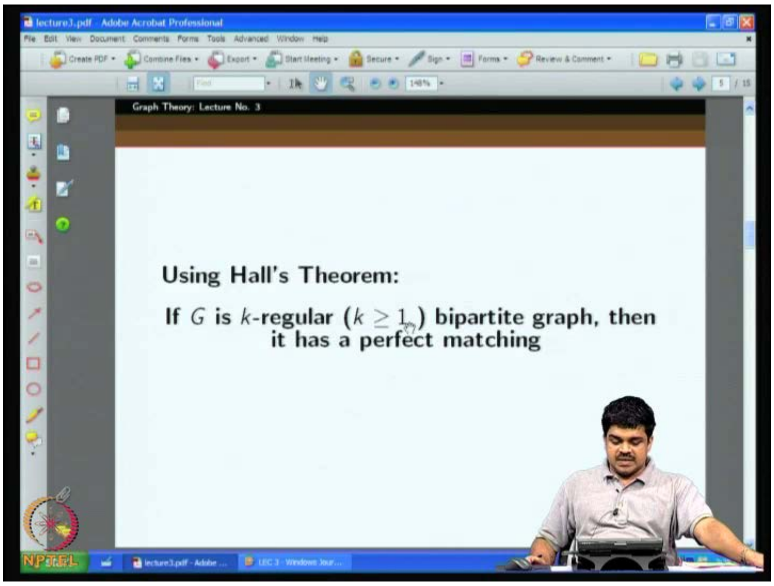

Now, the next thing is to see how we can use Hall's theorem to infer some interesting results. For instance, see when it is come to checking whether the Hall's condition is true or not. So, it will come to verify whether each subset of the a side, that a side mention there are two to the power cardinality of a subsets.

So, all of them satisfies the Hall's condition; that means, the neighborhood is greater than or equal to the cardinality of a, but that is a tedious job, but of course, there are some situations, some in some special cases, we can immediately see that bipartite graph will satisfied the Hall's condition; for instance one such cases when the bipartite graph is k regular. So, I can remained you what k regular means, k regular means k regular means - all the vertices of this bipartite graph has degree k, degree equal to k, degree means a number of edges incidents with each vertex is k; I is in k greater than equal to 1, otherwise definitely they would not be any Hall's condition also will not be satisfied, because they would not be neighbors right.

So, then we want show that that, here in if I for k regular bipartite graph and k greater than sorry. So, the what I meant to say is, if k was equal to 0 then definitely they want to be a perfect matching. So, if k is greater than or equal to 1, we want to say that there is a perfect matching; how do I show that.

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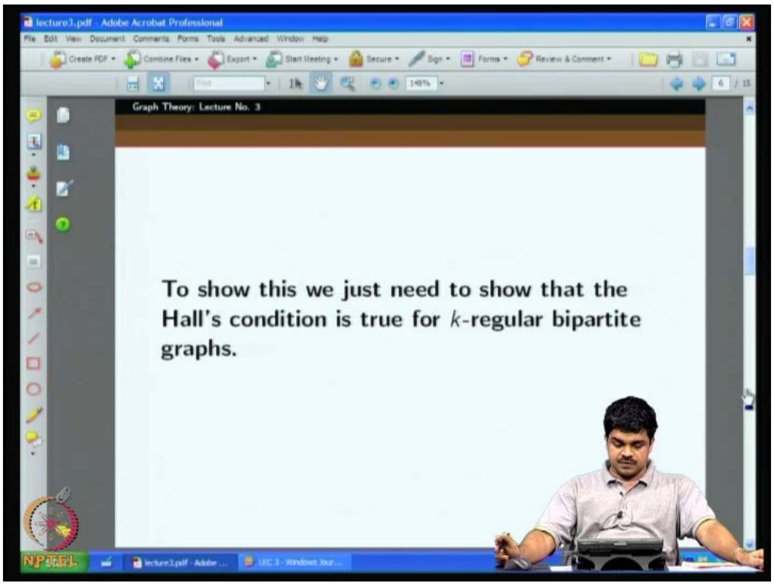

So, it is definitely from if you want to apply Hall's theorem do this things; it is very easy to see that what we have to show is the Hall's condition is true for this any k regular bipartite graph. If the Hall's condition is true automatically there will be a perfect matching; that is what Hall's theorem says us. So, how do we show that the Hall's condition is true for k regular bipartite graphs so, to show this thing.

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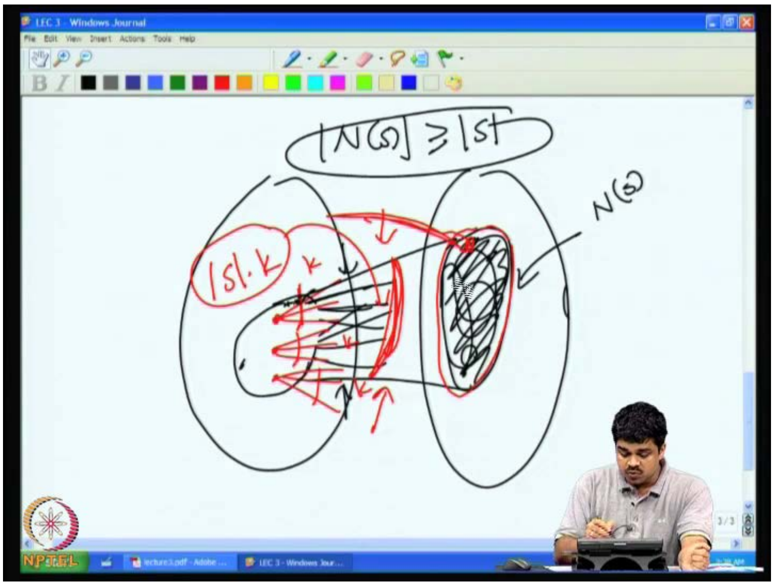

So, let us take up an example again a picture let us consider. So, here is the bipartite graph; it is suppose to be k regular bipartite graph. Now, take an arbitrary subset S of it; let say this is an arbitrary subset S of it. Now, I want to show I I is look at the neighborhood of this - neighborhood of this means, I collect all the vertices it is connected to at least one vertex in S. So, all the vertices which is connected to at least one vertices in S. This is what? This thing S, this is called N of S right, neighborhood of S.

Now, I want to show that the cardinality of this the N of S is the cardinality of this N of S is at least as much as S, we want to show that N of S cardinality is sorry. So, the cardinality of N of S, this is in the cardinality of N of S is greater than or equal to S is what we want to show, this is what we want to show.

To show this thing, the technique is the count number of edges that goes out of S; that means, we ask how many edges are such that it has one N point in S; naturally this can be easily counted, we ask how many edges of going out of one see this vertex from S and then another vertex you take. So, like that we can we can ask for instance. So, if I mark this, the vertices in this thing. So, this the vertices in this set S this one.

So, now I ask how many edges are going out of this, how many edges are going out of this, how many edges are going out of this. So, definitely there are k of them here - there are k of them here, and there are k of them here. And totally definitely cardinality of S into k edges will go out right; the number of edges you see here will be this much. So, this will be this many edges will go out. Now, where will these edges go. So, all these edges should go into this set. It may be possible that some vertex here may have an edge going somewhere else, but any edge which is coming from the this subset has to reach here, this subset has to reach here - subset has to reach here. So, that is the this thing.

So, but then how many edges can reach here. So you know that, because there are only N of S cardinality of N of S number of vertices here, and each vertex can aspect or expect means the only at most k edges can coming to it right. So, we know that the only k into N of S edges.

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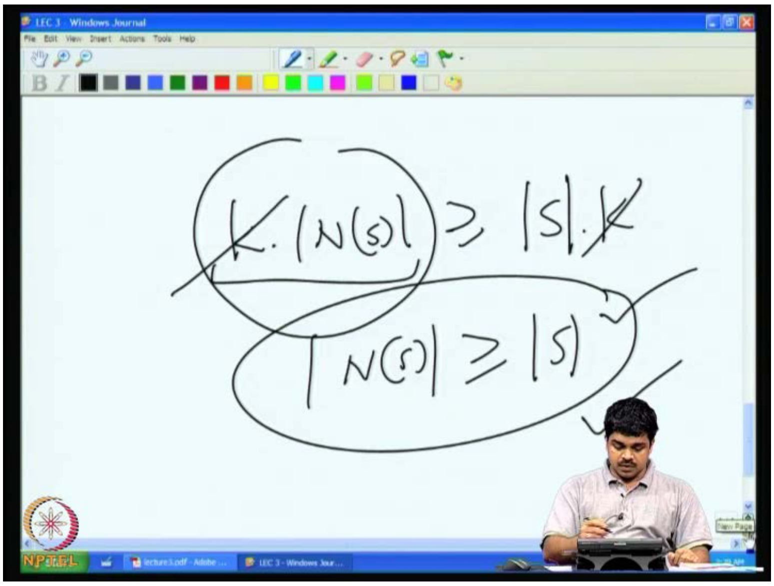

So, let us say we will say only k into N of S edges can getting do it. But then... So, definitely this number has to be greater than or equal to this number, because S into k edges has to getting to it, and only this many can coming to it right. So, which essentially means at this has to be greater than this; now cutting k and k we get N of S is greater than or equal to S this is what. So, we see.

So therefore, for any since S was selected arbitrary as an arbitrary set. So therefore, for any set - any subset S of a, we will get this Hall’s condition true. So, N of S greater than equal to S. Therefore, Hall’s theorem says there is a perfect matching in the bipartite graph. So, that is what we see.

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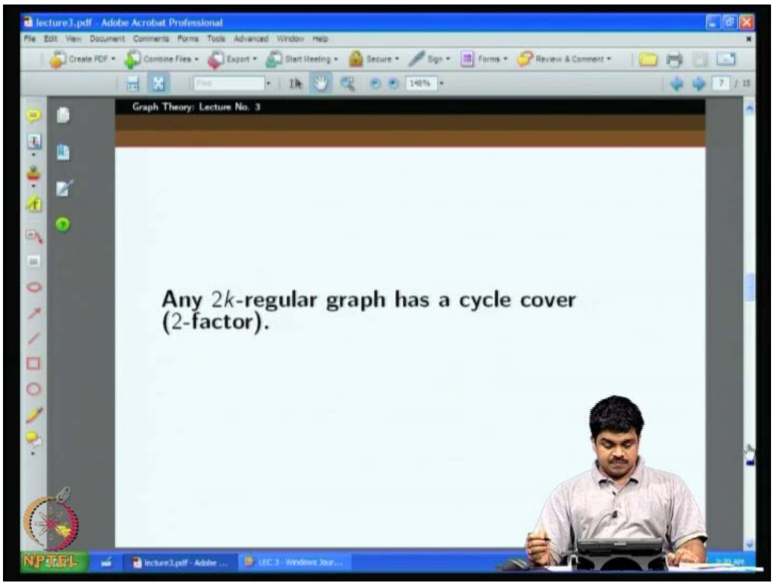

Now, we will look at another example, where we can apply the Hall's theorem. This is another statement which looks a little different, because it is a statement about a more general kind of graph; if say you take any undirected graph, see there is when why do I say undirected graph - usual graph which we were discussing the kind of graphs, we were discussing up to now are undirected graphs.

So, I will soon explain what is a directed graph? So, you consider any undirected graph which is 2k regular; that means, all the vertices have degree 2k. We want to say that, that will have a cycle cover. That cycle cover we have already discussed, it is also call two factor ; that means, you can get a collection of cycles, since go here what is a cycle cover?

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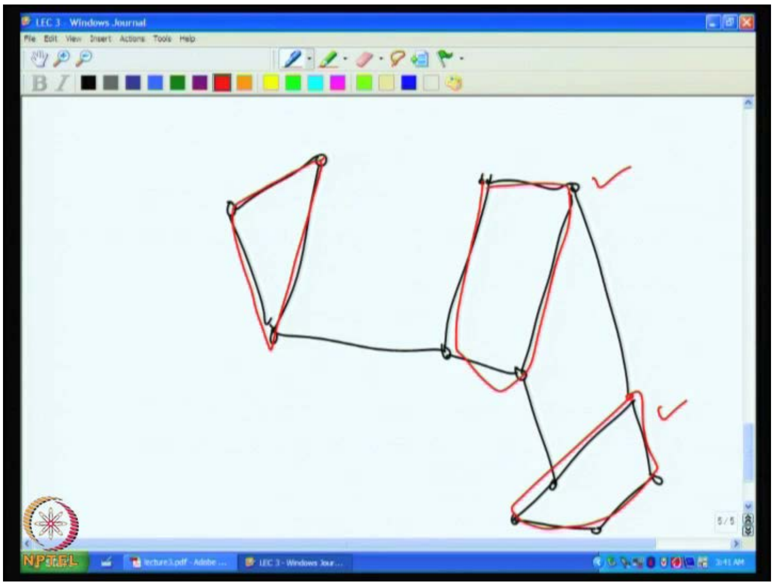

So, for instance; so, you will be able to as I had to (( )) for instance. So, look at this vertex. So, a cycle cover of this graph can be mark like this. So, one cycle, another cycle all the vertices are to come have to come in exactly one cycle, we should find out a few cycles, a collection of cycles in the graph, such that each vertex of the graph is contained in exactly one cycle .

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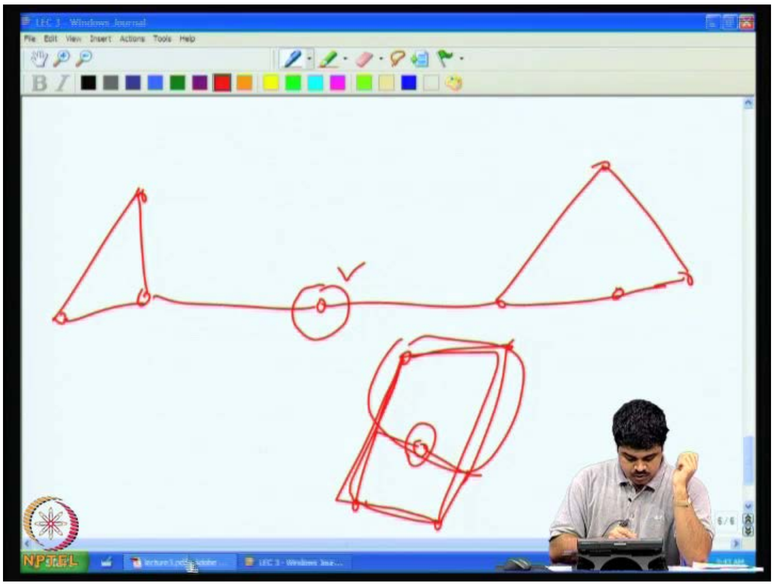

On the on the other hand, if you had taken a graph like this. So, for another example example, for instance if you take you take an a graph like this. Here we cannot find a cycle cover, because it is not possible to get this vertex in any cycle right, it is not possible to get a collection of cycles that this is in one of them or for instance, is it possible to have a cycle cover for this kind of a graph - is it possible to have a cycle cover for this kind of a graph.

So, yeah. So, so it is for instance if you are taken this cycle; then they would not be any cycle; if you are taken any more cycle which I can take; if you are taken this cycle then who will cover this. So, so there are graph which is does not have cycle covers. There are graphs which do not have cycle covers, there are graphs which have cycle covers. So, it is also called two factor, if you remember a two factor is a subgraph of the given graph which is spanning and to regular.

A k factor was a subgraph of the given graph which is spanning, and k regular. Spanning means, it has to contain all the vertices of the given graph; and it on top of that it has to be k regular, then is then we say that the k factor of the given graph. two factor is definitely a cycle cover, because if it is a spanning subgraph and 2 regular, then it has to be a collection of cycles. Now the statement says, if you give me any 2 k regular; that means, it is a regular graph, but the degree of each vertex is an even number, 2 k regular graph not zero; that means, 2 k has to be at least two; it has always got a cycle cover is what the statement says. How do we go about proving this.

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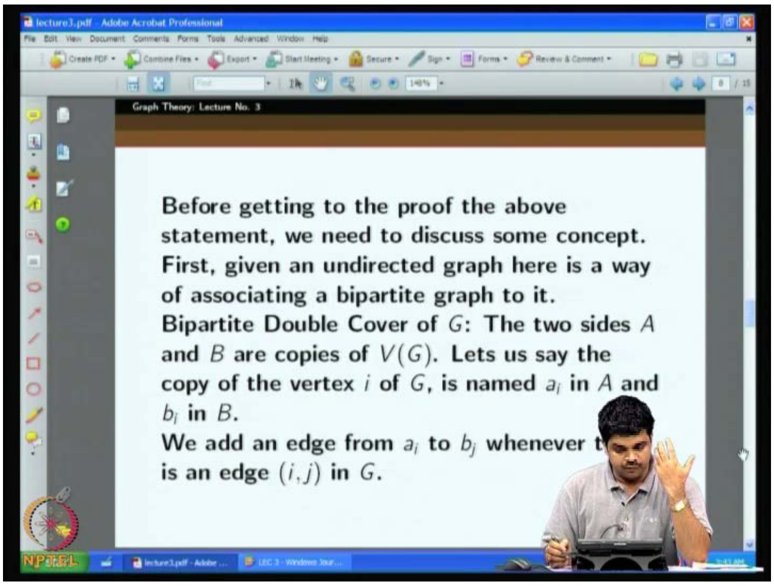

So, before getting to the proof of this statement. Let us consider some some some ideas which will need. One one idea is that of a natural way to associate bipartite graph to given undirected graph of also to in a directed graph. So, we will consider that.

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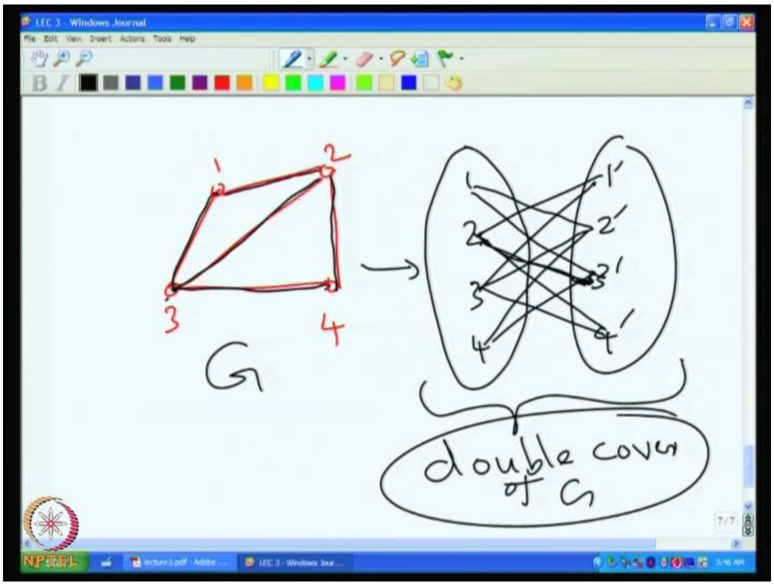

So, suppose just let take this graph. So, this is a simple graph. So, one way to associate a bipartite graph is like this to it, is like this. So, what I do is this 1, 2, 3, 4 are the vertices of this graph – 1, 2, 3, 4 are the vertices of this graph. And then I will make two copies of the vertices of this graph ; that means, 1, 2, 3, 4 again 1, 1 dash; let say 2 dash, 3 dash, 4 dash to distinguish that is another copy.

Now, we see that there is an edge here between 1 and 2. So, I will put an edge here between 1 and 2 dash, and also 2 and 1 dash. For 1 edge here, I put 2 edges. And so, I see that between 1 and 3, there is an edge. I put the corresponding edge - corresponding to this edge, I will put 1 and 3 dash and 3 and 1 dash. Similarly, there is an edge here between 3 and 4, I will put an edge between 3 and 4 dash, and 4 and 3 dash.

And here between 2 and 4 there is an edge. So, I will put an edge between 2 and 4 and 4 and 2 here, so 2 dash here right. Now, this last edge here between 3; and 2 I will put this 3 and 2 dash edge and 2 and 3 dash edge. This is this is the - this is the bipartite graph which is associate to this thing.

This this bipartite graph, we associate that we associate with this graph G is called a double cover, this is the word which we use double cover of G double cover of G double cover of G . So, this is a natural way of associating associating a bipartite graph a given graph; we just make two copies of the vertices, and corresponding to each edge of the graph we put two edges to the double cover. That means, between i to j if you see an edge in G, we put i to j dash and j to i dash, this is our strategy.

And. So this is one concept, and then an interesting thing for instance you see that, because this undirected big graph - undirected graph, corresponding an edge i to j we had to put two edges in the bipartite bipartite double cover, but if you had start up with the directed graph. So, what do what do I mean by a directed graph, which means that each edge has also go to a direction associated with it.

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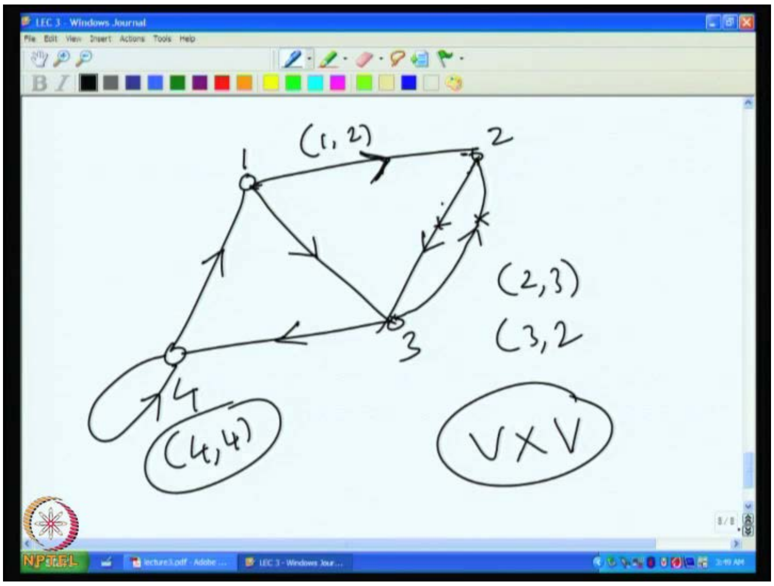

So, let us say if this is our graph. So, graph. So, we will also put an arrow direction associate. So, this edge is not just this, it is from here to here that is what we will say, form here to here. Similarly, so we will say this may have a direction like this. This is one from from here to here. So, if I number it 1, 2, 3, 4. So, it essentially order pairs in the graph k - undirected graph though we use a notation of ordered pairs that is essentially an abuse of notation. In fact, it is an unordered pairs, an edge is an unordered pairs while here in directed graph, it is exactly it is really an ordered pair. We have we can write it as (1,2) 1 to 2 there is an edge.

Similarly, 2 to 3 there is an edge here; similarly 1 to 3 there is this edge right suppose. So, like that. So, 3 to 4 we can put. So, some directions are associated the each edge. So, like in the the graph we we are undirected graph we are considering mostly simple graphs; that means, between any two vertices we have only one edge, and also there were no selfloops. So of course, in the directed, because you wanted avoid the repetition. So, it make thing simple. So, in the directed case, because there is a direction also, see it is natural have this allow this kind of edges also if necessary; that means, it is not this edge is very different - really different from this edge right. Because two end points are different (2,3), and the other is (3,2).

So, if there is a selfloop you can have this kind of. So, there is only one direction 4 to 4. It will corresponding to the pair 4 to 4. So, that also possible. Now, so this is this will essentially once you get a directed graph, you can write on the pairs and that will be a subset of case - subset of n to n sorry, subset of V cross V right. So therefore, it is it essentially represents a relation, it represents a relation and now coming back to our point.

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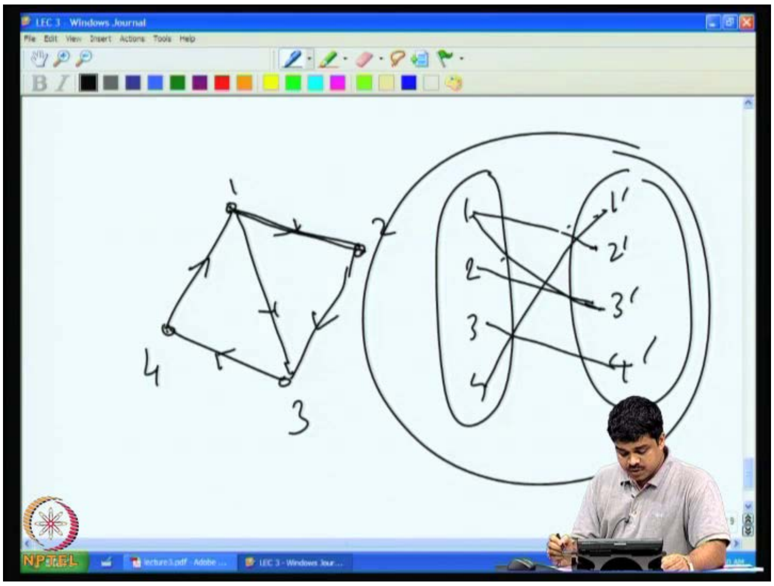

So, we could have constructed starting from a directed graph, also suppose we take this directed graph. We could have constructed a say, double cover of this directed graph also for instance. So in that case, what we will do is the same trick, but so we will have two copies of the vertex sets say 1, 2, 3, 4 say 1 dash, 2 dash, 3 dash, 4 dash.

And 1, 2 this is a directed edge. Therefore, we will put 1, 2 dash. We want put this edge will not put, in the undirected case I have put this also right. We will just put this here to show that we are we are only interested in, we are only interested in this directed edge. So, the direction using... So, similarly 2, 3 will be marked like 2, 3, 3 dash. And 3, 4 is will be 3, 4 dash and then 4, 1 will be like this, and then 1, 3 will be like 1, 3 dash. This the way we will we will we will make the corresponding thing.

So here, we are not putting two edges corresponding to one edge, because one directed edge will translate to one one edge in the bipartite graph here - this is the bipartite double cover of this directed graph. Now, so why did we explain all these things. Because see we were, we our plan was to make a statement about the existence of a cycle cover in a graph, and we told we are going to prove it using Hall's theorem. So, the Hall's theorem is about bipartite graph. So, we have to Hall's theorem is about bipartite graph, Hall's theorem about perfect matching. Therefore, we have to translate the the the the cycle cover problem to a matching problem.

So therefore, we should we should be some relation between the cycle cover in the directed graph, and also cycle cover in the graph; and then the perfect matching in the bipartite k, the corresponding bipartite graph. So, that is why we are constructing this this corresponding bipartite graph which we called a double cover right.

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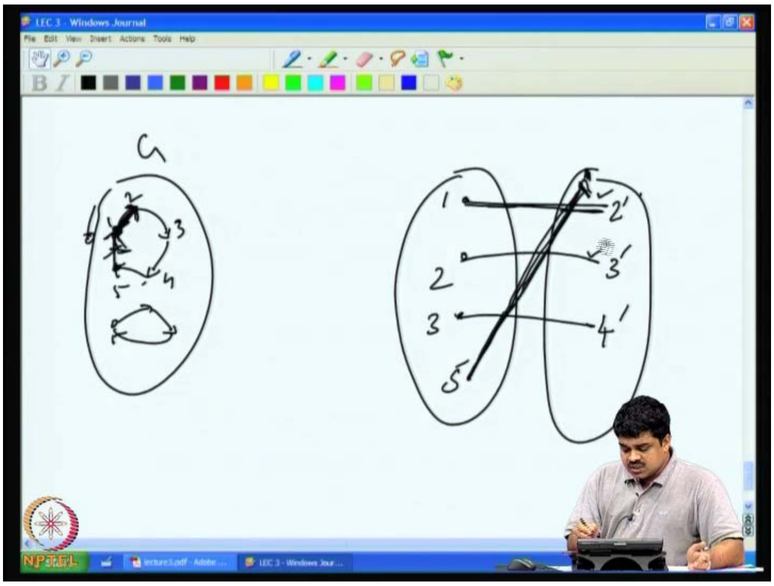

So, what I am going to do. So, let us say let us say there are if you if you constructed a bipartite cover of a graph, let for instance. So, suppose if we construct a bipartite cover of a graph, suppose we constructed a bipartite cover of a graph. So, and it so happens that, if this is the graph G and then we we see a it is a directed graph, and we see a cycle cover of this graph here; suppose something like this right. So, this is say 1, 2, 3, 4, 5. So, what do we see corresponding to this cycle cover here. So, this 1, 2 edge will translate to a 1, 2 dash edge here and 2, 3 edge will translate to a 2, 3 dash edge here; and then 3, 4 edge will translate to a 3, 4 dash edge here and like that.

So, this each edge will translate to a mutually disjoint mutually the edges which are independent here right, because you can see that any vertex here say one here on the cycle cover. So, I will I will take I will see there will be, there are 2 edges here on the cycle cover which touches - one is a outgoing edge, and one is an incoming for instance is 5, 1 edge will come like see 5, 1 dash.

So, this incoming edge here will translate to a 5, 1 edge which is which which will touch 1 dash, where the outgoing edge here on the on this things will translate to a 1, 2 dash edge here right or in other words, if I say this if if you consider a vertex here, it has 1 outgoing edge and 1 incoming edge. The outgoing edge will be the edge which is incident on that vertex - the copy of that vertex on the a side, the incoming edge will become the edge which is incident on the copy of that vertex on the b side right.

So therefore, here in the cycle cover though we see 2 edges touching a given vertex, we will in the corresponding bipartite curve we will see only one edge touching which vertex. Essentially, because we have may two copy for each vertex.

Now, again every vertex is there in once at least one cycle in cycle cover. Therefore, corresponding to every vertex we will be we will be having an incoming edge, and an outgoing edge. Therefore, so if for if for a vertex i. So, we will for the I will have the copy of i in a, and b both of them will see one edge touching them. So, all the vertices are touched by one of these edges, and also every vertex is touched by one of these edges and also, but one thing it we get a. So therefore, we get a perfect matching here. And this is a perfect matching, because no two edges will touch will be touching each other. Why it is so, because given any vertex the only exactly one edges touching that therefore, two edges will not be able to touching each other. If they are touching each other than there should be a vertex which has two vertices, two edges incident on it right. So therefore, we see that the cycle cover translate to a perfect matching here.

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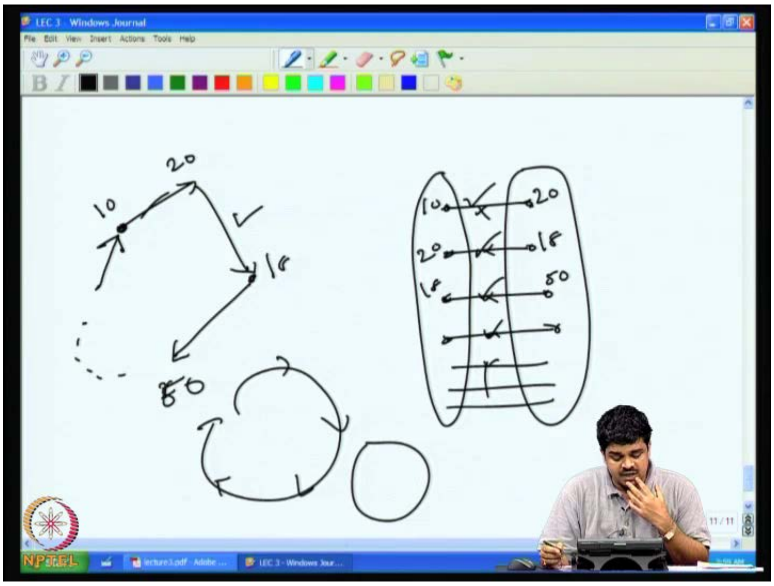

Similarly, once you had construct the bipartite cover and then look at the look at a perfect matching, and then try to see what it means in the in the in the original graph. You see that it again gets translated to a to a cycle cover why, because for instance if you are perfect matching is like this all right; it just have to follow for instance this number is 10- 10 goes to say 20, 20 goes to say 18, 18 goes to 50. So, like that you can you can tracks and you say that 10. So, will draw an outgoing edge from here to 20, and then will come to 20 where 20 it is there is an outgoing here edge from 20 to 18. And then you look for 18 here, and then from 18 to 50 you see an edge. So, there is an you draw an outgoing edge from 18 to 50.

So, like that where will it go an end; finitely it is way it is clear that is has to it will keep on forming a path, know vertex which is already met will be met again. So, will clearly it is like that that is the way we are constructing, and then we will the only way for it is stop is to comeback and hit them right, you will form a cycle sometime. So, this will be one cycle. And then we will restart with one of the new see this, this all matching just we are already considered, the consider with start with some of the other matching edges which we having take considered; and we will get another cycle from that. So, we will get a collection of cycles corresponding to that

So, assuming that we had first go to a directed graph, and then we converted it into a into a double cover, we got we constructed a double cover from that. And then we are tracking the perfect look at the perfect matching, and track the the corresponding edges in the original directed graph, we will end up with a cycle cover. So therefore, you can see that two prove our theorems; that means, given an undirected graph. So, the extra condition an undirected graph is that every degree is equal to 2 k, where 2 k is some even number of k, 2 k is an even number. And . So, what we can do is we can first give a directions to the edges of that undirected graph, and make it a directed graph. We say that we can orient the edges, we can give some direction to the edges of the each edges of the undirected graph, we get a directed graph.

And then from let us call it a G dash; and then from there we will from there we will what will do is - we will construct the bipartite double cover of that G dash. So, first we will give some direction some so, how we will we are going to give directions to the edges is another issue, we are going to explain it later. First the first step is give direction to the edges of the undirected graph.

In such a way that it becomes a directed graph now, and from the directed graph we will construct a bipartite double cover; this is the plan. And once we get a bipartite double cover. So we finally, want to prove that there is originally in the original graph, there is a cycle cover; of course, to prove that there is a cycle cover in the original graph, it is definitely enough to prove that there is a directed cycle cover in the directed graph. What is a directed cycle cover? Cycle cover we also insists that a cycle, we allow only cycles if it traverse the arrows along the cycle, it should be in the proper direction should be it should not be that (( )) vertex, and you see the arrow in the coming invert that vertex right, it should we should be by following arrows we should be able to traverse the cycle.

So, the we we we we can always look for a directed cycle cover; of course, directed cycle cover in the corresponding directed graph when we discard the arrows on the edges will become a cycle cover in the undirected graph is it not. So, now it is of course, the two questions come. So, how will I proof that in the there is a directed cycle cover. Of course, we see we have further transformed it to a bipartite graph double cover of it. So, looking for the cycle cover in this directed graph will be equivalent to looking for a perfect matching in the bipartite cover of it.

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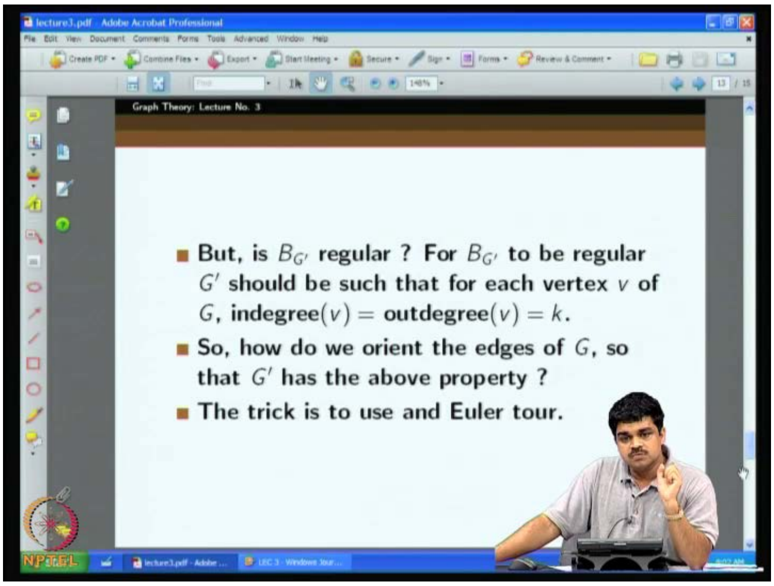

So, and we know that, if this bipartite cover that bipartite graph is a regular graph; if it is a regular graph, then we can if it is a regular graph then we can what you say. So, we can we can definitely in for that if there is a perfect matching it, that perfect matching will become a cycle cover in the directed graph, and that will become a become a cycle cover in the original graph also that is a tree.

So, but then now how do I make sure that there is a perfect matching in the bipartite double cover. Of course, so there for that perfect matching to be present; one good thing is this Hall's theorem. So, if we somehow can establish that the Hall's condition is true then there will be a perfect matching.

So, and as we have seen if the if it is a k regular graph - if it is a k regular bipartite graph, definitely the Hall's conditions will be true and there will be a perfect matching. But how can a how can I ensure that this bipartite graph if finally construct is a k regular graph; it is possible to guarantee that, to do that we should see what is what corresponding to the degree of a vertex in this this bipartite graph.

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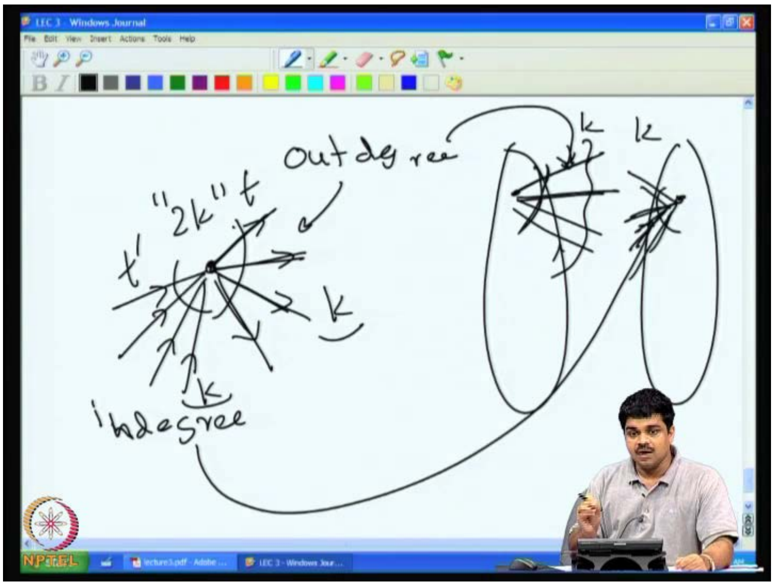

Clearly, as we can see in this construction, if we take any in the bipartite in the double cover construction you see in the graph if you have any vertex with say. So, many outgoings suppose some t outgoing edges, then the corresponding vertex in the a side will have t edges incident on it right, because the outgoing edges become the edges incident on this. Similarly, if there are some t dash incoming edges to that vertex, the corresponding vertex here will have t edges incident on it. So, the degree corresponding to degree of the vertex on a side corresponding to the outgoing degree - out degree we can call it out degree, out degree here that will corresponding to this number; and then the in degree here - in degree means incoming the degree of the number of incoming edges that is inward edges to that vertex, this will correspond to the degree of the corresponding the copy of it on the b side right.

So of course, if you want it regular then this out degree has to be equal to in degree for each vertex, and if I for a given vertex you know the total is going to be 2 k, because the original graph - the original graph has only 2 k exactly 2 k edges incidents on each vertex. So, if out degree equal to in degree; it has to be k and k each right out degree equal to in degree equal to k, here it has to be k, here it has to be k right. So, in other words the question is this, is it possible to give directions to the edges of the original graph, in such a way that the in the directed graph for each vertex the out degree is equal to in degree and that is equal to k, because each vertex a total number of a incident edges was 2 k. Can I make in degree equal to out degree equal to k for each vertex.

So, that depends on how we orient the edges of the graph - the original graph. How can we make sure that that will happen, in general if may not be possible right; sometimes what is the guarantee that you can do that. So, we have to come out with the strategy.

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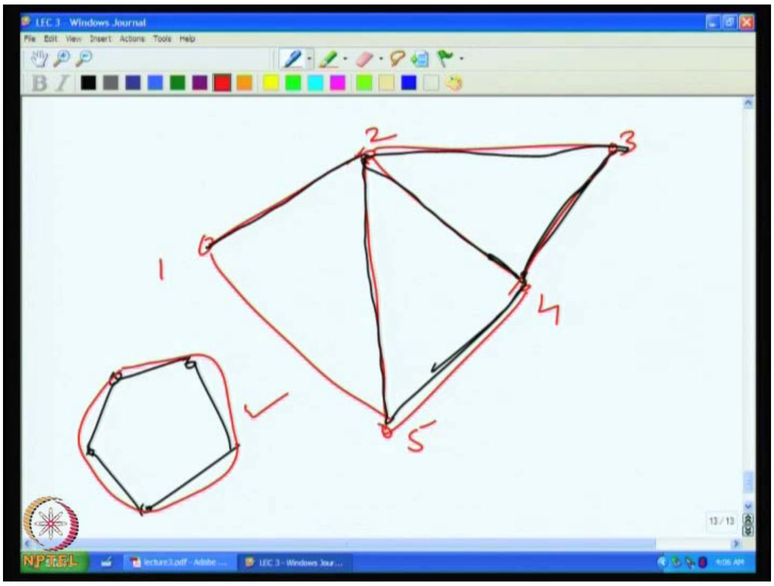

So, here the trick play going to use is to use a concept called Euler tours what is an Euler tour? The Euler tours means. So, you consider a graph. So, here is a graph . So, here is a graph . In this graph... So, I want to if we can consider these are all roads. So, this edges corresponding to some roads, and then some car is starting it trip from here. So, we travel like this, we reach from this city to this city; and we take another road and then we take another road. So, our intension is two somehow traverse each road, but exactly once; that means, we have to go through each road once, but only once we should not reuse the road, it should we we reached here, now we if you want to get out of this city at use this road, this road or we have to go back; that is not allowed right.

So in this graph, you can see that you cannot do that right; for, but on the other hand if you (( )) telling like this right. So, no every road is used, every edge is used and no edge is reused; it is only use to once. So, this kind of a tour is called an Euler tour; that means, it is a travel itself, traverse of the edges of the graph; in such a way that each edge is traversed exactly once. It is travels to once, and only once. And we have to reach backward you started it; that is why it is a it is a it is round type. So, this is called an Euler tour, but the question is how is it going to help us right.

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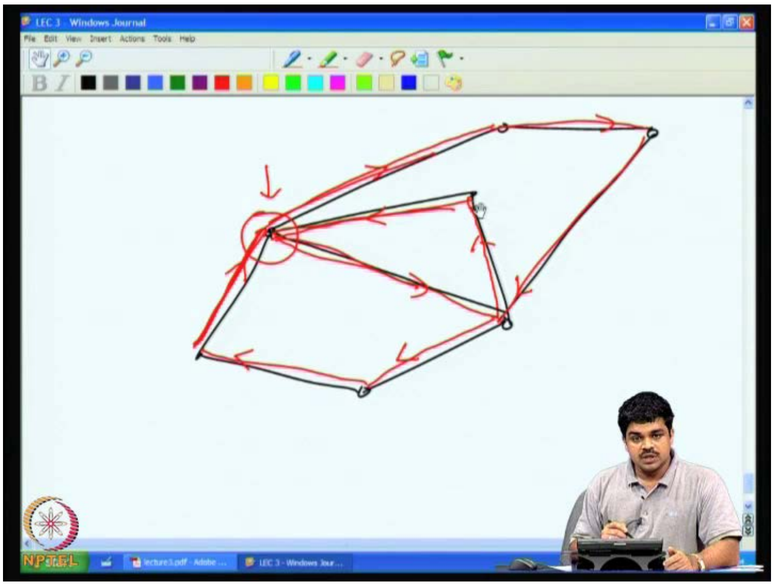

So, you can see that. So, let us see. So, you draw a graph like this. So, this is our suppose this is our original graph. So, let us say right, this is our original graph and then I draw this things. Now so, let me draw an Euler tour here. So, this is the Euler. So, this is the way the car is going our our car we are traverse in the edges. So, now what if I have an Euler tour. So, (( )) that this has an Euler tour, I will I will trace that Euler tour. But along with the the in the direction in with the car is going, I will give direction to the edges - I will first give direction to this like this, and then I will give direction to this and then I will give direction to this - like this, and then I will give this things.

And then this the way I am traversing; in which ever direction I am traversing I also direct the edges as it is cover, because it is only going to be covered once. So, I will give only one direction to the edges. So, this is one way of orienting the edges also. So, what you do is you follow the Euler tour, and then as we go through the edge, as we go through the edge, we will give a direction to that edge which which corresponding to the direction which we traverse the edge. See the good thing about it is, here we can see if we take any vertex, and you imagine that somebody sitting there, and seeing so that the car is coming in into the vertex and going out. And again sometime it will come into the vertex and will go out right.

So therefore, you see that each time the car is getting in it has to go out also. So, every time it will using a different road; which means that the in degree has to be equal to out degree, by so if we had followed an Euler tour. Either you will coming to the vertex and go out coming to the vertex and go out. So, come in will correspond to one in coming edge, going out corresponding to an outgoing edge. So therefore, if you get a Euler tour in the graph by following the Euler tour and giving the direction to the edges, in the in the way we traverse the edges, we can in the in the direction which we traverse the edges, we can get the corresponding the kind of orientations that we are looking for right.

So, the… So, the therefore, that is possible, but the final question is it sure that we have an Euler tour in our graph. So, there is this famous theorem in graph theory, that if each each vertex of the graph has even degree; that means, every vertex of the graph are of even degree, then there is a always an Euler tour. This is there in all the text books, you can quickly look through that it is very easy. So or maybe next time, we can just in the next class we can quickly go through that proof.

But so, even if we do not do that is not very difficult thing. So, you can quickly thing about the proof, and find out and read from any other preliminary chapters of this reference text books. But, because our graph was 2 k regular. So, each vertex has even degree namely 2 k and before there is an Euler tour; now ,we follow the Euler tour and give direction to the edges of the graph in the way we traverse, in the direction which we traverse.

And definitely now the orientation is such that the corresponding directed graph, the directed graph which resulted have in coming degree equal to outgoing degree. Since, the total numbers of edges incident on each vertex of 2 k; now the incoming degree equal to outgoing degree has to be equal to k.

Now, the corresponding bipartite graph - the double cover has to be a k regular graph, because the incoming degree will correspond to the degree of the vertex of the copy of the vertex on b side, and outgoing degree will corresponding to the copy of the vertex on the a side. Now, because it is a k regular bipartite graph, there is a perfect matching in that bipartite graph. And this perfect matching as we have shown will translate back to a cycle cover in the directed graph; now, we can discard the directions, we get a cycle cover in the original graph, this is a way of proved it.

The in this proof what we have we have omitted or we have discarded is the proof that if the degree of each vertex in a graph is e 1, then there is an Euler tour. So, the yes it is easy you can try to prove it yourself. So, this is the end of this lecture, thank you.