Graph Theory
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Module No. # 04
Lecture No. # 29
More on Hamiltonicity: Chvatal's Theorem

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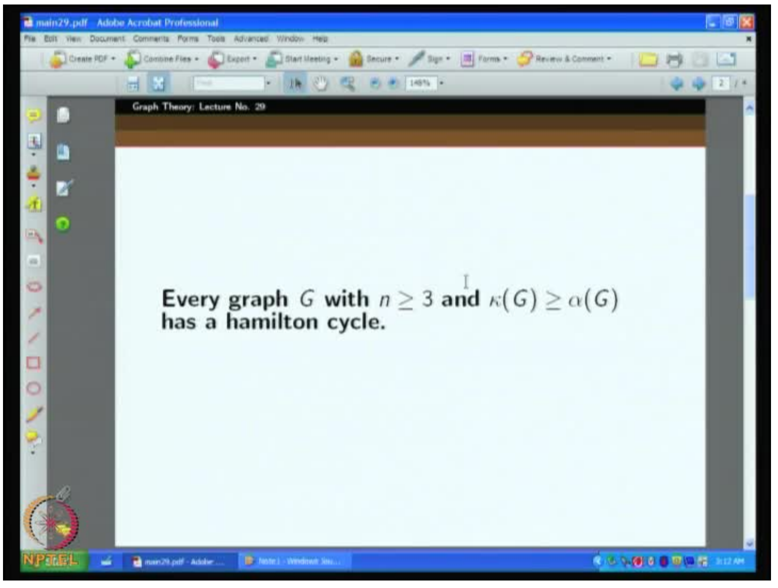

Welcome to twenty-ninth lecture of graph theory. In the last class, we will looking at this theorem; every graph g with at least three vertices and the vertex connectivity graph graph greater than the stability number, the independents number, cardinality of the biggest independent set, then it has a Hamiltonian cycle.

So, we told that if the independents set is small, say if it is less than equal to t, then that there is an n by t cycle - length cycle - is there in the graph so previous; but if the independents set size is less than the connectivity, then we can be sure that there is hamiltonian cycle; this is the poof we want to we trying do the proof for that.

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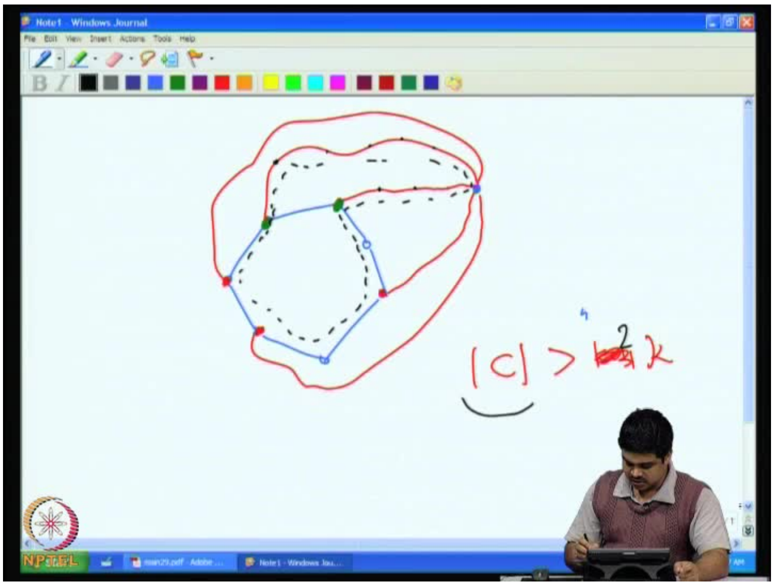

What we did is to take a longer cycle in the graph longer cycle in the graph. So, So, now, the when we looked the the longer cycle you have took, so you can assume that alpha is so so of case, so what will happen for instance? The connectivity can be assume to be 2, then of case there is a cycle, so, we can take that and then you…; now, we see suppose this is this is long is cycle is hamiltonian cycle is of length and then there is nothing to proof; therefore, we can assume that there is at least one vertex outside this and because the connectivity is k, they should be a fan starting from this vertex to this cycle; fan means, there is some paths of this, right some paths this; so, how many fans can be there? So, it is equal to the connectivity; fans means, it is starts from here and then goes through this paths and an exactly one vertex here; see this are the end of the fan paths and they will hit this cycle on only exactly one point. In fact, the first time it hits this cycle, it stops it is that you can have two cycles from one path, two vertices from this cycle on the same path, and there all disjoint path except the beginning point.

Now, we notice that this cycle length has to be greater than or equal to strictly greater than kapa sorry the connectivity, because if not then you can get path like this to all of them; in fact, c it is not even necessary to get paths to all this vertices on this cycle, but even if you can identify two vertices on the cycle which are adjacent in the cycle and that two fan paths like this to green vertices.

Then, we can easily see that there is a contradiction, because you know if you travel like this, and then go like this, and then come back like this, you will get a longer cycle than the blue cycle that we have drawn here; so, this black cycle, I have the dotted black cycle here is longer than the blue cycle, because there is at least one more vertex in that, that can be several vertices also, because the here also we can have intermediate vertices; but if even, if they are not there, it means that, we have one more vertex at least, so it will be a longer path.

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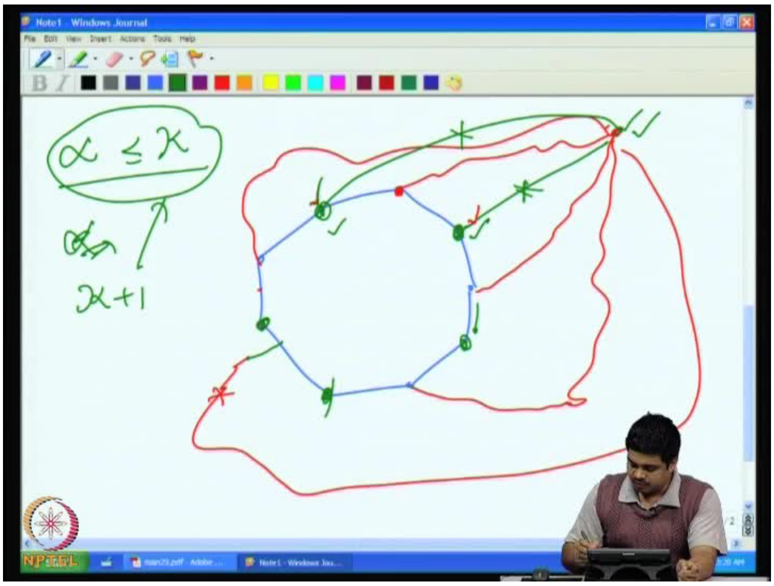

So, we can assume that it not possible for the fan paths to ending vertices that mean, adjacent vertices on the cycle; so, it means that c has to be at least sorry two times kappa because you know those we can identify all those paths kappa paths and then at least one vertex has to be left in between. So, the currently the picture that we see is like this. So, is the blue cycle and this paths from this vertex will come here, suppose it hits here, then it want hit in the next one, they want be next edge though then it can hit here and then it cannot hit here, so it can hit here and then another path can come in hit here, so like that another path can come and hit here, but if it is hits here, then the it should be some other node here.

So, the point is if a node here is the n vertex of fan path, then this vertex and this vertex, the adjacent vertices cannot be then n vertices of a fan path, more over in fact it cannot be directly connect to…, so, this edges cannot be at all; this kind of edges cannot be there at all; they are not neighbors of this vertex; this fans paths not there, it is it means that, they cannot be the neighbors of it also, because if there is an edge between them there is a fan path.

So, now, what we going to do is, to demonstrate independent set of cardinality at least kappa plus 1, we know by we our assumption alpha is less than equal to kappa, so if we show that the independent set with cardinality kappa plus 1, this will violate this assumption and then we get the contradiction.

So, the our candidate vertices are this, this alternate vertices this vertices which are just after the the end points of this fan paths; the end points of this fan paths on this cycle will.; so, those vertices will form a fan form an independent set; and see and we because there are this kappa of fan paths there us kappa of them and then this along with that I will take this outside vertex also; so, together they will form a k plus 1 independent set, that is what we want to say.

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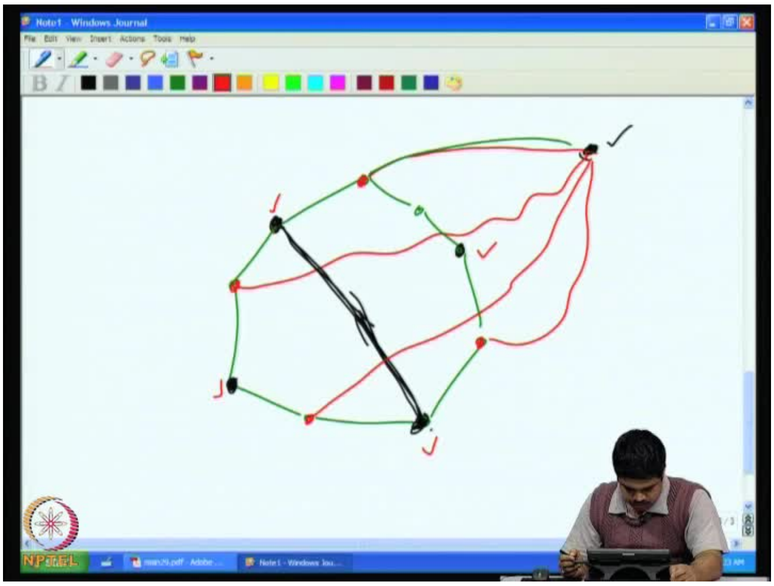

So, now, again we will repeat this. So, the pitch now is… So, we have this cycle; and this outside vertex, the fan paths are coming and ending at points several points say this was some some other points where the fan path…; and then we can mark these points; the points points just one two sides of the just after you say you can go in a clock wise direction and then mark the anti-clock wise direction and mark the paths this black; black in vertices are the vertices which is just after red vertex, red vertex is the end points of some fan path like this.

Now, we pointed out the that is black vertices are non-adjacent to this vertex. So, I will say these vertices are non-adjacent to this vertex, because if there is directed to them then we can, in fact, get a longer cycle then this cycle, because we our assumption is that was the longer cycle; we if we getting a longer cycle then it is a contradiction.

Now, suppose the this black vertices go forming an independent set, then along this black vertices I can add this also, because this are this is non-adjacent of them; and clearly, this are at least kappa of them, so we will get kappa plus 1 independent set size independent set to do this things. So, we we see what suppose, there is an edge between this two such vertices, for instance, suppose this edge is there here between these two vertices set to, we will show that such an edge cannot exist, that is why they we cleaned that there independent this black vertices, suppose it is there then what can happen?

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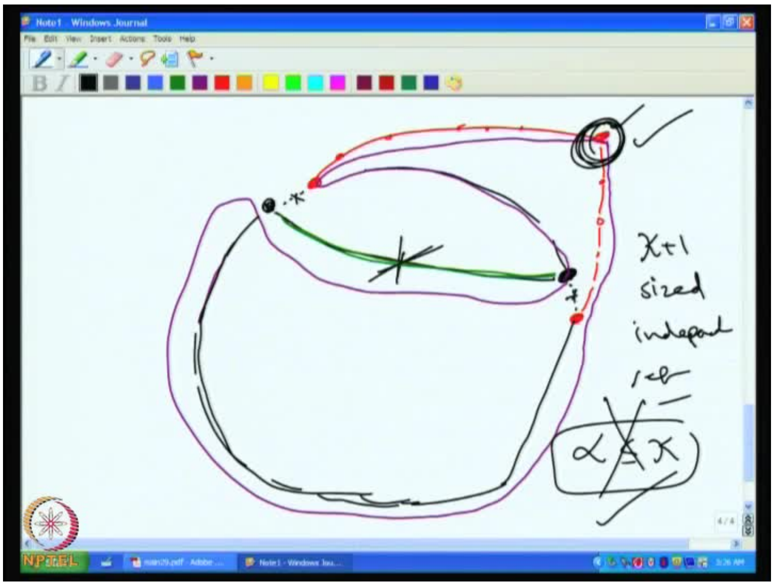

The the point is the, see this is a red vertex right and then sorry this is a red vertex, this a, this comes after that, and here this is also a red vertex this comes after that. Now, I will show a path which is which is bigger than…. So, here we draw like this; so, to clarity I will draw a figure and mark the red vertex here, and then I mark another red vertex here, and then I mark one black vertex here, and I mark one black vertex here.

Now, suppose this connection is there here. So, may be this blue connection connections is there here, sorry this green connection is there here; now, you considered this outside vertex, we will show how to get a bigger cycle including that; here we have this connection, this this is a path here, it can be so there is a path here; now, how will I get bigger cycle?

So, the point is I am just using this violet color to track the path; we will starts from here; so, then we will go like this, go like this, and then here we will take this to, and then here we will take this and then go back here and we will go back to cycle. So, we have touched this cycle; this cycle has touched all the vertices of all cycles and also included this new vertex; therefore, this a bigger, this a bigger cycle, this a bigger cycle; it is a contradiction to the assumption that they all universe belongs a cycles; see what edges will go away from here is this one and this one also will go from this here to look like this. So, earlier, this this was the cycle; now, we remove these two edges and then we we using this connection we could complete the cycle like this including the outer vertex including the outer vertex that was the trick.

So, before what we have shown is, this kind of an edges not possible because between the two black vertices here we cannot have an edge; that means, the black vertices that is the vertices which are just after the red vertices in a counter clock wise order or indeed independent set; and those black vertices are at least kappa of them because you know there are at least kappa red vertices, because the fan has at least the kappa's path the connectivity vertex connectivity number of paths in it. So, those mean black vertices will come plus this vertex, this vertex will form kappa plus 1 size, the independent set that is getting. So, therefore, it is a contradiction to the assumption that alpha was less that equal to kappa. So, it contradicts that.

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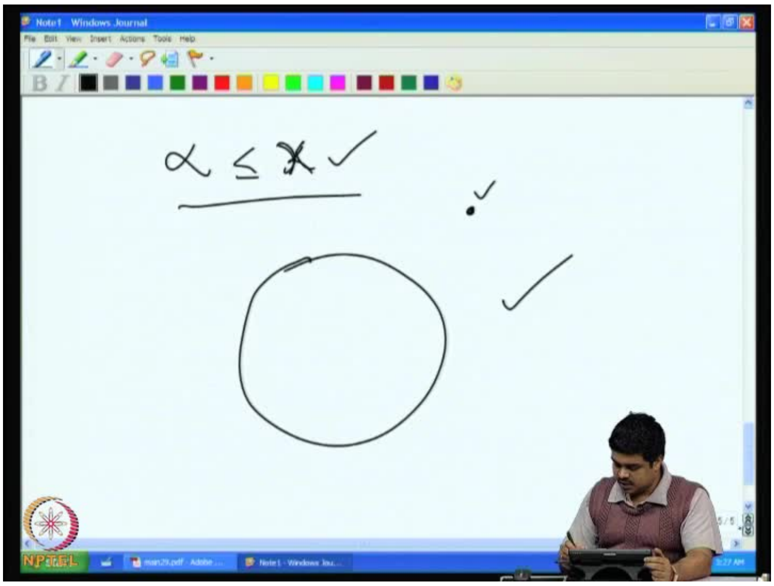

So, we claim that so where is the contradiction coming from? The contradiction is coming from fact that we after taking the longest cycle, we could find some vertex outside which is not covered by this; so, the longest cycle has to be such that, it covers all the vertices; that means, it has to be a Hamiltonian cycle itself; so, they should be a hamiltonian cycle in the graph if alpha is less than equal to kappa; this is the kappa being the vertex connectivity, this is statement.

Now, the next theorem about Hamiltonian cycles that we want to study is the following; So, this is about how the degrees of the…, see as we as we have seen the daric's theorem said if the degree of a graph is the minimum degree of the graph is greater than equal to n by 2, then there is a…; see if strictly greater than n by 2, then there is a hamiltonian cycle. So, now, the that is the condition on the degrees of the vertices, so this was in fact several people, several researchers of worked improving the result of daric like relaxing the degree condition more and more and more; and then finally, we came to this particular result of (( )) and which makes use of the property of the degree sequences to say that the hamiltonian where there there exists Hamiltonian path in the graph or not.

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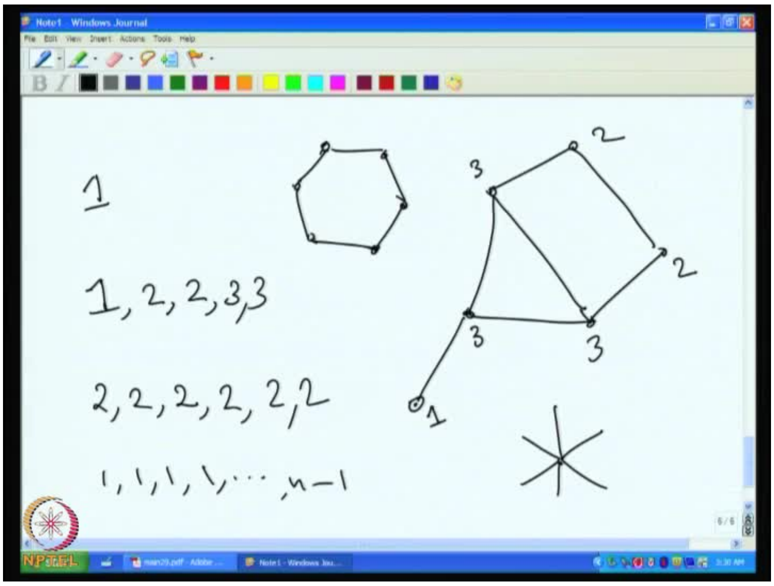

So, So, this is it is it is only again the condition will say that if the degree sequence satisfies certain property, then there will be a hamiltonian cycle; so, for that we need some definition; and first of all what is the degree sequence? What is the degree sequence? So, the given any graph you can write down the degrees of it, so for instance, for this graph, see for this graph, so the degree of the vertex is 1, the degree see I will write the degrees here, the degree of the vertex is 1, the degree of the vertex is 3, here is again 3, here is again 3, this is 2, 2; so, degrees sequence is written like this 1, 2, 2, 3, 3; this numbers are written in the increasing order that is all with the reputation; some numbers are repeating more than ones, we will write them once again, for instance, so can you imagine a graph with degree sequences like this of case it is this graph, so 1 2 3 4 5 6, if one more two is there that is the cycle.

So, say for instance, for this graph - star graph - total be the degree sequence, it will be a 1, 1, 1, 1, so n minus 1, because the last one will be n minus 1 and till then it will all ones; so, this is what the degree sequence of a graph is. So, what we write down the degrees and then order them in the non-decreasing order. So, this degree sequence, this is degree sequence of a graph.

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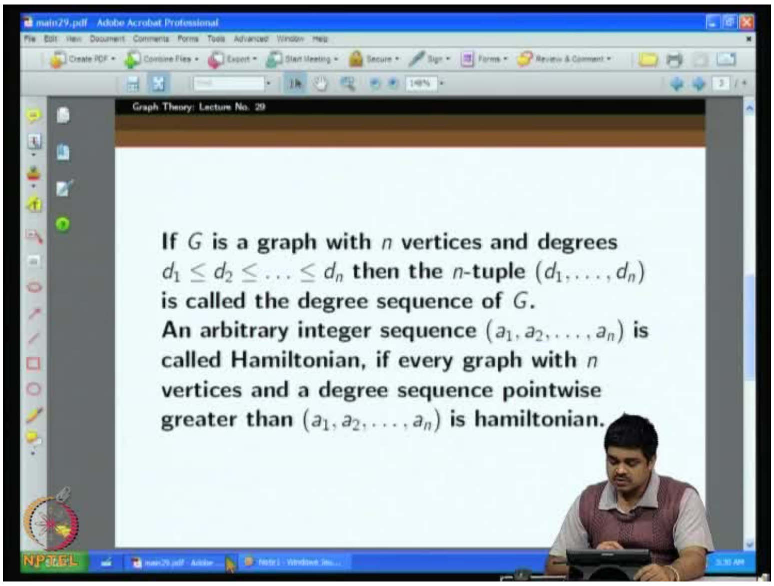

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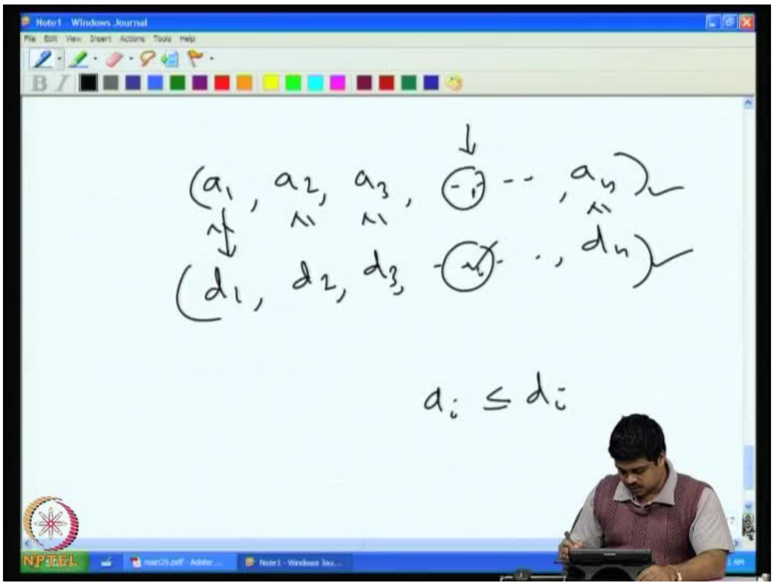

Now, suppose we are given degree sequence, so say suppose we given a degree sequence a 1, a 2, a 3, so a n numbers will be there, some of them may be repeating; we say that another degree sequence d 1, d 2 d 3, d n is point wise greater than this degree sequence a - the first degree sequence; so, if this for each i, a i is less than equal to d i; in other words, this is bigger than or equal to this is bigger than or equal to this is like that. So, each in each position - the ith each position - the number here is at least as big as the number here, then it is point five great.

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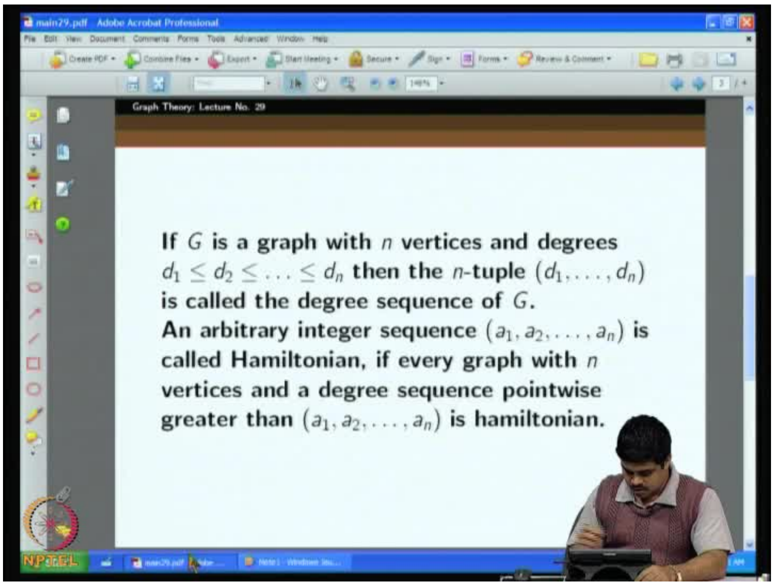

So, now, we say that the degree sequence is point wise, see suppose, so we we we are given a degree sequence a 1, a 2, a n, so we say that it is hamiltonian degree sequence; if any graph with that degree sequence or any other degree sequence which 0.55 is greater, than this is Hamiltonian.

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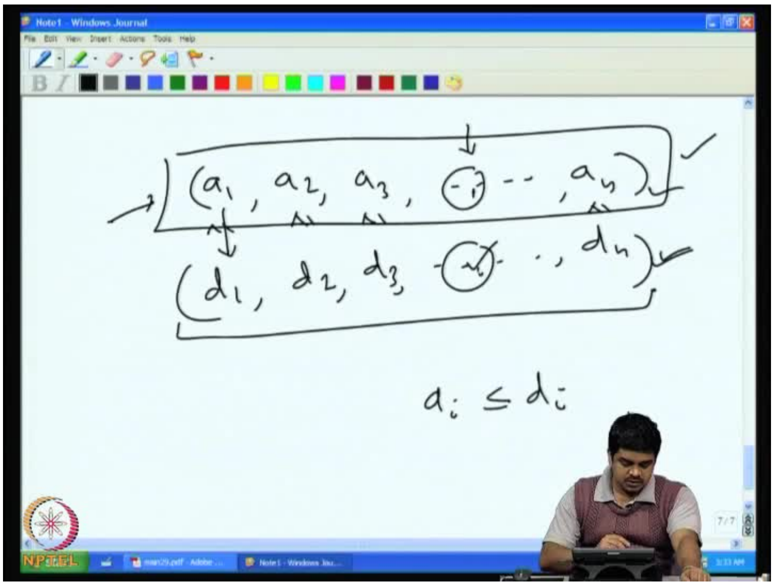

So, which means that, so when do we say that this degree sequences is Hamiltonian; so, not only that any graph with this degree sequences is Hamiltonian, also we need any graph with the degree sequence, this having all the values at least as much as corresponding value in this is also to be Hamiltonian, the degree sequences and all the point wise degree sequences of it should correspond to graphs the graph with such degree sequences should all be hamiltonian then we will say that this degree sequences Hamiltonian; in other words, it is in those numbers the Hamiltonicity is…, if the degrees are those numbers are higher, so we will get the Hamiltonian cycle.

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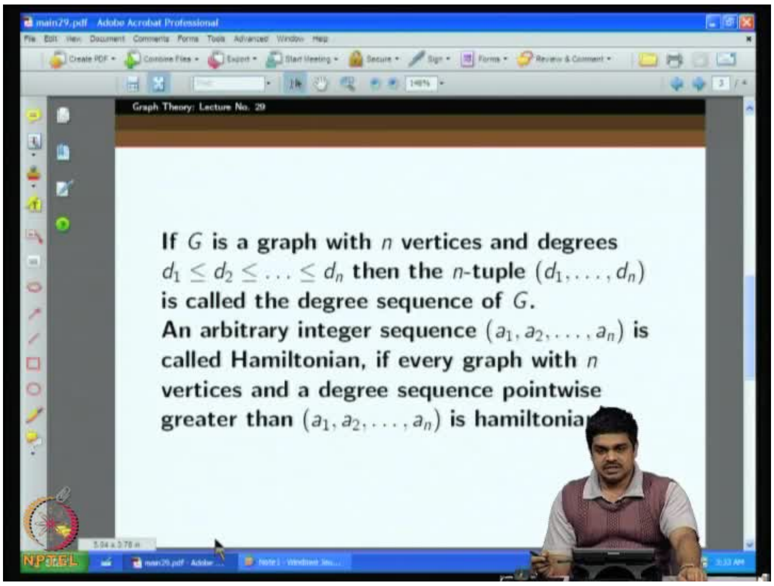

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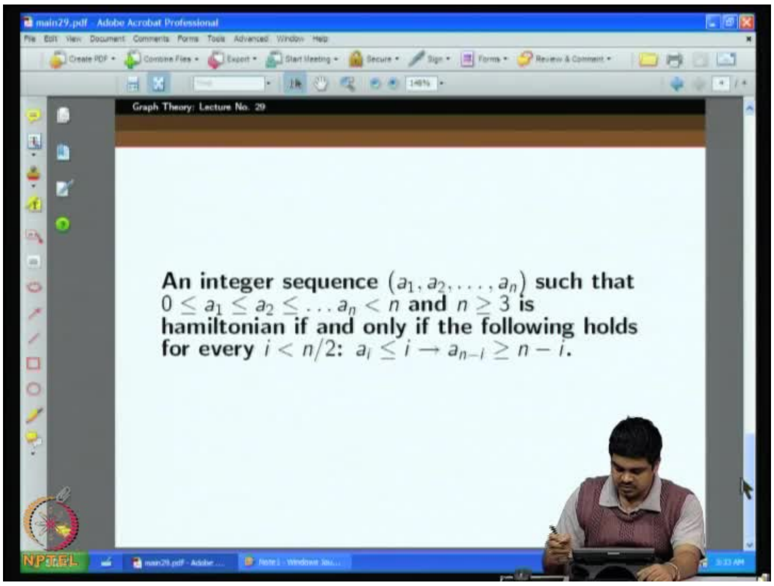

So, now the we will we will characterize the degree sequences which are Hamiltonian, that is the next intension; so, we will say that an integer sequence a 1, a 2, a n such that;

so, this is an increasing the non-degree sequence; so, of case the biggest number is strictly less than n, so lowest number can be 0 and number of vertices is assuming to be always greater than or equal to 0, otherwise there is no cycle; so, it such a degree sequences is Hamiltonian, if and only if the following holds for every i less than n by 2, what is the condition if a i is less than equal to i, then we should have a n minus i greater than or equal to n minus i, so this require some kind of this condition is little trick, so what is this the same.

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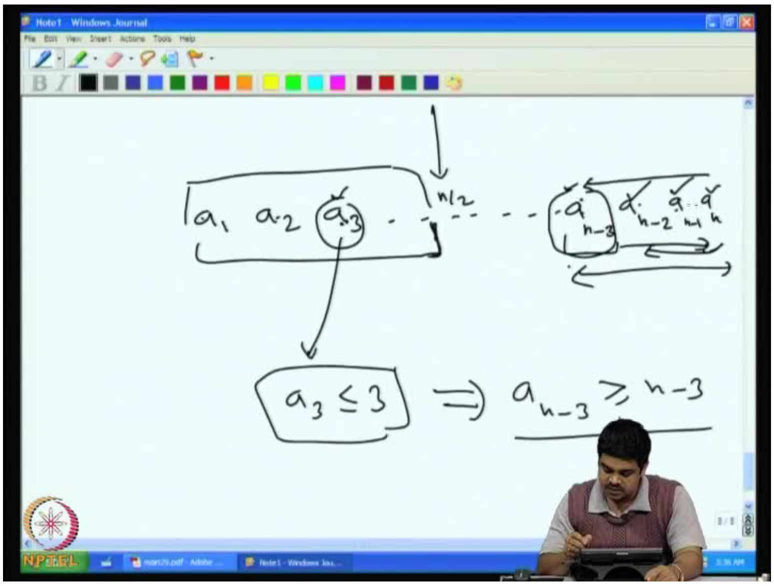

So, same that, so, suppose this is the degree sequence a 1, a 2, a 3, a n minus 3, a n minus 2, a n minus 1, a n; see we have an n by 2 here somewhere for all i strictly less than n by 2; so, if you picking up some i before this, this can be if it is odd number up to n by 2 trick, if it is an even number up to n minus n by 2 minus 1; so, you take any any any value here and suppose I take this one a 3 and then ask is a 3 less than equal to 3, this is the questions are asking the third number is the third number is less than equal to 3, so it is asking.

In that case, suppose, it is if it is not less than equal to 3 then there is nothing to bother about, but suppose is less than equal to 3 then we should get the corresponding number n minus 3; n minus 3 not the third number from here, it is actually the fourth number from here, it is essentially there is a it is not symmetric when you are looking at the third here, this is the fourth from this side; so, therefore, between we are matching this for this. So, we want a n minus 3 should be greater than or equal to n minus 3, this is the condition we need, which essentially means that the number here has to be at least n minus 3.

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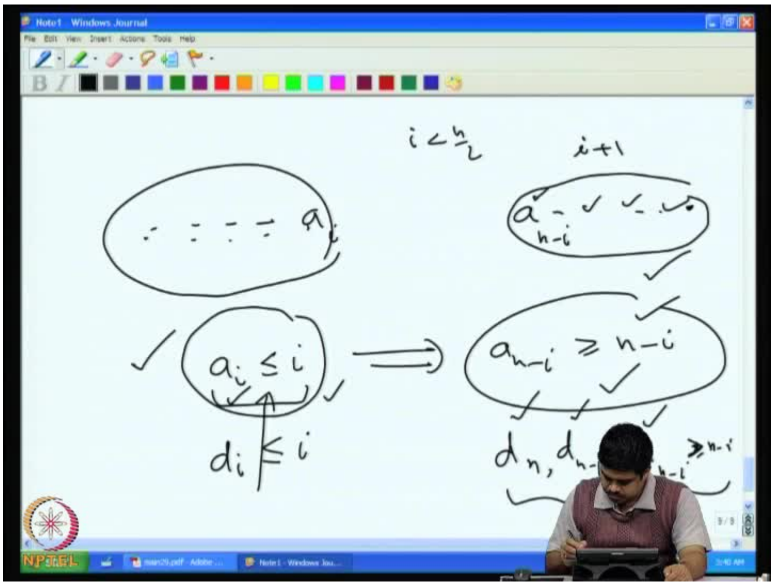

So, this also will be at least n minus 3, this also will be at least n minus 3 because this is an increasing order. So, therefore, we will if the third number is at most 3, then we will have in the end four numbers which are at least n minus 3, this one, this one, this one, and this one forth; in general you will take a ith number here of case i has to be strictly less than n by 2 and if it is so happens that a i is less than equal to i, if it is not less than equal to i, if it is greater than i forget about it we do not we do not have anything to worry about it.

Suppose, this condition is met then we should have, so we count from this side, i plus number i plus oneth number; that means, a minus i has to be such that greater than or equal to that is greater than equal to n minus i; in other words, here we should find at least i plus 1 numbers which are greater than n minus i; if we have here at least i numbers which are less than equal to i, because here all these things will be less than i; if a i is a i is less than equal to i at least i numbers less than equal to i then we should have at least i plus 1 numbers greater than or equal to n minus i, this is the condition slight as a symmetry there in the this is not i i but i and i plus 1.

This is what it the conditions say. Now, we will show that, if this condition is met then every one of the sequences which is point wise greater than this or this one or this sequence itself or any sequence which is point wise greater than this should corresponds to graphs which are Hamiltonian or in other words if a graph has a degree sequence which is either this or a degree sequence with which is point wise greater than this that graph has to be Hamiltonian, because this satisfied this condition, this is, this is what I am going to tell.

See the interesting things, so I am this satisfies the condition; suppose, we consider a point wise greater degree sequence, will it also satisfy the condition? So, of case that will satisfy the condition because you know if you take any ith vertex sorry a i here, so suppose you consider d i instead of this the point wise greater degree sequences.

If this d i has become greater than a i well and good we do not worry anything about because we does not we not ask whether this condition; this condition anyway says, if this is true then this to be true, if this is not true then we do not have to worry about anything; suppose, on the other hand, if d i is also less than equal to i then we just need all of the d and n d n minus 1 up to d n minus i, all of them should be greater than or equal to n minus i, that that will n by 2 because anyway all of these d n, d n minus 1 up to d n minus 1 increase the value not decreased, therefore, we still will have so many values which is of greater than equal to n minus i; So, this will be met even if this is met for an integer sequence any point wise integer sequence also will satisfy the condition. So, therefore, that will not be lost if we will consider a 0.5 greater integer sequence right.

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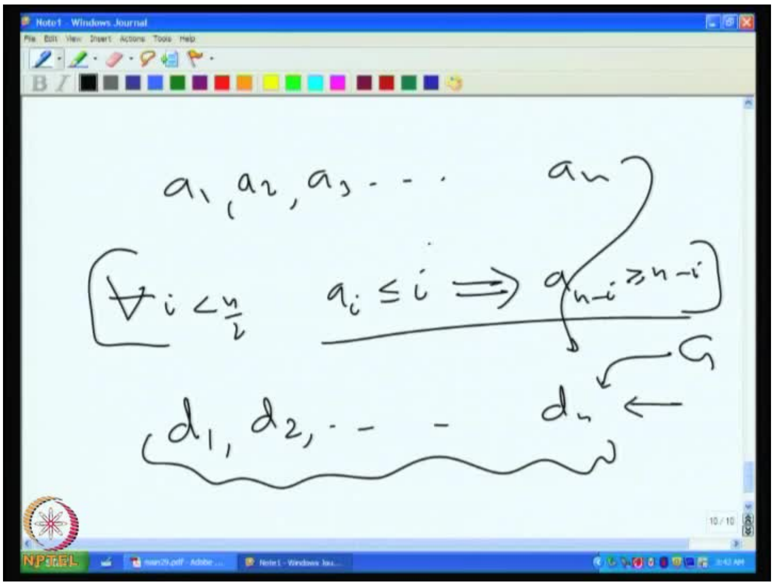

So, now, the the the the idea here is to the idea here is to show first, so what is our intension? We will we will show that if the particular degree sequence satisfies is condition, then any degree sequence which is point wise greater than or equal to this will be such that, if i graph has that degree sequence then it will be Hamiltonian, this is what we want; suppose, this suppose, this is not true, suppose this is not true, that means, so we have the degree sequence a 1, a 2, a 3, a n which satisfies our condition, that is, for all i less than n by 2, a i less than equal i implies a n minus i greater than equal to n minus i then in that case. So, if something is wrong, so if so, that means, they they can be expose there is another degree sequence not necessary this some point wise greater degree sequence like d 1, d 2, d n, such that, a graph corresponding to it some g corresponding to this degree sequence is not Hamiltonian, suppose is does not have the Hamiltonian city.

Now, what we are going to do is to find one such degree sequence which very suitable for us to work with, what is what will do we will consider all degree sequence which is point wise greater this and does not satisfy the theorem; that means, it is not hamiltonian because once it is point wise greater of case, this condition will be met, but suppose it is not hamiltonian; we will pick up that degree sequence and a graph corresponding to that; So, such that it is of the maximum number of the edge possible in all the possible degree sequences which point wise greater than the given degree sequences a 1, a 2, a 3, a n; we will pick up the one as such that, the corresponding graph is a maximum possible number of edges; that means, if you consider adding one more edge to it, it will satisfies it will have a Hamiltonian cycle why because if you add one more edge to it which degree will increase. So, when the degrees increases, the degree sequence, if the degree is only increasing may rearrange the degrees of cases very clear that it will be point wise greater than the all degree sequences, because some some degrees increase that is all they may go to a higher place, but what takes the current position will be definitely some bigger number than what it was earlier.

So, when we rearrange the numbers after making this exchange, so, therefore, so that will be if you add one more edge to such a graph, it is it is clear that such a graph will be such a graph will be somewhat saturated in the sensor; if you add one more edge, it is very it is it is maximal in the sense that, if you add one more edge then immediately Hamiltonian cycle will appear.

So, it is about to become Hamiltonian, it is it is in the it is just a very saturated very extreme situation, that means, any more edge will make it Hamiltonian; that is the type of graph g, that is why we are we are taking a graph with maximum number of edges which among all the possible violations I mean among all possible graph which does not have Hamiltonian cycle and having a degree sequence point wise greater than the given degree sequence.

So, therefore, if we if we add one more edge the that the degree sequences of new graph is also will be point wise is greater than this; and then if still there is no Hamiltonian cycle, we could as we taken that, because we when for the biggest number of edges among all the possible violations the bad examples, which does not satisfies the theorem.

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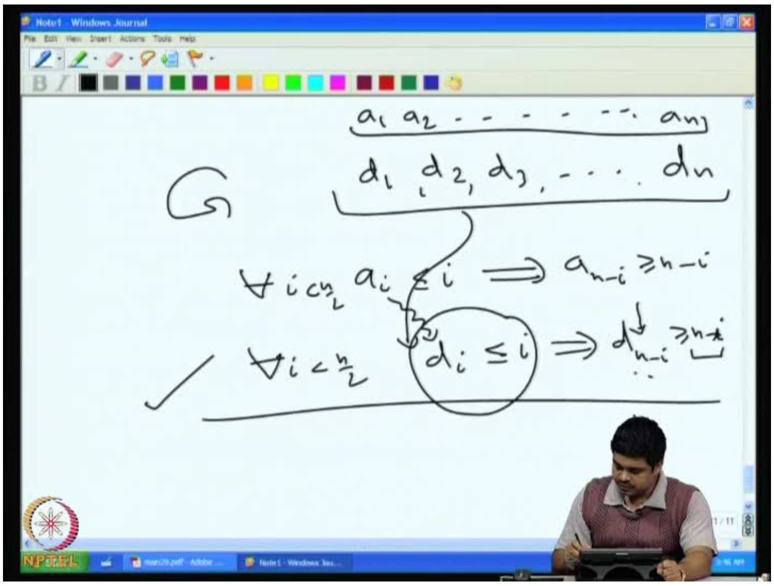

So, then. So, we pick up that graph, let us say, g is that and the corresponding degree sequence can be written so, let us say, d 1, d 2, d 3, d n is the degree sequence; and as we have already noticed this point wise degree sequence compare to the original degree sequence given and also because this original sequence satisfies the condition, that means, a i less that equal to i implies a n minus i greater than or equal to n minus i, this sequence also will satisfies that; that means, i less than n by 2 less than n by 2 if d i less than equal to i, then it will imply that d n minus i greater than or equal to n minus i.

So, this is the this condition also will be met; if this satisfies this, this also will be satisfy this, and I will explain why it is because a everything is only increasing, if it as if this is not satisfied well and good, if this self is not satisfied well and good, but if it is says satisfied even after increasing this thing; and it is very clear that, all the these number n minus i has only increased, so this is not changing; so, therefore, i will still satisfies, so that is the thing.

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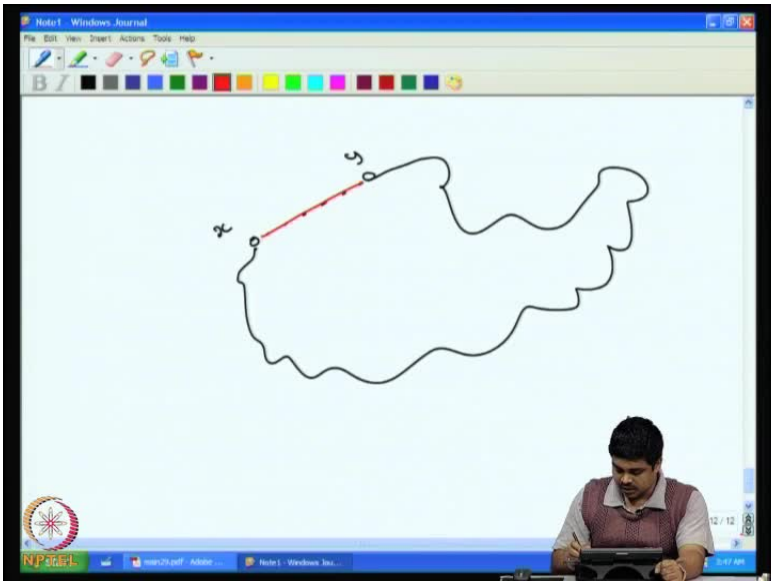

Now, this this is the graph. So, you can always add any edge and hamiltonian cycle will appear; in other words, if you take any a x y in g such that, it is non-adjacent, it should be an hamiltonian path already between this x and y, that is why when add this new edge you get Hamiltonian cycle, otherwise how can you get it. So, between any pair of non-adjacent edges, you should already have a Hamiltonian path ready. So, so the therefore, therefore, when you add new edge should it will become a become a Hamiltonian cycle.

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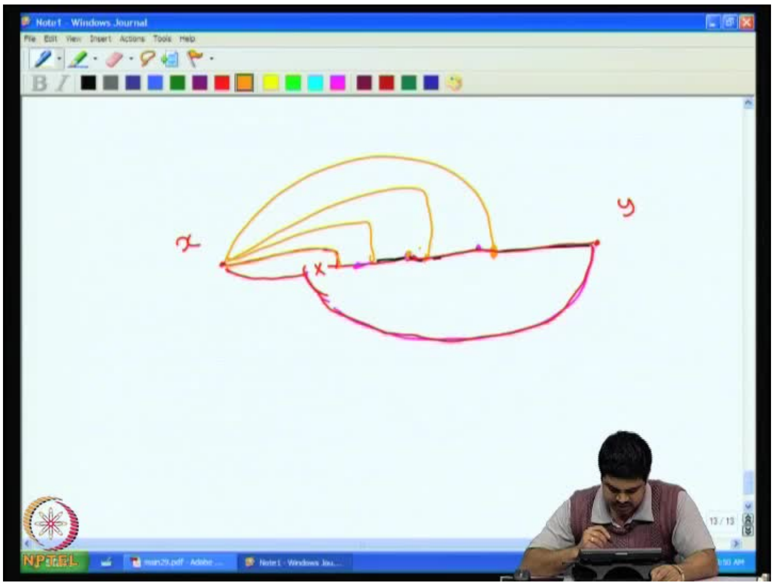

Now, let us consider in x y. So, an interesting thing is the question we will ask is that, so there is Hamiltonian path between x and y here we know that. So, it is possible that we can get a Hamiltonian cycle immediately from that, for instance, when we studied the Dirac's theorem and we consider the proof of that; we had seen that, if we if we a considered these vertices, so, these are the neighboring vertices of x; suppose, these are the neighboring vertices of x and then we consider the vertices which are which are the which are just behind the neighboring vertices of x, this violet vertices which I marked just behind it. So, here suppose you have these vertices, one of them if even one of them is adjacent to y, if one of them adjacent to y, then we can convert this Hamiltonian path into a Hamiltonian cycle, how do we do that? That is by just that is by…. So, this is the path of this, this was the path, this was the path; so, them so, this suppose, if this was adjacent to this even one of them and then we can delete this edge, we can here and then what we can do. So, here using there is connection between this and this I am working at here, and then we can follow like this, and then come back through this edge, and then we can go here right. So, we will complete the cycle here; what the only thing that we that we done is to remove this, this portion right this particular sorry, so this connection is there, this connection is there. So, we just cut this edge and then making use of the…, there is a connection like this and there is a connection like this, we could we could we could complete the cycle.

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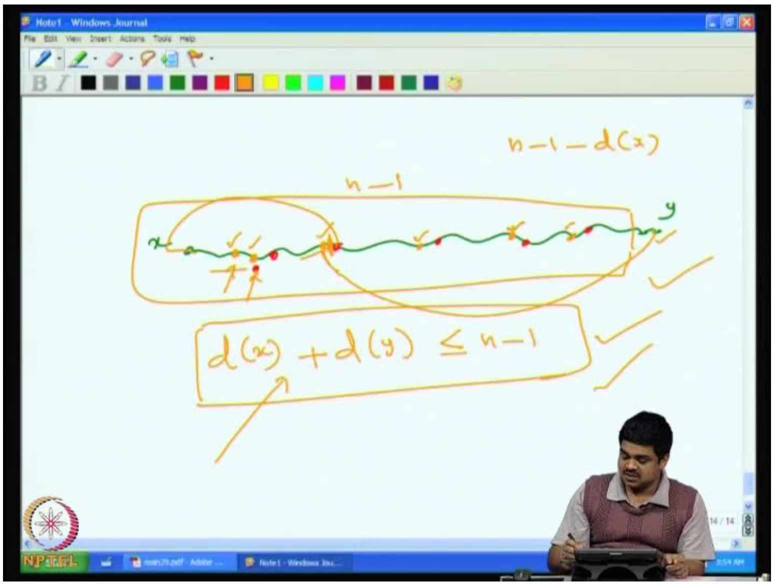

So, what we see is none of it means that, if we do not have a Hamiltonian cycle in the graph none of the vertices which just behind the neighbor of x can be a neighbor of y. So, in other words, again this is x this is y x and y any arbitrarily any pair of non-adjacent vertices; and if you consider this red, this red vertices is the neighbors of x; if we consider the red vertices is the neighbor of x and then let this orange vertices be the vertices which is just behind the red vertices, it does not means that orange cannot be a red, there can be it can be red and orange together, but that case there will be one more orange behind this that is all.

We just consider the vertices which just behind the neighbors of x and none of this orange vertices can be then the neighbors of y. So, if you count though orange vertices there how many of them, there there are dx of them right dx of them, because for each neighbors of x we will get 1 behind. So, dx of them and then the none of those vertices are taken by the neighbors of or neighbors of y; therefore, those neighbors of y as we know because this a Hamiltonian path all the neighbors of y also should be present in this path only they should sit somewhere in the path not only this orange orange marked vertices somewhere else.

So, but then they will get only the remaining n minus 1 minus dx positions, because this dx for all from here to here, because y cannot be anyway…, because there is nothing beyond y. So, this one of this orange vertices cannot be y itself. So, we have n minus 1 vertex in here and then y's neighbors also have to be in this thing. So, if you add dx plus d of y this has to be at most n minus 1 is in it at most minus 1, this has to be at most n minus 1, because otherwise this orange is some other orange vertices should be a neighbor of why they can be if this is not true. So, if together than make more than n minus one, that means, i will have to share, they will have the cannot be disjoint. So, then we will get that situation here this is connected to this and this is connected to this and then we can brake here and then we can brake here and then we will get a cycle jumping here and then like this and then like this we will get a Hamiltonian cycle, that is a contradiction we know that the graph does not have a Hamiltonian circuit.

So, for every non-adjacent pair we have this property also; this is remembering how derive did his original proof we see, but this is dx plus dy is less than equal to n minus 1 is true for a every non adjacent here, because there is already a Hamiltonian cycle, between Hamiltonian path, between any non-adjacent pairs x and y.

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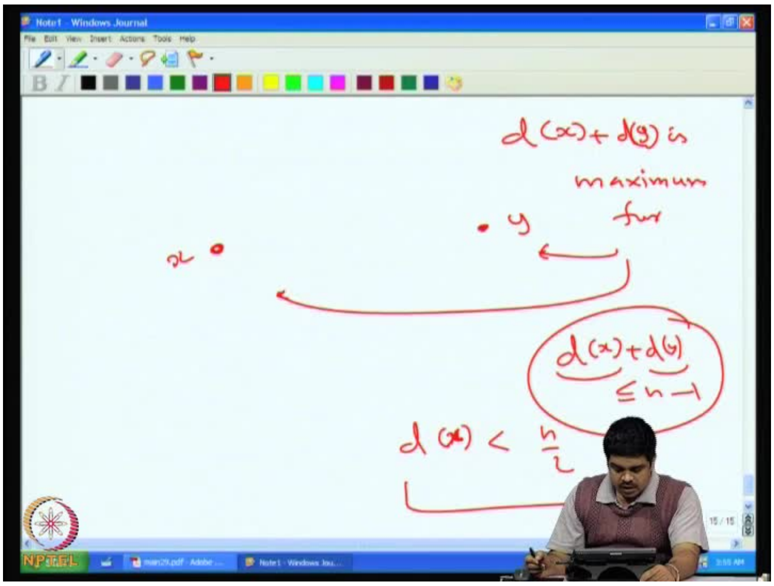

Now, see somehow we want to find out a contradiction; somehow we want to find out a contraction. So, what would be some place, suppose we will search for some x and y, some pairs x and y non-adjacent pair, x and y such that, this gets violated it. So, it is very natural to this get violated or it will create some trouble somewhere. So, it is very natural, because this this saying that this plus this has to be less than equal to 1, then it can very natural to seek that pair of x and y, where dx plus dy is maximized, because if it is maximum may be there is a chance that will go beyond n minus 1, because the biggest ah x and y such that, dx plus dy maximize is the god candidate. So, we will go for that. So, let x and y this select it such that, this is x this is y let such that among all the non-adjacent pairs dx plus dy is maximum for this pair maximum for this x and y this maximized.

Now, of case, we know dx plus dy is indeed less than equal to n minus 1, otherwise we already have a Hamiltonian cycle; now, we will see what the conditions are condition can say about these vertices now. So, the point here is, if you consider the smaller of these two things dx and dy, because it at most together the at most minus 1, this smaller then has to be strictly less than n by 2 is in it, because if both of them are at least n by 2, then together they will make. So, the smaller of dx and dy has to be at least n minus one. So, let lets the of generality let as say x is the smaller the vertex is the smaller degree and then dx is less than n n by 2.

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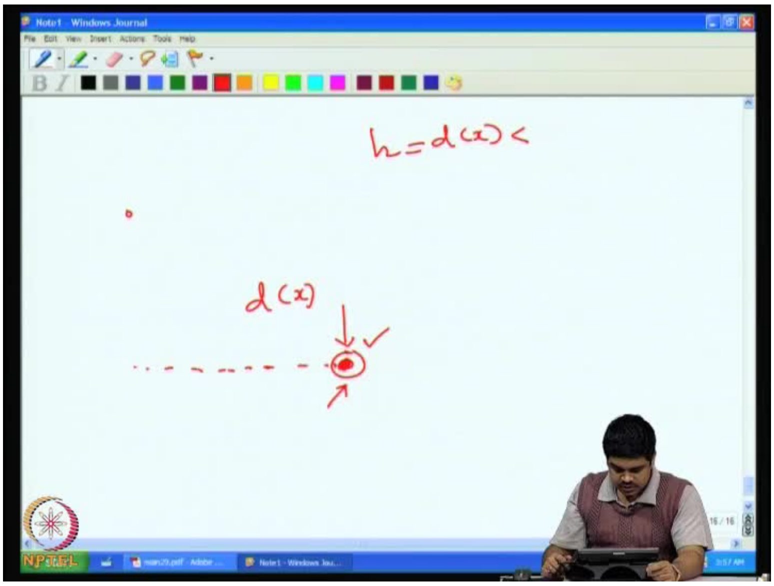

Now, dx has to satisfy the…. So, let us see, let us see, whether dx is satisfy the condition or not that is what we want, because if dx satisfy the condition, suppose somehow dx satisfy the condition.

Then we will have dx; suppose, put dx equal to some…, see the the edge right. So, in the position dx, when when dx comes right, so that number. So, this vertices will suppose this position. So, we will we will get some condition from this things.

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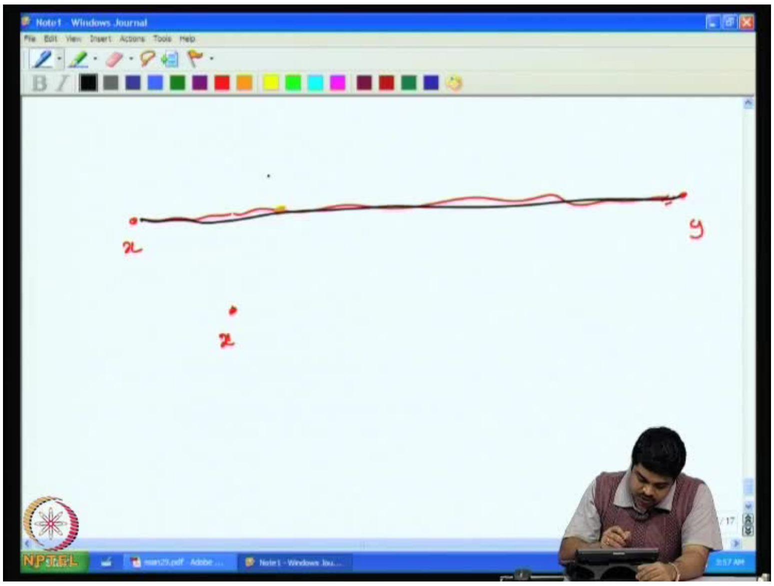

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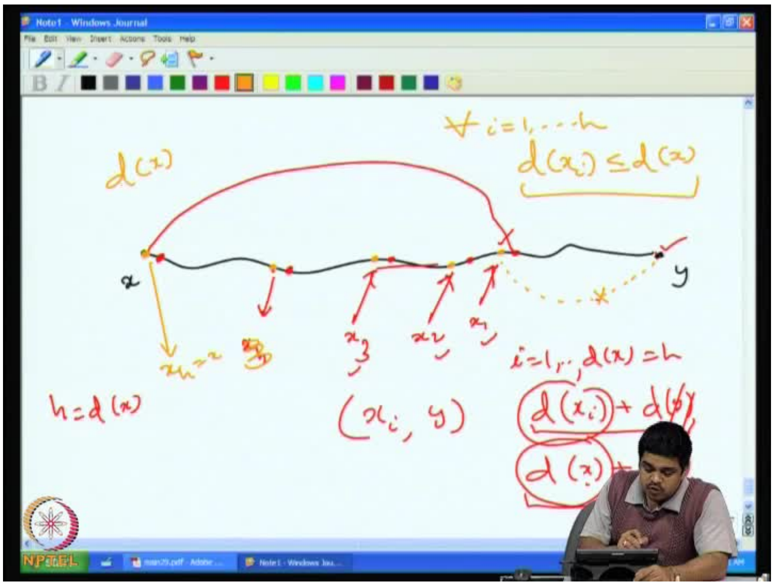

Now, let as let us see the yeah if yeah. So, this is what we can do is, we will look at the x and in the sequence in the Hamiltonian path we have seen x and this we have seen y. So, remember that, we had this path and then so we had marked certain orange vertices. So, let us say, let us draw the path is in black and then we marked. So, we have drawn the path using this is x, and this is y, this is x, and this is y, and then this is the Hamiltonian path. So, let us say, this y the neighbors of x some some other some neighbors of x and we have marked this orange vertices which just behind the neighbors of x; and we had already notice that, this kind of edges cannot be present right because if this edges present, then along with this edge it would from Hamiltonian cycle like this, this already seen, therefore, this kind of edges will never to be present; in other words, this orange edge vertices, each of this orange vertices along with y. So, if I call it say x 1; this is x 2; this is x 3; and this is x h, where h equal to d of x; each of this x 1 to x h are candidates to become candidate for a non-adjacent pairs along with y, because any x i y pair y pair is non-adjacent pair. So, we could have considered that. So, we could have taken the degree of dx i consider the degree of dx plus y d i instead of dx plus dy, but we show that this x and y we are selected such that, dx plus dy some was maximized. So, what can we tell about this? This some is definitely smaller than or equal to this some for every i for i equal to 1 to dx, that is true, 1 2 dx means such. So, all of them if any of this vertices, if I consider if some the degree of this plus the degree of this thing that should be less than the degree of this plus degree of this, otherwise we would a taken that pair instead of x and y right y and that see x i would have been taken.

So, dy be in the same here; we can say that, this has to be smaller than or equal to this. So, each of these vertices x i has to be smaller than equal to the corresponding vertex x. So, we see this clear for one of this vertices x 1, x 2, x 3, if this vertex may even be x. So, let us say, so, therefore, do not worry to worry about whether this is disjointed from, because x h this vertex may be this this may be a neighbor here. So, this, In fact, x is always there, because you look at this vertex is already an neighbor; therefore, this will orange vertex so. In fact, this x h has to be here. In fact, of them x h will be equal to x. So, x will be also part of them, but whatever it is each of them will have a degree see dx i should be less than equal to d of x four i equal to 1 to h any of each of them will be like that.

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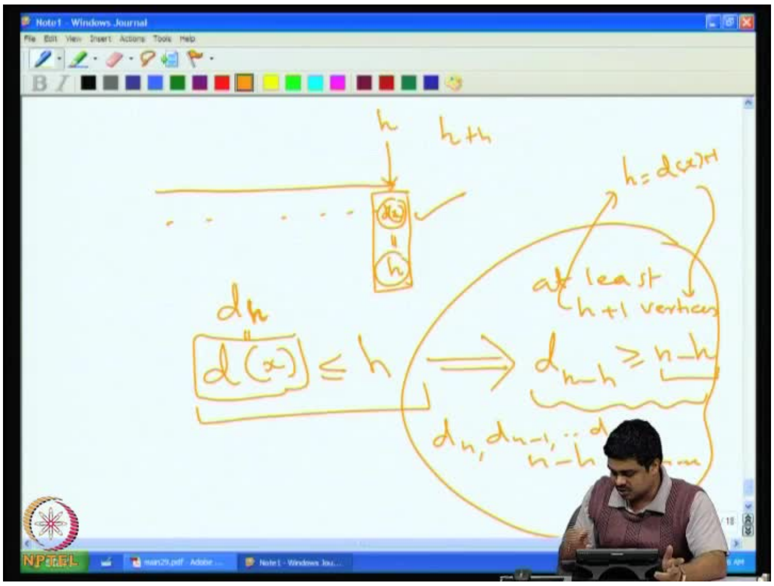

Now, if you numerate the degree sequence, all these numbers that is d of x 1, d of x 2, d of x 3, d up to d of x h, that means, all of them will come before d of x, it is before equal to d of x because d of x itself part of them. So, therefore, there are h numbers, there are h means d of x numbers, which are of degree less than d of x. So, when actual d of x comes it will be at least the hth number, we cannot below h d of x equal to h, this one be write the number d of x here, that means, we that is h. So, already we are the h position or even bigger. So, we can that d of x is less than equal to h this is the point this little tricky argument, but we have told this this the all that orange vertices on the path are such that, there non adjacent to y; therefore, that degree of those any of those orange vertices plus degree of y should be less than degree of x plus degree of y, because the degree of x plus, degree of y was maximized when that x y pair was select it we would take, otherwise in another orange vertex and the y.

So, it means, because d of y is same in both the some. So, we can that the degree of each of the orange vertices have to be less than equal to the degree of x degree of x be h then, which those numbers are less than equal to h; now, the degree sequence those numbers will get listed first right than before the degree of x is listed; that means, h degree x equal to h is listed, all those numbers will get listed first. So, how many such numbers are there? There are at least h of them, because h orange vertices are all including x. So, h hth number can be dx right or dx can come even after that. So, in the thus case or the earlier case the d of x appear is the hth the number. So, there are many repeated h as, but at least I can say that the hth number has to be has to be one of the edges, it cannot be greater. So, therefore, we will we will see that sorry, we will see that here we have an h this h number. So, this number is less than equal to h.

So, it means that the condition says; so, in the graph the condition this d this sequence d d 1, d 2, extra satisfies the condition, therefore, it tells us that we our d of…. So, it says that d of n minus h, this is essentially the d h number d h, this is the hth number of d h d h is equal to d of x here, the hth number here. So, that happens to be dx is less than equal to h. So, now, it is not there, it has to be h because is it if there are many vertices which are of lesser degree, you can even have this less than strictly less than h, then it means that d of n of h should be greater than equal to n minus h; so, it has to be greater than or equal to n minus h.

So, which means that, there are at least h plus 1 vertices with degree n minus h or more n minus h or more there are which are the vertices, this essentially corresponding to d n, d n minus 1, d n minus h, these are h plus 1 of them are there, so these all these vertices have degree at least 1 minus h and however, the vertex x has only degree h. So, it cannot it can be adjacent only h vertices is it possible for it to be adjacent all the h plus 1 vertices. So, thus definitely not possible, because if it has only h neighbors, h plus 1 means this h recall h is equal to d of x.

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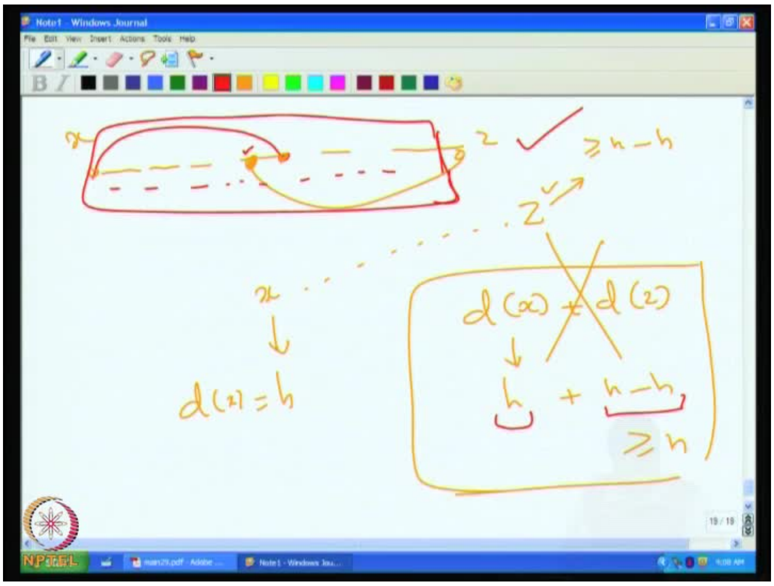

So, we have d of x plus 1 vertices with degree at least n minus h; so, out of that our vertex can be adjacent to only only h equal to d of x vertices; so, one vertex among them has to be… some set has to be search that it is not adjacent to x right; so, it is not it is not adjacent to h, but on the other hand h degree is at least n minus h and this degree is h d of x equal to h.

Now, for this non adjacent pairs, if I consider d of x plus d of z, what will I get I get h for this and here at least n minus h that is greater than or equal to n, this is the point, but we know that this is wrong, because if this happens the if you consider the Hamiltonian path between x and z; we will have a red vertex and then orange vertex just side by side there is a connection here and there is a connection here, because you know a together the sets neighbors cannot be placed without sitting a one of the without taking one of this vertices which just behind the neighbor of x, because we do not have enough there are more than n of h neighbors of set h neighbors of x together; they already make more than n minus 1, because in there is n minus 1 places that to find in this n minus 1 places, all of them have to find location. So, therefore, this will happen and will get Hamiltonian cycle. So, that is the contradiction we are getting a Hamiltonian cycle that is a contradiction.

So, its means that, so, if the condition is satisfies for a for every point wise greater degree sequence, the corresponding graph has to be to Hamiltonian. So, we have shown that to summaries the proof what we did is, if it is not true we considered one such sequence corresponding to graph maximum number of edges among all possible such cases, which violate our theorem that they will be Hamiltonian cycle.

And then among all the non-adjacent pair that is a key point among all the non-adjacent pair, because between non-adjacent pair we already have a Hamiltonian path, because of them maximally among all the non-adjacent pair; we maximized dx plus dy and then we show that if it is maximum and then we applying the condition we can find a contradiction, because we will always have another pair dx and z. So, that dx plus d z goes beyond that another already be…, and then it to be violations this is what bit it.

So, in the next class we will study the second part of the theorem namely, if it is a Hamiltonian path, this condition should be satisfied, in fact, some all the, so, if it is Hamiltonian sequence. So, that means, if this condition is not met, then we will find some point wise greater sequence which is violating the theorem. So, that is what in the next class we will show in one example. Thank you.