Graph Theory
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Lecture No. # 28
Boxicity, Sphericity, Hamiltonian Circuits

Welcome to twenty-eighth lecture of graph theory. In the last class, we were discussing about a special kind of intersection graphs, namely the intersection graphs of axes parallel rectangles on the plane.

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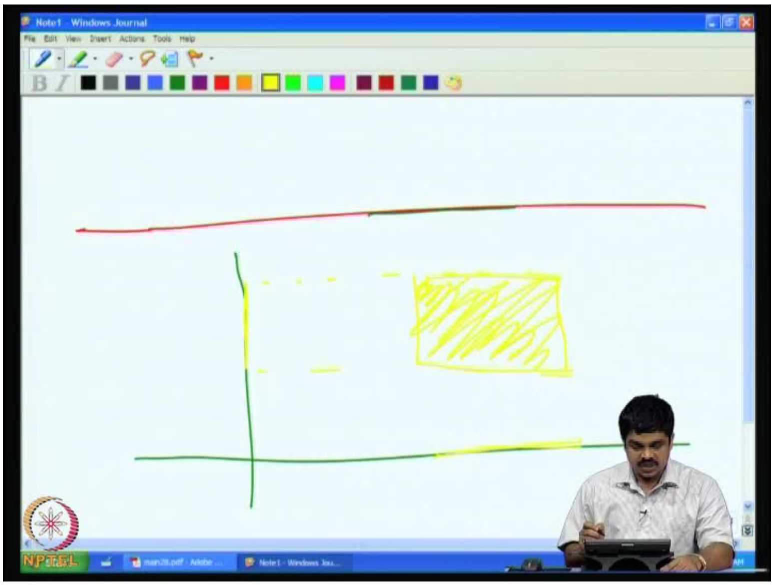

So, this were again considered as a generalization of interval graphs, because axes parallel rectangles can be seen some kind of two-dimensional rectangles, because an interval, so, it has a two-dimensional intervals, because an interval is like this. So, if you have two dimension, and if you consider an interval here, and interval here, so combined effect is this rectangle, this we take the Cartesian product is see this thing.

So, in other words, this rectangle projection is this interval, this rectangle projection is this interval; so, in that sense, the these are kind of two-dimensional intervals, so that way we can thing that this an generalization of interval graphs. We had consider several other generalizations of interval graphs earlier like circular graphs, when the intervals are considered on the circumference of a circle rather than on the straight real line, and eighty free graph which which try to capture this linear structure of interval graph made identifying that it is that… absence of the… are there reason for that and have to how it generalized to a much general structure called eighty free graphs; chordal graphs, where we identified the interval graphs are graphs without cordless four cycles and then we define the super class, chordal which contains the interval graphs.

So, there are several such generalization; here is an obvious generalization, where instead of considering the intervals intersection graph of the intervals on the real line, you would rather consider axes parallel rectangles on the plane and then consider the intersection graphs.

And in the last class we had also seen that, so there are graphs like cycles and all big cycles which can be represented as the intersection graph of axes parallel rectangles, but which cannot be represented as the as an interval graph; there are not interval graphs: for instance the cycles are example.

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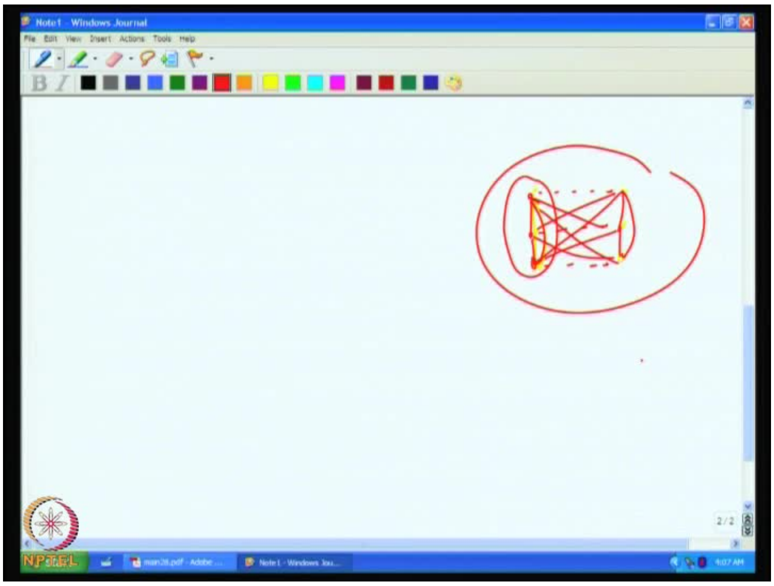

So, and then we ask this question is, is it possible to represent any graph as the intersection graph of two-dimensional rectangles, so axes parallel rectangles, it is not possible also, because we told that there is this graph, we promise that there will be a graph which is drawn like this three vertices, three vertices, and if it is so i will three vertices, three vertices. So, this kind of rectangle, see this graph cannot be represented as the intersection graph of axis parallel.

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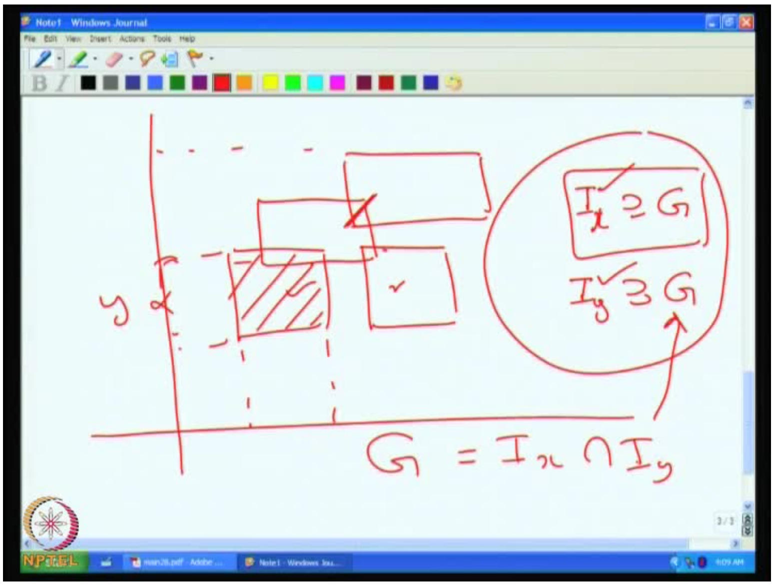

Today, we will explain why it is so; and to understand this we want to understand how to analysis this rectangle graphs has the intersection graph of two interval graphs; we had shown that, if you get a rectangle representation, I mean, if you get a graph and somebody gave us a representation of the graph as the intersection graph of the axes parallel rectangles, what we can do is, we can drop project each rectangle to x axis; and if you take the intersection graph of the intervals corresponding to these projections, we will get an interval super graph of the given graph, means, i 1, i x, which is the super graph of the given graph, the same vertex it, but the edge set can be more.

Similarly, if you do the projection on the y axis and take the interval graph resulting from the projections, then that will also be a super graph of the given graph that will also be a super graph of the given graph.

So, here the vertex set is same, but the edge set can be more, but every intersection here we will correspond to an intersection for the projections also, but sometimes you may get an intersection even when the rectangle are not intersecting; for instance, if you had a rectangle here and then if you project it here and it here, both of them are intersecting in the projection in the y axis, but they are not actually intersecting, that is why these extra the edges are coming; but the crucial properties is that, if we taking intersection of these two interval graph that we created; that means, i x intersection i y, when i say intersection i y, I mean intersection of the edge set we will get back the edge set of the original graph; because it is not possible for a two non-intersecting rectangles to have intersection in both their projections, because if both they projections have intersection the rectangles have to intersect, because there after all these are Cartesian product of this thing the projections.

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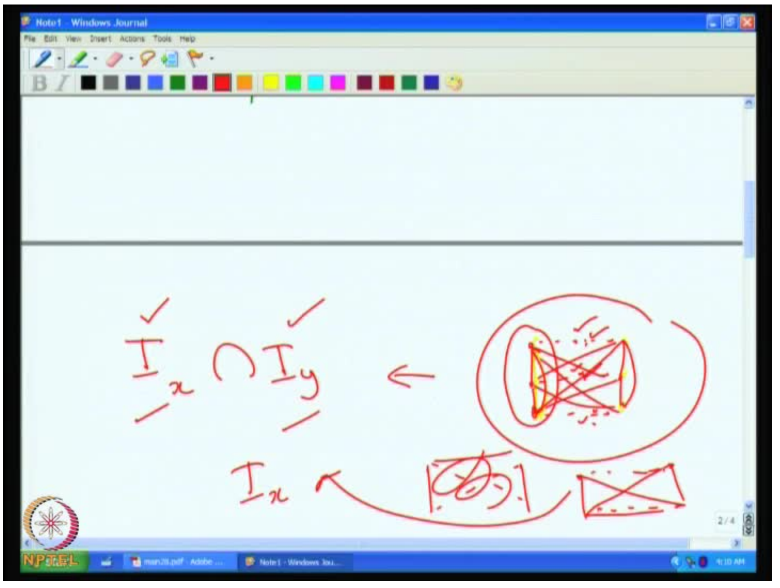

So, we can see that, if you have a representation in terms of the rectangles then we can find two interval graphs, such that, g is the intersection of these two things. So, the point here we told that now that particular graph we considered, so, this graph, this is three missing edges, and if this is represented as the intersection of two interval graphs i x and i y, then this should go away in i x or i y, this should go away in i x or i y, this also should go away in i x or i y; so, that means, because there are two interval graphs and three of these missing edges absent edges to disappear in one of them. So, we applied (( )) principle and told that they should be one of this case may be i x in which two of them misses, say let us say, this and this is missing.

That told as that, it will create two sorry, so it will create a four cycle for us; this kind of a four cycle will come from that. So, this is the four cycle in the interval graph, but in the interval graph we cannot have a four cycle and then that is that is have the contradiction.

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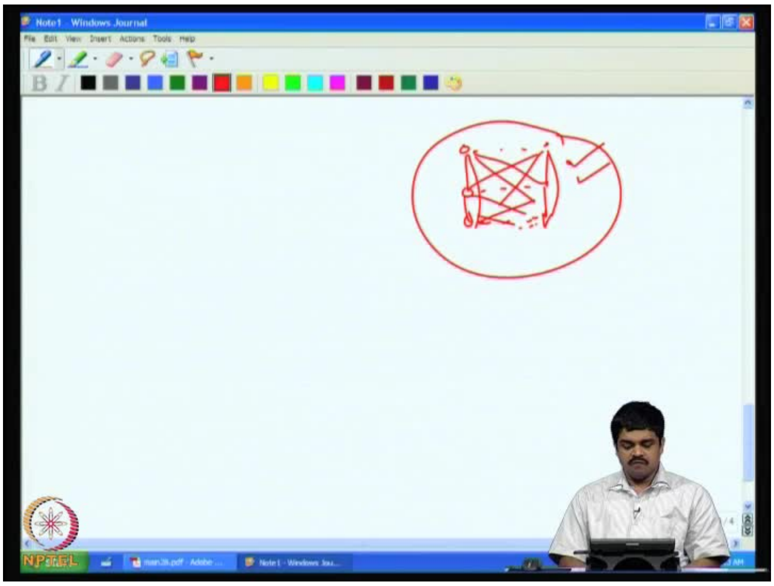

So, this was a simple argument to show that there are graphs which cannot be represented as the intersection graph of axes parallel rectangles; this is simple enough graph. So, we have a small point here on three vertices and we have a small point here on three vertices and we connected except this matching, so like this and except this perfect matching of them.

Now, we can generalize this concept further; we can ask what if i consider the intersection graph of axes parallel boxes in three-dimension, so, two-dimensions, we can go to three-dimension. So, then, so, of case we can it is a super class now. So, it can capture all the rectangle graphs the intersection graph of the rectangles on the plane, but then it can capture more for instance this can be represent it.

So, and now, we will ask whether it is possible to have graphs, which cannot be represented as the intersection graph of axes parallel cubes in three-dimension. So, happens that there are still graphs which cannot be..

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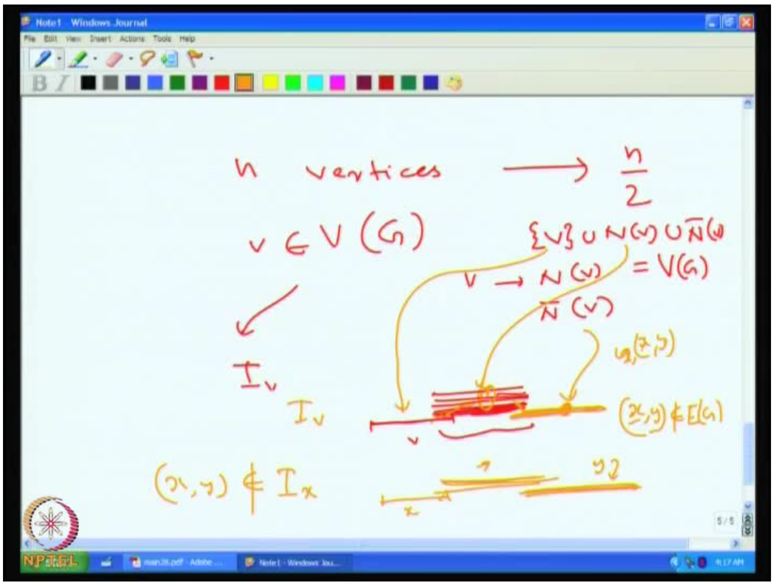

So, it gives us two question, suppose I go to next dimension - fourth-dimension - can I do it here, who given a graph n, is it possible to represent it as the intersection graph of axes parallel rectangles in some dimension - some finite dimension. So, this is the first question, which comes to mind. The second question is, in that case is there what is the best value of that, so what is the smallest value of that. So, the first I will show that it can be done on some dimension. So, this is done by (( )) in the nineteen sixty's, he showed that any graph on n vertices can be represented as the intersection graph of some axes parallel boxes in some n by two-dimension. So, we want get to the details of n by two-dimension; I will show you how to represent in n dimensional at least.

So, what we can do is, for each vertex v of the vertex set of g, we construct an interval graph of i v like this; so, we associate an interval for v and then v has some neighbors let say n of v and v has some non-neighbors n bar of v, let us say, this is the entire vertex set total v union n of v union n bar of v will give you the total vertex set.

Now, so, if this is v…; now, all the neighbors of v can be placed here, see one top of the other, because same interval can be given to them like this touching v and then so though I am drawing like this, it is all same interval like this; and then all the neighbors of n bar v can be placed after that may be like this; n bar of v comes here, n of v comes here, and v comes here, this is an interval graph; this is now an interval graph and you can easily check that is an interval super graph of g, why is an interval super graph of g is because v if you take any edge u v u sorry x y in g if both x and y are in the neighborhood of v, then they are intersecting here; and if one of them is v, then the other has to be in n of v because their an edge and then they are intersecting; and then one of them is in both of them outside n of the n bar of v, then there also intersecting here; and one of them is in n of v and one of them n bar of v, there intersecting here.

So, therefore, all the edges which are present in g is present, so it is a super graph, the more edges that is, but as far as v is concern. So, this is what for each vertex v, we can draw a graph, we draw such any graph, i for each vertex, i v for each vertex, v i can construct i v like this.

Now, I claim that, if taking intersection of all these interval graphs, I should get back the edge set of the original graph, because everything is a super set. So, when you take the intersection we make it a super set; it is possible that some edges which have present, which is not present in g may still remain in the intersection, suppose such an edges there x y which is not present in e of g is present in intersection.

So, let us say, for x and y, for corresponding to x we have constructed a graph like this. So, now, where is y here, y cannot be in these group, because y is in n bar of x, because it is not a neighbor of it is not a neighbor of x, so y should be n bar of x, so, y is here, this is x.

Now, you see here x and y are not intersecting, so therefore, even x y is not an edge of this interval graph i x is not an edge of i x, so when you take the intersection of all these interval graphs we should get back the original one; So, the key point here is, one thing is we we have we have shown that every edge which is presently of g is present in all the graphs, because that is what is meant by saying that all these interval graphs are super graphs of g.

Now, second, we have shown that whenever there is an edge missing in g x y, say, it should be missing, it should be absent in one of the interval graphs we consider, which means that when we take the intersection of all these interval graphs the original edge set will anywhere remain in the intersection and nothing more will remain, because every extra edge, that means, an absent edge in g will be absent in one of the interval graphs and when we take the intersection that has to disappear. So, that is the key point. So, this is the idea.

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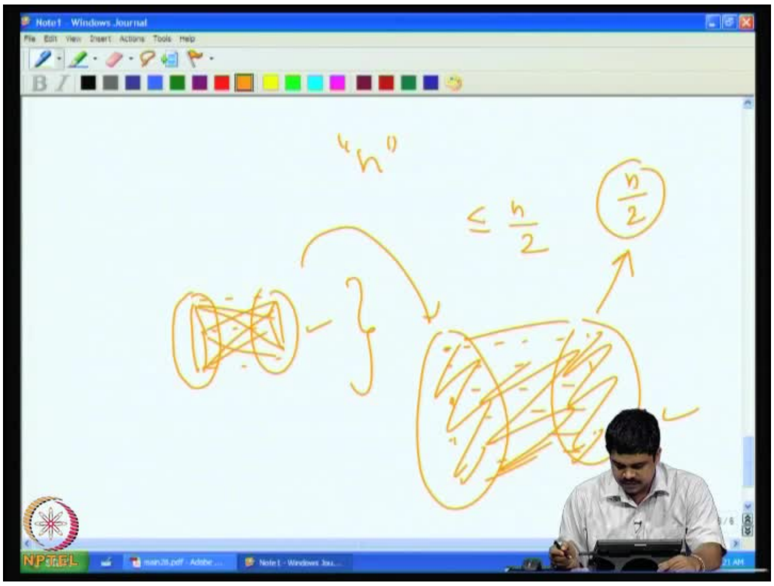

So, what we have seen is, there is always a given a graph it is always possible to represent the graph in n dimension, what do you what do why do I mean by, I repeat that, when I get n interval graphs, such that, the given graph is the intersection of these an interval graphs; it means that, you can draw the corresponding box representation in dimension by taking that each of the interval graph in different axis's and reconstructing the boxes; what how do you reconstruct the boxes? For instance, two-dimension if you know the projection corresponding to x in x axis and the projection corresponding to y in y axis, then definitely we can draw the corresponding rectangle back; then because the intersection will happen between the rectangles only when all the projections and all the axes are intersecting; so, therefore, essentially the edge set that results in the in intersection graph corresponding to the boxes draw will be essentially will correspond to the intersection of the interval graph; we consider therefore, it is enough to show that we can get so many intervals graph, such that, the given graph is the intersection graph of so many interval graph.

So, the key point here is, anyway the idea is even a graph, you can always find some value which is less than equal to n, such that, it can be represented as the intersection graph of axes parallel rectangle so many dimensions; so, that boxes of that dimensions space; now, robots had shown that, in fact, we need only n by 2 and extending the argument which we gave for this graph, this is the triangle, this graph extending the argument that we gave for this thing to say we can put n by 2 vertices here, and n by 2 vertices here, and then we can put all the edges excepted perfect matching here, all the edges except perfect matching here. So, this is exactly this graph instead of three vertices we are taking n by 2 vertices here, and n by two vertices here, we can argue that at least n by 2 interval graphs are required, if I want to represent it has the intersection graph of so many interval graphs.

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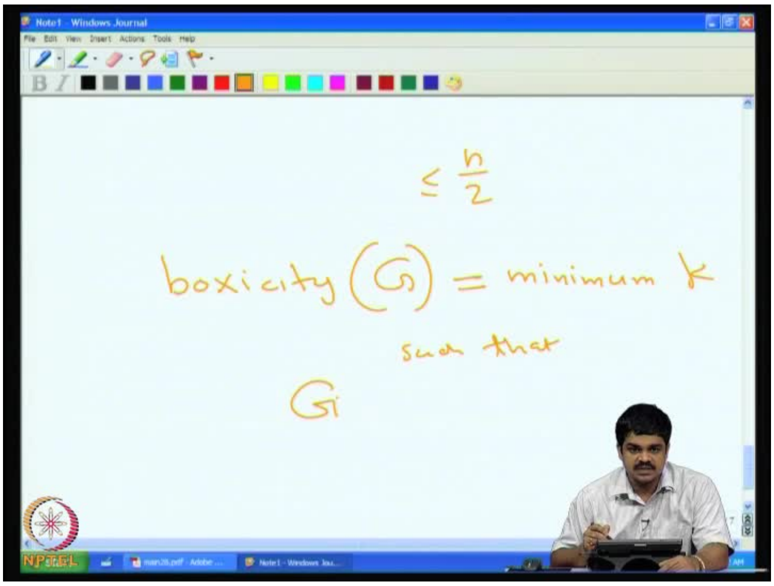

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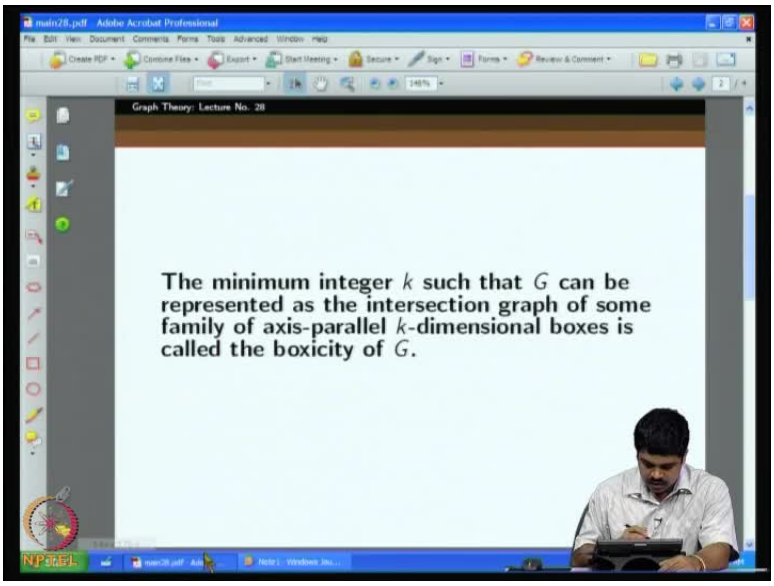

So, therefore, he has shown that n by 2 is an upper bound for the number of dimensions required and then for some of these graphs we need n by 2 also; and of case the interesting question is, what is the smallest value, and we also can call this parameter as boxicity of the graph; that means, the minimum integer, it which is the minimum k such that g can be represented as the intersection graph of axes parallel rectangle boxes k dimensional space. So, will a minimum integer k, the g can be represented as the intersection graph of some family of axis parallel k dimensional boxes is called the boxicity of g.

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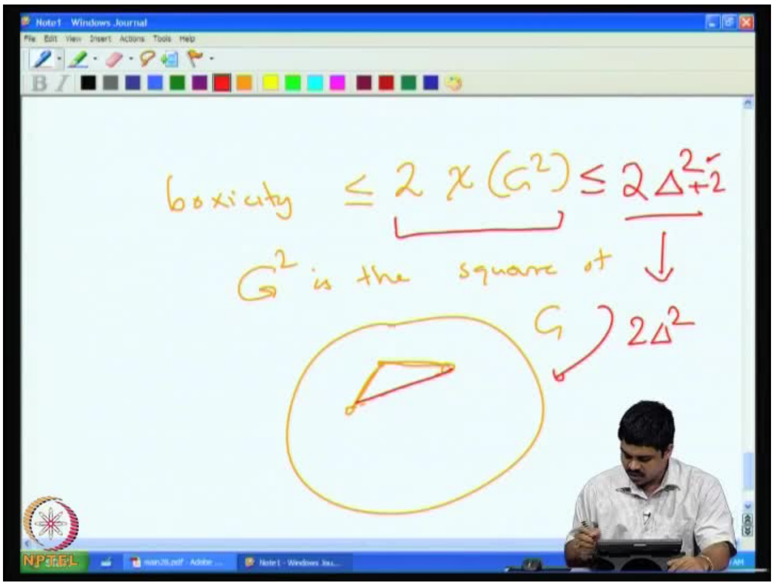

So, later, in fact, many other upper bounds and non n by 2, so for instance it can be shown that, in fact, 2 into chi of g square is an upper bound for the boxicity, where g square is the square of the graph square of g; square means what, that means, you take the take g and along with that add a humor edges; whenever u and v are non-adjacent at a distance of 2 in g, then we add this new edge, so, there is resulting graph is called g square. So, this is chromatic number of g square times to will be an upper bounds of the boxicity; this will be two times delta square plus 2, easily you can easily show that plus 2 and by some little work you can show that 2 delta square is also an upper bounds; later it can also show that it can be improved, but it is just find the information; but point is to show that, this are all generalizations of interval graphs in the various place, there are several other interval graphs these are all interesting special classes of graphs. So, like boxicity you can also consider intersection graph for axes parallel cubes say cubes in two-dimensional will be square; so, that means, it is the restricted case of boxes representation; that means, every box has to be half of equal, so side length equal side lengths.

And all different boxes, each box has to be of a unit side length lets a unit cubes that then if I can ask the same question, what is the smallest dimension integer case such that, I can represented that graph g has the intersection graph of k dimensional axes parallel cubes.

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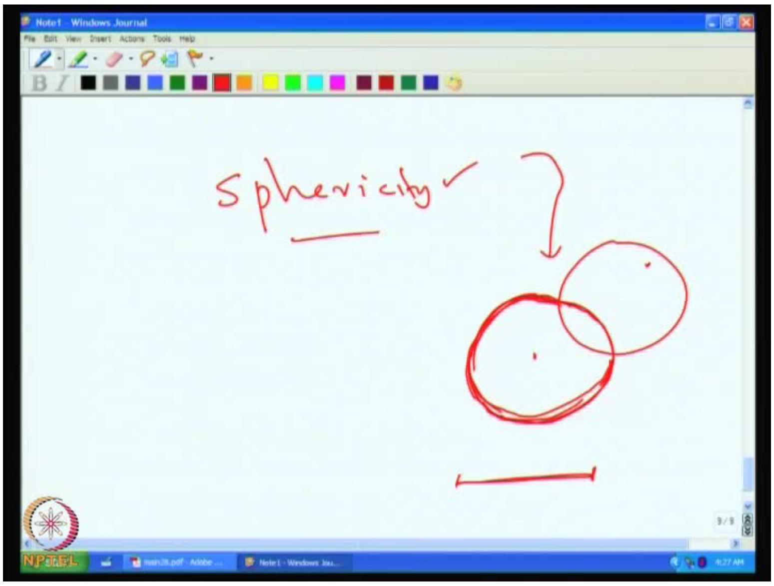

This parameter the minimum integer is called the cubicity; cubicity is always greater than equal to boxicity, because the cubes are especial restricted kind of boxes and they can be somewhat bigger, sometimes even i log n factor bigger than boxes representation. And another parameter similar parameter is called a sphericity; sphericity where we use sphere instead of boxes of cubes and we are supposed to use some k dimensional spheres instead of just three-dimensional or two-dimensional sphere will be a circle; when coming back to one-dimensional, it all become intervals, the cubes will become unit intervals, while spheres in the see in the again unit intervals; but we are when the case of spheres we have two options, one is we can alone sphere of same radius only or we can allow spheres of different radius; sphericity is used for the corresponding dimensions when allow only spheres of unit radius.

So, these intersection graphs are can have the applications, for instances, we can consider the molecule; and then so the molecule is a graph, because when two atoms are close together they have some attraction; and then, so, we can capture the molecule as a graph like will two spheres around the nucleus and then there when they closed in of the intersect, otherwise the one intersect. So, the intersection graphs of spheres. So, anyway, so, this are all to introduce of some concepts, which are related to interval graphs and generalization of that and essentially the interesting special classes of intersection graphs; now, with this thing we will be leave this topic, because though there are lot of material that we can study about this thing, but because this we have other things to see in this lecture series. So, we will we will leave it to the interested student to explore this parameters leisure time. So, of case that is lot of material in the literature.

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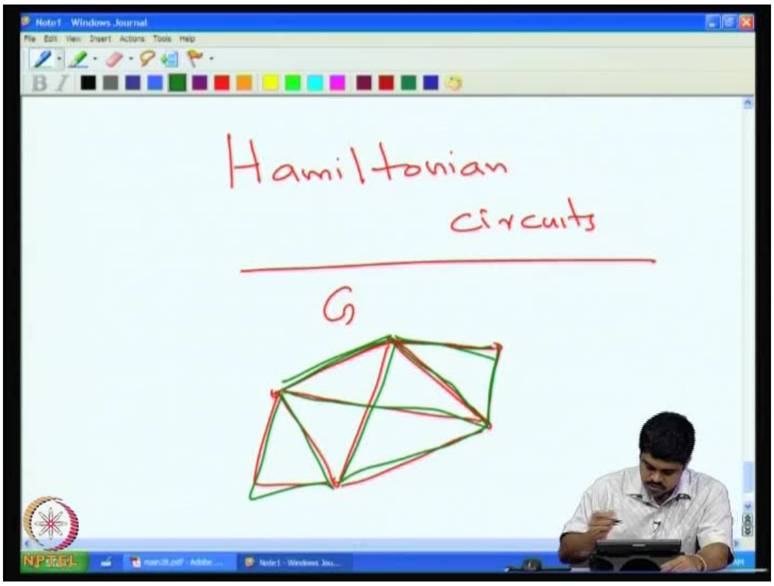

And now we go to a different topic called the Hamiltonian circuits. In fact, we had seen a certain type of circuits earlier which is known as the Euler circuits. So, what was that, given a graph g, so we are taking a closed walk in such a way that, every edge is covered ones and only ones. So, for instance, so, if this is these things, so, this was the Euler to write you just go from here and then go from here anyway.

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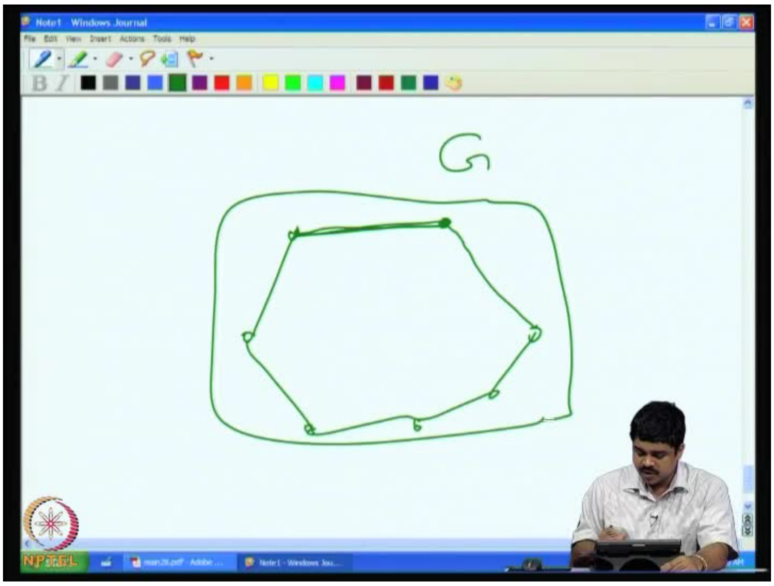

See you can go like this. So, you have studied it earlier. So, if you go like this, you have covered every edge exactly ones; now, the Hamiltonian circuit is, its Hamiltonian cycle is the contour sorry is a the corresponding question for vertices. So, given a graph g, so, we are interested in a closed walk, such that, we meet each vertex ones and only ones, so it should be like this. So, it can be several. So, we should come back to the original place, but never walk in our circuit, we should meet each vertex each vertex of the graph exactly wants.

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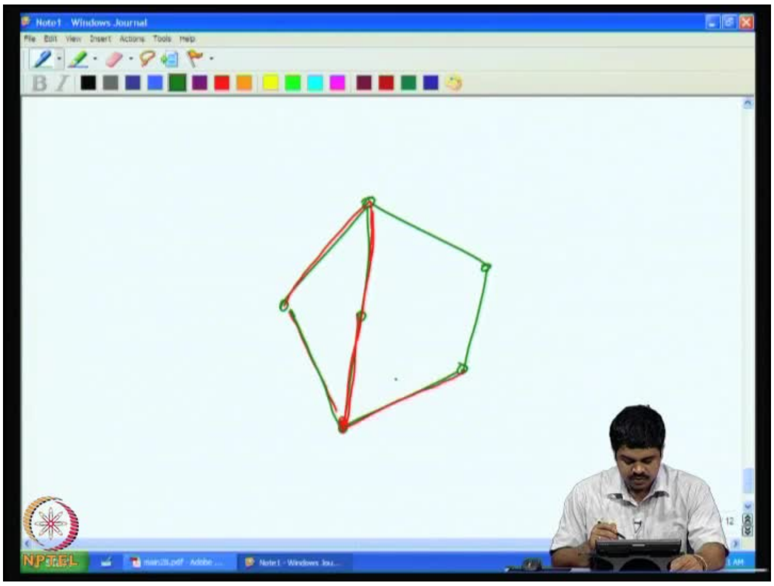

So, sometimes such circuits may be there, sometimes this circuits may not be there; let us see, we have such a circuit in this graph here, you can try to do this thing, because the problem is if you go like this, then if you get into this you cannot see meeting it for the second time or if you do not start from here; now, so, if you go from here, and then go then will an end up coming here, so its some trial and error we can see that it is not possible to find a hamiltonian circuit here, because every time without reentering the vertex, one of the vertices you cannot come back to the same place.

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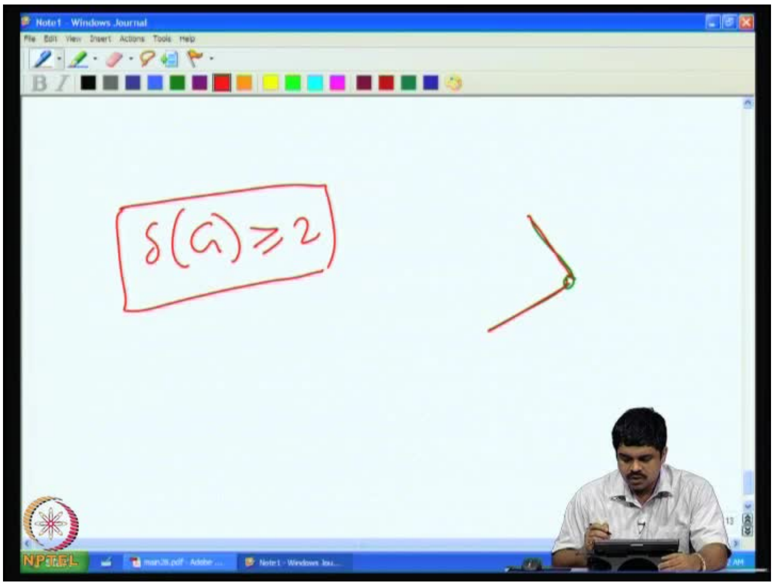

So, what are some obvious situations where Hamiltonian circuits cannot be present, for instance, the very fact that we are saying that every vertex should be visited ones and only ones and we should come back to the place where we started; it means that, the vertex each vertex in the graph, each vertex should have a degree of at least two, because the circuit should enter it and come out, otherwise we cannot have a Hamiltonian circuit; that means, the minimum degree of the graph should be at least two, otherwise that is no way for a Hamiltonian circuit. So, a graph is called Hamiltonian graph, if there is a Hamiltonian circuit in the graph.

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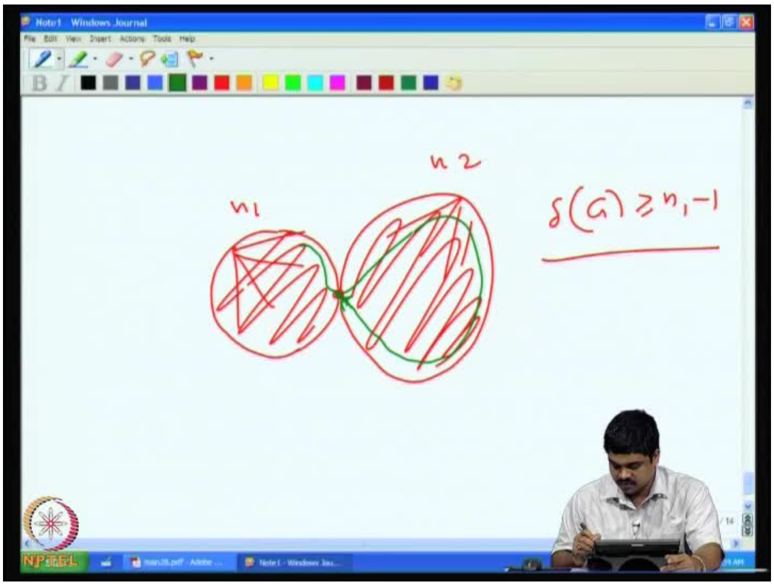

So, does it mean that, if we increase the degree of the graph you can get a Hamiltonian circuit always that is also not correct; for instance, you can construct a graph like this. So, for instance if you have n 1 nodes here, and n 2 nodes here, so every vertex here have degree n 1 minus 1, so, here n 2 can be bigger than n 1; so, minimum degree of the graph is such a graph is suppose if I make it clique, and if i make it a clique, and then paste them an one vertex it is a connected graph, but delta of g can be greater than equal to n 1 minus 1, so there is no Hamiltonian circuit in this graph, why is it so? Because some point if you want cover all the vertices here to cross from this side to this side, then you cannot come back, because there is a cut vertex here we cannot come back. So, if there is a cut vertex then we cannot have a Hamiltonian cycle, because two connectivity is a requirement to have a Hamiltonian circuit in the graph. So, it is not to have Hamiltonian circuit in a graph when there is a cut vertex.

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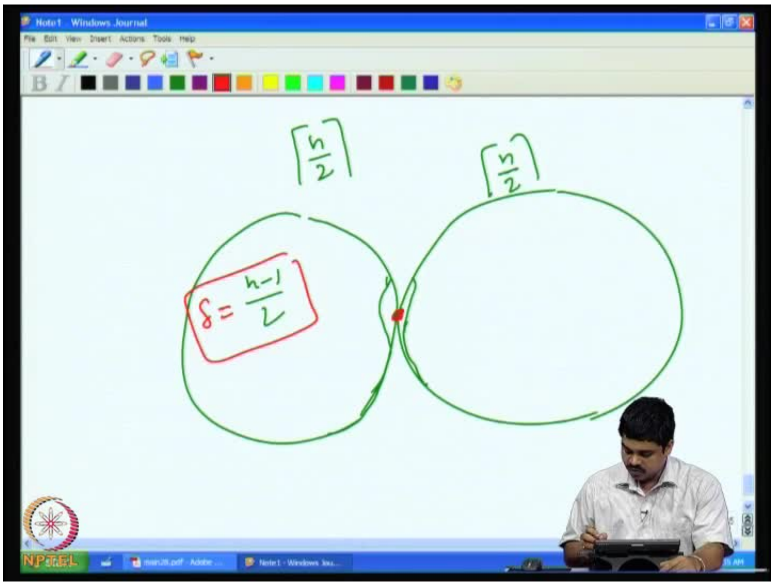

So, it is says that even if we try to keep the minimum degree as much as possible to great extent, you cannot achieve a Hamiltonian circuit in the graph just by that assumption. So, the worst case may come from this example, the worst case may come when we consider an n by 2 size to clique and then an n by 2 size clique here, because you know this a common vertex; so, we have if n is an odd number, so n minus 1 by 2 especial exclusive vertices here, n minus n by 2 exclusive vertices here, and then one extra right that will make it.

So, that means, even so, then the minimum degree of this graph will be n minus 1 by 2, so the n by 2; even then we see that it is not allowing as to have a Hamiltonian cycle, because it is the cut vertex, because to avoid the cut vertex it is not enough to have the degree high degree has minimum degree high minimum degree has to be very really high, then only it can go to the sit level that, it can guarantee that, there are no cut vertices right in this example, tell minimum degrees even, if minimum degree goes to this level right it is not.

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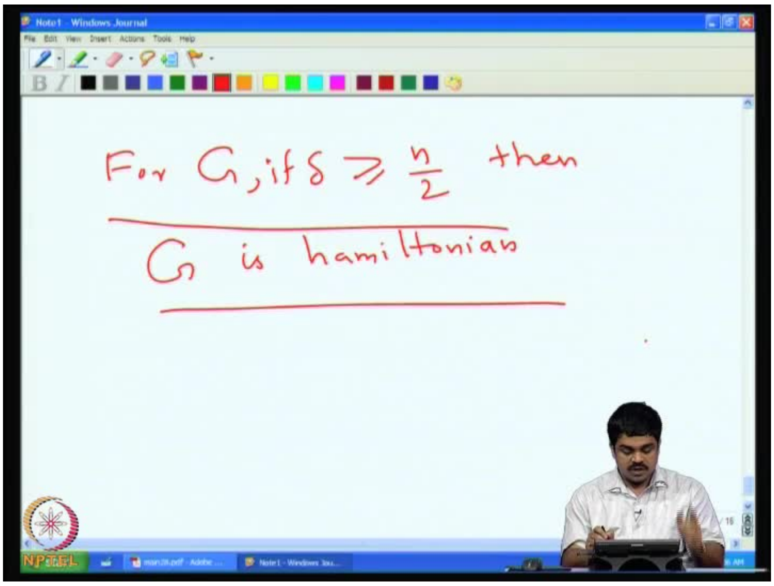

But, then very interestingly if the minimum degree is above this, for instance, is we know that minimum degree is greater than equal to n by 2, then the g is Hamiltonian; this is the first theorem we want to study in this section.

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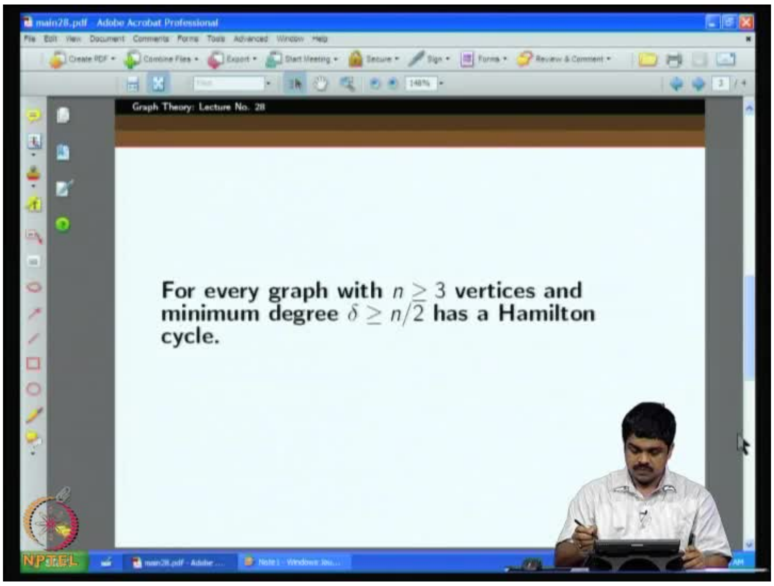

For g if minimum degree is greater than equal to n by 2, then g is Hamiltonian and also of case, so that is not just enough because if the g has the two vertices, it cannot have hamiltonian cycles. So, we will say that, if the number of vertices is greater than equal to 3, the minimum degree greater than equal to n by 2, then it has a Hamiltonian cycle.

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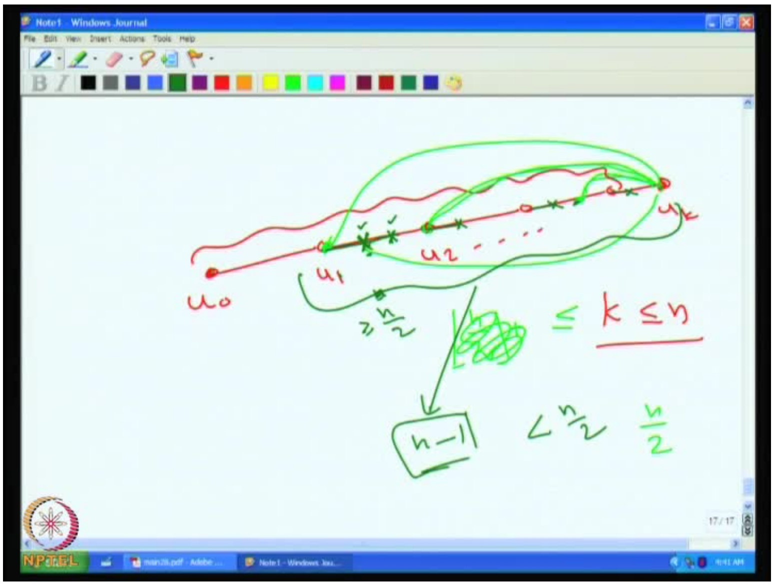

The technic to prove this is like we have already encountered this kind of ideas before, though again this proof is one Dirac; so, what we consider is a long longest path longest path in the graph. So, we consider a longest path in the graph so, but when we see a longest path the interesting thing is that all the neighbors of this n point has to be on the path; suppose, the longest path has some k vertices, then k can be less than equal to n it can be strictly less than n also, suppose the longest path has some k vertices in it.

So, let say or may be say k plus 1 vertex in we, let us mark it as u 0, u 1, u 2, so this is u k; now, what can I tell about the neighbors of u k, the u k's neighbors are all in this collection, because it cannot it cannot have a neighbor outside by this; suppose, if you have neighbor outside, this collection then this cannot be longest path, it will you can extend the path from here to this node also, so it cannot be longest path; so, therefore, it is not possible to have a neighbor outside this, so, this neighbors of u k are all inside may be, it can be like this, it can be like this, it can be like this, but we know that u k has n by 2 neighbors. So, this green edges that I have drawn from here goes to n by 2 of this places, which immediately says that, this has to be some n by 2, n by 2 4 plus 1, 1 more than this thing suppose right, I cannot say because if it is an even number n by 2 plus 1 vertices, k should be at k u 0 also should be the n by 2 at least right.

So, the neighbors of u k n by 2, neighbors of u k has to be present somewhere here; now, it can be here; now, suppose I consider the places, so I identify the neighbors of u k here and then I just I am just interested in the vertices just above each of this neighbor, see this vertices they marked it green.

So, how many such a vertices are there of case therefore, each neighbor of u k i identify one neighbor which is just about, so at least n by 2, n by 2 or more because there are n by two neighbors here; and each of them see does not mean that, we are saying that they want to be an a green edge from here, we adjust collecting those vertices, which adjust up if there is this thing; we will collect this also not only not only this vertex, we will also collect next vertex also, this two vertices I will collect.

So, I am collecting all the vertices which are just after a neighbor of u k and there are at least n by 2 or more such things; now, the question is, where are u zeros neighbors? So, is it possible to have u zeros, all neighbors are in this, in this collection, of case, u 1 to use u k among the u 1 to u k, all the neighbors of u 0 should be present.

Now, is it possible for u zeros neighbors to avoid all these places, which just above the neighbors of just after the neighbors of u k in that case, because we have already consider at least n by 2 such positions and u 0 is 1 and u 0 itself cannot be a neighbor of that. So, it is k can be maximum n minus 1. So, sorry n, so, when u one to u k can be total n minus 1 among the available n minus 1 position; if you avoid n by two or more things then you will have only less than n by 2 candidates for then placing the neighbors of u 0; we cannot do that, because u 0 also have n by two neighbors. So, there it is clear that, it is not possible for u zeros neighbors to avoid all these green these places dark green places which I have marked means the places which is just after a neighbor of u k.

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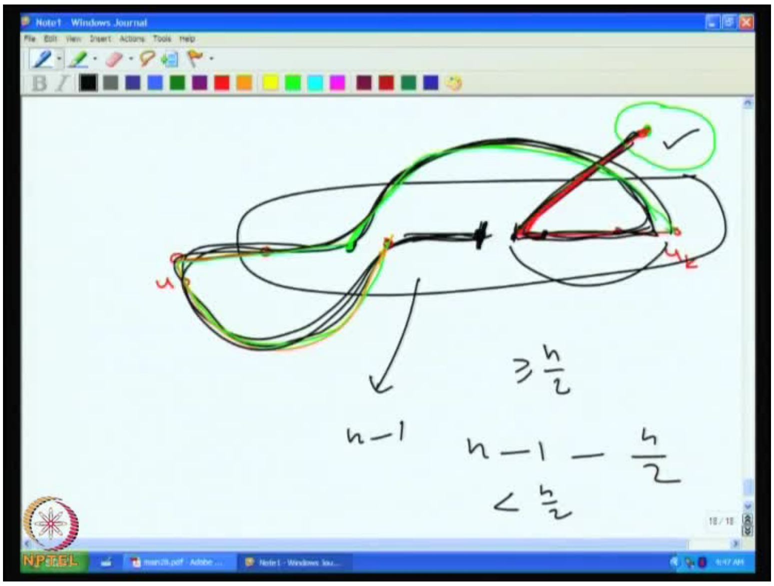

Repeating the argument, so, we considered a longest path the interesting thing about the longest path is both u 0 and u k; the n vertices should have its neighbors in this path itself we that there existing neighbor of u k in the path, such that, just after this the next neighbor next vertex in the path is a neighbor of u 0 like this; there is something like this is what we want suppose, it is not there, then u zeros neighbor should be avoiding all these places, which are just after the neighbors of u k; that means, they are avoiding greater than or equal to n by 2 places from the n minus one available places for them.

So, how many of them can be there u 0, how many neighbors u 0 can have only at most n minus 1 minus n by 2, that is strictly less than n by 2, but we know that u 0 has at least n by two neighbors. So, it is always its guarantee that u 0 at least one neighbor of u 0 will sit just after a neighbor of u k some neighbor of u k on the path.

So, this clearly tells us that, this path can be converted into a corresponding cycle this way. So, for instance if i draw it with green. So, i can go like this, then i can take this jump to u k, and then come back and then take this jump back to u 0 the green cycle which I drawn here will be a cycle of length k sorry k plus 1 here. So, means if the number of vertices in the path is k plus 1 here, the cycle also has the same length that path is converted to a cycle now contain u all the vertices.

It is possible because we could identify an edge in the path, such that, below the earlier n vertex of the edge with respect to this path was a neighbor of u k and the other n vertex is the neighbor of u k that why we could cut it here right. So, that is why we could make it. So, we just get rid of this thing we we get a cycle like this.

Now, if this u 0 to u k was the entire set of vertices; that means, this path contained all the vertices, we are done we go to the hamiltonian path here cycle itself here. So, suppose, it is not suppose, it is not then they should be some vertex outside the graph, they should be some vertex outside the graph, outside this cycle in the graph, such that, it is not part of this, but then this is a connected graph. So, this cycle cannot be totally disjoint from the remaining part of the graph. So, if there are some they should be some vertex outside, which is adjacent to something here in the cycle. So, may be it is adjacent to here, say it is adjacent to here, it has to be because it connected graph you cannot have this cycle not adjacent to any vertex in the remaining graph.

In that case, what will we do, we will start from because here suppose if this is the this thing. So, then consider the vertex which is just below the cycle, we could have started from here followed this cycle like this, and then followed here, and then we could have extend it, because it is a cycle any vertex adjacent to the cycle can be brought into a path, which contains all the vertices cycle; and that because we can jump into this, and then follow the cycle like in whatever order until here and then we will we will just cut here that is all.

So, this will be a longer path, because now it contains more vertices see u 0 and u k and this vertex. So, the original path we considered was not the longest path. So, it is a contradiction. So, we can in for that. So, if then minimum degree of the graph is greater than equal to n by 2, then we should have a Hamiltonian cycle in the graph.

So, the argument is to consider the longest path, and then we argued that the longest path can be converted to a cycle containing all those vertices in the path; and if this longest path contained all the vertices of the graph, then we got a Hamiltonian cycle, because now one cycle has all the vertices on it.

If not then they should be some vertices outside this cycle, which is adjacent at least one of these vertices cycle, otherwise the graph will get disconnected; graph is connected graph, why it is a connected graph, because the minimum degree is n by 2 and you cannot have a graph with minimum degree n by 2, but disconnected what is the reason for that the reason; for that, suppose, it is disconnected, this should be a smallest component the smallest component will have number of vertices less than equal to n by 2 and even if it is a clique the degree of the vertex has to be strictly less than the cardinality of the component less than n by 2. So, it follows that it cannot be disconnected, it cannot be disconnected. So, therefore, in the connectivity is assured every vertex outside the cycle has to be adjacent at least one vertex on the cycle; and now, we can cut the cycle at that point and elongated the path make the path longer, that contradicts the assumption we have taken the longest path in the first place.

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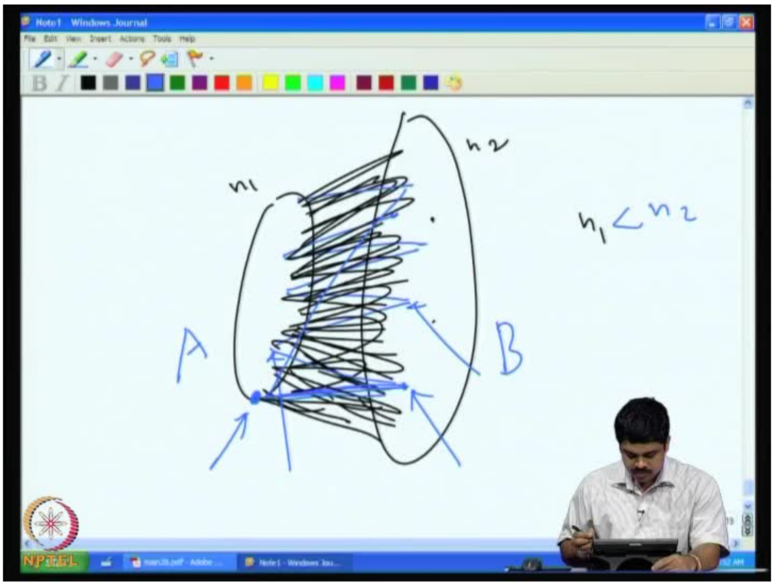

So, now, let us look at the hesitation. So, it is possible that, it is the connectivity that creates the problem, because we have seen that an example, where even n by 2 degree of minimum degree of n by 2 is not helping us in further, there is the Hamiltonian circuit in the graph so, but then there was a cut vertex that was the problem, suppose we assume that the connectivity of the graph kappa is greater can it help, for instance, is it some two three may be little high. So, large value, but even then it will not help, because if you consider a complete bipartite graph. So, let may be n 1 vertices here, n 2 vertices here. So, it is a complete bipartite graph the connectivity of the graph is definitely n 1, because you can see any two vertices if you want disconnect. So, you have to remove all the vertices here because as long as even one vertex is left on the other side then you have a path still through that vertex right. So, the smaller of the twos parts is the connectivity of this complete bipartite graph.

So, then, so, then what you can say is, what how pick Hamiltonian how big a cycle can be there in this graph, is it possible to have a Hamiltonian cycle? So, some thought we can easily see that is not possible, because see if you go say any cycle can one have to zig-zag between the two parts the edges are only across the two parts. So, suppose, if you go like this, then you can come here, then go here, come here, go here, like this you can zig-zag between the two parts, but it like the cycle is taking every time it is taking one vertex from this side taking one vertex from this side also and then it takes one vertex from this side and one vertex from this side.

So, it is the finishing of the vertices on the both sides in equal amounts. So, definitely n 1 is since n 1 is smaller than n 2, n 1 will finish of faster than n 1, and then we will not able to follow the cycle right. So, the cycle cannot be bigger than two time's n 1. So, that is finally, when you finish this, come back here, and finish. So, the biggest two n 1 may be the biggest cycle that is possible.

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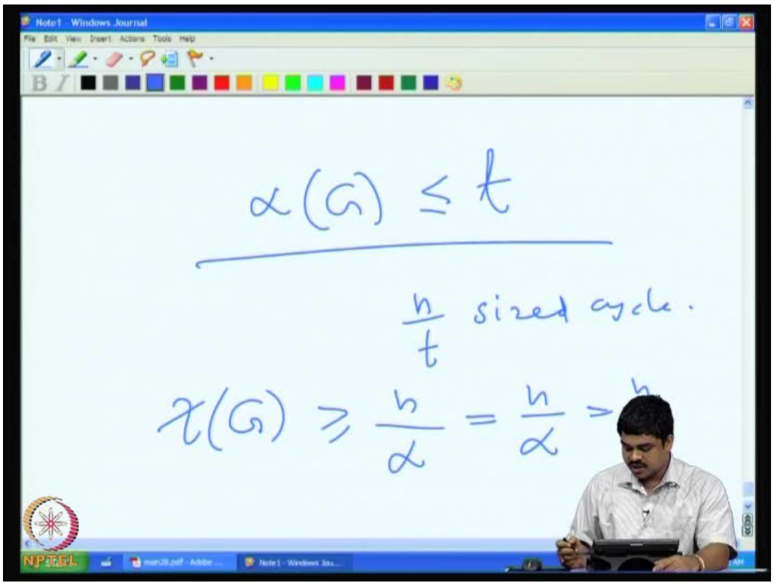

So, you cannot you cannot zig-zag more than the n 1 time, because every times it comes in to the this side, a side it will this is a side, and this is b side it will finish of one more vertex from the a side. So, the connectivity cannot help because this complete bipartite graphs of need high connectivity, but still it is not allowing has to do that there are Hamiltonian paths in its. So, connectivity is also note the…; so, but there is a interestingly suppose you can assume that, if you can assume that the largest independent set size alpha of g is less than equal to some number, say, some t, then we can show that there has to be a large cycle; in the graph like you can show that, there should be a n by t sized easily, you can show that cycle in the graph, how do you show that because alpha is less than equal to t; what should be the chromatic number? Chromatic number has to be greater than or equal to see n by alpha, because the any color class can have at most alpha number of vertices in it t number of vertices in it. So, they should be at least n by alpha n by t color classes. So, the chromatic number has to be at least n by t.

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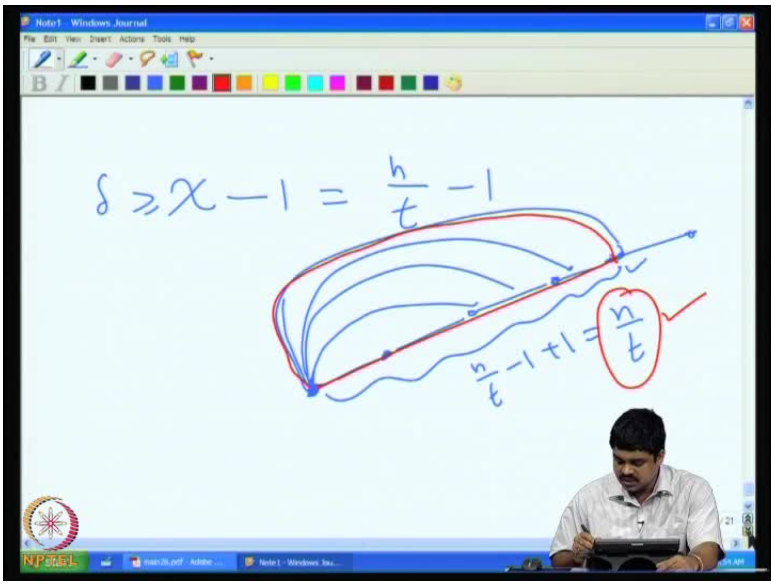

And now we has studied earlier that, if the chromatic number of the graph is some n by t, then they should be a critical graph color critical graph, such that, this sub graph has chromatic number minimum degree is greater than or equal to chromatic number minus 1 here, n by t minus 1.

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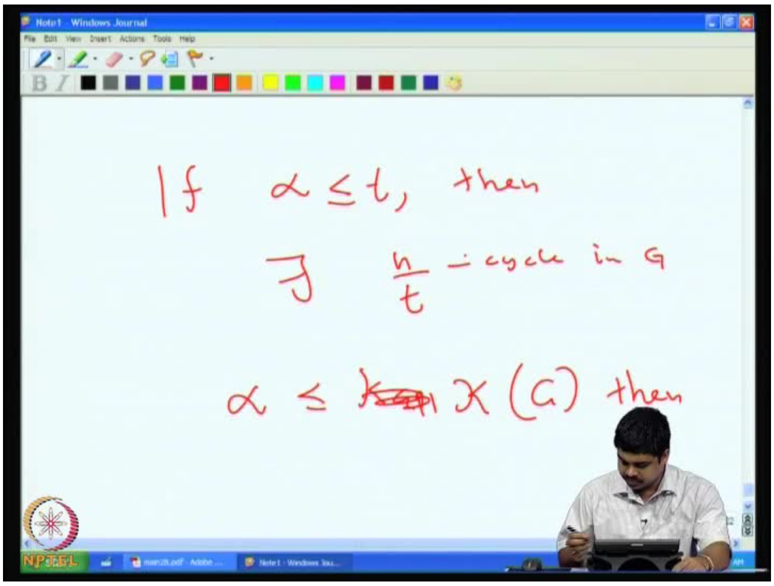

Now, again taking the longest path argument, if it considers the longest in that sub graph, then we know that in the longest path should have all see, for instance, if you look from this side. So, all the neighbors of it should be on this path, because otherwise you can make the path longer than this; now, if you consider the last neighbors among this thing, so this entire part should contain n by t minus 1 neighbors of this thing may be more, but at least those many and this vertex this plus 1 that is n by t vertices in it and here we have because we can jump directly to here and then come back here via this. So, there is a cycle of having n by t vertices right. So, if we can in for that, if the independents biggest independents set size, if alpha is less than equal to t, then there exist n by t cycle in g, where exist n by t length cycle in g, but it is not a Hamiltonian cycle, it is only a long peek cycle; it is biggest Hamiltonian cycle means, it has to t of length n itself; that means, it has contain all the vertices of g all the vertices of g.

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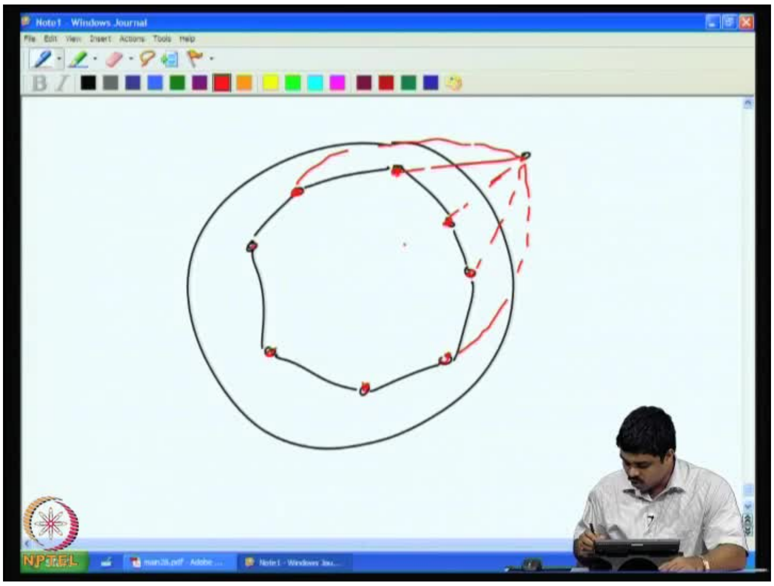

Interestingly, so, this is alpha being small enough to have a Hamiltonian cycle, but suppose the alpha is less than the connectivity of the graph kappa kappa of g connectivity of the graph, then we can in for that there is a hamiltonian path. So, we will the next time of this aim is to prove that how do I show that. So, now, the setting is like this; suppose you have a…, so you consider the longest cycle this time longest cycle is in the graph, if it is the Hamiltonian cycle well and go we are done. So, it is not a Hamiltonian cycle.

Now, there are some vertices outside, the cycle longest cycle; now, we can consider it to this cycle, we can consider a fan, because is the k connected graph, there is a fan k fan. In fact, so, for instance, if the cycle length is more than k, we can find k paths independent paths, its starting from here internally disjoint paths and ending at different vertices here somehow.

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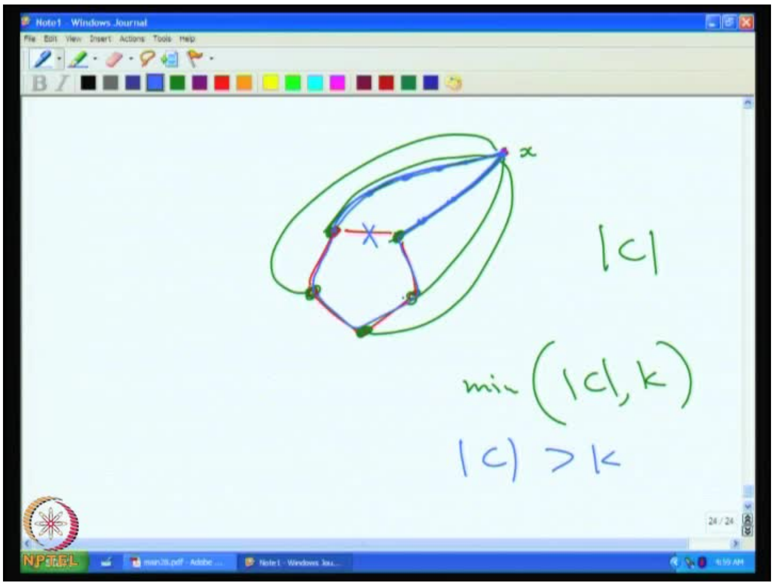

Some where it ends we have to analyze, one the other hand if the cycle was smaller than k; then what will happen? Suppose it was the small cycle very small cycle compare compared to k, then what will happen is, you will able to get fan paths starting from here, to here, to here, to here, to here,, and after reaching all the different n vertices, reaching all the n vertices; that means, the cardinality of the fan will be c.

So, therefore, we can see that starting from any vertices outside x outside we will able to identify if fan paths of cardinality minimum of c, k, where k is the connectivity of it right the fans, but is it possible that c is the smaller of k no; because in that case what will happen? Because this all to all vertices is e, there is a fan path and d then what will I do, then I will start from here, then will follow like this, and then come here, and then go here, and then catch this x and come back. So, this is the longer cycle than the red cycle, which I assume the was the longest cycle available in the graph.

So, it is not possible to have this picture; that means, every so many fan paths coming from x and ending at all the different vertices are this thing that is not possible, because in that case I can elongate. So, this cycle make a make this cycle bigger by not going through this edge, but go here and come back it will be bigger up please one more vertex even more.

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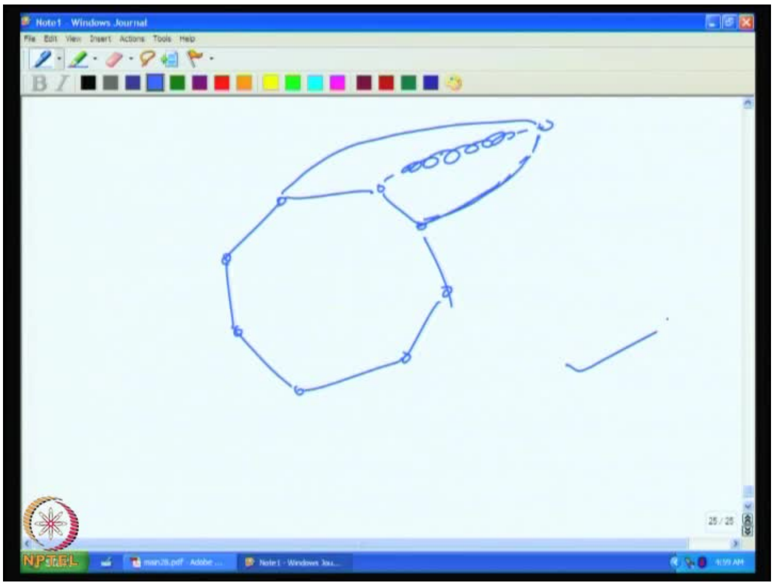

So, it is not minimum of c, k. In fact, c is bigger than k. In fact, we know that c is bigger than k. So, therefore, k is the smaller of this thing. So, its all the k path should be there that c. So, small compare to k definitely not because as this argument, we just told essentially tells us that, we cannot have two, one paths from x and ending at adjacent vertices of c. So, because we can if this is ending here, then we cannot have an ending, it cannot having ending here. So, it can only end here right from next, because otherwise we will just elongate here; and we will continue this proof in the next class, because this need some time. So, we will finish it in the next class. So, thank you.