Graph Theory
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Module No. # 04

Lecture No. # 25

Proof of weak perfect graph theorem (WPGT)

Welcome to the 25th lecture of Graph Theory.

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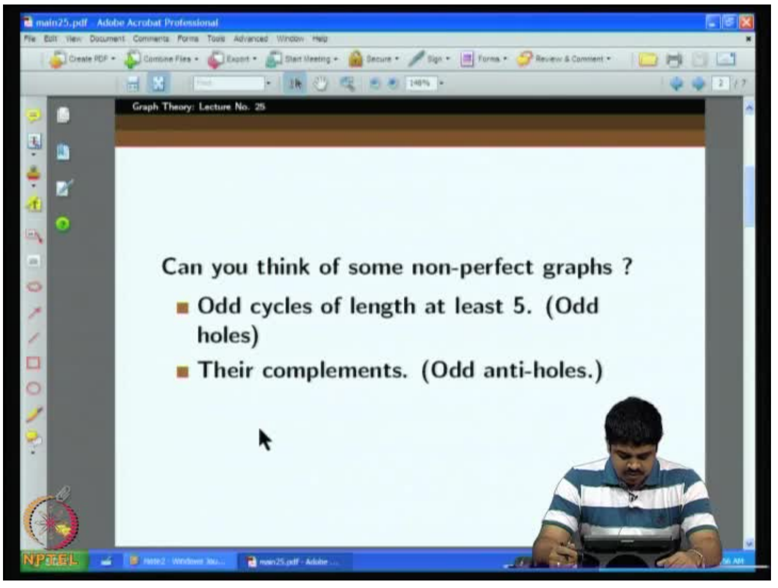

So, in the last class, we were looking at several classes of perfect graphs. Now, let us ask the other question, can you think of some non perfect graphs; graph which are not perfect? So, after some thought, everybody will come up with the following example, odd cycles of length 5 at least 5, why are they not perfect.

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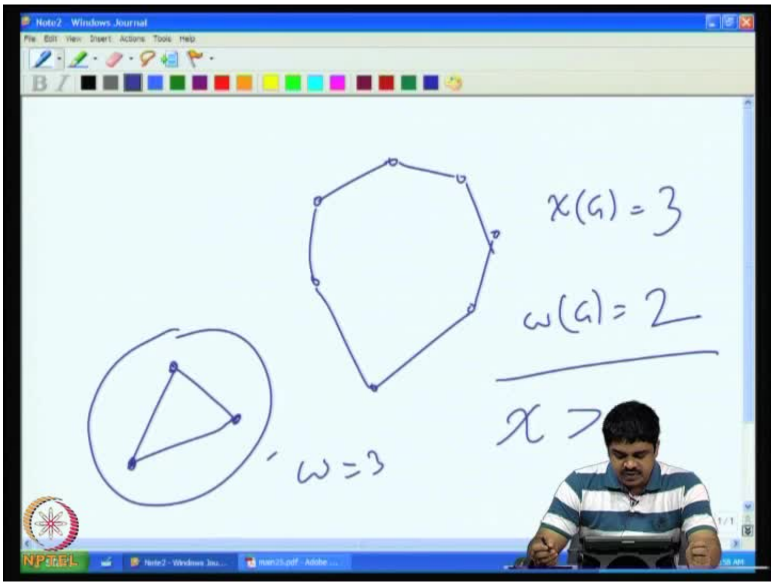

Because if you consider an odd cycle, so then, how many colors are required so this is an odd cycle right how many colors are required to color it, psi of (G) is equal to n by 2 chi, so for instance if it is 7 we need 3 here. Now, omega of (G) is equal to 2 1 sorry, not n by 2 to color the odd cycle, we need 3 colors. So omega of G is equal to 2, therefore chi is strictly greater than omega.

So, for odd cycles are not going to be, odd cycles of length 5 or more 5, 7, 9 like that are not perfect, because the chi is strictly greater than omega. See, odd cycle of length 3 is not an example, because here 3 colors are required to color it and also the chromatic number is equal to 3 here, so therefore they are not examples (Refer Slide Time: 01:59).

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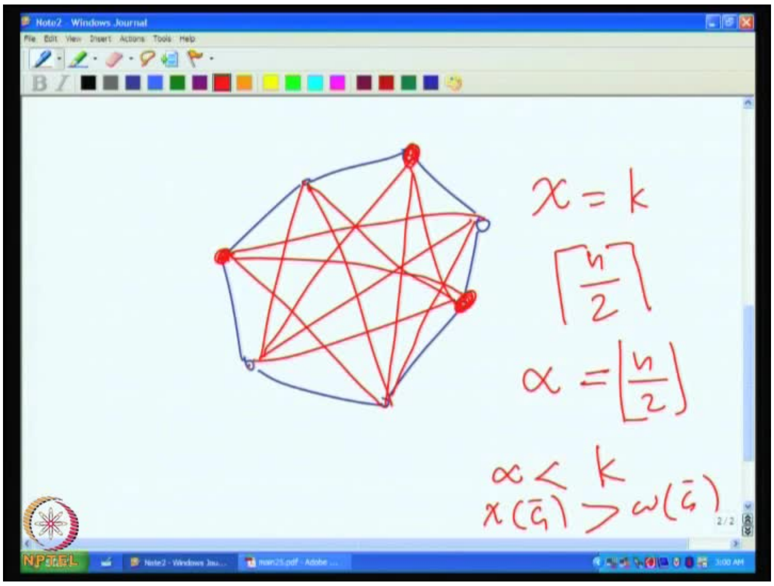

Now, so the case obviously we will look at the complement of it, there also examples of non perfect graphs, why are the complements of an odd cycle, say for instance is an odd cycle. And you can consider the graph which is complement of it, so like these graph, the graph defined by the red edges, this graph is it (No audio from 02:39 to 02:49). This complement graph, why is it the an example of a non perfect graph?

So for instance, if you try to color this chromatic number, that will become the clique cover number for the cycle and because there are 2 vertices in a clique. So we need how many cliques, it is essentially n by 2 seal cliques are required. So, therefore clique cover number for the cycle is n by 2 and we know that, so the chromatic number of the complement is going to be n by 2. On the other hand, if you look at the clique number of this omega for this graph that will happen to be the alpha of the cycle and that we know that is n by 2 floor only.

For instance, if in a 7 odd graph you could have taken this vertex, this vertex and this vertex and then I cannot take this or this (Refer Slide Time: 03:50), only you can take 3 for 7. So, here alpha is strictly less than clique cover number that is the chromatic number of the complement is strictly greater than the omega of the complement. So, these two graphs are the typical non perfect graphs, they are the obvious examples of non perfect graphs.

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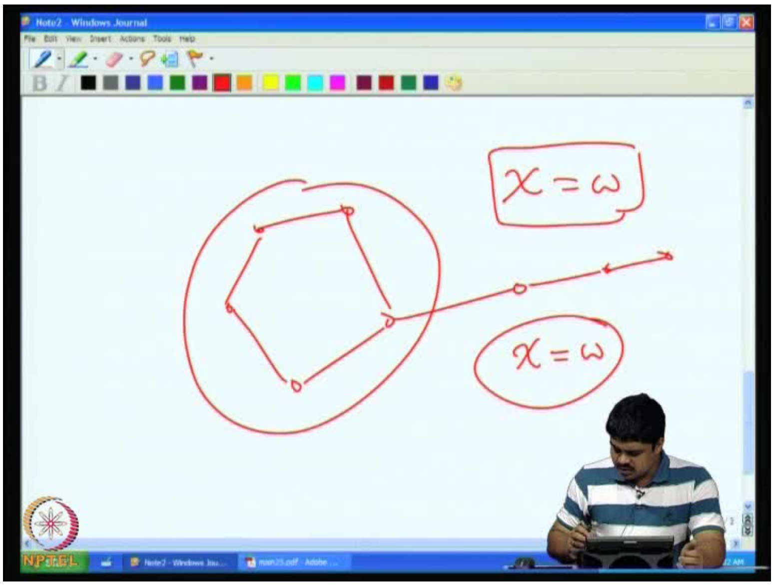

Are there other examples of course, there are other examples like, you can take a odd cycle that you can take an odd cycle of length 5 or more and add something like this. It is non perfect, why is it non perfect? Because this odd cycle is there, as an induce sub graph unit and this induce sub graph will not satisfy the chromatic number is equal to clique number condition for this sub graph.

Because, you remember the perfect graph requires for that, for a graph theory to be perfect, we require that every induce sub graph satisfies the chi equal to omega condition, every induce sub graph satisfies the chi equal to omega condition. So, this brings us to this question of what are the minimal in perfect graphs, because that induce sub graph, if you take. So, you can always ask, is this an perfect graph, is this a non perfect graph, there should be some induce sub graph which violates the perfectness condition.

So, you pick up that induce sub graph, you can look into even further like you can ask, why is this imperfect, is it because of some induce sub graph; and then go for the smallest induce sub graphs. So, the minimal imperfect graph in the sense that, if you take any induce of sub graph of it the condition is satisfied but, for this graph it is not satisfied.

So, for instance the cycles are odd cycles and of it is complements are one such case, if you take any induce sub graph of this odd cycles, then they satisfy the chi equal to omega condition, while this graph itself does not satisfy the chi equal to omega condition. Therefore, interesting thing is, once you see these are the two minimal imperfect graphs and then are the other minimal imperfect graphs.

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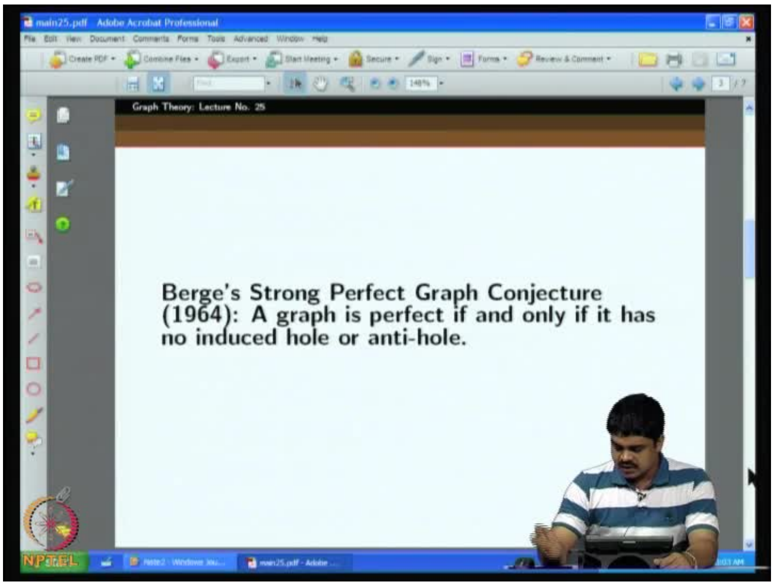

So, Berge conjectured that a graph is perfect if and only if, it has no induced hole or anti hole, anti hole means the complement of the anti; hole means an induced cycle of length 5 or induced odd cycle of length 5 or more. An anti hole means the induce cycle length complement of the induce cycle of a length 5, 7 or 9. The hole and anti hole so here we can say odd hole or odd anti hole or may be depends on the author.

So, the graphs is he conjectured that a any minimal imperfect graph should contain an induce sub graph an odd induce cycle of length 5 or more, that is 5, 7, 9 like that or the complement of it, one of the two things are required, this was his conjecture. Then of case it was a difficult conjecture, somehow nobody could prove it for a long time.

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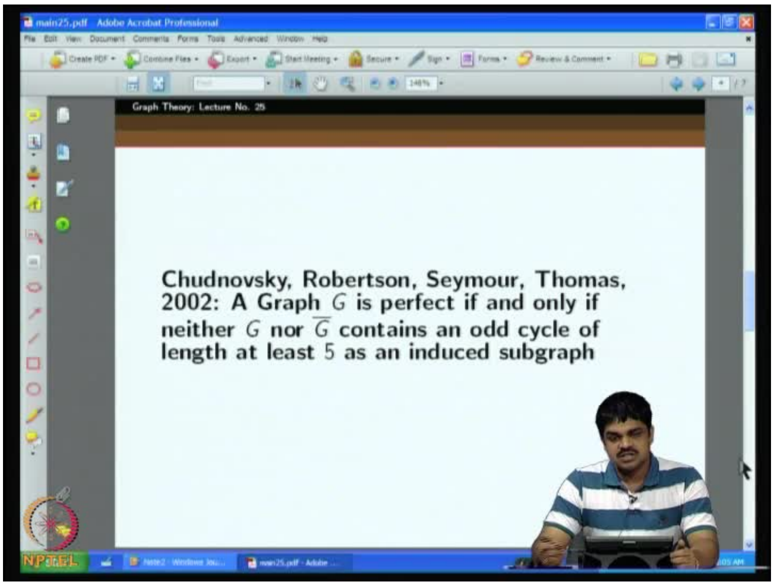

And then very recently in 2002 Chudnovsky, Robertson and Seymour and Thomas they proved the conjectured. They say that a graph G is perfect if and only if, neither G nor G bar contains an odd cycle of length at least 5 as an induced sub graph. Or equivalently it is like saying that G does not contain a hole or an anti hole, that means it does not have an odd induce cycle of length 5. Or it does not have a the complement of the induce cycle of length 5 or more or equivalently it is told like this.

So, it is either G or G's complement does not contain in induce cycle of length 5 or more, this is what it is now known as the strong perfect graph theorem. Historically there was a weak perfect graph conjectured also, which was proved by Lovasz, which said that if G is a perfect graph, then it is complement is also a perfect graph, we have been verifying this for several graph classes in the beginning. And then whenever we saw that a graph classes is perfect we saw that its complement class is also perfect.

The general statement infact whenever G is perfect, G complement is also going to be perfect. You see once the strong perfect graph theorem is known, that means this statement, if G is perfect, if and only if, neither G nor G bar contains an add cycle of length at least 5 as an induce sub graph. So, it is clear that if G is perfect, G bar is also perfect, because when G is perfect, G does not contain an odd induced cycle, G bar also does not contain odd induced cycle. And this statement is symmetric for G complement, so does not if G bar is equal to h, h and h bar will not contain odd induce cycles.

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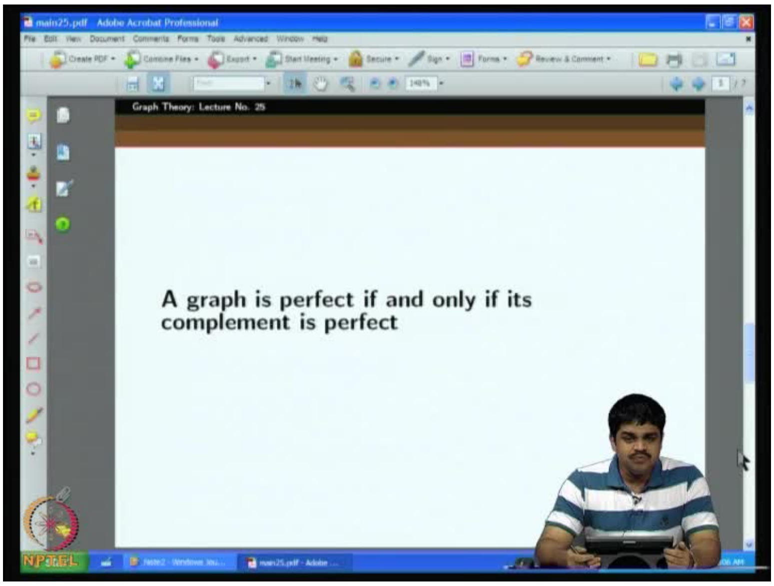

So, that will follow from that, so taking the proof of this strong perfect graph theorem by Chudnovsky, Robertson, Seymour and Thomas is too long and also it cannot be taught in a class. Therefore we will to get some very nice intuitions about this perfect graph and to appreciate the beauty of this perfect graph topic, it is a good idea to look at the weak perfect graph theorem. Namely the graph is perfect if and only if it is complement is perfect and it is proof by a Lovasz and it is shorter proof by a gasparian, that is what we will aim to do in this class.

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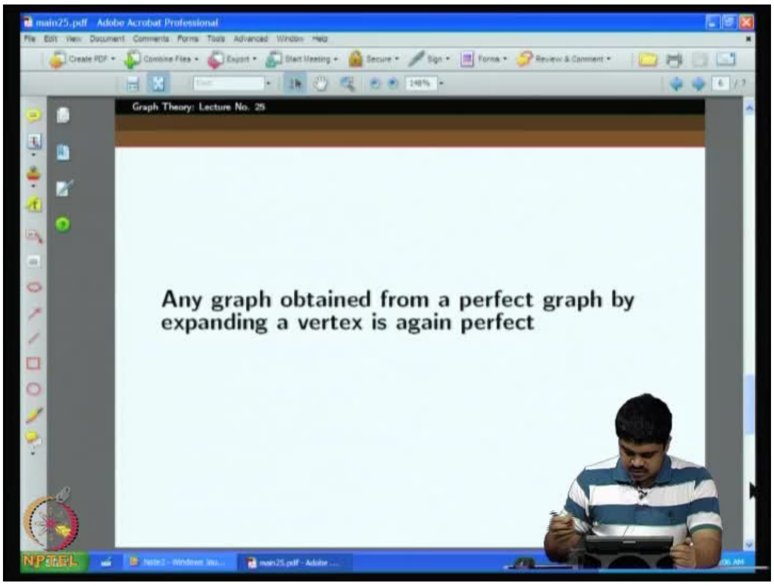

So, now our intention is to look at this one, to do that we need some lemma, which is developed by Lovasz. So, it says, a graph obtain from a perfect graph by expanding a vertex is again perfect, what do you mean by that.

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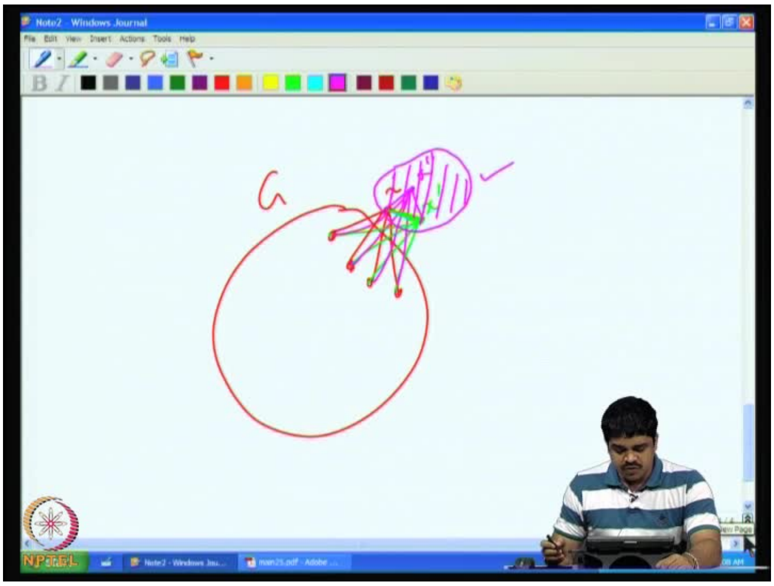

So, first of all we have this operation called expanding a vertex, suppose you have a graph here, G. And then suppose this is the vertex of the graph, these are the neighbors, then I say that this is x, this is expanded it to an edge x dash. So introduce a new vertex x dash, we connected to this and also you connect this new vertex to all the neighbors of x. In other words x is some kind of a copy of x dash is some kind of a copy of x in the sense that it is connected to exactly the same set of neighbors S x and also it is connected to x, this is called the expansion operation.

So, you can see that you can keep on expanding, for instance you can, suppose if I expand again with an x double dash, what will happen, It will be connected to this, now it will be connected to all the neighbors including x dash. Now as you keep on expanding here you will get a clique, which is connected to the same set of neighbors and so the clique, such that all the vertices in it is connected to the same set of neighbors. This is what will happen when we expand a vertex.

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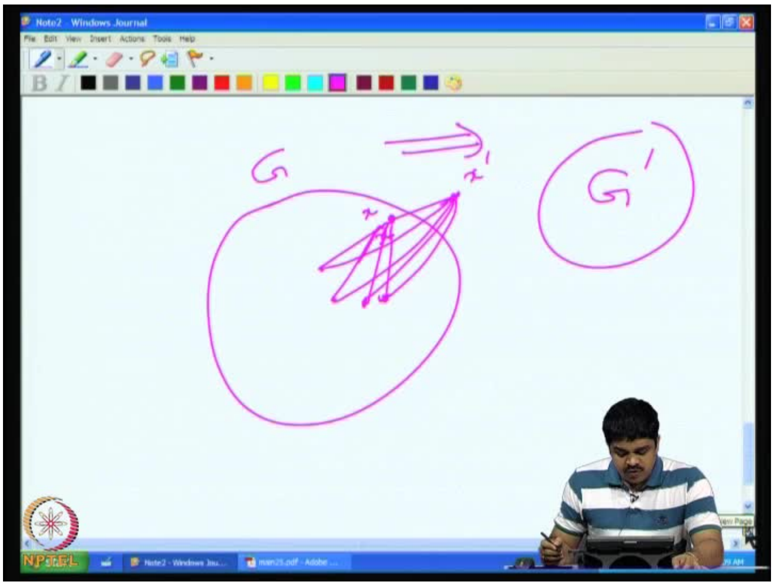

So, the lemma says, if you have a graph perfect graph G and then you select some vertex x, now you expand it to an edge by adding a new vertex x dash, that means this is a situation, these are the neighbors and this is connected to this. So, if this is what we are need to do, then lemma says, let this new graph called G dash after expanding, so G dash is obtain from G by expanding the vertex x to x x dash.

The edge new vertex x introduced and x dash is made adjacent to x and made adjacent to all the neighbors of x. So, now we will say that G dash is also perfect, we want to prove that G dash is also perfect after expanding whatever graph you obtain is also perfect, how do we prove it.

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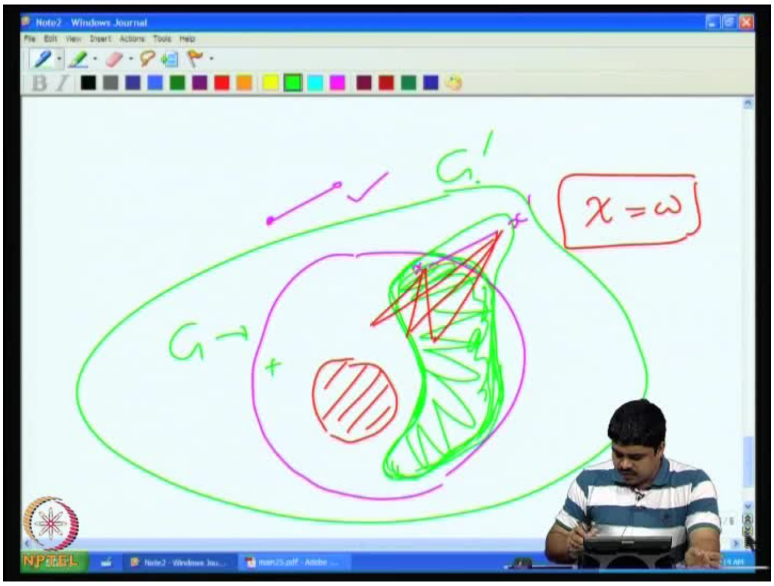

So, we will prove it by induction, for instance, if it is 1 node graph, G was a 1 node graph, then there is nothing to prove. Because expanding will give an edge and then it is perfect, there is nothing to be proved. And that small even for 2 node graphs you can easily verify that it is the perfection.

So, therefore lets inductively assume that, if for all smaller graphs the theorem is proved whenever you expand a vertex, the perfection is not lost, the property of being perfect is not lost. Now, you take a new graph n node graph and you identify the vertex x and now you are expanding into x x dash, The question is, these are the neighbors is it possible that this becomes imperfect, we will show that it is not possible, it is always perfect.

Now to do this thing, so we will to prove that a graph is perfect, it is not just enough to show that chi is equal to omega for the entire graph. We also have to prove the statement for all the induced sub graph, but now, if you look any induced sub graph it is either completely coming from here. So, or see, it can be something like this, in which case we know this graph itself was perfect. Any sub graph of it is also perfect or otherwise it can contain two of these things and then it can be something like this.

In which case you will be living out some other vertex, this being a smaller graph than G, this being a smaller graph than it is like you have expanded x in the smaller graph. And therefore for all smaller graph this expansion, we know that, if you expand a vertex the perfection will not be lost. That is the induction hypothesis, therefore we know that, induced sub graph also will be perfect. Again I repeat, so we have to verify that the chi equal to omega property is satisfied not only for the entire graph but, also for all the induced sub graphs.

So, we will claim that infact we can discard the proper induce sub graphs, may be we can concentrate on the entire graph only because, suppose it was an induce sub graph which is strictly smaller than the entire graph. Then, that induced sub graph is either a sub graph of G, this original graph, in which case its perfect by definition, because an induced sub graph of a perfect graphs is also perfect.

Or it contains x as well as x dash and some other vertex is left out from the original G, original G was this, this is this the entire, these G dash, this is G, this violate is G and this entire thing is G dash (Refer Slide Time: 16:50).

Now, you know that some smaller graph of G was selected and the vertex x in it was expanded to x x dash, that is how we got this kind of an induced graph. That is why it is perfect, because for all smaller number of vertices, we had already assumed the result is true, so it is also perfect. You may ask, what if x is not there only x dash is there, but it does not matter, because it is as good as it is isomorphic to a graph where x is contained and x dash x not contained it.

So, it is just another, instead of x, you can take a x dash, so that belongs to the former case. So, therefore our point is that we can we need not worry about all induced sub graphs of G, because all induced sub graph of G are automatically taken care by the induction assumption or because it is a subset sub graph of G.

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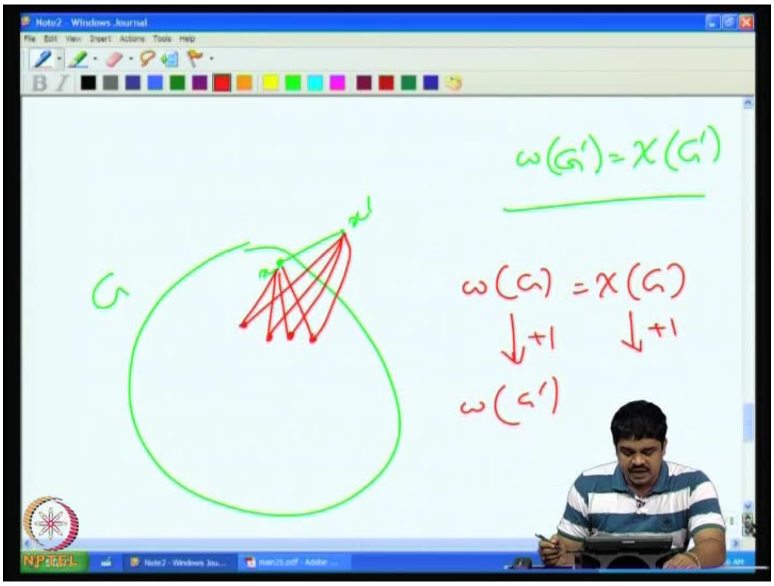

Now, we will consider the entire graph G dash, only we want to prove that omega of G dash is equal to chi of G dash. How will you do this thing, so this is the G and here we have this x and we here, we have this x dash, so some neighbors are here and some neighbors are here, so this is the situation.

Now, the question, what we are bothered about here is, what is the chromatic number of it, what is the omega of this. Now, the first point is, when you expand it, is it possible that the omega increased, if the omega is increased, that means the clique number increased by 1. Then we only have to show that, we can color the entire G dash with one more color than which was needed for G. Because, the chi of G, we know that, for G a perfect graph omega of G equal to chi of G, therefore and then, this became plus 1 for omega G dash.

Now, here also we have one more color available, we can color it, we can use that extra color to color the new vertex. And therefore, it is of case omega G dash and chi of G dash will be equal, because an extra color will be available. Because the omega increased you retain the omega coloring of this thing and then use the new color.

So, we can assume that, by expanding the clique number does not increase, that means the maximum clique remains omega. But, we have added one more vertex though G was colorable using omega colors, is it possible that after expansion we may need one more color? This is the question. Now, so if it happens what can I tell, I have assumed that omega is not increasing. If omega is not increasing what can I tell about x and the maximum clique, so you take any maximum clique is it possible that x is present in that maximum clique.

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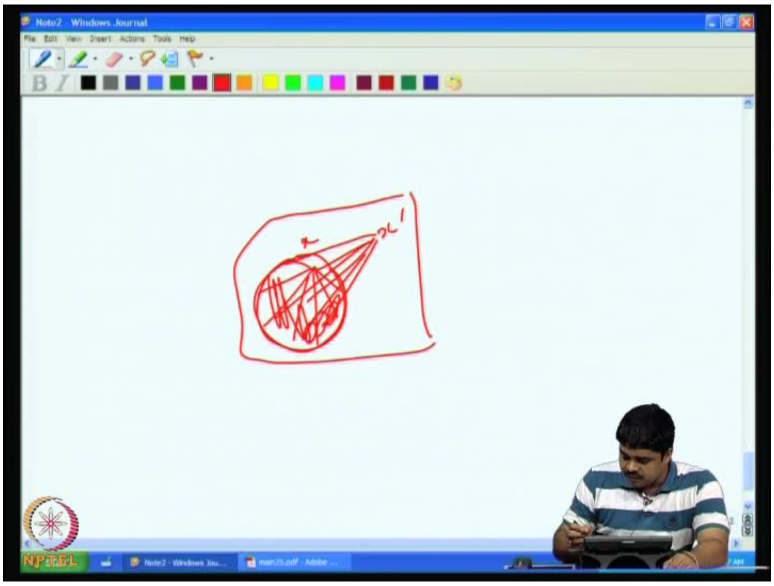

Suppose, this was a maximum clique and suppose x was present in that, that means it is neighbour of all the other vertices in the clique. Now if you introduce x dash and you make this adjacent to all the vertices here. Now of course, then there will be a new clique containing this entire thing and the newly added x dash. So, it means that our clique number will go up, so it means that x was a vertex, which was not part of any maximum clique in G, x was always outside any maximum clique.

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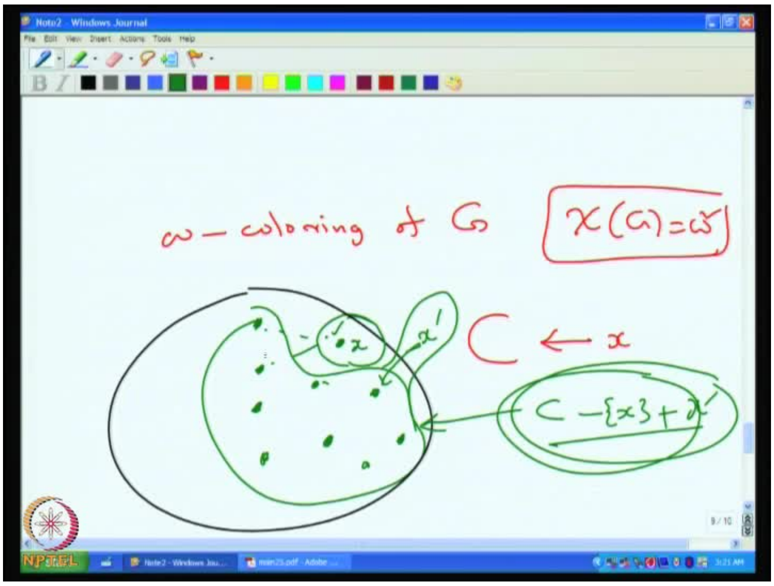

But, on the other hand if you consider a coloring of G, omega coloring, we know that there is omega coloring of G, since chi of G equal to omega. Because, G is a perfect graph by assumption and chi of G has to be equal to omega, so there is an omega coloring of g.

Now in this omega coloring, x is also in a color class, so let say x belong to a color class, say we can say for the sake of C is the color class, where x belongs to. C may be the color class in which x is part of or it can be a green color class, may will I can identify in the graph, I will identify the green color class. So, see x belongs to the green color class, this x maybe here, x maybe this but, then interestingly x does not belong to any maximum clique.

But, then if you consider any maximum clique of G, because it contains omega vertices and each of the vertex has to get a different color, there should be a representative vertex in each maximum clique for each color. That means if you take any color in the omega coloring, there should be 1 vertex in the maximum clique having that color, because the maximum clique has omega vertices each of them should get different colors. So they should take up all the colors, so for every color they should be a represented in a given maximum clique.

So, in particular this green color class also should have representatives in each maximum clique. But, then we know that x is not such a representative for any maximum clique, because x does not belong to any maximum clique. So these remaining case, this is C minus x, these remaining green vertices, will intersect with each maximum clique. That means, there is a representative from this collection this C minus x in each, now what will happen if I remove this collection of vertices this green vertices other than x from the graph.

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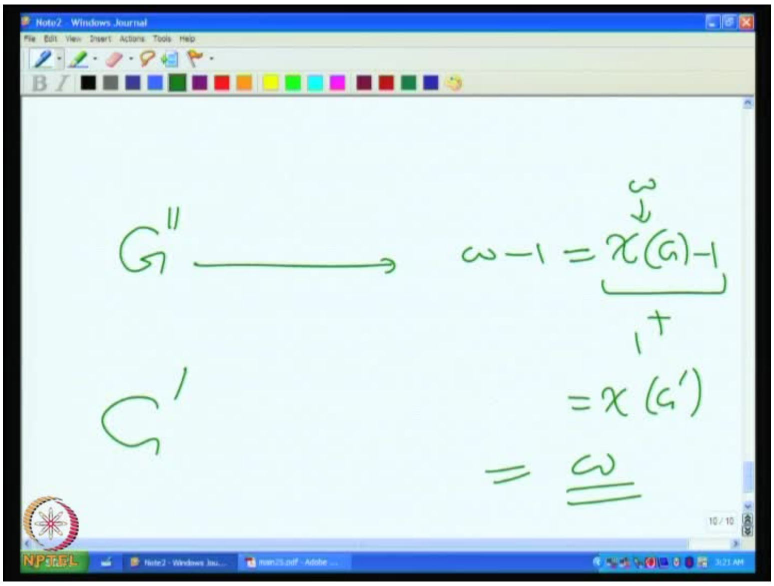

That means every maximum clique will reduce it is size by 1 after removal of this set the new graph, let me call it a G double dash, this omega will be omega minus 1. Now, G double dash is a subset of the original graph G, therefore it is a perfect graph. Therefore its chromatic number also has reduce to chi G minus 1, because the clique number is omega minus 1, the original clique chrome coloring number was omega. So it should be it should become one less now, it can be colored with one less color.

So, one color we have free, now you know that if this color if be removed, this an independent set. Now x is already there we removed only this thing other than x but, we have this x dash instead of x, we can add it into this collection because x is not adjacent to any of this case. So x dash also will be not adjacent to any of this case, so together with x dash, so that means this plus x dash will be an independent set.

So, now we can add this new independent set to the vertex with just one more color, because all of them need only one color, because that is an independent set, you can add it. So, therefore this plus one more color will be enough to color the entire graph G dash, chi of G dash will be equal to omega, after expansion whichever graph was there we colored with omega colors. So this completes the proof that, after expansion the perfectness will not be lost.

So, the sequence of arguments was that, you first noted that, if you are expanding a vertex x to an x x dash by introducing a new vertex x dash. Then to establish the perfectness of the new graph, that means expanded graph, you only have to prove that chi is equal to omega for this new bigger graph. Because any induce sub graph will be either an induced sub graph or isomorphic to an induce sub graph of the original graph G or it will be obtained from a proper induced sub graph of G by expanding the vertex x.

So, referring both cases, we can apply either that the fact that G is perfect or the induction hypothesis, to assume that, the induced sub graph satisfies the property chi equal to omega they are perfect anyway by assumption, now we only have to establish omega is equal to chi. Now we considered two cases, first case was easy, suppose after the expansion the clique number increases, if the clique number increases the chromatic number also can increase by 1. So, we have enough colors to accommodate to the new vertex.

So, we can assume that the clique number does not increase, if the clique number does not increase, then the key property is that x cannot belong to any maximal maximum clique, in the graph G that is a good thing. Because the color class C of x, when you consider an omega coloring of G, will have to cut all the maximum cliques. But a representative from this color class to a maximum clique can never be x.

So, you can rather leave x in the graph and remove the remaining vertices of the color class, it is an independent set. You do not remove the entire color class you will leave x there in the graph x as such remove the remaining things.

Now, the graph become a smaller sub graph of S sub graph of G a perfect graph, so the key point is you manage to bring down the size of the maximum clique from omega to omega minus 1. Because, one representative was remove from each maximum clique, the good thing was that representative was not x, so therefore we could leave x there itself.

So, now that maximum clique is omega minus 1, so you just need omega minus 1 colors to color. Now, we have one more color if you want to color using omega the entire expanded graph. Now, we can give back whatever we removed that was an independent set, not only that that x dash new vertex after which we use to expand x dash that also can be added, because x x dash behaves, let say x and if x dash is added along with this color class it will not become not independent, it will also become an independent set.

So, together with x whatever we removed from the graph can be added back but, with one color, because an independent set only one color is required. So with omega colors we can color the entire graph, this is the idea. So, therefore we manage to prove that in both the cases chi is equal to omega, so therefore it is a perfect graph. So, this is a lemma which will require later, now we will come to the proof, so suppose again it is an induction proof, we assume a proof that if G is a perfect graph and then the complement is also going to be a perfect graph.

To prove this, we will again use induction, so for 1 vertex graph it is obvious because, if you look at the complement what is that it is a same graph and or if 2 vertex graphs you can take all combinations and check whether if it is perfect it is complement is also perfect or not, it is very easy thing. So therefore let us assume that all smaller graph that is known and we take the new n, right, for which we have to prove.

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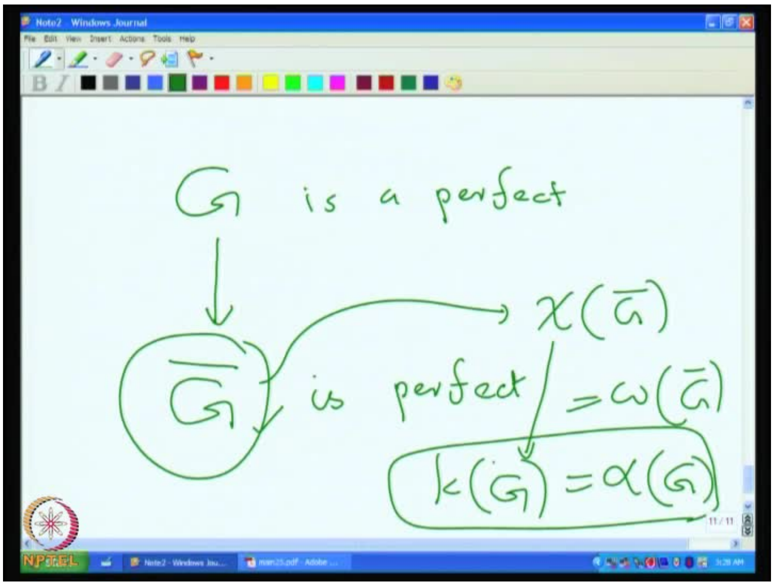

G is a perfect graph any sub graph of G is known to be must satisfy the statement also is perfect, therefore it is complement is also perfect. Now our intention is to show that G complement is perfect. So, it to show that G complement is perfect, we have to show that every induced sub graph of it also satisfies the chi equal to omega property. But then we again notice that it is not very important.

Because, if we take a proper induced sub graph of G complement that is the complement of some proper induced sub graph of G. And then their induction hypothesis already valid, because you know, that says smaller number of vertices there, therefore induction hypothesis already valid. That means any perfect graph should satisfy the fact that its complement is also perfect. G's induced sub graph are perfect therefore it is complement is also perfect.

So, any proper induced sub graph of G complement satisfy the chi equal to omega property, therefore we only need to bother about this entire G complement name, we have to show that its chi is equal to it is omega. Now we will work on G rather than working on G complement, so that means we have to prove that this chi will become the clique cover number in G and the omega of G bar will become the alpha of G, so this we know. So, now the point is to show that, we have a clique cover of G using exactly alpha of G number of cliques.

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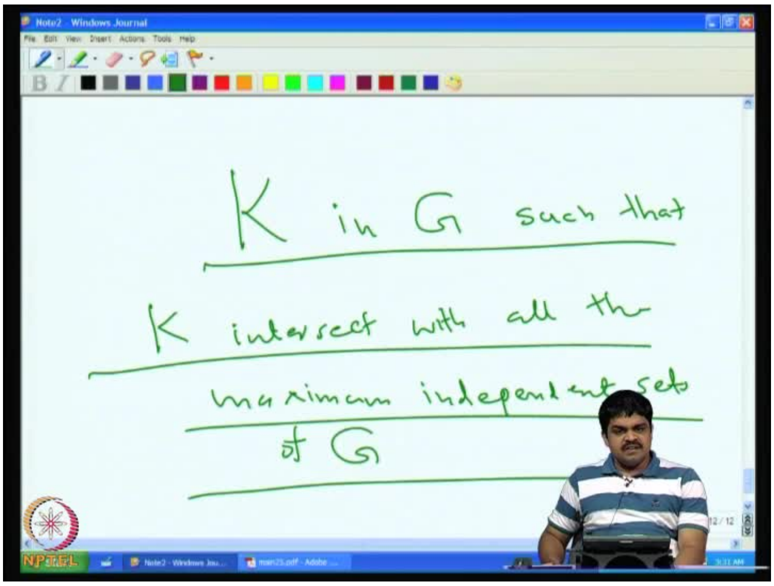

Now see the key point is to notice that, suppose we have one clique in K in G such that K does not intersect with one clique in G such that all the K intersect with, all the maximum independent sets of G. Then what does it mean it means that, if you remove that clique from G the maximum independent set size itself will reduce. If you have one such clique, which will cut which will intersect with each maximum independent set, then when we remove this clique from G the maximum independent set size cardinality itself will reduce, alpha will reduce to alpha minus 1.

Now, the smaller graphs satisfying already the perfectness property, because the smaller graph is already perfect, so it is complement also have to be perfect by induction hypothesis. So there we will get the fact that alpha is equal to clique cover number, so you will get a collection of cliques which will cover the entire the alpha minus 1 cliques which will cover the entire graph. In other words in the complement we will be able to color using alpha minus 1 colors, the complement can be colored using alpha minus 1 colors.

Now, you have only removed 1 clique, you can reintroduce the clique to the graph with one extra. So alpha minus 1 will become alpha, the clique cover number will be alpha even for the entire graph, so that will work. In other words in the case in terms of the complement, we are saying that this clique is introducing G mean, an alpha is introduced an independent set is introduced in the complement, so we can color with one extra color.

So, alpha minus 1 coloring was available so with alpha colors we can manage the entire complement, alpha being the clique number of the complement that alpha is the independent set number of the original graph G. So, the point here is that this is enough to complete our proof that means if you can find out one clique in G which can intersect with all maximum independent sets of G, then we are done.

Because, we remove that clique from the graph, so thus we got a smaller perfect graph. Because G is the smaller perfect graph, G is a perfect graph. And after removing K you will get a smaller perfect graph an induced of graph G is perfect not only that, induction hypothesis can be applied now for the smaller case. That means in the complement, we get a perfect graph, the complement of this smaller graph will be a perfect graph.

That means, they are the chromatic number and clique number will be equal, which means that here alpha, the independent set number is equal to the clique number. Good thing is independent set number reduce by 1. In other words if you can if this can be covered with alpha minus 1 cliques it is one more clique is added back, so alpha cliques are enough to cover the entire thing, this is the point.

So, somehow easy argument infact that point, if this works, then our proof is done, but then what we are asking for is too much. In fact we are saying that we want to find one clique which will intersect with every independent set of G, every maximum independent set of G, why should it intersect with every maximum independent set of G is it possible. There can be several maximum independent sets of G, how can you somehow adjust one clique to cut with all maximum independent sets of G.

So, it looks like a too much of a demand that your making but, it so happen that its true, because it is true then our proof is done, so why is it true that is the only think we have to verify. So now, to prove that such a clique exist in G, what we will do is to assume suppose that a clique does not exist in key in G.

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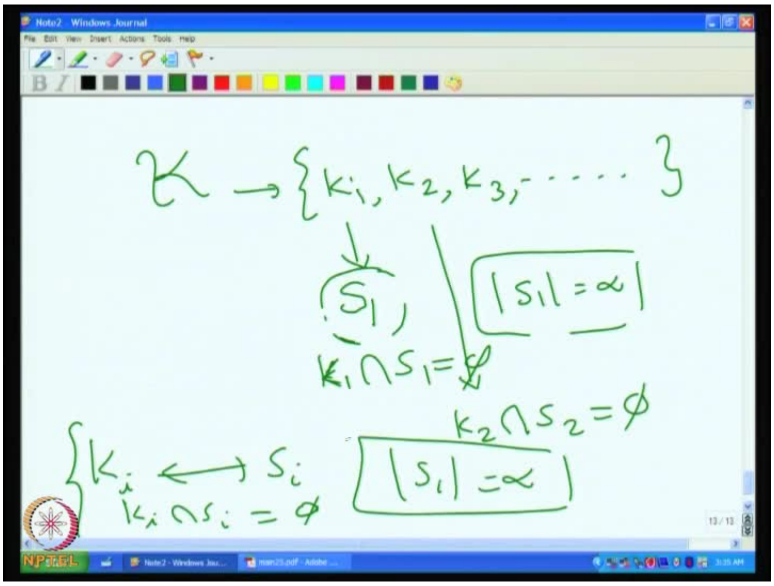

So, let we will collect all the cliques of G, so that means we will say K 1, K 2, K 3 this is the set of all cliques of G, so there can be several cliques all the clique are collected. Now, you know that if you take K 1, we know that this clique is such that it cannot cut all the independent sets of G, that is a maximum independent set; there exists a maximum independent set S 1 corresponding to K 1.

Look S 1 cardinality is equal to alpha, it is a maximum independent set, moreover this K 1 intersection S 1 will be empty, because you know you can find one S 1, because there should be at least one maximum independent set which it cannot cut, similarly for K 2 also they should be one, S 2 such that they does not intersect with it.

So, here we are taking S for each S i, so for each K i you can associate a S i such that it is a maximum independent set, that means it is cardinality is equal to alpha. And also K i intersection S i is equal to phi, it does not intersect with it, this is the point. So, it means that with each K i I can associate a maximum independent set S i, such that S i does not intersect with this K i.

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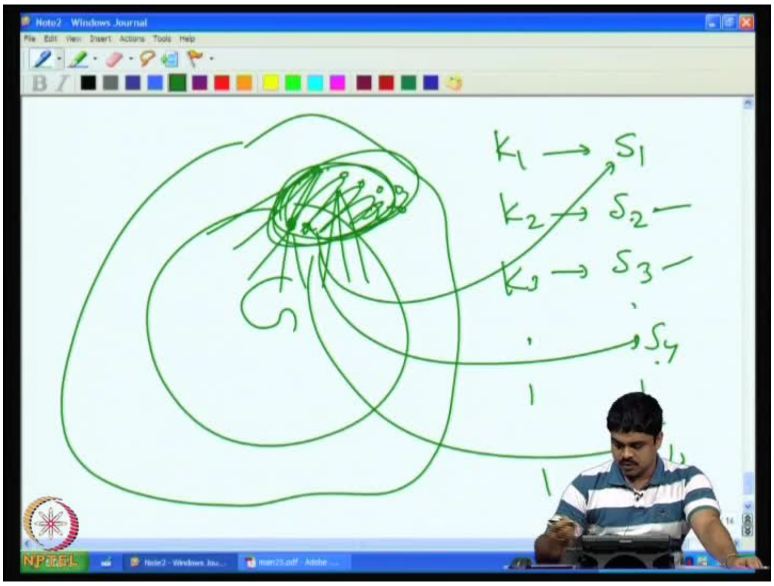

Suppose, this happens, we will show that there is a contradiction, what is the contradiction, what we are going to do is we are going to expand G, we will take up any vertex x of G.

Now, we will see in how many of this S I, we can write down this S 1, S 2, S 3 corresponding to each clique we have this K 1 corresponding to S 1; K 2, we have this some of them may be same but, it does not matter, we will consider different copies of that. And suppose this is x it may be part of this, it may not be part of this, it may be part of this some S 4 and it may not be part of S 2 and S 3 it may be part of S 10, so we will count in how many S is it comes.

Suppose, x comes 20 times, then what will I do, I will make 20 copies of x here and total including this original x, total 20 copies and I will replace this x with a clique of cardinality 20. And each vertex of this clique will behave like x in terms of its adjacency, for instance, to which all neighbors x is adjacent they all will be adjacent to an non neighbor of x they none of them will be adjacent. As if x is multiplied it into 20 copies of itself is. So, it is x friends all the friends of neighbors of x are the neighbors of all the vertices in the clique, the non neighbors and not the neighbors of any of these vertices in this clique.

We can see that such a clique can be obtained by repeated expansion operation, you can keep on expanding it 19 times, so then it will bring that entire clique. So you can do it for every vertex one by one. So, that will get us a graph which is much bigger than the original one but, we are replacing every x by so many copies that is all, and then between the copies we are making a clique.

And of case this entire graph can be obtained by repeated expanding, this select the correct vertex and expand it, select the correct vertex and expand it. Now, the good thing about this graph, because we obtained it by series of expansion operation from G. G was a perfect graph to begin with, so this new graph is also perfect graph.

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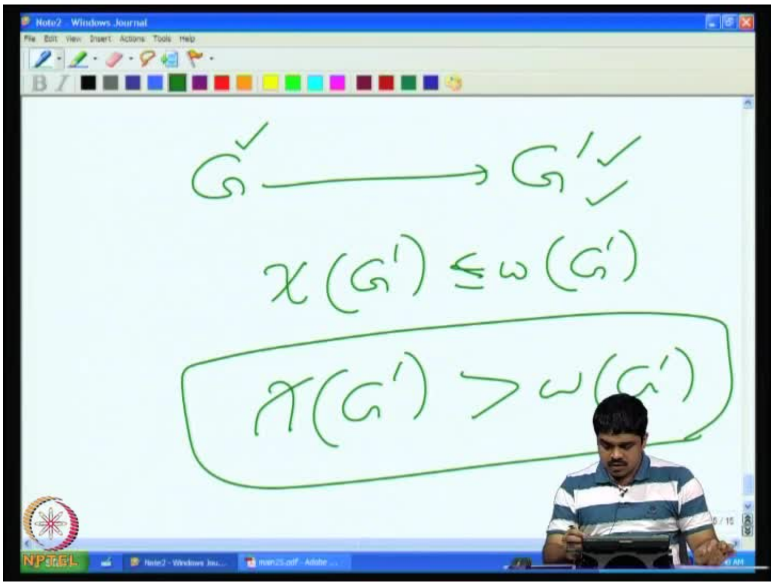

So, we can call it G dash, from G we obtained G dash, since G is perfect G dash is also perfect, because the way we obtained G dash is by expanding vertex after vertex several times, sometimes.

Now, it is even possible that from vertex may become extinct from the graph or can be removed from the graph, because that vertex does not appear in any of these things, so the number will be 0. So it is like we remove it, so that means, we are essentially expanding sub graph of G and getting all these thing not are necessarily G, because some vertex may disappear also.

So, now we have G dash a perfect graph so above G dash what I can say is because it is perfect graph chi of G dash has to be less than equal to omega of G dash. You remember it. Chi of G dash can never be strictly less than omega of G dash, because chi of G dash which means that this is equal to omega of G dash, because for the perfect graph, we can put it as equal to.

Now, we will show that by counting in a different way we can show that this is wrong by contradiction we can show. But, in other words, by counting in a different way we will show that chi of G dash has to be strictly greater than omega of G dash which will be contradiction; it will be a contradiction, because how can G dash be perfect in that case.

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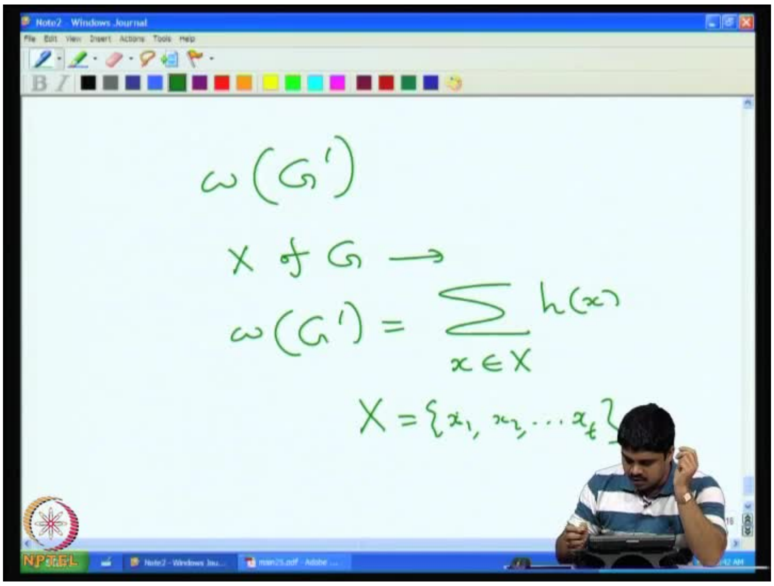

See the key observation is, to count both I mean, so let see how much is omega of G dash. See this omega of G dash is essentially a clique of G dash, you remember to construct G dash from G what we did is we replaced each vertex with a clique having so many copies of the corresponding vertex and whenever two vertices x and y where adjustment in the original graph you may put all edges between them, that is the expansion operation.

So, therefore if we had selected a clique or a original graph G and when you expanded all the vertices in the clique, we will get a big clique right. And it is easy to see that any big clique, any clique which in G dash is obtained from G like that, by selecting some clique and expanding. So, therefore, we can say that this omega of G dash, the biggest clique will correspond to some clique say x of G, such that the number of vertices is omega of G dash is equal to sigma of x element of x, how many times it was multiplied. We can see how many times it is multiplied so we can put say h of x is the multiplicities of that x. Now, we know what is this? This is essentially, this clique vertices will be, you know what, for if for instance if I write X equal to x 1, x 2, x t.

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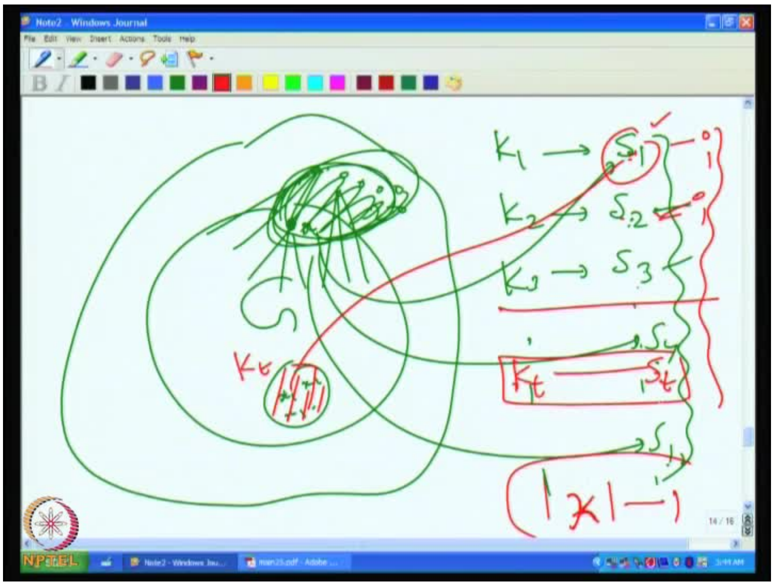

Then x 1, if you remember the way it was counted, this x 1 suppose this was a clique so x 1, x 1, x 2, x 3, this clique this counts in which all this things I am part of. See one point we can notice that, suppose x 1 is part of S 1, then x 2 cannot be part of S 1, Why? Because, S 1 is an independent set here it is a clique.

So, this clique, only one vertex can be part of S 1, so suppose you can think that these S 1, S 2, S 3 etcetera are counting the total multiplicities due to this clique, so that is the sum of h of x 1 plus h of x 2 plus h of x 3, if it is to be counted from the point of view of these sets S 1, S 2, S 3 etcetera. S 1 will say that I will anyway count 1 or 0 for that this entire clique. Because, only one vertex from this clique can be inside me, if they are counting no two of them will count me only one they will count.

Therefore this S 1 from S 1 you may get 0 or 1, so from S 2 also the same thing 0 or 1, it may get the total count for this entire thing. So the entire sum is contributed by S 1 either S either 0 or 1 maximum 1. So the maximum contribution can be total is cardinality of this the total number of these things, that means the total cliques, that means it can be the if K is the total collection of cliques it can be K. But, then there is there exists one among these sets which does not contribute, which is that, we know every clique is such that it has a pair.

Because, this clique itself this clique is called say K t, K t will be appearing somewhere here and it will have an S t, such that they are intersection between them is 0 that S t at least that S t would have conduct 0 for it so that subset would not, that independent set would not contribute to the count.

So, we will only get K minus 1, because all these cliques will not contribute all of them can contribute at most one and there exists one among them who will not contribute. So K minus 1 is the maximum.

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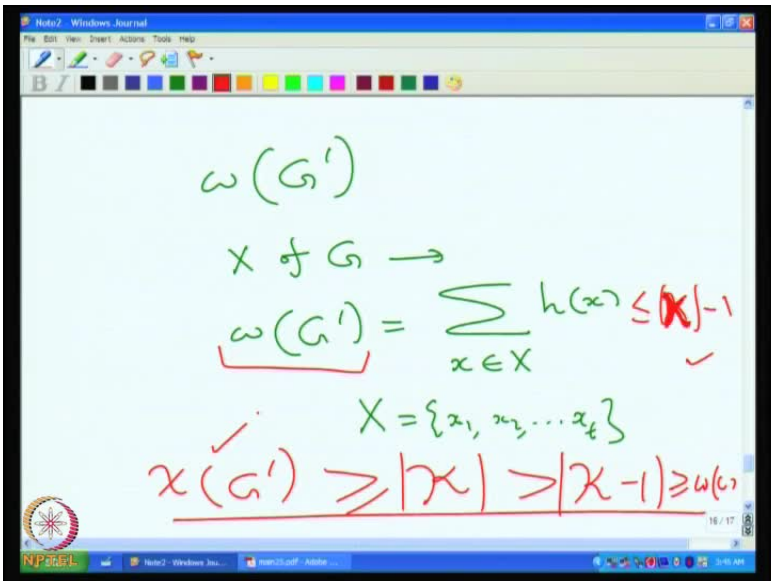

We can say that, this number is infact K minus 1, where K is the entire collection of cliques, the set of all cliques in G, K minus 1 is in it. So, then now so we got an upper bound for omega of G dash, we will show that chi of G dash is strictly bigger than that. We will show that chi of G dash will be bigger than or equal to K, so which will be definitely strictly bigger than K minus 1, which is greater than equal to omega of G.

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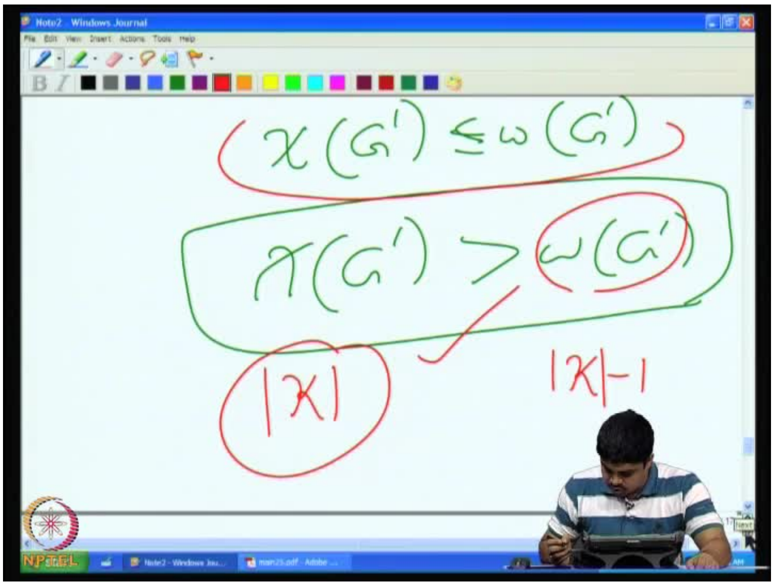

So, which is a contradiction for our, this thing, because we knew that, because it is a perfect graph this is correct chi has to be equal to chi of G dash has to be equal to this thing. But, it so happens that this turns out be true, because this is this is only K minus 1 and this we will show that this is at least K, so that strictly bigger.

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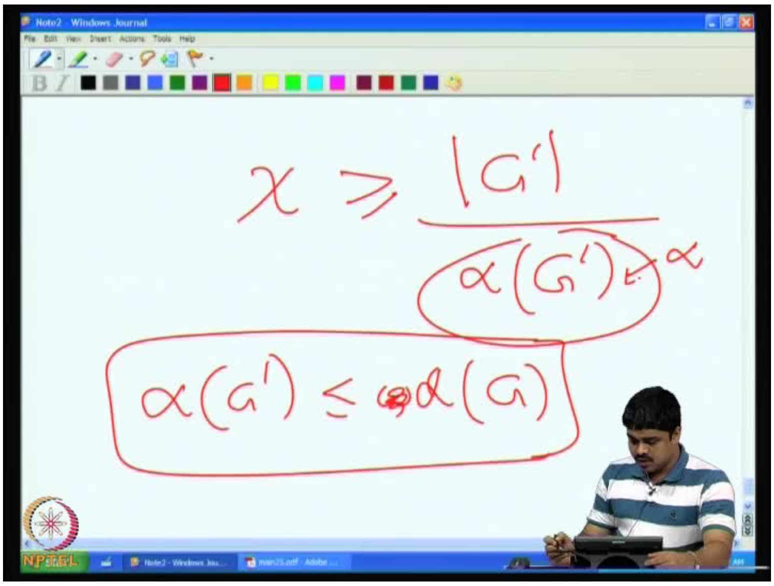

So, how do I show that this chromatic number is at least K, To show that the chromatic number is at least K, because to show an lower bound for chromatic number what we do was, we will divide the total number of vertices. So, that means n of G dash, the total number of vertices, so let say or we can just use this notation G dash, the total number of vertices divided by the maximum independent set size of G dash, this is enough. Because, this can never be lower than this, because this color class is which color glass can contain at most this.

So, therefore, we need at least G by omega G dash this thing but, interestingly this omega of G dash can only reduce, because our operation was to substitute each vertex by a clique. Now your omega cannot increase, because the original biggest independent set can be only bigger than or equal to this alpha of this thing, so this can independent set size cannot increase if we replace each vertex of the graph by a clique. Because, any two of the same clique cannot participate in a independent set.

So essentially it is the original independent set or therefore we can say that chi of G dash is less than equal to omega, so instead of this thing we can just put original alpha.

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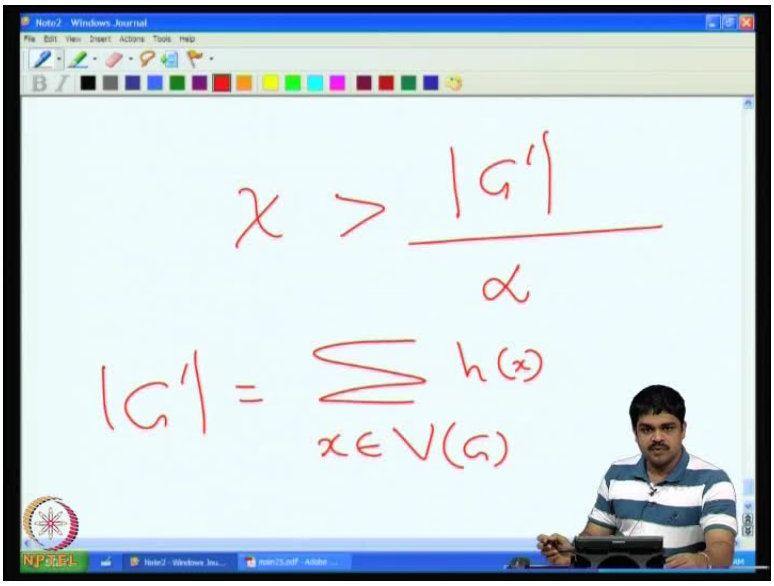

Now, how much is G dash? We know that chi is greater than equal to the cardinality of G dash, the number of vertices in G dash divided by alpha. Now how much is G dash, G dash is essentially the sum of all vertices in V of G. So, how much is the multiplicities? Because, every vertex was substituted by an h of x size clique the multiplicities, so many copies of x, so we just have to sum up the multiplicities.

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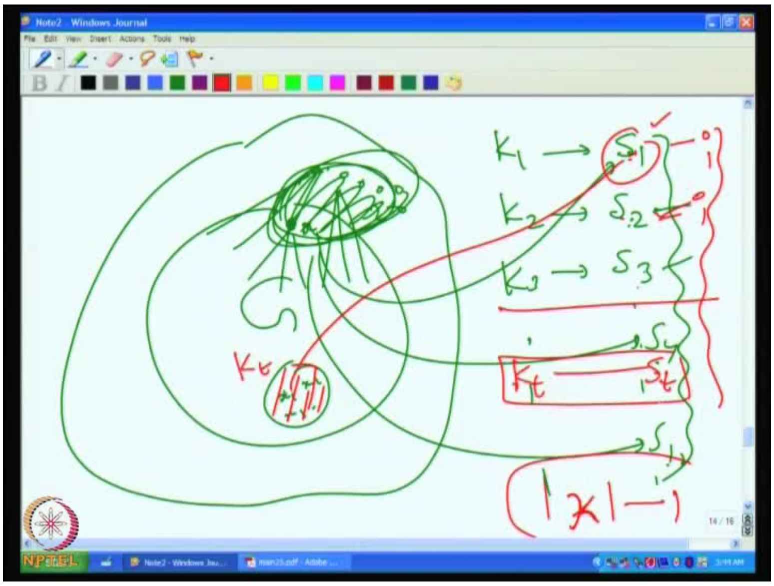

Now, again going back to this picture, how this multiplicities, if I want to sum up the multiplicities, the sum is essentially, because the total see for again we can come from the side of S 1, because how much S 1 will contribute total sum of multiplicities. S 1can only contribute the total cardinality of it because every vertex in S 1 will count once that S 1 not twice. The same vertex will not count S 1 two times, so all the vertices in S 1 will count it once. So, therefore S 1 will contribute total its cardinality that namely alpha, S 1 can contribute an alpha to the total count say S 2 also will contribute an alpha to this thing.

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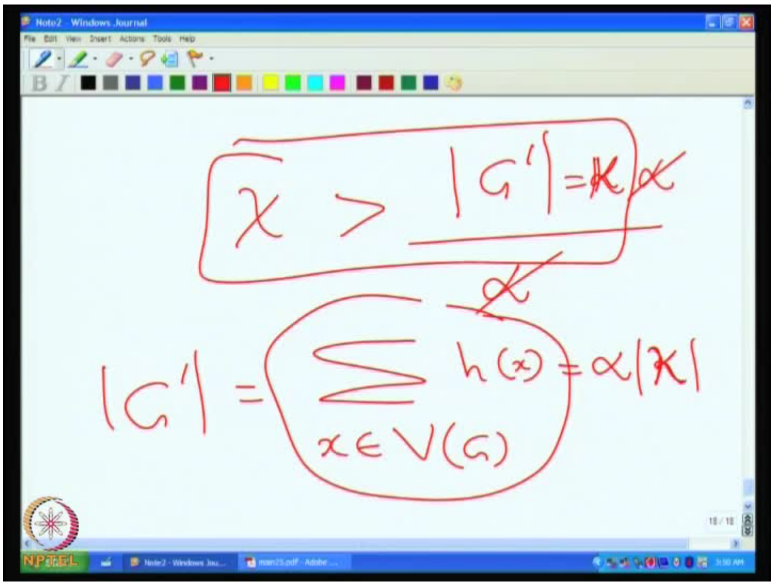

So, similarly we can say that the total contribution will be equal to alpha times K, why because, if you want to find out this total sum, you ask each of the S size that is S 1, S 2, S 3 etcetera. How much you have contributed to total sum because one S i will say I have contributed alpha, because it is cardinality is alpha. Because, for each vertex inside it will contribute 1, so alpha it will total contribution will alpha from each inside.

Since it is not (()) K sets are there, so this many sets are there K into alpha. So this can be substituted by K into alpha, so canceling this alpha and alpha we get chromatic number has to be at least the number of cliques in the graph, so chi is greater than equal to K.

So, we see that chi is greater than equal to K, omega is less than equal to K minus 1, so chi is strictly greater than omega. So, we have shown that this is not a perfect graph G dash, but G dash has to be a perfect graph because G dash is obtained from a perfect graph G by repeated expansion of vertices. So it has to be a perfect graph, so this is a contradiction finally.

So, what is the info of that, what is it contradiction? Essentially the contradiction came from the fact, that we did not have any clique in the graph which intersect with every maximum independent set of the graph G, because if they was even one, then in this calculation we would not have got the contradiction. Because, we show that in the total sum there was one clique which did not contribute. So, because that was possible because every clique could identify one maximum independent set, which is not intersecting with it.

So, this contradiction came from that therefore we can infer that, there exists a clique in the graph, which will intersect with every maximum independent set of the graph. Now the proof is obvious, we just remove that maximum clique, now the remaining graph satisfies the induction hypotheses, that means, the complement can be colored with a alpha minus 1 number of colors. Now put back this clique that means in the complement there is one big independent set is added, so one more color is necessary, so alpha colors as enough to color the entire thing, so this finishes the proof.

In the next class we will consider a different proof of this thing by this we perfect graph theorem by a person called Gasparian, so the original proof is by Lovasz we will proof is a little shorter and we will consider that thank you.