Graph Theory

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Module No. # 04

Lecture No. # 24

Interval Graphs, Chordal Graphs

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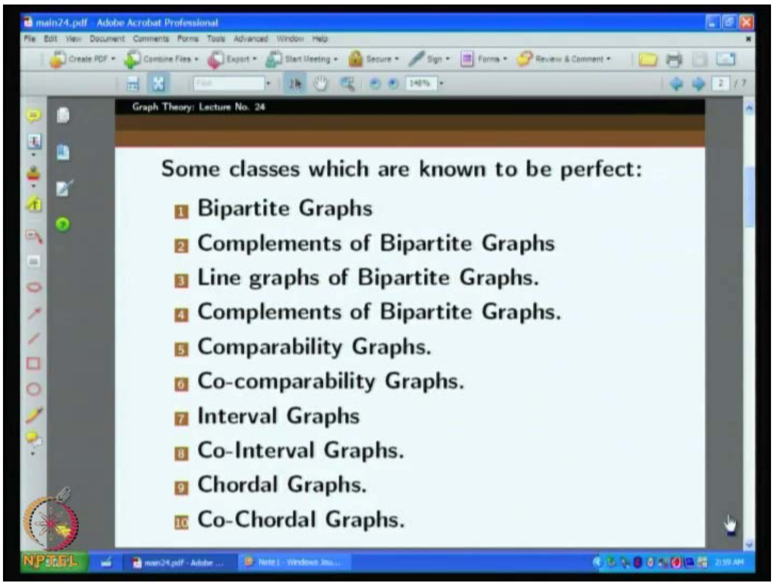

Welcome to the twenty-fourth lecture of graph theory. In the last class, we had introduced the class of perfect graphs and we had considered several examples of perfect graphs. First was bipartite graphs and complements of bipartite graphs, then we considered line graphs of bipartite graphs, complements of line graphs of bipartite graphs, comparability graph, their complements - co-comparability graphs, and finally, we had reached interval graphs.

So, we introduce the class of interval graphs as the intersection graph of a family of intervals on the real line; that means, for each vertex, we will be able to assign an interval on this real line such that two vertices are adjacent, if and only if the corresponding intervals intersect.

So, for a given graph, such an interval representation is possible, if we can find intervals on the real line for each vertex of the graph, such that two vertices are adjacent, if only if the corresponding intervals intersect. Then, we say that the given graph it is an interval graph.

These interval graphs are also very useful graphs in several practical applications it arises. So, of course, they are defined because of their usefulness and our intention here is to verify that this interval graphs are also perfect graphs; that means any interval graph is a perfect graph. To remind you what was a perfect graph, it meant for the graph and each of its induced sub graphs not only for the graphs, we have to also consider in every induced sub graph for the graph, the usual inequality khi of G greater than equal to omega of G; that means the chromatic number greater than equal to the clique number should become equality. That means chromatic number has to be equal to the clique number not only for the graph, but also for each induced sub graph of a graph.

Then we have to explain why probably this slightly strange definition arises because otherwise, the question was not very relevant. That is why we have to introduce this induced sub graph thing also in it in the definition and then we verified in most of this cases, induced sub graph also, for instance, in the case of bipartite graph, line graphs, in most of the cases, we have seen the induced sub graph also belong to that class.

Therefore, as long as we prove that the clique number is equal to chromatic number was not enough. It is not true in every case because some cases, the general properties may not that is there for this entire graph, may not hold for an induced sub graph. Therefore, it is not enough to always consider just for the entire graph, the chromatic number and the clique number; some cases, we have to go into the induced sub graph and prove it. Here, again, integral graphs have this property that if you consider an induced sub graph it is again an interval graph. Therefore, we do not have to worry too much about this special property being satisfied for every induced sub graph.

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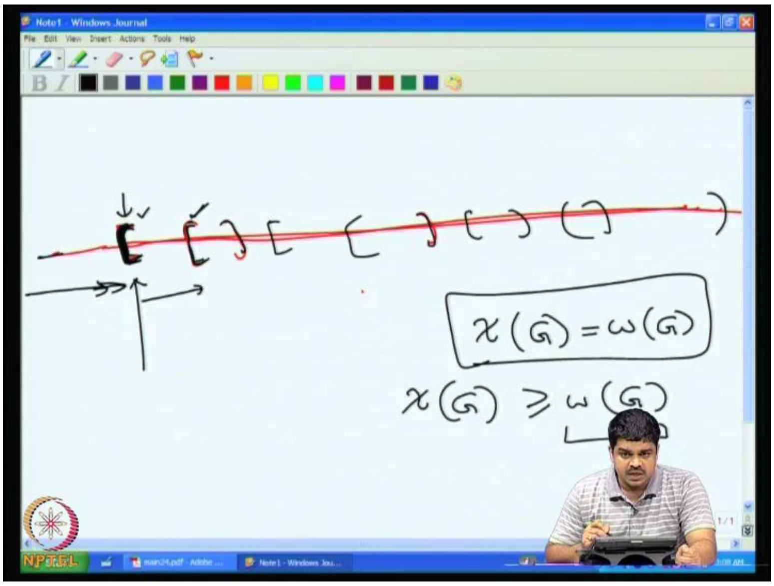

So, we will just prove that the chromatic number is equal to clique number for any interval graph G. This is our next aim. So, why do you show these things? We will consider this interval representation; because it is an interval graph, we have an interval representation. So, let us say, this is the real line and see I will mark an interval like this. This can typically be the interval representation of some graph.

Now, what we do is we search from the left hand; that means, we can come from this side and we find out the first interval graph that opens. So, we are going to prove, Our intention here, is to prove that the chromatic number of this graph is equal to the clique number. As we all know, this is always true for any graph greater than or equal to omega of G.

Now, it is enough if you find the colouring of this graph such that the number of colours used in the colouring is equal to the clique number. This is what we are going to do. That will imply that this is an equality. So, for that matter therefore, we are going to colour the graph. How am I going to colour? I will search for the first opening interval; the first starting interval does not mean that the interval which starts first should end first; that is not necessary.

So, we just find out the starting interval and give it a colour and then we look at the second starting interval and we will give it a colour in the greedy way. What is the greedy way? We will look at its neighbours which are already coloured other than we have to avoid those colours, of course.

Now, among the remaining colours that we have already used, if anything is left, we will take the lowest numbered colour, lowest colour; otherwise, we will go for the next colour. That is the plan. That means, we will reuse a colour, if possible, otherwise, if only in the case of a reusing not possible, then only we will take a new colour. then only we will take a new colour.

So, this is the greedy strategy way of very familiar with it in the case of in the Earlier, when we introduced the colouring, we had introduced this concept of colouring number, degeneracy, all those things. We consider certain ordering and that is how initially, we proved that maximum degree plus 1 is a upper bound for the chromatic number and later, we told that mini graphs that can be much smaller because if we consider the correct ordering, we may see that the ordering may allow us such an arrangement in such a way that for instance, whenever we see a vertex, its lower number vertices with respect to that ordering may be much smaller than its maximum degree. That is what that plus 1 was the That was the degeneracy; that plus 1 was the required number of colours. We called it colouring number.

So, the same strategy here; but here, see how it works. You know in those cases, we were not guaranteed that we will get an optimum colouring. The number of colours that we end up using may not be optimum, but here it will become optimum. Why? It is because of this reason.

So, we are taking this is the ordering; ordering, we are considering is with respect to the arrangement of the intervals, the left end points. We will consider the intervals or the vertices in the order in which the left end points of their intervals appear and we scan from left to right on the real line. So, this is considered first, then this is considered and of course, the colouring strategy is greedy strategy and may be third, we will be consider like this.

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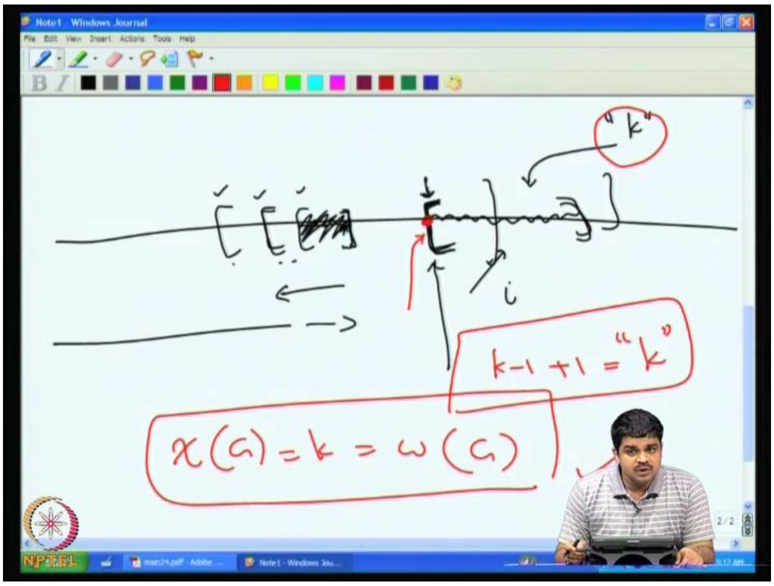

Now, we want to say that. When we colour the ith interval, suppose, we are colouring the ith interval, so we will. in the entire process, we might have used say k colours in this way. Now, there will be an interval say, the ith interval, ith vertex which got the colour k.

How did he get the colour k? Because there were k minus 1 neighbours of it, which were coloured differently and we have to take kth colour. That is why it happened. You see it starts here, the ith interval starts here and it has to be given the kth colour - this interval.

So, now, how will their neighbours be? Because all the neighbours which are already coloured, started from somewhere here, before this - somewhere here because we are scanning from left to right. Therefore, we will see these neighbours of this, starting from here, but if a neighbour started from here and ended here itself, so, a vertex started somewhere before this and ended before it, then it would not be a neighbour of it because this intervals are not intersecting.

So, only interval which intersects with it, will be a neighbour. So, any neighbour of this which is already coloured, which starts before this, but ends after this point - may be it can end here or it can even go further, but the key point is all those intervals corresponding to the neighbours of this, which are already coloured neighbours of this has to pass through this red point, which I have marked red because they start before and have to end before because it is a neighbour of it. So, if all of them go through this point, then what does it mean? They all intersect with each other - pairwise intersect because this point is in all of them.

So, it is a clique and this ith vertex itself is part of the clique. So, that means, we have this k minus 1 plus 1 vertices belonging to a clique corresponding to this red point. So, we have a k clique there and by assumption, we have only used k colours. What does it mean? So, we have coloured the interval graph using k colours. On the other hand, there exists a k clique in the interval graph. So, the clique number can only be bigger than this k because there is a k clique; it can be bigger than equal to, but it cannot be strictly bigger because if it is strictly bigger, we will need more colours also, but we are saying that we have used only k colours. So, how is it possible?

So, we conclude that k is indeed the chromatic number also. The k, the chromatic number is indeed the clique number also. That we get that khi of G is equal to k equal to the clique number. This is what we get; this is what we want. Now, for an interval graph its chromatic number and clique number is equal. Any induced sub graph of the interval graph is again an interval graph; there also the chromatic number and clique number will be equal. Therefore, interval graphs are perfect graphs.

So, this is the reason why interval graphs are perfect graphs. Now, of course, as we were checking in all the previous cases, whenever we considered a class, we also considered whether the complement, the class of the complement graphs was also perfect. Here also, we will do that. What about co-interval graphs? What do we mean by co-interval graphs? Just that, its co-interval graph is a graph whose compliment is an interval graph that is all.

So, now, we know how to check this. How do we do that? We just have to consider the independent set of the interval graph because this independent set of the compliment interval graph will become the clique number of the co-interval graph we are considering. Like, we have done it before also. In the case of all the previous classes, we have done this thing; alpha of its compliment will be the clique number of the original, the co-interval graph we are considering.

Now, we have to show what will be the chromatic number convert to in the compliment interval graph? The chromatic number is a covering using independence set. So, it will become a covering using the cliques in the corresponding compliment graph; that means in the interval graph. So, that is called clique cover number; in the compliment, the chromatic number becomes the clique cover number because we are trying to cover using cliques, all the vertices; you want to cover all the vertices of the graph using as small a number of cliques; that is what the clique cover number and then we want to show that for an interval graph, the clique cover number is equal to alpha, the independence number that is the stability number, the biggest independent set size.

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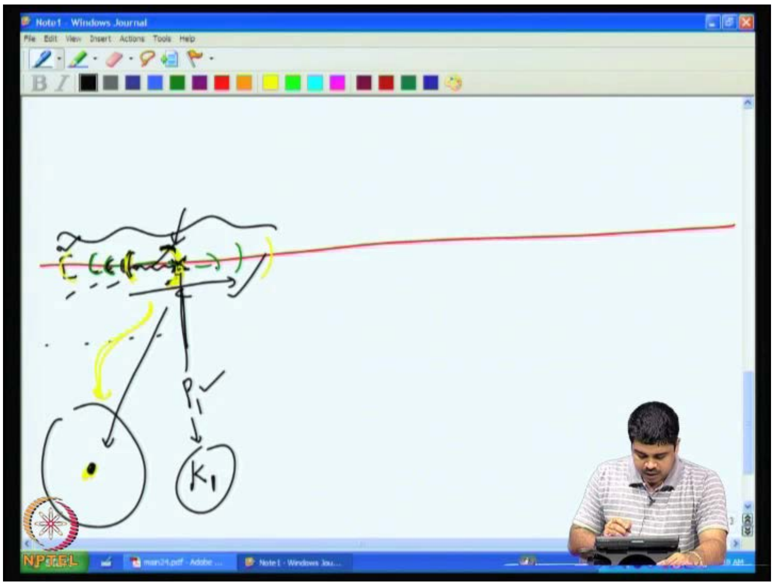

Now, if you show this thing, for all the induced sub graph also, this will be true, which means that for the compliment, the khi of G is equal to omega the omega of G, that the chromatic number and the clique number will be equal for all the induced sub graphs.

So, let us show that for the interval graph, this is true; that means the clique cover number is equal to the independent set. This is the method. We again consider our real line and then we first consider we consider the interval, this time we consider the interval which starts sorry which ends first; we scan from left to right. We look at the interval corresponding to the earliest closing brace; that means the interval, which closes first.

So, there can be several things which start before this thing, but this is the one which closes first. It does not mean that it is an interval corresponding to the earliest starting bracket. It can be, see, this may be that. So, all these things may go further, it may be this may go like this; this may go like this or this may go like this. So, just like that we are considering the interval with the earliest closing brace, the first closing brace. That interval we will add to our independent set. that interval. This is what we are going to do first.

Now, what do we do? We will remove all the vertices, which pass through this point which goes through this point because this is the first closing brace. Anything which starts before this thing has to go further because they do not close before this. So, they all have to at least touch this or go further. So, this point P 1, we can define a we define the clique corresponding to this point, means all the intervals which cross through this point has to be adjacent to each other in the graph. That means, they intersect, the intervals touch; the intervals intersect with each other. Therefore, they correspond to a clique in the graph.

Therefore, that clique we can collect. Let us call it K 1, the first clique. So, that we put one in the independent set. This an independent set we are forming, this namely, this interval which ended there and then we also collected all the intervals, which pass through that that correspond to a clique and that is added to the clique cover.

So, we are going to construct an independence set as well as a clique cover for the graph side by side. One in the independent set, one vertex in the independent set, a clique in the clique cover number like that, we keep adding and show that by the end of the procedure, we will get an independent set and a collection of cliques so that the cardinality of both the sets are equal. That means, the number of cliques in the clique cover will be equal to the number of vertices in the independent set.

This collection of cliques will cover the graph and moreover, we will show that yes right they cover the graph; that is what because that is enough. Why, because as we see, the clique cover number has to be always greater than equal to the independent set and what we are demonstrating is a clique cover, whose cardinality is equal to the cardinality of some independent sets. So, the maximum independent set cannot be more than this because if it is more than the clique cover, we will not get a clique cover of this cardinality.

Because clique cover number has to be at least as much as the cardinality of any independent set; in particular, greater than equal to the maximum dependent set. If you demonstrate a clique cover, whose cardinality is only equal to some independent set; that is enough; that means that the independent set, we have got is also a maximum independent set; that is what we are going to do.

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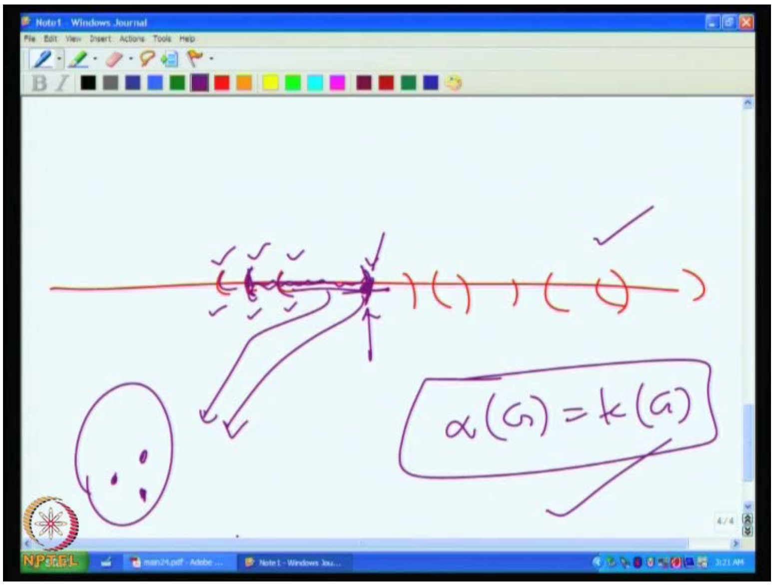

So, we got one here; after that we remove this interval, as well as all the intervals corresponding to this clique. So, we just remove it. That means, now we will see after the removal, we will see some intervals are removed. Now, the remaining intervals will be seen here as if nothing has happened because again it is an interval graph after removing something. Just that we removed one vertex and added it to our independent set and removed all the cliques, all the vertices which pass through the closing end point of that interval, which formed a clique; that was added to the clique cover.

Now the same thing we will do. From the remaining interval graph, we pick up the first closing brace; maybe, this is the first closing brace and it may correspond to this and then we will add this to our independent set. This will be the second vertex in the independent set and then corresponding to this point, again we can collect all the vertices which pass through this; in particular, everything which starts before this thing has to pass through this point. This will correspond to a clique and this clique is added to the independent set.

You remove all the vertices now, which belong to the clique. Now, you see that by this procedure, by the end of it, you will remove all the vertices because you see whenever I am removing the vertices, we are sure that the starting opening braces are before this and we are not arbitrarily removing anything.

So, each point corresponding to which we remove this thing, we are actually removing the intervals, which starts before that and ends after that and because we get points across this thing, every vertex, every interval will be covered by these cliques. So, it is indeed a clique cover number and what we are making is indeed an independent set because we added something, a closing brace and then we know that we have removed all the vertices, which started before this. So, any vertex which is left starts only after that point, it will be in fact independent of the already added vertices because that will start strictly after the last the end point of the last interval that we added to independent set. Therefore, we are going to get an independent set.

So, we see that we will indeed get a clique cover, whose cardinality is equal to the cardinality of the independent set we make by the end of the procedure. Therefore, here alpha of G is equal to k of G, the clique cover number. That is it, fine.

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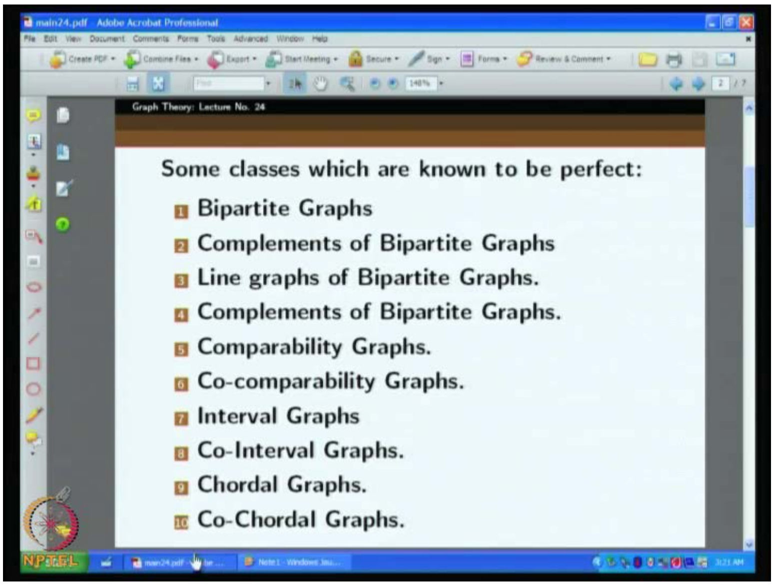

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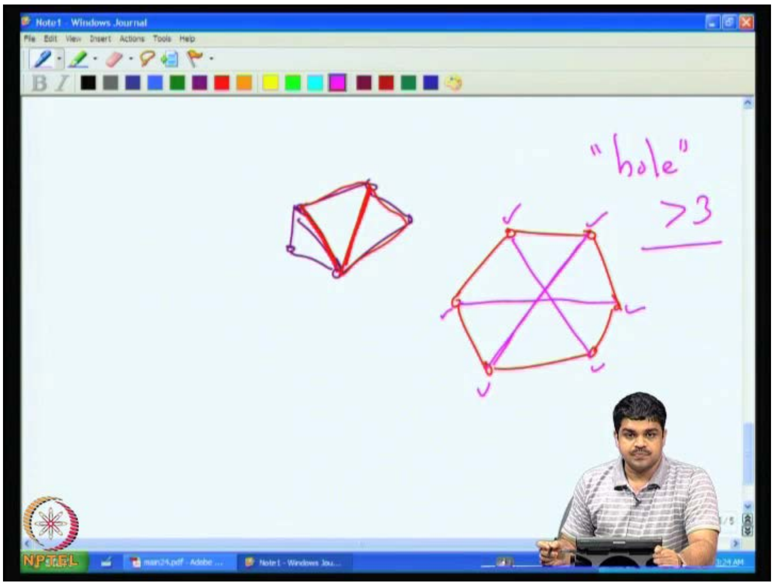

So, this is the proof for that, proof that any complement of an interval graph is also perfect. The final thing is very well known class of graph called chordal graphs. What is a chordal graph? It is defined like this. Again, these are also coming from application because of the usefulness in practical situations.

So, the chordal graph means you consider a graph and you consider a cycle in it. Suppose, the cycle is of length 4 or more say, this is 5 cycles here, then we should get a chord like this, chord for that, it should be a chord here. It should not happen that now we see a 4 cycle here, we should get one more chord here; this is a chord. That means, in other words, there should not be any cycle in the graph of length 4 or more, without a chord; no chordless cycle of 4 or more should be there.

So, if this is the cycle, and other than the adjacent vertices in this cycle, in this order, nothing else is connected; so chordless cycle. A chord for the cycle will be like this, for instance, this is a chord; this is a chord. If no chords are present, then it is a chordless cycle. If you consider the vertices of this cycle and take induced sub graph of that, then if it becomes a simple cycle, when you take the induced sub graph, that is a chordless cycle or it is also called an induced cycle. In the perfect graph literature, it is sometimes called a hole; if the number of vertices participating in this cycle is more than 3; 3 cycle is a triangle; it is not a hole, in fact.

In other words, we do not allow any holes in chordal graphs. So, we do not allow that. In other words, any cycle of length 4 or more should have a chord or any induced cycle in the graph, has to be a triangle; no induced cycle of length 4 or more. This is what the chordal graph is.

So, it may look like the definition of the chordal graphs does not seem to imply, to tell too much because it is not it is talking about the cycles, it is talking that there is no induced cycle of length 4. How does it help us to prove that it is perfect or not perfect, whatever. Does it make any sense? Is it connected to the chromatic number or is it connected to the clique number somehow, but it so happens that the chordal graphs have many properties. Just simple looking fact that there are no cycles of length 4 or more without a chord; no chordless cycle of length 4 or more will allow us to infer a lot about the chordal graphs.

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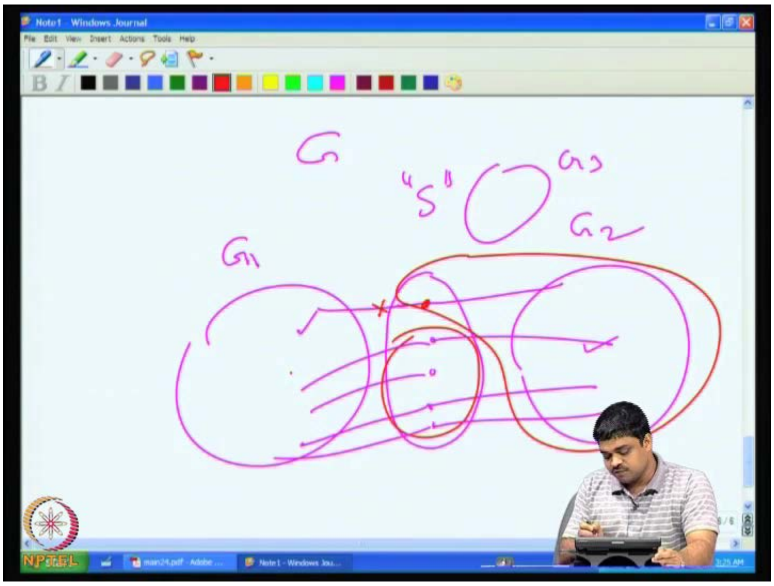

Let us look at some of the immediate properties. We have studied what are the separators. So, let us consider this suppose, we have a chordal graph G and we consider a minimal separator S of that. What do you mean the minimal separator? It is a collection of vertices and if you remove this set of vertices, the graph will get disconnected; that means there will be at least one piece here which we can call G 1, at least some piece here G 2, may be there are more also, G 3, but the point is if you remove any subset of this vertex, we cannot disconnect the graph; in that sense, it is minimal, that is the point.

In that sense, it is minimal. So, you cannot, no proper subset of this can separate the graph.

Now, if you consider a minimal separator of the graph. So, that means at least two vertices because it is a minimal separator, we also know that every vertex has to be adjacent to both sides by at least one vertex because if there is a vertex here, suppose, this vertex was not adjacent to this side, then you could remove this portion and it will separate this part from the remaining. This we have seen before. Therefore, every vertex should be adjacent to one vertex on both sides and otherwise, removal of that, in fact, to each component and otherwise, a subset of that, you see, if that vertex, which is not connected to both sides can go along with some other component and the remaining vertices themselves can manage to separate one piece from the remaining.

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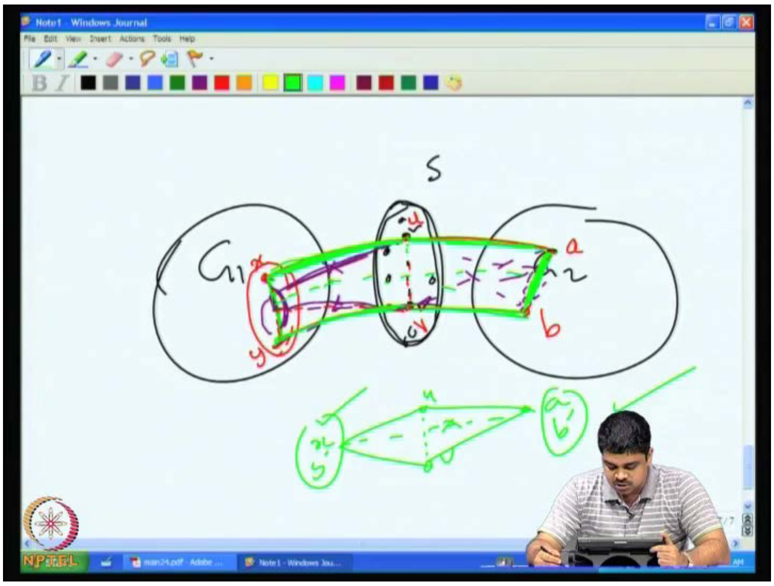

So therefore, So, the minimal separators have this property. This is the one separator, we are considering, let us say, without loss of generality, there are this component G 1 and G 2 and suppose, we want to claim that this induced sub graph on this, if G is a chordal graph, has to be a complete graph. It should be a clique. Suppose, it is not a clique, what can happen? Suppose, it is not a clique, then see there are these two vertices there because if it is just one vertex, it is a clique. Moreover is possible to have some two vertices It is not a clique. There should be two vertices such that they are not adjacent to each other.

So, let this be the two vertices, which are not adjacent to each other. Mark it with red. Yes, so, two vertices. Now, you know this has a neighbour here; this has a neighbour here; this has a neighbour here; this has a neighbour here; there can be equal the neighbours but, there are some neighbours. So, what we do is here, I will say this is x; this is y; this is a; this is b; this is so called (( )) vertices u and v.

Now, we can see because this is a connected component from y to x, there is a shortest path; there is the path here. Similarly, a to b also, there is a shortest path. When I say shortest path a and b can be equal, in which case, the shortest path is that a to b, single vertex and similarly, here.

Now, we claim that so Among all the neighbours of v, if you had considered this, y and x to be the shortest, I mean the closest such neighbours; that means their distance is minimized, then we see that then there will not be any more edge from here to here. So, we know that there will not any more edge like this or like this into the path because we could have taken these two pairs as a shorter pair then.

Therefore, we can rule out this kind of edges also because this is a shortest path. There will not be any edges of this sort; this kind of short circuits will not be there because it is only a short path. So, that would not be there and similarly, so of course, then x, v, u, y such edges will not be there because in that case, we could have taken x and y to be the same.

Similarly, similar things can be told about it. This edges, this edges, this edges, this edges will not be there; similarly, this kind of edges will not be there. So, this is indeed a This route, this one, this edge u, x and the shortest route here x to y and then y, v and then v, b and the shortest route here and then u, a is indeed a chordless cycle or induced cycle in the graph, but then here 1, 2, 3, 4, 5, 6 vertices are there; need not be six vertices, if x and y are same, then it may turn out to be something like u, v, x here. This is a here because a and b can be same; here, this a and b, this x and y can happen to be the same vertices. So, this will and Of course, you may ask is it possible to have any chord of this sort - cross code, this way. No, it is not possible because it is a separator here; it is not possible to have a code of this sort; it is not possible.

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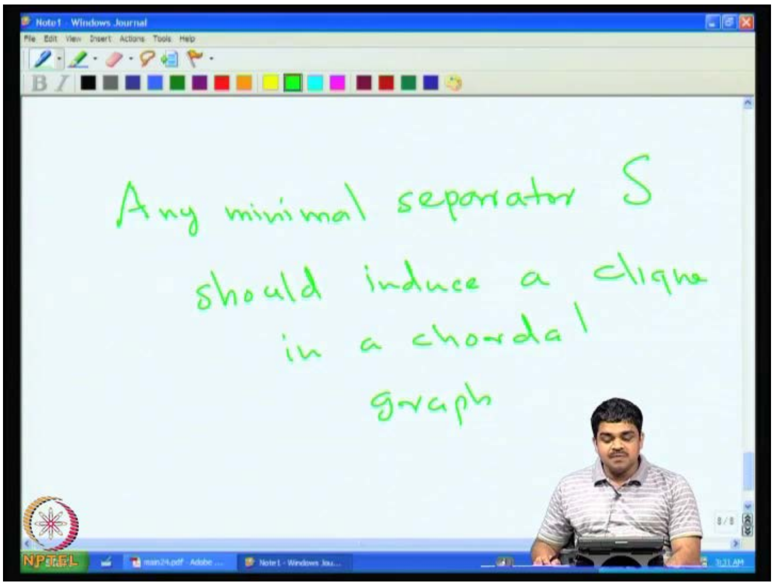

So therefore, it is indeed four cycles. We infer that it is not possible to have some u and v in the minimal separator such that they are nonadjacent. So, every pair of vertices in the separator, minimal separator has to be adjacent; that is what we say. We infer that in a chordal graph, any minimal separator S should induce a clique in a chordal graph.

See Note that we have proved it by using this simple fact that in a chordal graph, we do not have induced cycle of length 4 or more. That implies that any minimal separator S of a chordal graph should induce a clique in it. That is a good thing. How does it help?

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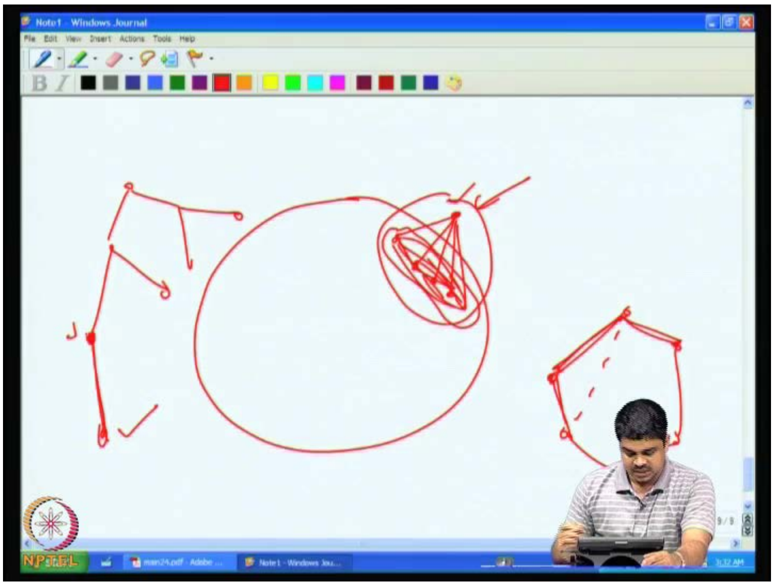

There is this concept of simplicial vertices, which is very useful in the study of chordal graphs. What is a simplicial vertex? In a graph, a vertex is called simplicial, this is a vertex, if you look at its neighbours, if they form a clique, if this neighbours form a clique that means they are all pairwise connected. Of course, including this it will be a clique which has one or more vertices. Such vertices are called simplicial vertices.

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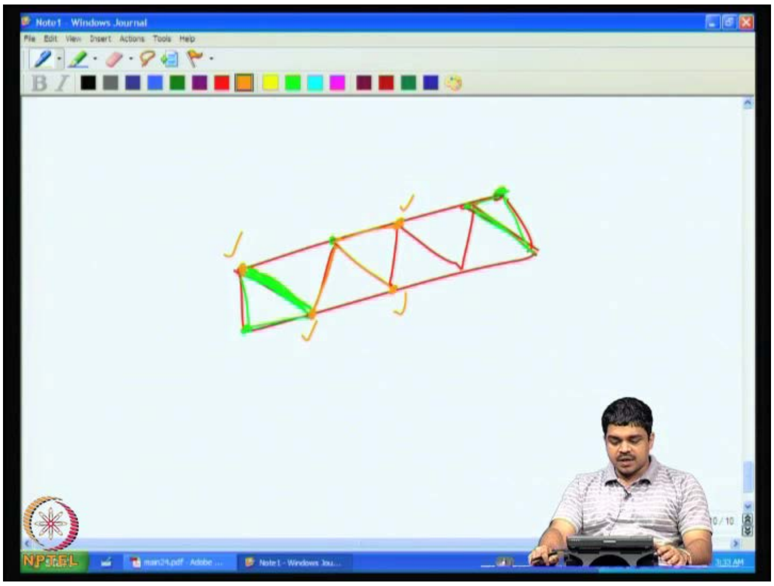

Here, in a cycle, do you see any simplicial vertex? No, definitely you do not have a simplicial vertex. that is why this so this is There is no simplicial vertex because if you take this vertex, its neighbours do not form a clique; if you take this vertex, its neighbours do not form a clique, but on the other hand if you take a tree, do you see any simplicial vertices? Yes, of course, if you take this leaf, see here, its neighbour is only one and then its indeed a clique - one singleton clique or it can for instance, you can see a graph like this, simple graph like this. Here, we have several simplicial vertices. For instance, if you take this vertex, this is simplicial; neighbours form a clique here. So, this is a simplicial vertex because its neighbours form a clique here.

But this is not a simplicial vertex. Why? Because its neighbours this, this, this and this and this, this, this and this together do not form a clique. Some graphs have simplicial vertices; some graphs do not have simplicial vertices. In a given graph, all vertices need not be simplicial vertices; some vertices can be. So, if all vertices are simplicial vertices, now, what does it mean? so you can

So then see In our case, we are talking about chordal graphs. In the chordal graph, we will say that there will always be a simplicial vertex. So, this is what we want to prove now. In a chordal graph, the property that how does it allow us that the property of not having a four cycle for induce cycle of length 4 or more.

Now, how does it allow us to infer that there is always a simplicial vertex. In fact, we will prove a slightly stronger statement, which is not that there exists one simplicial vertex; in fact, most of the time, there will be two simplicial vertices. In fact, there will be two simplicial vertices. Essentially, if the chordal graph is a complete graph, every vertex is a simplicial vertex, is not it? So, if the chordal graph is not a complete graph, we will show that there will always two simplicial vertices, which are not only simplicial vertices which are non-neighbours in fact, nonadjacent - two vertices which are nonadjacent and also simplicial; such simplicial vertices will be present. That is what we are going to prove now.

How do we prove this thing? Let us say, to prove this we will consider an induction for a small graph, for one vertex graph. So, it is a complete graph. There is nothing to prove for two vertex graph; it is obvious. So, like that smaller graph, we can prove it for induction; for three vertices graph, we can check.

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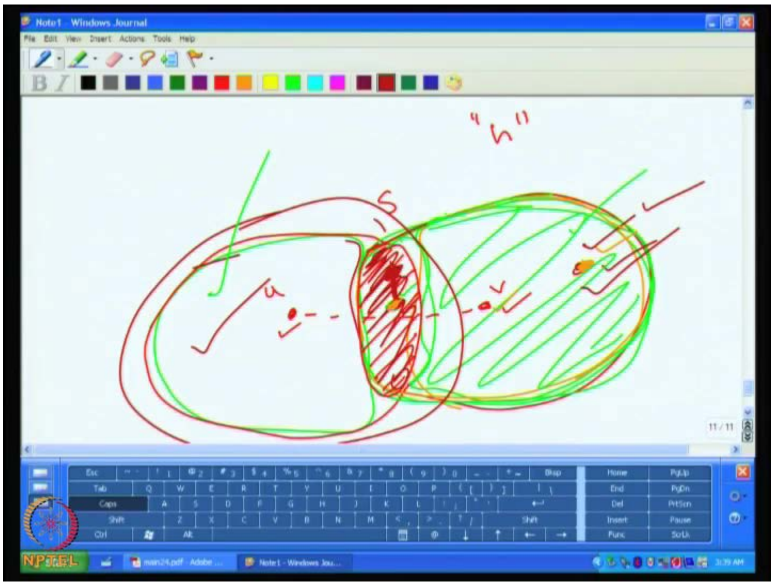

Now, let us say, suppose, you take an n vertex chordal graph. Now, for all smaller chordal graph, we know the statement; that means, if it is a non complete chordal graph then there exists a pair of nonadjacent vertices such that both of them are simplicial in the graph.

Now, what do you do? This is a graph; this graph is given. So, there is this two vertices u and v, which are nonadjacent. Therefore, there is no See, if two vertices are there in the graph which are nonadjacent, we can always find a separator for them that is nonadjacent. Let us call it S and if there is a separator, there is a minimum separator. We can always consider the minimum separator, separating u and v.

Now, we have two, we can consider two graphs. So, this side of S which containing u and the side of S which containing v. So, there can be other thing which we can add to this part or this part. For instance, it may not be total number of components that we are interested.

There are at least one more vertex here because v is here, at least one more vertex other than S. Now, the question is what about So, we look for one; see, this portion there is at least one vertex u. So, this portion there is at least one vertex v. What if we can pick up one simplicial vertex from this side and one simplicial vertex from this side because they are on different sides of S, the separator and they are going to be nonadjacent. If such simplicial vertices exist then indeed our statement will be a true, but is it a guarantee that we will get a simplicial vertex in this side. For instance, if I consider this graph is it guaranteed that on this green side, I will be able to find a simplicial vertex?

See, we know that if there are simplicial vertices because it is indeed a chordal graph, a smaller chordal graph and we can apply the induction hypothesis, but there is a possibility that this simplicial vertex take, because it is just a chordal graph together, it may end up here inside S, but we know that we have a chordal in simplicial vertex here, but then if it is a complete graph, this happens to be complete then we could have taken any vertex. You could have taken this v itself; it will be a simplicial vertex or any other vertex on this side would have been enough.

We can assume that this is as such not complete. Therefore, there are two nonadjacent vertices, which are simplicial in this part itself. If one of them is here, do not worry take the other and if the other happens to be here in this side, we are done; we got a simplicial vertex.

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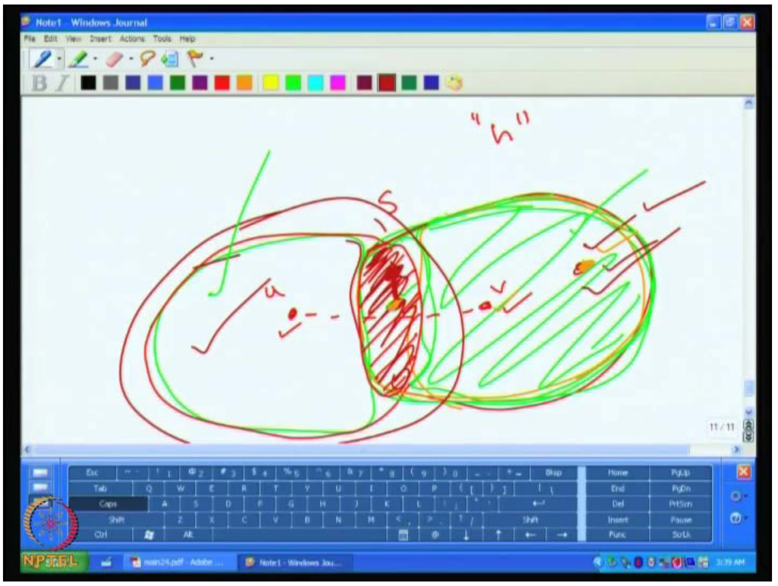

Now, what if the other vertex, out of this two nonadjacent simplicial vertices is also here. In fact, it can also happen to be here inside the clique. is it possible? No, it is not possible because this is where we are using our previous lemma because this is indeed a clique, this portion is indeed a clique and we have this edge between them because it is a clique, but then we know that two simplicial vertices we have are nonadjacent. We have two nonadjacent vertices which are simplicial. So, if one of them is here, the other has to be outside this. That means, it has to be somewhere inside.

So, we get one; indeed, we will definitely get one in this region. Similarly, we will get one in this region also when we consider the induction assumption on this portion. Therefore, we will get a simplicial vertex here and simplicial vertex here. That will mean that we have two nonadjacent vertices, which are simplicial in the chordal graph.

So, this is a slightly stronger property than we usually need; usually, we need only one simplicial property, this from the simplicial vertices in the chordal graph; that would have been enough for most of the purposes, but then if I try to prove that we have one simplicial vertex, we will one it would be a good exercise to try to prove that by putting an induction hypothesis that there exists for up to n minus 1 vertex chordal graph, we have a simplicial vertex, one simplicial vertex and then we try to extend induction, take the induction forward, you will find it a little difficult.

This idea that we are strengthening the induction hypothesis by saying that we have two simplicial vertices, which are nonadjacent, helps us to take forward because that is strengthening the induction hypothesis which helps us to push it fast easier. So, there are some, We had studied a classic example of such strengthening of the induction hypothesis to prove the theorem, when we considered the five choose ability of planar graphs. How in that proof also it was by a clever strengthening of the induction hypothesis that we manage to rather than what we put as inductively, we assume what is more than required because we just wanted to prove that the planar graphs of five choose able we assume something much more a stronger and then proved it.

Because when you strengthen the induction hypothesis, it so happens that we have more things to use that helps us, but we have to manipulate it very carefully. This is another such nice example. So, this is also much easier. Therefore, we should try doing in the other way and try to make it work without the strengthening and see it, appreciate such a thing, such a technique and now, we know that this simplicial vertices always available in a chordal graph.

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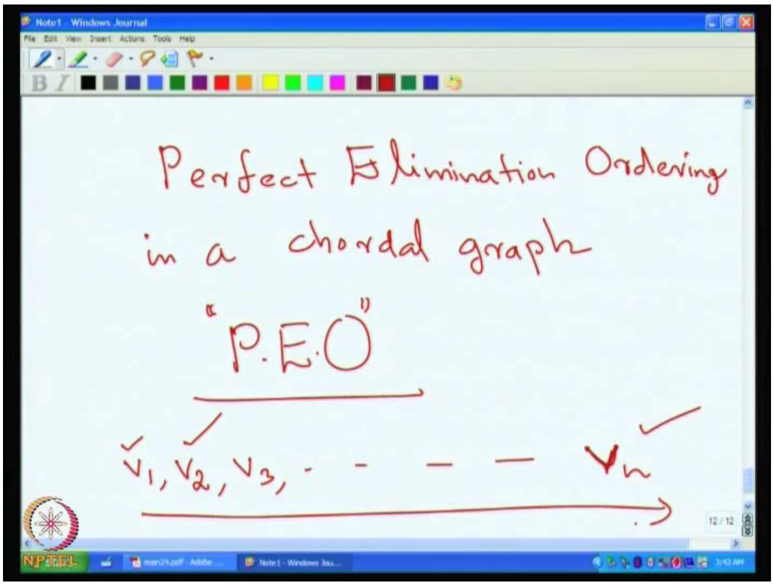

This gives rise to the idea of perfect elimination ordering. This is what, perfect elimination ordering in a chordal graph. So, it is called What is a perfect elimination ordering? Also, it is called PEO; that is the short form of PEO - perfect elimination order.

So, perfect elimination ordering is a sequencing of vertices in the following way. First vertex V 1 will be a simplicial vertex of the chordal graph. Now, we remove V 1 from the graph and then we consider, because we get a smaller chordal graph, because all induced sub graph of the chordal graph is again a chordal graph, we pick up V 2 such that V 2 is simplicial vertex in the remaining graph and V 3 is picked up after removing V 1 and V 2 in this chordal graph that remains. It will be a chordal graph because induced sub graphs are always chordal. So, we pick up a simplicial vertex V 3 and so on until V n.

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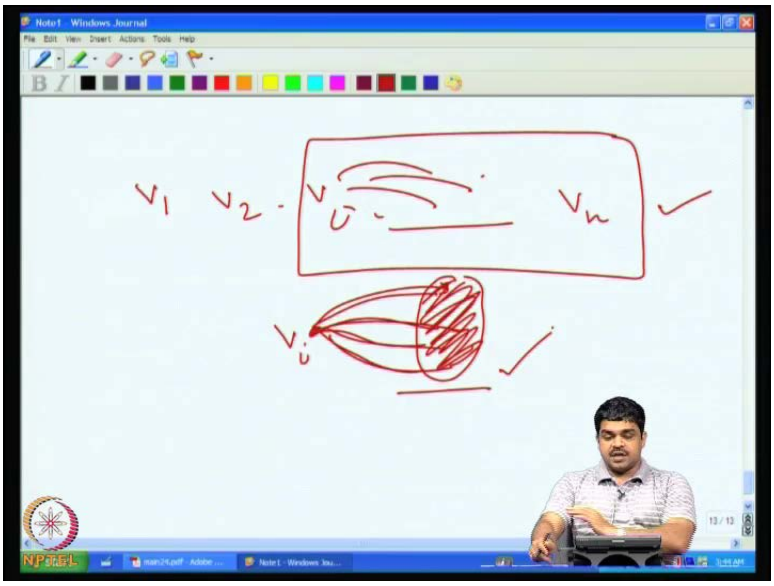

Every time, we will get a simplicial vertex because in a chordal graph, there is always a simplicial vertex. So, such an ordering of vertices is called a perfect elimination ordering. The property of these perfect elimination ordering is that if you write down V 1, V 2 up to V n and then look at V i, the ith vertex and you look at the neighbours of V i, which are higher numbered than, that means if V i is here, we look at the neighbours which are higher numbered than V i. So, those what will you see on the neighbourhood? Of course, because V i is a simplicial vertex in the induced sub graph on V i, V i plus 1 to up to V n. Therefore, this is going to be a clique; that is the special thing.

So, it is very much like when we considered the degeneracy or colouring number or anything, but here in the property that we are looking is not that the higher numbered neighbours are small, but we are saying that the higher numbered neighbours will induce a clique; that means V i will be a simplicial vertex in the graph induced by V i, V i plus 1, V i plus 2 up to V n, the induced sub graph on the higher numbered, higher or equal numbered vertices.

So, such an ordering is called perfect elimination ordering. In a chordal graph, as we can see a perfect elimination ordering is always available as a sub (( )) because in any chordal graph, there is a simplicial vertex, remove it, the remaining graph is again chordal because the induced sub graph of a choral graph is chordal, we get another simplicial vertex and this sequence of elimination that will be a perfect elimination order. Now, it is not very difficult to show that this is also a characterization of the chordal graph. This is a characterization of the chordal graph. So, I will leave it to you to prove that if a graph has a perfect elimination ordering, it has to be a chordal graph. Yes, it will have; yes, it will turn out to be a chordal graph. It is not possible for any other graph. If a graph is not chordal, it is not possible to get a perfect elimination ordering, I leave it to you to verify that.

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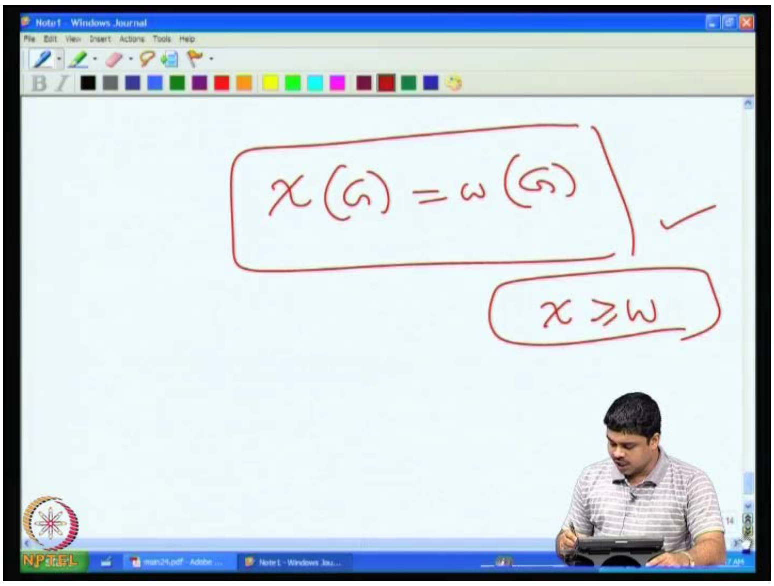

Now, we will go back to our main aim. So, this is all told in order to study the chordal graphs and get some ideas about the chordal graph, but then these theorems will also help us to prove that the chordal graphs are indeed perfect graphs. How do we show that? See to show that the chordal graphs are perfect graphs, again we need not worry about the condition on induced sub graph because an induced sub graph of a chordal graph is again a chordal graph and therefore, if I simply prove that for any chordal graph G, khi of G is equal to omega of G; for induced sub graph also, the property will hold.

So to show that a chordal graph is perfect, we just have to prove that the chordal for chordal graph, the chromatic number is equal to the clique number. How do we show that? As usual since chromatic number is in general greater than equal to omega, we just have to show a colouring of the given chordal graph using number of colours equal to the clique number or number of colours equal to the size of some clique; that is enough because the chromatic number, if the colour, number of colours used is only equal to the cardinality of some clique, it has to be the maximum clique because if there is a bigger clique then we could not have done it because that bigger clique would not have allowed a colouring of a graph using lesser number of colours.

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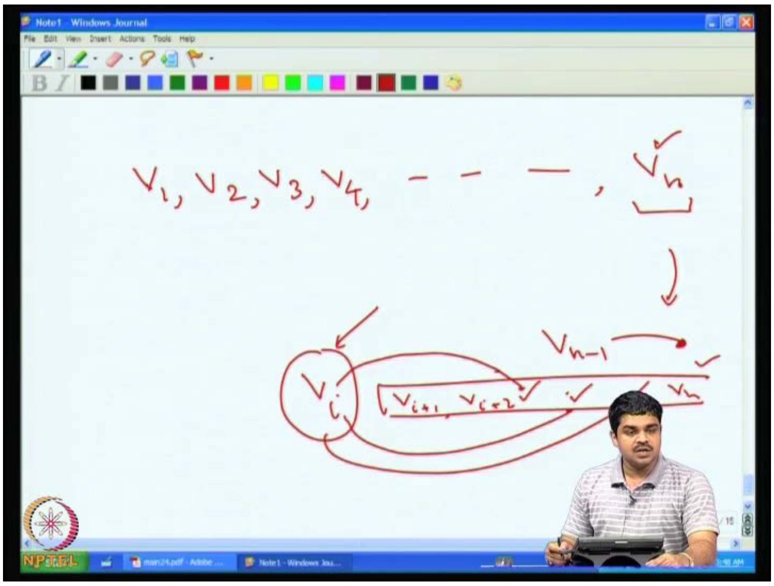

So it is enough to show that the number of colours used is We show a colouring with the number of colours used equal to the number of vertices in some clique; that is what we are going to do.

We will use the notion of perfect elimination ordering to do this thing. So, we will consider a perfect elimination ordering of the chordal graph say V 1, V 2, V 3, V 4, so on up to V n and then we first colour V n rather than colouring V 1. We will colour first V n, give a colour. So, the same greedy strategy like we did for interval graph, we will use here. In fact, essentially that idea of the interval graph is being extended here.

So, V 1 will be given colour number one say, 1 colour and V n minus 1 will be given the colour, it will see, whether its V n is a neighbour or not, then we will use the second colour. In general, when I am colouring V i after colouring all the higher numbered vertices V i plus 1, V i plus 2 up to V n, suppose I have coloured and we considered V i. What I do is I consider its neighbours, which have already coloured that means neighbours which are higher numbered than this and then see the colours, which are already taken up by its neighbours and then you say a colour which is available, which is the lowest colour available from the pool; that means, if some colour is unused then among the unused colours, we will use this smallest.

If no colour is available among the pool, up to now use, then we have to go for the new colour. That is again same greedy strategy, we are using. The only thing, we have to show is the number of colours used is only equal to the cardinality of some clique. How do you show that?

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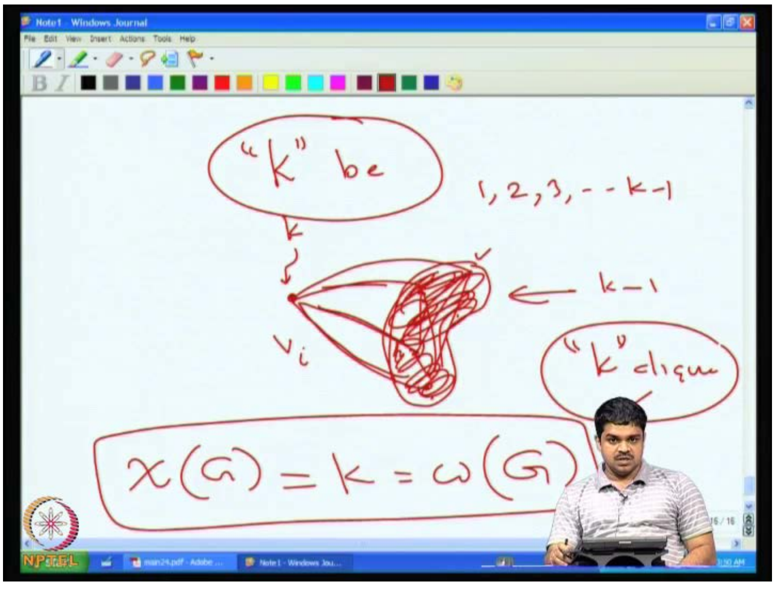

Suppose, k, as usual, k be the highest number of colours used. So, there is a place, there is some time when the k appeared for the first time. Suppose, this is V i, ith vertex used k colour, the colour number k for the first time; that is the largest colour we have used.

Now, we look at its higher numbered neighbours. So, what do you see? Because higher numbered neighbours Because it is a simplicial vertex in that graph, so, you see this is going to be a clique, this is going to be a clique and this is going to be a clique. So, k colours are used. Of course, the k if I had to use the kth colour, that means all the colours below it 1, 2, 3 up to k minus 1, all of them are already used up by the neighbours of this thing. That means exactly k minus 1 higher neighbours are there here. Otherwise, how can they use up all the colours?

Now, these neighbours form a clique and this if you add to it, you will get a k clique. So, you have demonstrated a k clique, where k is the maximum number of colour used, the maximum colour used. Therefore, we see that chromatic number of this graph is equal to k equal to the cardinality of this clique, which essentially means that it has to be the cardinality of the maximum clique because there cannot be a bigger clique and in that case, we could not have coloured with k colours. So, it happens that for the chordal graph, chromatic number is equal to the clique number. That is we want. Therefore, chordal graphs are perfect. Now, the final thing we want to verify is whether the complement of chordal graph is perfect or not; so, the same strategy.

So, we just have to figure out what is the so called co-chordal graph, the complement of a chordal graph. So, given G be a co-chordal graph, now, G bar is a chordal graph. What do we have to show to show that? Again, if it is a co-chordal graph, induced sub graph is not a problem because it will again be a co-chordal graph. Therefore, we just have to prove that for a given chordal, any co-chordal graph, the chromatic number is equal to the clique number. So, that will amount to showing that a chordal graph, the clique cover number is equal to the largest independent set size; that means the independence number, the stability number.

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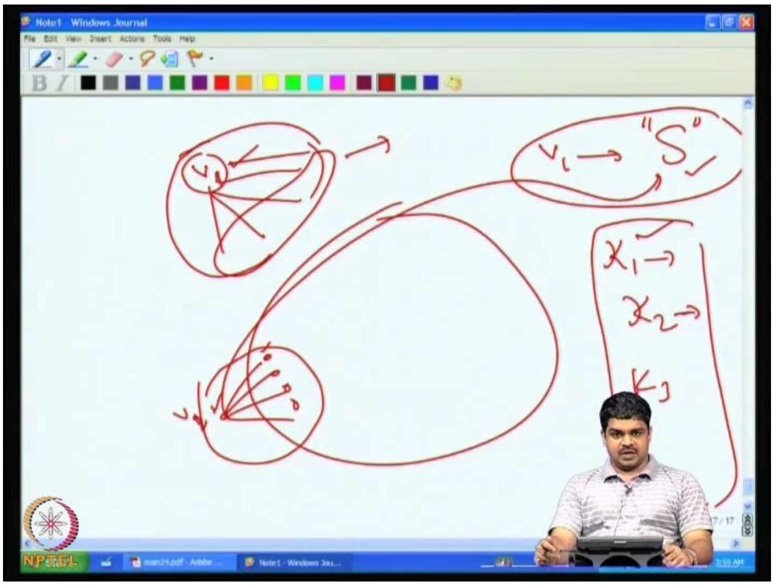

As we did in the case of interval graph, we will do the same technique here. What we do is we start from a simplicial vertex. We take a simplicial vertex V 1 and then we will remove So, we will add V 1, this simplicial vertex to the independent set that we are creating. We will collect some independent create an independent set by adding vertices 1 by 1 to it and then alongside we will collection cliques and by the end of this procedure, we will show that what the cliques we have collected is forming a clique cover number and the vertices we have collected is an independent set and the cardinality is same. So, we have a clique cover, whose cardinality, the number of cliques in it is equal to the cardinality of the independent set size. So, some independent set size; that is enough.

Therefore, the clique cover number has to be strictly be greater than or equal to the maximum independent set size. So, this has to be a maximum independent set and that has to be minimum clique cover. That will prove the same thing as we are repeating the argument that we have done for the last, the intervals graphs or many of the earlier cases here.

We have The technique is slightly generalized from the interval graph. We are just picking up a simplicial vertex first and adding it to this thing and then what we do, we look at because it is a simplicial vertex, its neighbourhood forms a clique. We remove all the neighbours along with that vertex; that together will form this, its neighbours and itself; that simplicial vertex will form a clique in the first clique in the clique cover number.

Now, in the remaining graphs, it is again a chordal graph. We again have a simplicial vertex. We can collect the vertex and its neighbours and add the second clique to the clique cover we are creating and this vertex itself can go to the independent set, we are going to create. See, this new vertex I am adding and this V 1 and V 2 will not be adjacent because you have already removed all the neighbours of V 1. So, this V 2 will not be a neighbour of that. Therefore, V 1 and V 2 are whatever I am going to add to the set S is going to be independent. Every time I am adding a vertex to that, but I am removing all its neighbours. So, what I collect there is going to be an independent set; that is very clear.

Now, on the other hand, similarly, every time I am collecting the neighbourhood of my vertex is a clique and the along with the vertex, it is forming a clique and by the time, I finish up all the vertices, I have coloured all the vertices because I am removing from the graph only the vertices, which go along, go inside some clique because I am always when I am removing, I am making sure that I am removing a clique. Therefore, I have this k 1, k 2, k 3 etcetera covering the entire vertex set of the graph. So, it is indeed a clique cover of the graph and the number of cliques that we have collected here is indeed equal to the number of vertices, I have collected in the independent set.

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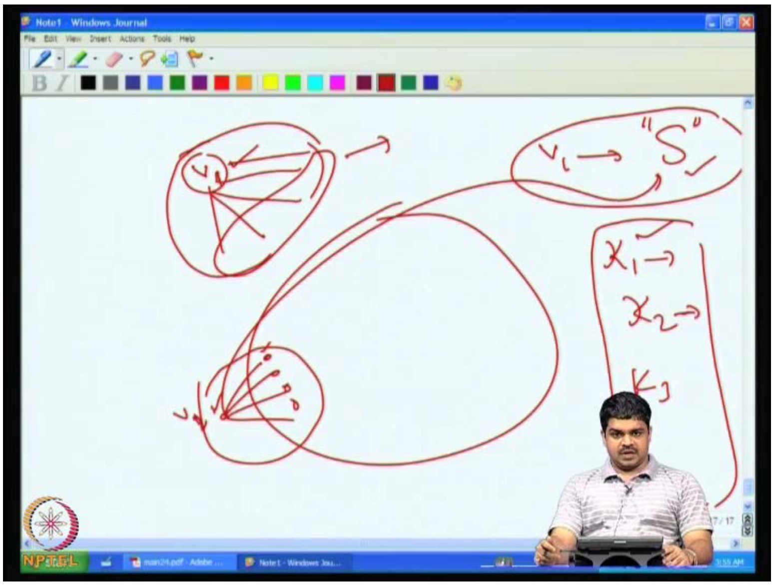

So, this S and this is equal, and then which means that this clique cover, this is the clique cover number, is indeed the minimum because it is equaling the cardinality of an independent set and this has to be a maximum independent set because it is equaling the cardinality of some clique cover. So, they are both minimum clique cover and the maximum independent set.

So, we have shown that in the chordal graph, the maximum independent set size is indeed the clique cover number, which essentially means that in the co-chordal graph, the clique number, the maximum clique cardinality size is equal to the chromatic number. So, co-chordal graphs are also perfect graphs. So, we have considered several sub classes of perfect graphs now, means several well-known graph classes and their complements and we showed that, not only the graph class, but also the components class is also perfect.

So, this essentially is a common theme, common property for all perfect graphs. That means, if a graph is perfect, its complement is also perfect. We will give a proof for this theorem in next class and we will explain it a little further. How we can understand it better in a general point of view? How we discussed all these things, this many several this many classes of perfect graph is to illustrate that the several important classes of graphs which are indeed perfect. So, we will continue with perfect graphs in the next class. Thank you.