Graph Theory

Prof. L. Sunil Chandran
Computer Science and Automation
Indian Institute of Science, Bangalore

Module No. # 03
Lecture No. # 21
Chromatic Polynomial, k - Critical Graphs

So, welcome to the twenty-first lecture of graph theory. So, in the last class, we were considering the list coloring to prove some theorems used regarding list coloring using Allen's combinatorial nullstellensatz.

So to remind what was happening…. So it was about…. So, we were trying to use the combinatorial nullstellensatz on a polynomial called the adjacency polynomial of the graph. What was the adjacency polynomial?

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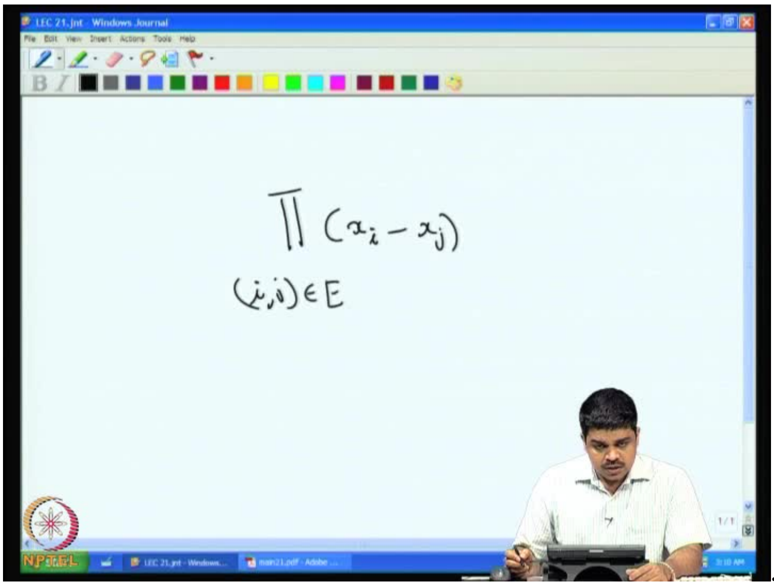

So you had… we had multiplied terms of the short x i minus x j for each edge, so that the product for i j was an edge here, right? This was the adjacency polynomial, and what was combinatorial nullstellensatz? It told, so it told, this is what, the… if f be a polynomial over a field F in the variables x 1, x 2, up to x n.

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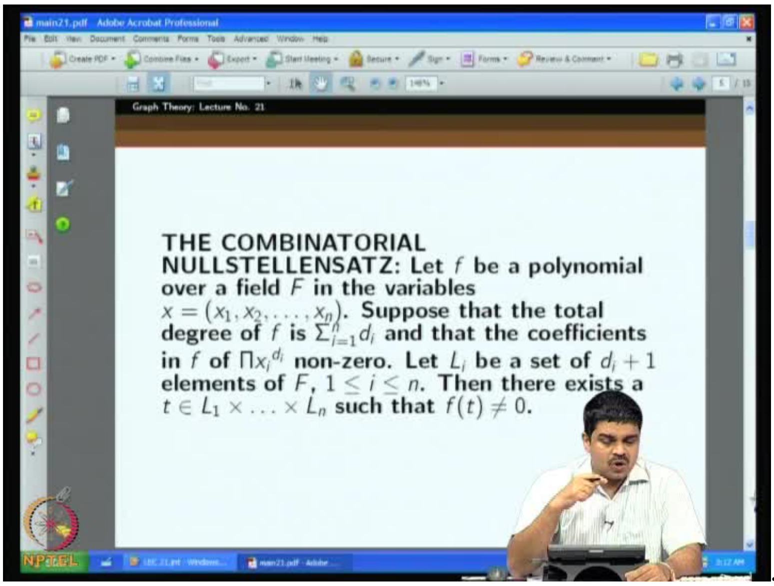

Suppose that the total degree of f is sigma d i. That means, the degree is the total is d i; that means, it is not guaranteed that each x i is of degree of d i, but the together the total degree of each term will be at most sigma of d i and one term will be there with that degree. And also, also there is term with x i; each x i having coefficient sorry degree d i x i at raised to d i together they have to form this thing, right? So one term will be there. That particular term has coefficient non-zero; non-zero coefficient.

In that case, suppose we are given a list l i for each variable, so l i for x i of length of cardinality d i plus 1, then we can get a value for x i from l i, such that, together if you assign those values to each of the x i, then the polynomial will evaluate a non-zero value. This is by somewhat modifying the theorem about the non-zero roots of sorry… so a corresponding theorem about polynomials.

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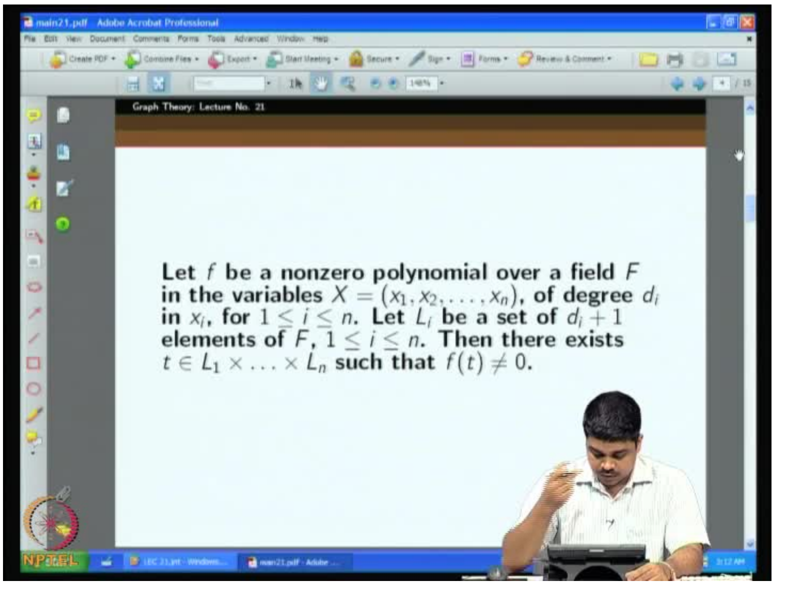

So there, in fact there are some differences. Here there is slight modification for the purpose of getting proofs, more proofs. So we are going to use it to prove that there exists some this list coloring, if the lists are of certain cardinalities.

So given a graph, we will take that if the lists are of certain size, that means, for each vertex, the corresponding list is of certain size, then they will be a list coloring. Typically we will, we are going to give the size of the list in terms of the orientations. We will say that here is an orientation of the edges, that means we giving direction to the edges of the graph and then we look at the out degree. So, what we look at, is the out degree sequence, for each vertex, its out degree plus 1 is the cardinality of the list associated it that vertex, then we will say that, we want to say that, there is a list coloring of it, but then, not in every case we can do this thing, there are some special cases where this happens, what kind of orientations are required.

So one may wonder – wow, what is the connection between this orientation and saying that there is a list coloring from the lists associated with that, right? So, because the cardinality of the lists we are getting by looking at the out degree with respect to a certain orientation. What is the relation between that orientation and this list coloring? So the relation is that when we say that there exists a coloring from this lists, we are going to use the combinatorial nullstellensatz on the adjacency polynomial, and you see, the adjacency polynomial evaluates to a non-zero value, then it corresponds to a proper coloring of the graph, and then, of case the solutions for each variable x i will correspond to a color of the corresponding vertex, and of case, we are picking up this value from the corresponding list. Therefore, it will be coloring from the list. So that is the connection; that is a connection to the lists coloring and a solution which evaluates a non-zero value on this polynomial.

Now, we just want to show that if certain, if a graph has certain property, then the corresponding polynomial will evaluate to a non-zero value. So we can use the combinatorial nullstellensatz to show that there exists some situation to, situations in which it evaluates to a non-zero value.

So, here we can see that it will happen if the total degree is the sigma d i. So with respect to some orientation, we will have to come up with this number d i. So you see that will correspond to the out degrees of some orientation. So, and of course, some numbers if d i can be produced and if our polynomial has the total degree almost, always, at most this much, a total degree is this much, and there is a term with each x i having corresponding d i degree and non-zero coefficient. That is all we need to show, that there exist lists coloring then, right? The connection to the orientation is that the polynomial, the adjacency polynomial, can be expressed as a sum of several terms based on the orientations. So this was what we saw.

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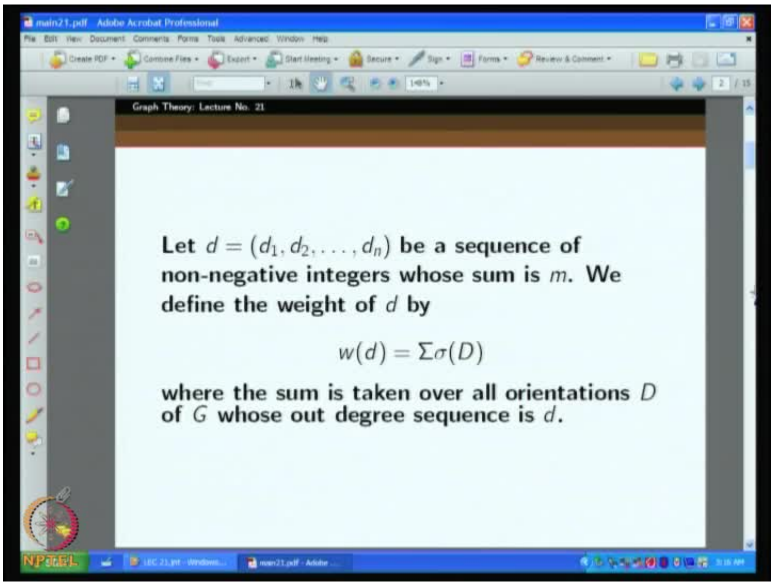

So we saw that when we multiply them, each monomial will correspond to an orientation. It has a sign. So the sign may be negative or positive depending on which term is selected. We discussed all those things. When for a given orientation there is a sign, and then a given a degree sequence - out degree sequence - we can collect all the orientations with that out degree sequence and sum up the signs; that will be the sign of that monomial finally. We can say that, that is a sign of the degree, weight of the degree sequence that need not be a sign (( )); it can be a bigger number than 1 also.

So but, typically what it will become 0, only if there are equal number of negative and positive terms, because negative 1 and positive, because the coefficient is either 1 or minus; if they, if they have to cancel, then the number of negative and positive terms has to be equal, right?

So we can see that for a, suppose you get an orientation. So then, and then, you look at the degree - out degree - sequence of that, say let it be d 1, d 2, d 3, so right, d n. And when will this monomial corresponding to this disappear? So only if… so the coefficients add up to 0; that means, the negative terms and positive terms happens to be equal in number. So suppose the number of such orientations with degree sequence d 1, d 2, d n, even degree sequence - even d 1, d 2, d n was odd in number, then that will never cancel of right, because when you add them up, you will get an odd number, then 0 is not an odd number. You will… therefore, it will not cancel off.

So if you take a sequence - degree sequence - so if you take an orientation such that the degree sequence is d 1, d 2, d 3, d n, and the number of orientations, which has the same out degree sequence is an odd number, then we know that the corresponding term will not cancel. So we can use this in combinatorial nullstellensatz. So here it is asking for then the coefficient of x i raised to d i to be a non-zero, that will happen in that case, right, for the particular term.

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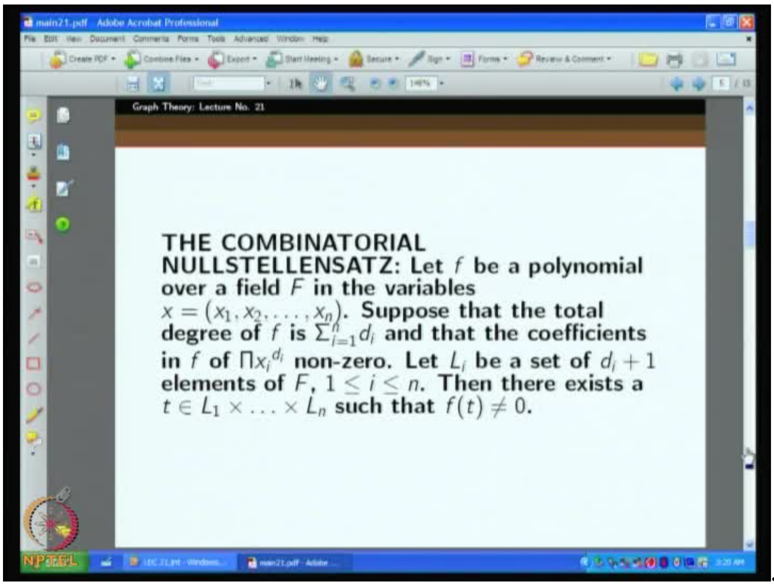

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And the other condition, that means the total degree sigma d i will be true, because every term corresponds to some orientation. If you take the total out degree of the orientation, because the out degree of each vertex will be the corresponding x i to the power d i; ith vertex will have x i variable and the corresponding out degree will be its power. So when you sum up, that sum is going to be always the number of edges on the graph. Therefore, it is not going to change. Therefore, that will remain to be… though these values d i is may be different there, but the total sum will remain same, because it is actually the number of edges.

So we can be sure that this condition for this combinatorial nullstellensatz will be satisfied in that case. And then we will be able to… if the number of orientations corresponding to this degree sequence happens to be an odd number, then definitely we are sure that we can get values from the corresponding lists, as it is l 1 one cross l n, right? That is the… from… we can get a value for x 1 from l 1, x 2 from l 2, x 3 from l 3 and so on, such that the polynomial evaluates a non-zero; that means, it is a valid coloring, right? So this is way we use it.

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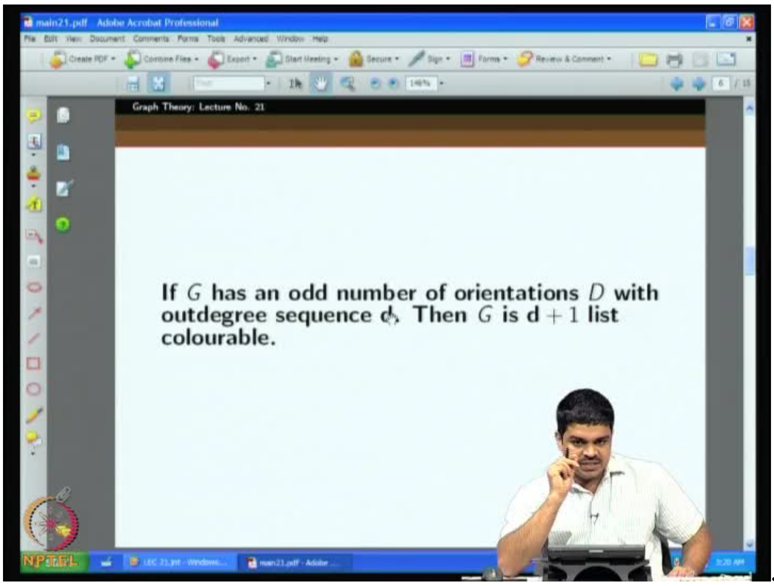

Therefore, we can state this statement - if G has an odd number of orientations D, with out degree sequence d then G is d plus 1 list colorable. This black one d essentially is the is a short form for the degree sequence. It contains the degree of each vertex d 1, d 2, d 3 like that.

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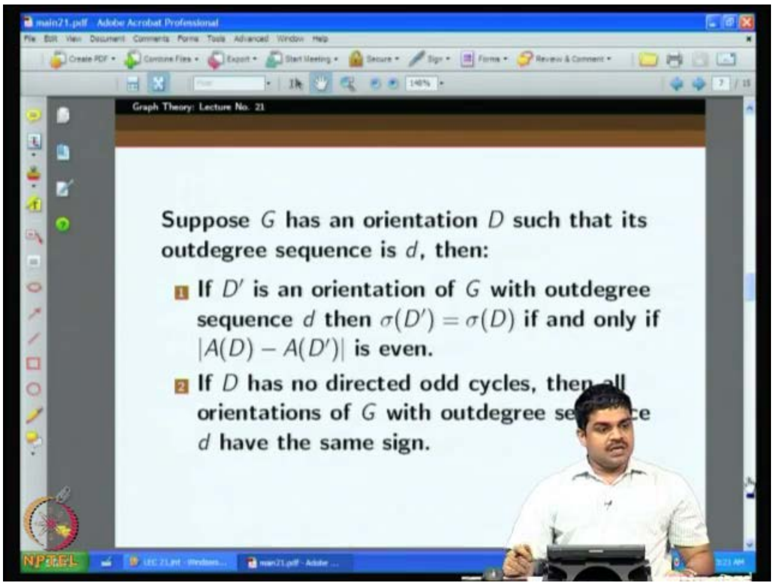

Now we will, but then, so this is asking for a little complicated condition. It says that first of all, if you are given a degree sequence and you have to count the number of orientations with this, with that degree sequence. So if it is an odd number, then the there is a list coloring, there is a coloring; it is possible to color from the lists, if each list l i has cardinality d i plus 1 is what I told.

But again, it does not look so appealing, but so we can, we can get a slightly better, interesting, more interesting statement here, this way. So suppose G has an orientation, so suppose G have an orientation D, such that there are no directed odd cycle in it. Suppose you can give direction to the graph in such a way that there are no directed odd cycle in it. For instance, if you take a bipartite graph, you can always do that, right? Because there is not odd cycle set at all; how can directed odd cycle come, right? So, it need not be a bipartite graph. In some cases, you can, you can make sure that they can be odd cycles, but then, we can make sure that there are no directed odd cycles. So then, in that case, if we can do that, we are claiming that it is lists colorable from the corresponding lists.

So from lists, if the lists satisfy the cardinality condition, namely, each list l i, ith list, the list corresponding to the ith vertex has cardinality at most d i plus 1, the d i is the out degree of that vertex with respect to this orientation, where we have claimed that there are no odd cycles, so directed odd cycles; how do I prove that? So it is simple. This is the way it is.

So you see, when you take this orientation, what we have to do is, as we can see from the combinatorial nullstellensatz, the total degree condition is not a problem, because anyway it is going to sigma d i and d i for all the terms. So, it is not a big… not a issue. So we can, what we are more bothered about is to keep the coefficient of the term x i raised to d i, x 1 raised to d 1 into x 2 raised to d 2 into x 3 raised to d 3 into this coefficient should be non-zero; that means they should not cancel each other. So how does it cancel? Because you have to collect all the orientations with that particular out degree sequence, namely d 1, d 2, d 3, d n, and some of the coefficients, and you have to see that this is non-zero. So how do we do that?

So first the trick is this thing. We will argue that all the orientations with the same out degree sequence, will have the same sign; if they have the same sign, how can they cancel, right? They will not cancel; they will simply add up; they will give a big number; the actually the either plus the number of orientations or minus the number of orientations, because all of them will have the same sign; this is what we are going to argue.

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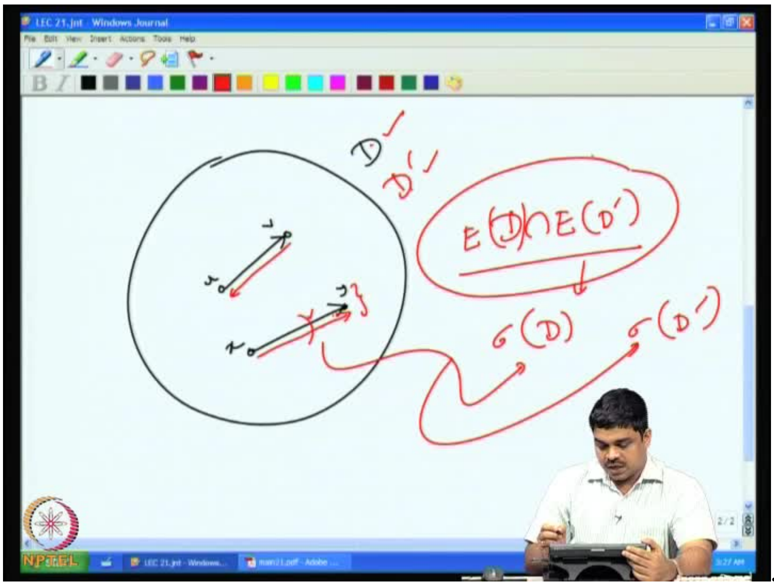

So you take two orientations D and D dash, two orientations D and D dash, with the same out degrees sequence. Now, so for instance, this can be for with respect to D, this vertex u, the u v the orientation may be like this; with respect to, this is with respect to D, right? D, black one corresponds to D. So with respect to D dash, the orientation may be, it may be like this for this vertex, right? Or it can also possible that for in some other case. So the both… say for instance, another edge x y may be oriented in the same way in both D and D dash, right?

So, you see, if you remember the sign of D, an orientation was the product of all the signs, which we got from the edges, right? Each edge will give a certain sign and then we are multiplying the signs. So in that when you consider the sign for this and this, in the product terms, there will be two products terms - a product term for this D and then there will be a product term for this D, right?

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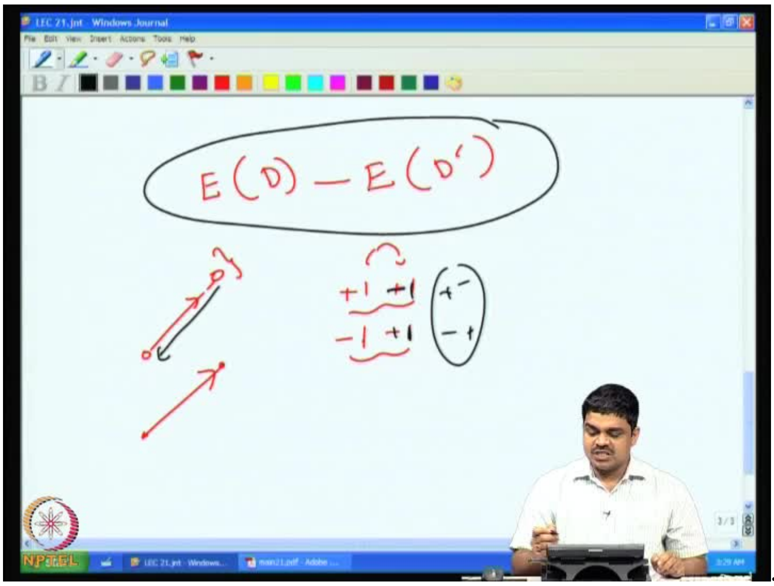

So these edges, which have the same orientation; that means, the intersection of D and…. So if you consider D intersection E, E of D dash, right, the edges in it, namely the edges of this type, they will contributes the same to, in the same way to both the product corresponding to, that means sigma of D and sigma of D dash, right? These kind of edges will contribute the same, right? So these things will not make a difference.

While these edges - one of them will contribute plus, one of them will give a plus to its term, its product, and other will given minus to its product. So, they will, each time such an edge comes, they will try to make the sign go in two different directions, right? So let us collect all the edges of the graph of G, apart from those case, who got the same orientation; that means, we are considering E of D minus E of D dash, right? Essentially that edge is there; the edges were one got this direction; the other got this direction, right? So the directions were different, right?

So we can write down the contribution. So when you consider the first thing, so 1 plus 1 minus 1 will be the contribution. The next one, see if here, if another edge if you take what will happen? In the first time if it is plus the contribution, the other will be minus; then together they will come to the same, right? Plus this was plus and minus, the second time it will, it will turn out to be same, because plus into plus, so big that is plus only; so minus into minus into minus also became plus.

So, on the other hand, if this was minus what will happen? This was minus 1 and this was, then this will be plus 1. So then, also you see plus into minus became negative; minus into plus also negative. So in the second time always it will be, it will come to the same, the product will come to the same sign. Now the third time, again it will become different signs, because if it is plus here, this will be minus here; if it is minus here, this will be plus here.

Therefore, the third time, we will get different signs. Fourth time, again, we will get the same sign. And fifth time, again different signs. So every even number of edges are considered, we will get same sign and every odd number of edges are considered we will get an even sign. It means that…. So essentially how many edges are there in this thing - in this, in this E D minus E D dash, that is all what matters, right?

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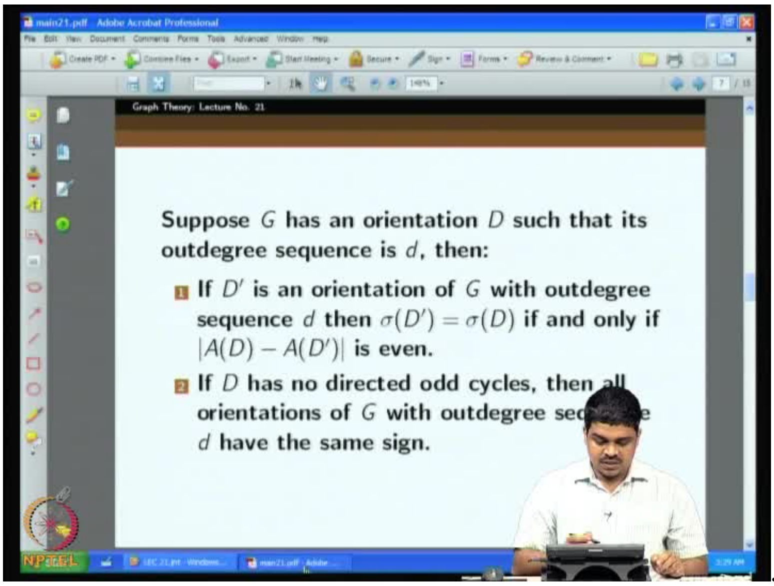

So we can say that if D dash is an orientation of G with out degree sequence D, D is the out degree sequence of this capital D. So then, the signs will be same, if and only if, this A is used for the edge set of D not in E, because that is the directed graph A of D minus A of D dash is even, if D has no direction.

Now the next thing is a condition which ensures that this difference will be even. So this is, if D has no directed odd cycles, then all the orientations of G with out degree sequence D have the same sign. How, why is it so?

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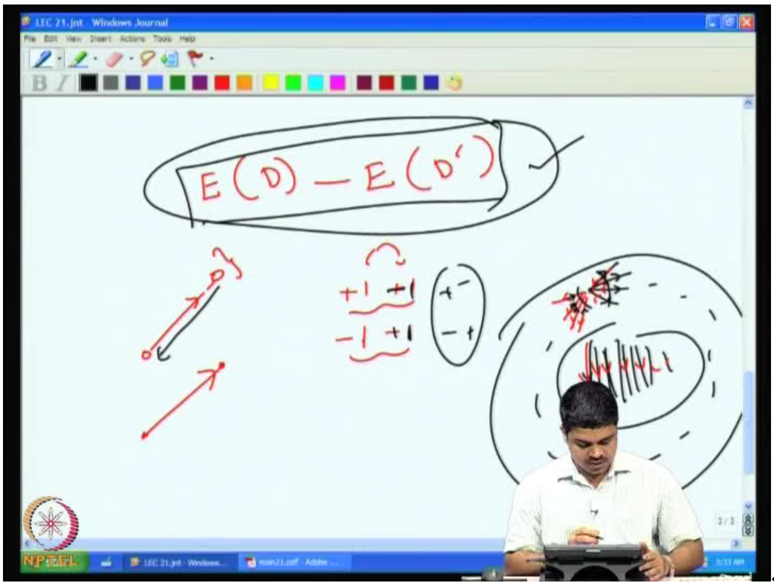

Because, see if you consider this thing, suppose we have removed the intersection part and outer part remains, which is not this was intersection part, that means, these were the edges, which have same orientation; they were of the same orientation, right? Black and red were same orientation. But the other the remaining things, if you take a vertex, you see that the out degree, if you look, the degree sequence is same. So if it look at the out degree with respect to the black, that means, with D, so the same out degree will be, should be there with respect to red, but then there the out these all edges are incoming with respect to the red, right? Therefore, these should be the outgoing. The remaining should be the outgoing, where they are incoming with respect to the black one, right?

Therefore, these numbers have to be same, if the degree sequence should be same for both red and black D and D dash; this vertex should get this, get the same out degree with respect to D as well as D dash; then this red outgoing edges and the black outgoing edges, have to be same and they are the different one; so they are disjoint.

So essentially, this has to be an even degree, not only that the in degree has to be equal to out degree. With respect to D, this red outgoing degree will be the incoming degree. So in degree and out degree has to be same for with respect to the black one, right? That means, with respect to orientation D, which means that we can decompose the edge set of this E D minus E D dash - that means this edge set here - with respect to the black set, we can decompose into cycles, because it the out degree is equal to in degree for each vertex. So we will be able to get a collection of directed cycles, which will the union of which will be, the disjoint union of which will be the total edge set, but then there are no even cycles. So each cycle has to contribute an even number, right? So together this has to be an even number.

So, if there are no directed odd cycles, the E of D minus E of D dash has to be a even number, and therefore, the sign of both E D and D dash has to be same. So this is true for any orientation. All the orientation with the same out degree, will have the same sign. Therefore, they will simply add up; they will not cancel with each other. So that particular term, namely x 1 raised to d 1 into x 2 raised to d 2 into x x n raised to d n this; so where d 1, d 2, d n is the out degree sequence of this orientation D, so will be non-zero; will be non-zero. That is that is…

Therefore, we can apply the combinatorial nullstellensatz, nullstellensatz sets, and say that there exists an assignment of values for x 1, x 2, x 3, up to x n from the lists l 1, l 2, l n, such that the polynomial evaluates a non-zero value. That means, these assignments corresponds to a proper coloring of the graph; that means, there is a list coloring from the given list, and here, the lists are of cardinality at most d i plus l; l i is of cardinality d i plus 1, d i being the out degree sequence, d i being the out degree of the ith vertex with respect to the orientation D; this is what we get.

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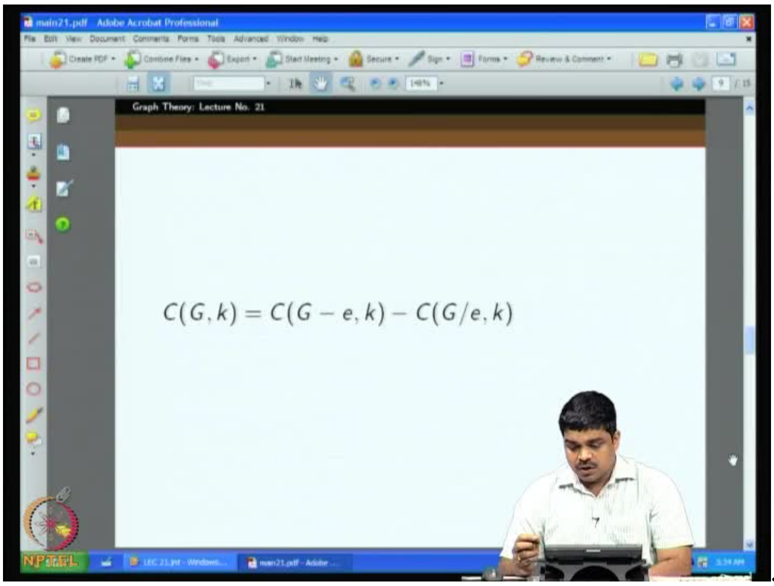

So this is one example of how to use the combinatorial nullstellensatz. So we will leave adjacency polynomial here, and then, we will look at another interesting polynomial regarding the color, coloring, the vertex coloring. So here is a parameter that we design, given a graph G, C of (G, k) C of (G, k) will represent the number of k colorings that G has. So, for instance, if G is not k colorable, if it is the chromatic number of G is greater than k, then it will be 0. So this is defined for non-negative integers C of (G,k), right?

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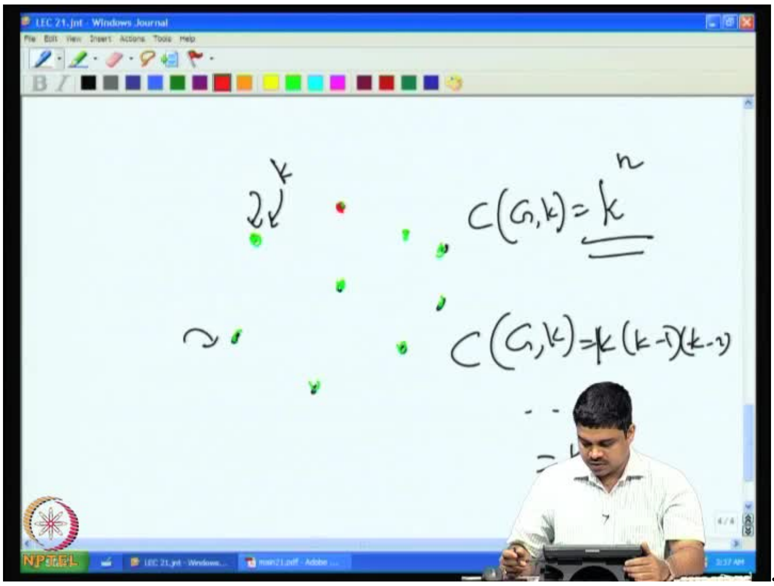

So, it is now, for instance, if there is a loop in the graph, that will be 0 by definition, because if there is a loop, it is not possible to proper color; the both end points are same, so, they cannot get two different colors. And now, C of (G, k) for also, you can see that C of (G, k) is…. See if the graph is just a collection of vertices, no edges in it, then every vertex can get, every vertex can get any of the colors, k colors, that if they are how many. Therefore, and then, k raised to n will be the C of (G, k) here, right? If n is the number of this thing,.

If it is a complete graph, what will be C of (G, k)? Because the first vertex can be colored in k ways, and then, once it is colored using a color, only k minus 1 possibilities are there for next one, and then k minus 2 possibilities for the third one, and so on. So the final thing will have only one possibility. So this is k factorial, for a complete graph. So this is a essentially how many ways we can color - k color - a graph the given graph G? This is the, this number is C of (G, k).

See one should understand that this k coloring means, see when do I say two k colorings are different. So if any vertex, if we fix a vertex, if it is a labeled vertices in two colorings, if they get, even at least one vertex get different colors, then they have to be consider different.

So you cannot say that the, I am just calling red by the name green; such arguments are not allowed. If you… so for instance, if in one coloring this is green, and suppose these are all green in that, and then in another coloring, suppose these are all green, but this one becomes red, we have to consider it as different colors. That is the way that we count, right? So exactly each vertex, when we say that two colorings are same, each vertex should have the same color, then only we will say that they are the same; otherwise, we will have to count each of them, right? So that is the definition of C G k.

And now the interesting polynomial, there is an interesting polynomial called chromatic polynomial. So P, so this polynomial is P G x you can say, a polynomial in x.

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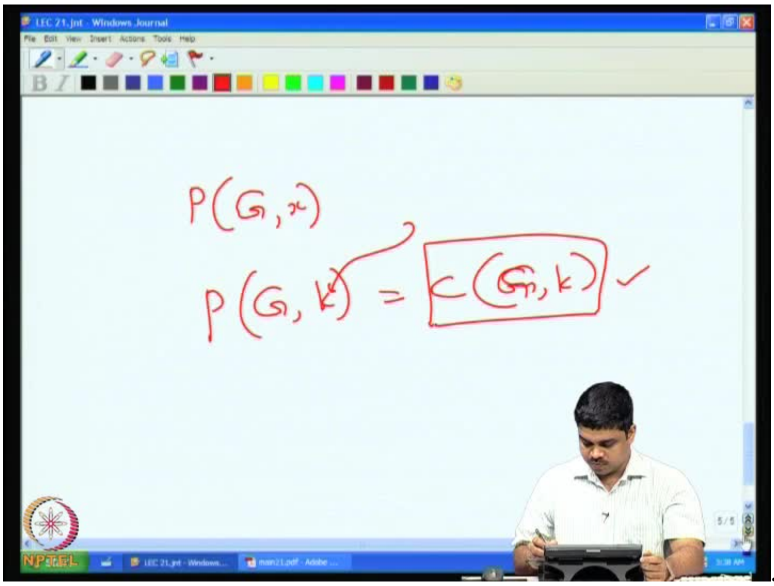

So, if you, if you substitute for x, say k, then what we get is C G of k. This is the interesting fact about this polynomial. You can, you can put the value of k in the polynomial, x equal to k in the polynomial, and C of G k, this number will come out. So, for instance, if it is a chromatic number was t and any number below t should give you 0; while so above the chromatic number, it should get the how many times we can color it, right?

So, how many, how many ways you can color? Sorry. How many ways you can color the graph? So this is, this kind of polynomial exists. So this is what we are going to prove now. So the… but then, to prove this thing, we need to observe a certain fact about this number C (G,k) namely. So I am telling this is the fact we need. So this is C G k can be recursively expressed as C (G,k) equal to C minus (G minus e, k) minus C (G e bar,k).

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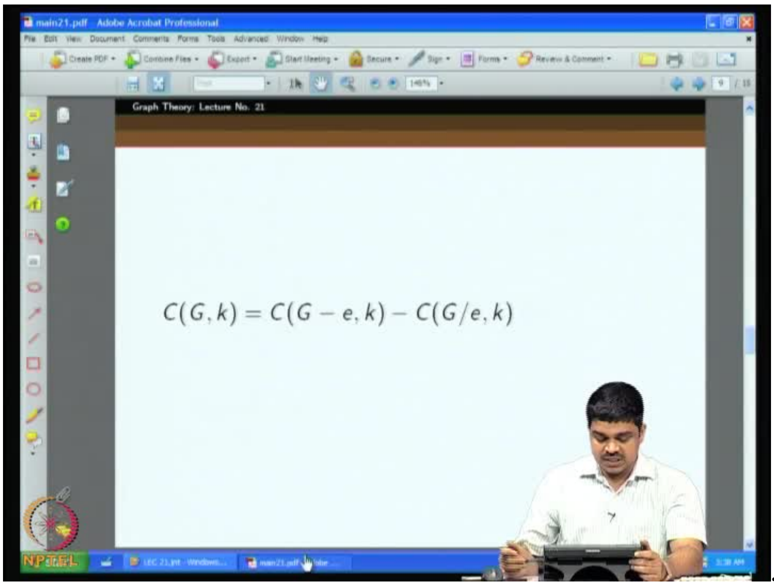

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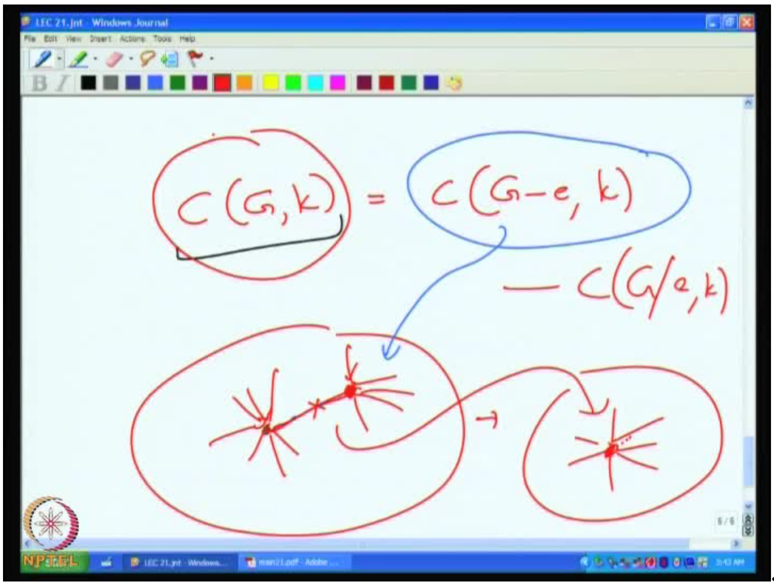

So which is essentially, means that, see to calculate the C of G,… how many times you can color a graph? k colors. This is essentially equal to C of G bar e… so this is… sorry.

I will write it as C of G minus e; that means, you minus the e k minus C of g bar e; this is the contraction operation. So what does it mean? So then sorry (( )). So contraction is written like this. So contract it. Now this one, suppose this is the graph, now you take an edge e in the graph, this is an edge e in the graph. Now you can ask so the… what is the… how many ways you can color this? So there are two types of coloring C G k, right?

You can, you can take each coloring of G, using k colors and you can categorize ita name like this; so, sorry not like that. We can consider, you can first try to remove this edge. For instance, this edge can be removed. So this edge is removed and now you consider all the k colorings of C G minus e k that, may after removing these, how many? These colorings can be considered in two different ways. One - is the colorings, among them, among these, the colorings of G minus e, some colorings will give the same color to both end points. So this and this will get the same color.

Now, you see, those colors cannot be converted to coloring of G k, because once you put this edge, that will not correspond to coloring of this thing, but then the other types of coloring, namely these two end points of different colors, can be converted to coloring of G minus k, because when you put this edge there is no problem.

In fact, essentially any of the colorings of G minus e k, which gets two different colors on these vertices correspond to a coloring of G also; but similarly, if G has any coloring, you can remove this edge and that will correspond to k coloring of G minus e. So, essentially they are same. So the number, the, actually the, the number of k colorings of the G is the number of k colorings of G minus e, where the colors at this end points of e happens to be different. So then, how do I get this value from… if i just know C G minus C k? What is to be minused? Essentially the number of k colorings, which have colors, same colors, on the end points of e should be minused. How will I get that? That is like this, for instance, if we had contracted these things, it would happen like this; these end points will become one vertex, right?

Now, you know, if we consider the coloring of this, that is essentially, so that coloring could have been the coloring of this also, because , that color can be given to both these vertices and from the neighbors it will be different also. Tterefore, any coloring of G minus e, where these two colors are same, correspond to the coloring of G contracted e, and that color will be given to the contracted vertex and this… the other way if any coloring of this contracted graph, graph after contracting this edge e will correspond to a coloring of Gg minus e.

So essentially the colorings of G , k colorings of G minus e, where both the end points, sorry, this end points of e, which we have, we are removing, get the same color correspond to the k colorings of the number of k colorings of G contracted e. Therefore, essentially, from the number of k colorings of G minus e, we can minus of the number of k colorings of G contracted e, that will give the number of k colorings of G.

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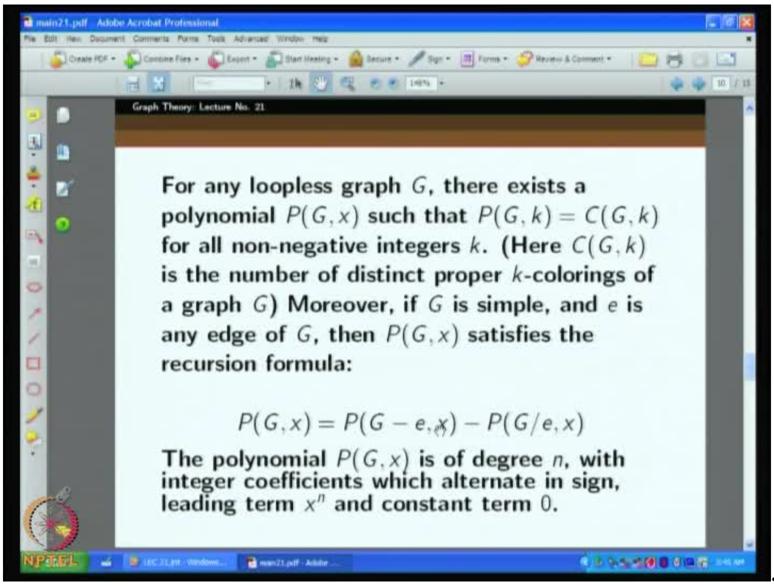

So this we have to remember, and then we will say that - so a polynomial exists, of the type of polynomial we are talking about, where you substitute k, non-negative integer k, then the number of k colorings of G is the value to with the polynomial evaluates.

So here is the formal statement: For any loopless graph G, so loop, there exists polynomial P (G,x) such that P (G,k) equal to C of (G,k) for all negative, non negative integers k. So moreover, if G simple…. So if we are, if we are considering simple graphs, so of case, so this C of (G,k) is defined for all non simple graphs also, but it does not matter, because essentially the number of colorings is the number of colorings of the underlining simple graph.

So, moreover if G is simple, and e is any edge of G, then P G x satisfies the recursion formula P G x equal to P G minus e x minus P G contracted (e,x). Then it is the polynomial of G minus e x minus the polynomial of G contracted e x.

So, you should understand that, this corresponds to the recursion formula we just showed for C (G, k). And moreover this polynomial has some properties, that polynomial is of degree n, n being number of vertices, with integer coefficients which alternated signs. That means, first will be positive, next will be negative, and next positive, negative positive, negative. So alternate in sign and the leading term will be x raised to n; the constant term will be 0; they would not be any constant term - non-zero constant term, right? This is the, this kind of a polynomial will exist and this polynomial will be called the chromatic polynomial; this is called chromatic polynomial.

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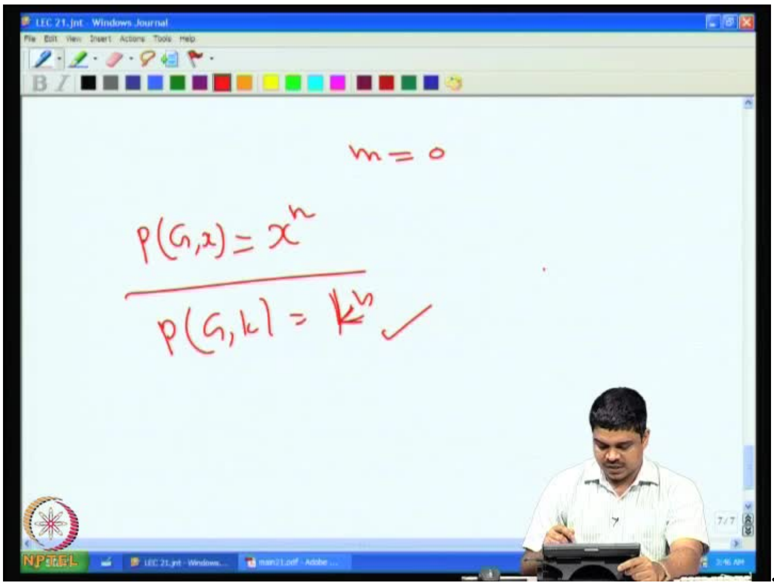

So, how do I, how do we show this thing? So the proof is simple enough. So, essentially you start an induction on the number of edges. So, if there are no edges in the graph, so that means, m equal to 0, let us say, then as we saw, we can take the polynomial equal to x raise to n; P of (G, x) can be taken as x raised to n.

So you put k. So P G k will be equal to k raised to n. Then as we know, if it is an empty graph, empty graph means there are no edges, then, of case, k raised to n is a number of colorings, right? Any vertex can take any color, k possibilities for coloring of vertex k raised to n. So this is, and then all the other conditions are satisfied. The first, this starts with x raised to n known non-zero term, and we can say alternate negative and positive terms, the coefficients are 0, in fact. So, then, so all those conditions are also satisfied, right?

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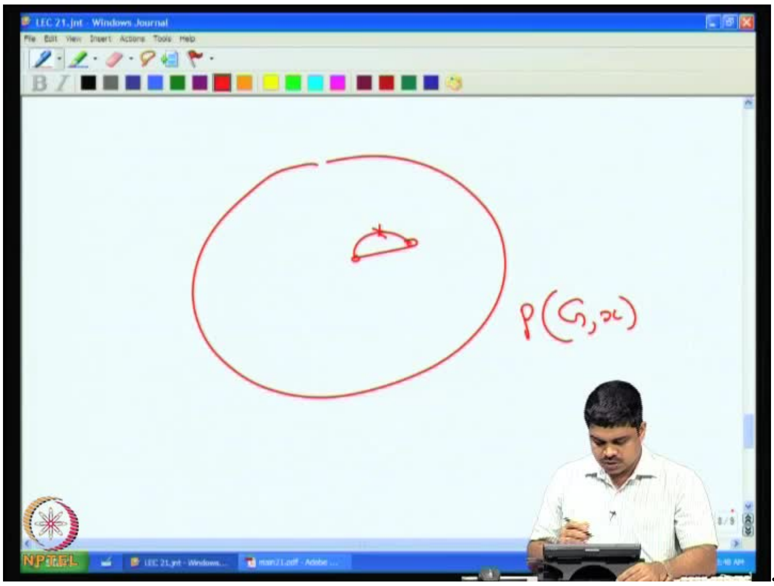

Now we will take a graph with one edge. First we ask - is it a non-simple graph? So in that case, you can always, sorry. So essentially, so essentially we remove in the right, so we can remove an edge. So, if it so happens that the, what we are removing is a… see after removing that edge, it is only a multiple edge that we have removed, then obviously that is true. There is nothing to prove here, because by induction we know the number of colorings C of P of G comma, the same polynomial for underlining graph will be, because there is no change, in fact only the extra edge we are removed. So, this will be given, the same polynomial of the remaining graph. And, if you, it is very clear that the number of coloring remains same. So you can use the same polynomial.

Now all the properties also will be same, because the other properties are about the edge; it should only satisfy for simple graphs, right? So need we not worry about it.

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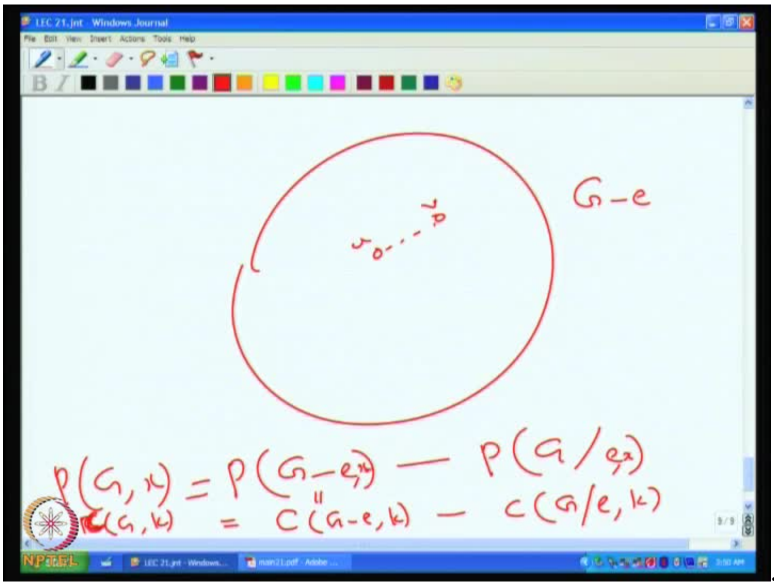

Now suppose if you are dealing with simple graph, so what we do is, we see that you removed a edge, Right? So, now this u v edge we have removed. Now we have, we see that there are there is this graph P G minus G minus e has come. Now we will consider the polynomial of G minus e and also the polynomial of G contracted e. So of case comma x comma x, so we minus, this minus this we take, and this will be our polynomial of P of (G,x).

If we take this, then you see that if you put k for x right non-negative integer k, then this will evaluate to C of G minus (e,k) and this will evaluate to C of G contracted (e,) sorry this contraction is written wrongly. So G of (e,k). So, and then, we already seen that this is essentially P of G C of (G,k). So we see that this polynomial also will evaluate, because of the recursion for C of (G,k). So the same recursion formula was written for P of this, this thing. Now if you substitute, we will get k, we will get the value for the number of k colorings of G from P also, right?

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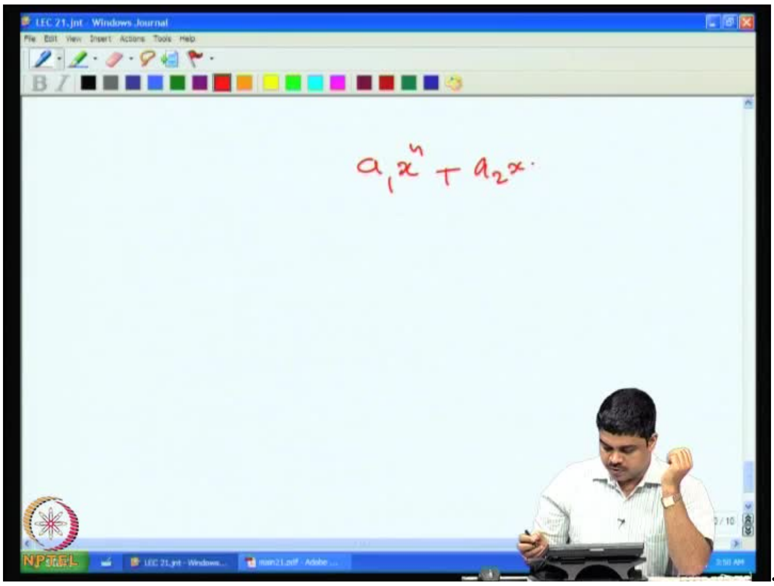

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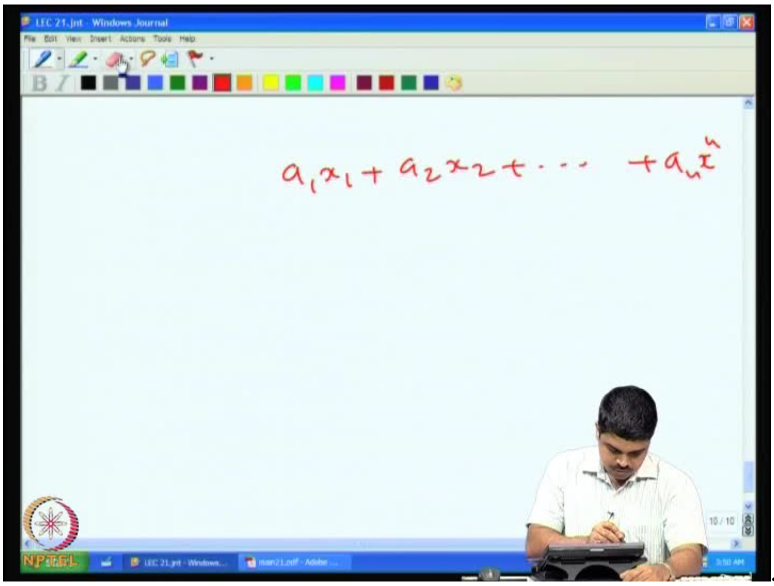

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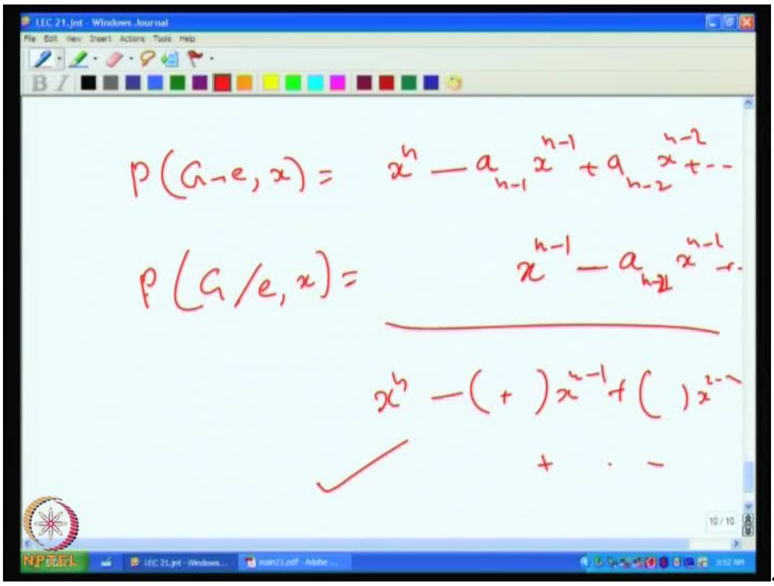

Now, the other things are easy to verify, because if this and this by induction, because they are… they have one edge less. We can use this induction. They do have the… they do have the format, say a 1 x raised to n right, plus a 2 x. So, maybe, we can, we can it write it like a 1 x 1 plus a 2 x 2 plus. So, finally, a n x raised to n will make n be the polynomial. So this is polynomial for by alternate positive and negative terms. So you can you can also write like, maybe we can write like, so because this first one P of G minus (e,x) can be written as x raised to n minus a n minus 1 x raised to n minus 1 plus a n minus 1 x raised to n minus 2 and so on.

The next one P G contracted e x can be also written. Here see the number of vertices have reduced. Therefore, this will be x raised to n minus 1, starting from x raised to n minus 1. Here it will be minus and a n minus 1, right? Sorry a n minus 2 will go here, right? x raised to n minus 2 and so on.

Now, you see, when you minus this, the second term is minus from this. So here again x raised to n will be first; this will be a negative term minus minus a positive term. So total will be a minus term. So then x raised to n minus 1 will become, this will be a plus term minus and negative term will become a plus term plus something x raised to n minus 2 and so on. They will not be, naturally they will not be any non-zero term also.

Therefore, all the conditions will be satisfied here. So this polynomial, so we have proved it, that if it is a simple graph, we just have to consider the polynomial of G minus e and G contracted e; that means, after contracting e whatever graph is there its polynomial and you minus the polynomial for G minus e sorry polynomial for the contracted graph from the polynomial for G minus e, then we will get the polynomial for… we can take the polynomial, that polynomial, as the polynomial of the original graph G. So this satisfies all the conditions; this will be chromatic; this is the definition of the chromatic polynomial.

This chromatic polynomial is indeed an interesting thing. So you can see that if the chromatic polynomial is non, then the question of whether the graph has a k coloring or not, the chromatic, what is the chromatic number of the graph? This is easy to do, because we just have to substitute value from 0 onwards till it evaluates a non-zero value, right? Because if it evaluates a non-zero value, that means, there exists some coloring, right? The number of k colorings is what it gives. So initially it will give 0, because they would not be zero coloring, they would not be one coloring, they will not be two coloring until the chromatic number; then once a chromatic number comes it will starts evaluating.

So if you know the chromatic polynomial, it is easy to find out the chromatic number. So, but unfortunately, even the chromatic polynomial cannot be found out in polynomial time. So, but of case, to study the chromatic polynomial in itself is interesting. So, there are even interestingly, if you give some values for x other than non-negative integers, like minus 1, that will also give some interesting information about the graph. So, any way, so this is, this is a useful polynomial in the study of vertex coloring of graphs.

So we will, now we will move onto, today what we will consider is a concept called critical graphs, so of case.

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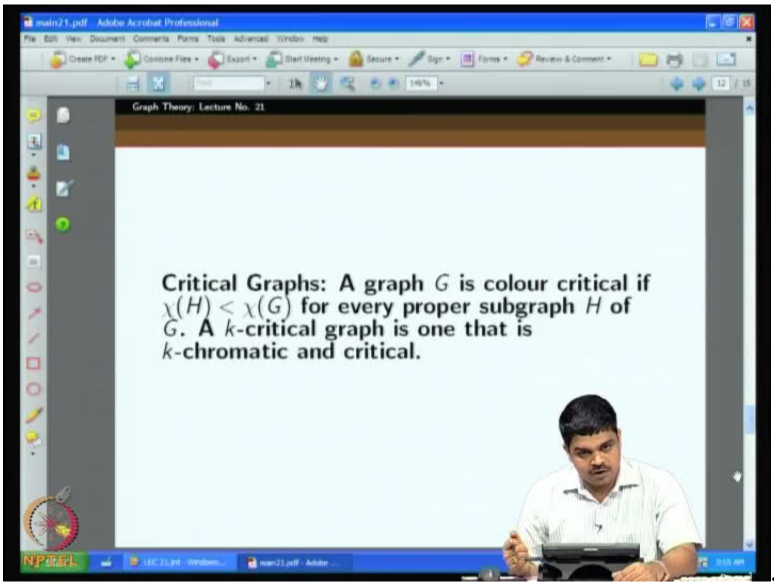

So this is also very useful in the study of coloring problems, because many times when you want to do a proof, you can consider a critical graph and try to do something. What is a critical graph? So if graph G is color critical, if chi of H is less than for chi of G for every proper subgraph H of G, so that means, if you remove one vertex, then the chromatic number should reduce, if you remove one edge the chromatic number should reduce, while this is the critical structure in that sense.

So when do I say its k critical, k color critical, I will just say k critical dropping the word color, critical color, from that. So when I say k critical, it means that the chromatic number of this graph is k, but if you remove even one edge, then the chromatic number will drop one vertex or one edge, any sub graph, any proper sub graph will have lower chromatic number. So that kind of a graph is called color critical graph.

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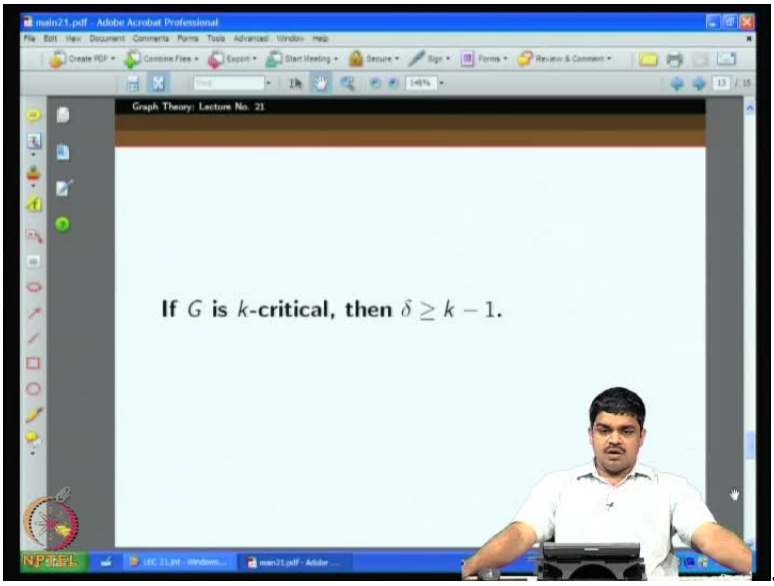

Now, so one if you observation about k critical graph is just minimum degree has to be at least k minus 1, why it is so? Because you take any vertex, and you will remove it, when you remove that vertex, what will happen? So it will suppose the graph is this; this is supposed to be k; you took a vertex here and then you removed it.

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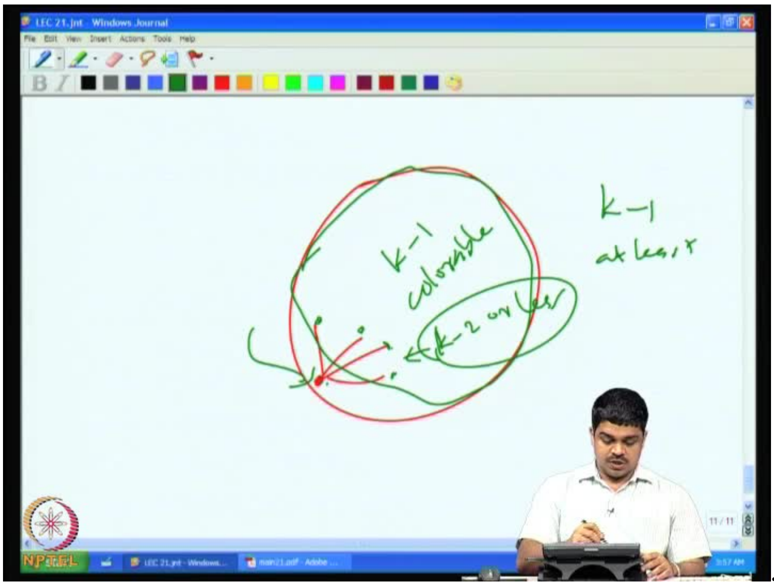

So the remaining graph, namely this graph, right, remaining graph is this graph, sorry the remaining graph, namely this graph right, after removing this is k minus 1 colorable, why? Because it is, it was, the original one was k critical graph; if you remove it, the chromatic number should reduce. Now you color it with the k minus 1 colors.

Now, suppose the degree of this vertex was k minus 2 or less, suppose it was k minus 2 or less, then this k has one color from the k minus 1 colors itself, and then, you can use it to color this, because only neighbors are only k minus 2, then one color is free from the k minus 1 colors available. So this entire thing can be k minus 1 colored, but we know that it requires k colors. So which means that every vertex should have degree k minus 1 at least, right? So that it cannot, its k minus 1 coloring of the remaining graph cannot be extended to it, right? So this is the reason for it to be, have minimum degree greater than equal to k minus 1.

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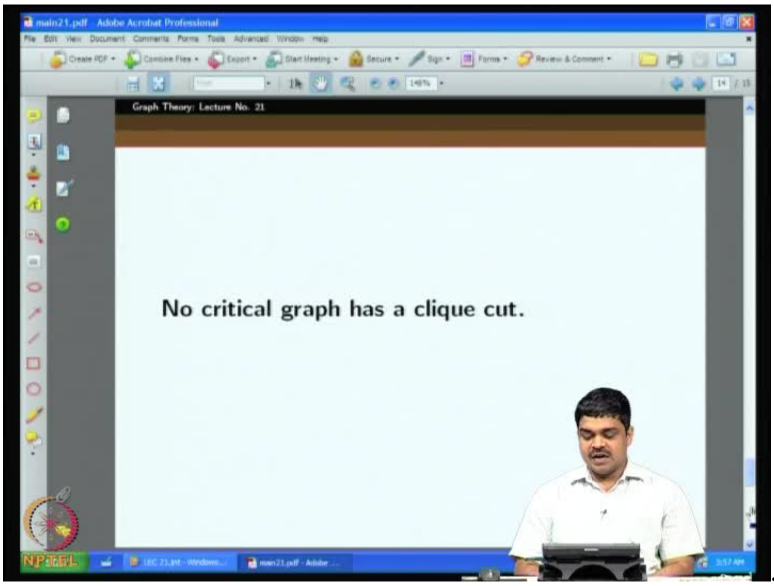

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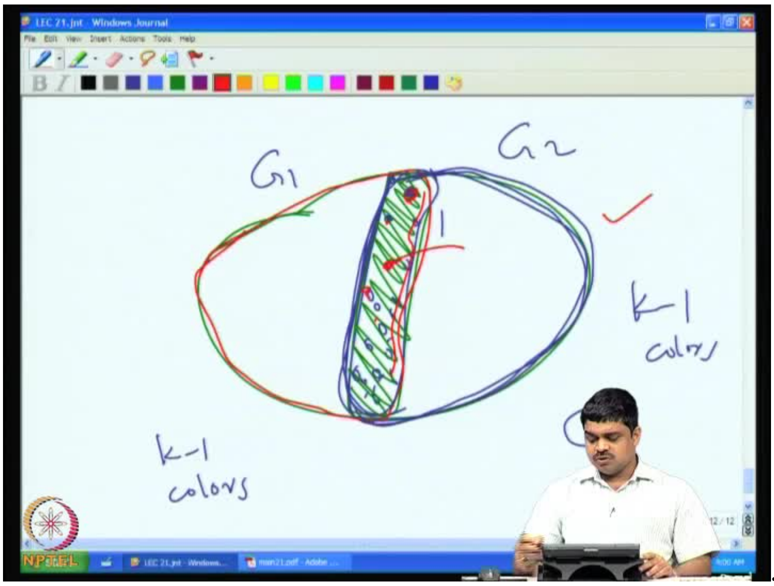

Now another property of a critical graph is that no such critical graph can have a clique cut; no such critical graph can have a clique cut; why is it so? Because the reason is if you have a clique cut, so suppose this the graph, and then you have a cut here, which is a clique, which is a cut, which is a clique. So now we can consider these two graphs, right? So this is smaller graph and this is another smaller graph, we can use this one, it is another smaller graph, right? This is G 1, this is G 2, so that together they are, they form G.

Now, because the original G was a critical graph, so say k critical graph, now this G 1 can be colored using k minus 1 colors, and G 2 also can be colored using k minus 1 colors. One thing we can say is, because this was the clique, all these vertices have got with respect to the coloring of G 1, the k minus 1 coloring of G 1, they all got difference colors here within this clique, is it not?

Because it is a clique they, should get different colors whatever, if it is a valid coloring, they should get different colors. So essentially means, that if it is k minus 1 colored, the cardinality of the number of vertices in the clique, also, the cardinality of the clique, the size of the clique has to be less than equal to k minus 1, right?

Similarly, about G 2 we can say, the coloring with respect to G 2, will give different colors to the clique vertices. Now, you can try to paste them together, because there all different colors. So we will, we will take G 1 coloring, and then, we will try to paste G 2 on that, but then it may see that here G 1 has red, given red color, then G 2 is trying to give green color to it; what will we do? We will, G 2, we will exchange red and green; that means, which ever vertex was given red color will be green color, and green color will be red, so that they will be coincident here, right?

Similarly, here there will be another color. If there is a clash that color will be change the name. So, and then, every color we can make, so the they can be a consensus about the colors that G 2 and G 1 gives two these vertices, and then, we can paste them together, right? That is why we can get a coloring of G using k minus 1 colors, by just pasting them together, right? So, that is why it cannot have a separator which is a clique. A clique separator will not be there.

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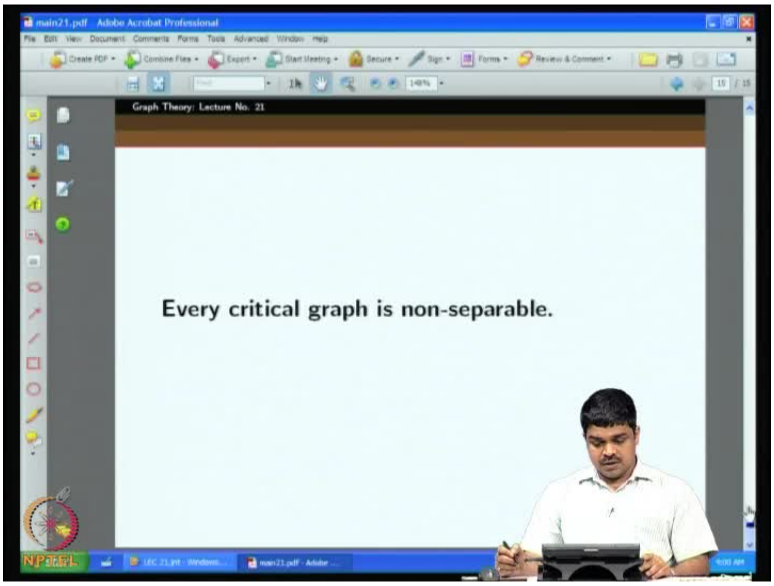

So, another immediate consequence of this statement is that no critical graph, right, can have a vertex cut, sorry cut vertex. That means, it is non-separable. So it is, it has to be two connected it, right? Why is it so?

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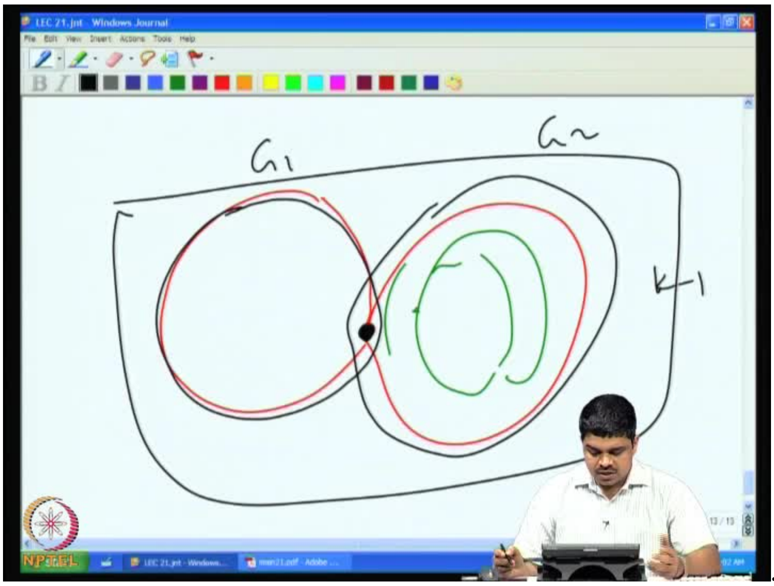

Because suppose it is not to connected, they will be vertex cut somewhere. So that is like vertex cut somewhere, and if there is a vertex cut, this is the same issue, this is the clique, right? Now, so to repeat the argument, what will happen? So this graph can be colored using k minus 1 colors; a color will come here, say black color, came here, right? Now this graph G this is G 1, this is G 2; G 2 can color it ;may be possible that there is some other color came here, but then we can always rename the color; the this green can be renamed black in this entire thing, and then, black can be renamed green here.

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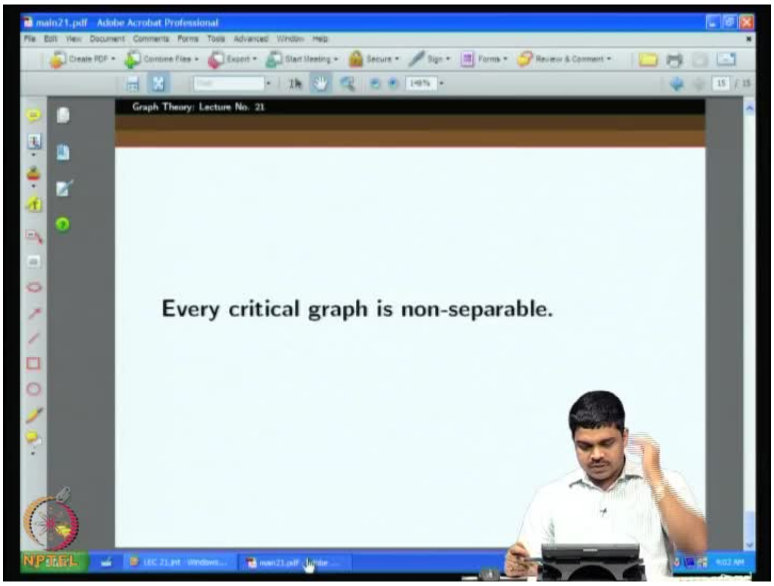

So that, so that, this will also here will give black to this vertex. So we can paste them together with the same coloring. Then, no more problem will be there. This entire thing will get a k minus 1 coloring, which is a contradiction, because we have already told that it requires k colors, it was a k critical graph. Essentially… so a k critical graph cannot be, cannot be, the connectivity of a k critical graph cannot be less than 2; it has to be two connected. So this is, this is, this is, about critical graphs.

So, then why are these critical graphs interesting? Because, of case, critical graphs are interesting, because, so when we want to prove something about chromatic number, we can typically reduce the problem to critical graphs and term. Therefore, people study about this structure of critical graphs, also. So essentially, we will get extra properties for critical graphs and we can make use of that in such proofs.

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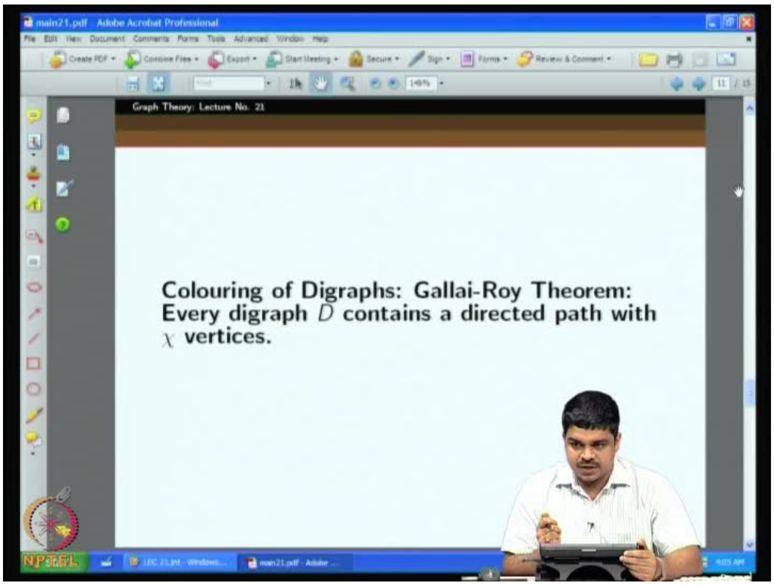

And so, the another, of case, the last the thing we want to discuss. So, another property so we would, we would also like to consider the coloring of digraphs. So what about directed graphs? So directed graphs, so there is no other notion of coloring. You have to come up with the with the coloring of vertices in such a way that whenever there is an edge - directed edge - here between u and v, the vertices would be same colored; colored differently, right?

But then, why do we study the coloring of directed graphs? So but, then still so, some how this coloring can point out some something about the substructures in directed graphs.

So here is this Gallai-Roy theorem; it says, every digraph D, contains a directed path with chi vertices. So one should thing whether we consider a undirected graph; is this true? So is it true that undirected graph has a path with at least chi vertices? It is obviously true, because if you consider a critical graph, what we do is, we we consider a chi chromatic graph, and we make it critical by throwing away vertices; we remove vertices until, as long as by throwing away vertex if the chromatic number does not reduce, throw away that, and then, similarly, edges can be thrown away.

So finally, we will come to a situation what the throwing, away throwing any more vertex or edge will decrease the chromatic number. So that will become a critical graph, so every k chromatic, k chromatic graph has a k critical sub graph in it. So this k critical sub graph has minimum degree, at least k minus 1 as we know, and we know that by Dirac's theorem, there exists a path of minimum degree plus 1, right?

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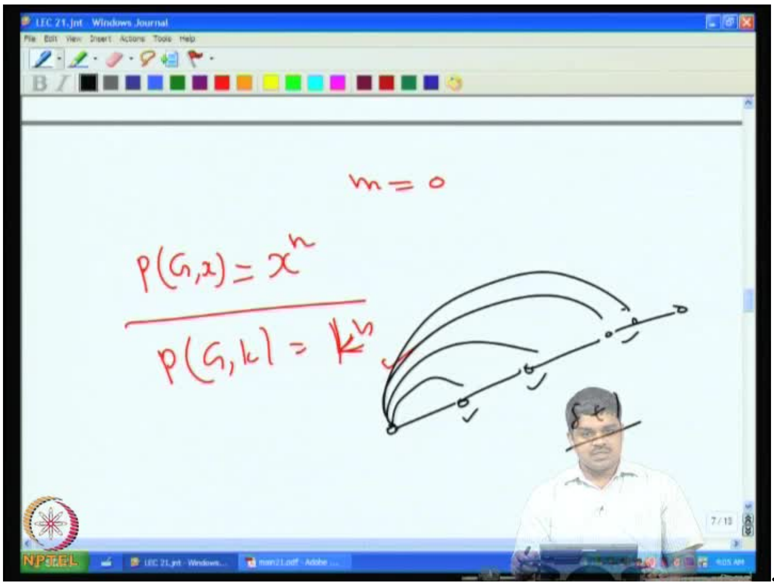

Because we can, we can remember that we have done, so there is a if you consider the longest path in the graph, so the minimum degree, all the neighbors should be in this path; otherwise, the path can be longer, and including this, we will get delta plus 1 length path in it, right?

Therefore, therefore, a undirected k it is easy, but here it says, give me a undirected graph and give any direction you want to the edges, orient the edges in whatever way you like, still you will get a directed path now, not necessarily an undirected, a directed path with a length number of vertices chi in it. Why is it so? So, this is what Gallai-Roy theorem says. We will give a proof of this in the next class. Thank you.