Graph Theory
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Module No. # 03
Lecture No. # 18
Proof of Kuratowsky's theorem, List coloring

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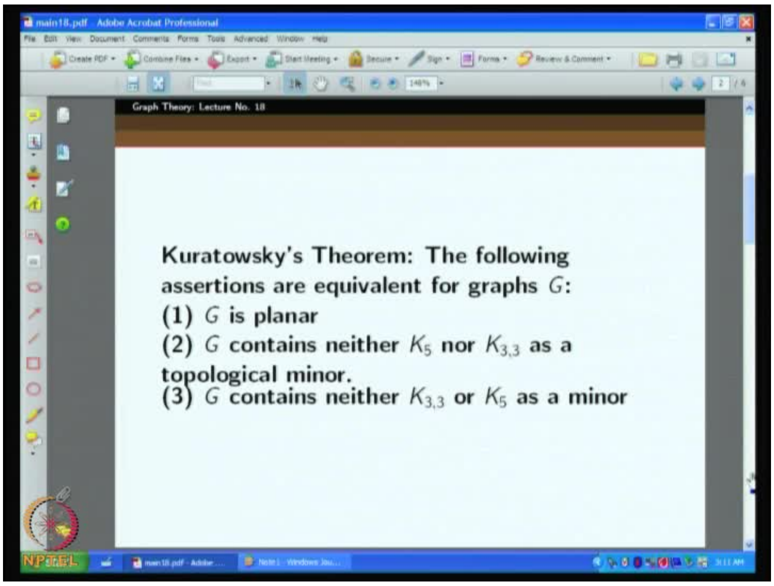

Welcome to the 18th lecture of graph theory. In this class we will continue with the proof of Kuratowsky's theorem. The theorem says the following assertions are equivalent for graph G. G is planer, G contains neither K 5 nor K 3 3 as a topological minor, G contains neither K 3 3 or K 5 as a minor. The… in other words, if G is planer graph it is clear that you cannot have K 3 3 or K 5 as a minor in it. Definitely it cannot have K 3 3 or K 5 as a topological minor. This is a base because in that, because a planar graph cannot have non planar minors or topological minors in it. K 5 and K 3 3 really non, definitely non planar. Now the more difficult part to proof is that if a graph does not have a K 5 or K 3 3 minor then it is planar graph. In other words any non planar graph should have one of these minors, either K 5 or K 3 3 minors. As you are seen in the last class it is equivalent to saying that any non planar graph will have either a topological minor of a K 3 3 or K 5 because if a we have a K 3 3 or K 5 minor then definitely we have the corresponding topological some of the topological either a K 3 3 or K 5 topological minor also if K 3 3 comes as a minor then K 3 3 will be there as a topological minor also on the other hand if

K 5 comes as a minor you may get either a K 5 topological minor or K 3 3 topological minor therefore, it is enough to show that any graph without K 5 or K 3 3 topological minor will be non planar sorry will be planar. So, the but, we will prove it for 3 connected graph first; that means, if G is a 3 connected graph and it does not contain K 5 or K 3 3 is a minor in it then ah

It is planar this is what will proof then we will we can extend into 2 connected or even other cases the connectivity less than 3 cases will be easier then (( )) now how do we do this thing. So, we this is the theorem we want to prove let G be a 3 connected graph and it does not have K 5 or K 3 3 is a minor then we want to show that it is planar. So, we will just show a planar drawing of it this is our plan. So, we will use induction let say that for all graphs with smaller number of vertices it is true. So, with a number of vertices K four also four or less it is very trivial because if K four of case it is the first 3 connected graph we can start with and a K 5 or K 3 3 minor is not there in K four now it is a planar graph it is it has a planar drawing therefore, it is a true now by induction let us assume that for smaller number of vertices this statement is true and now we need to consider graphs with n vertices a number of vertices in and a 3 connected and without K 5 or K 3 3 minor

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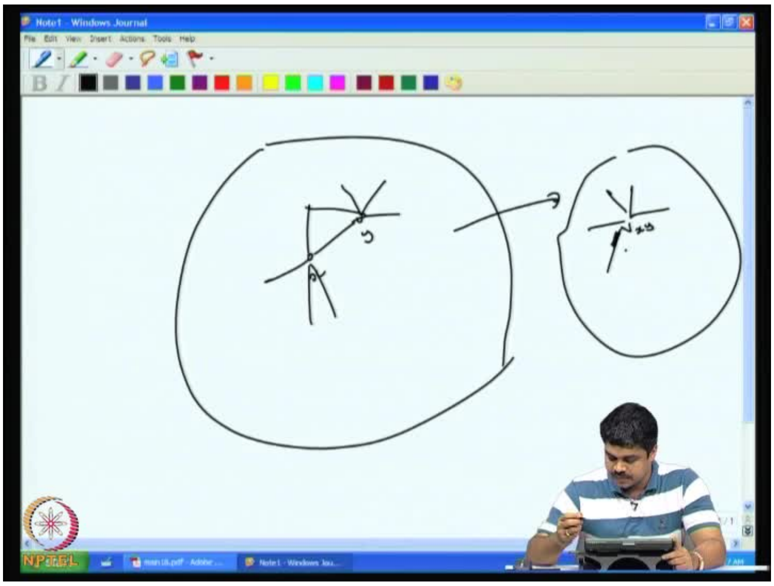

Now, the the point is because it is 3 connected graph there is one edge in the graph such that if you contract that edge the resulting graph is still 3 connected this theorem we had studied in the when we studied connectivity. So, this is theorem by (( )) and a we can use this theorem here how do we use it. So, we pickup this edge we know that such an edges exist. So, x y lets x y and. So, such an edge if i identify and then contract that edge right. So, this such an edges there x y the point is if you contract it. So, we will get a new graph. So, with that contracted it will be v x y you know after contracting what will happen is. So, the neighbors of x and y together will be the neighbors of v x y right the neighbors of x and y now we can definitely being this is a smaller number of vertices here

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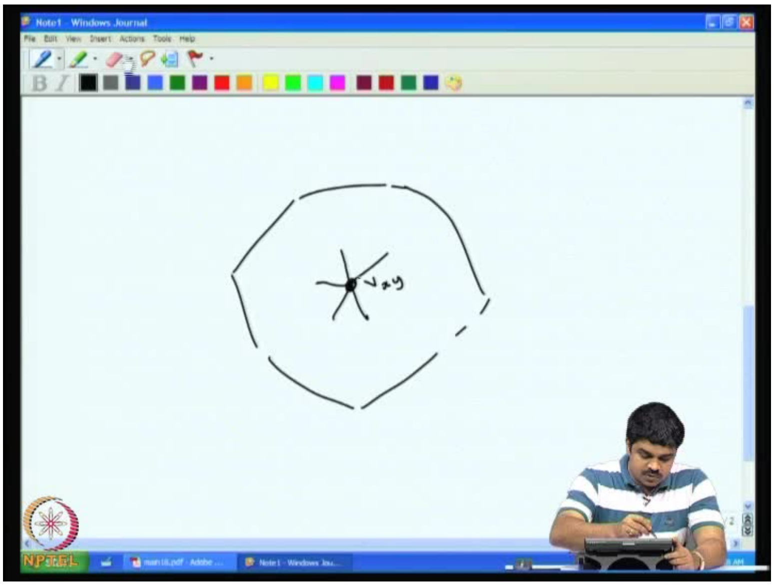

The number of vertices smaller here and it is 3 connected and of case K 5 or K 3 3 minor cannot appear after contraction because after contraction the graph the resulting graph has if the resulting graph has K 5 or K 3 3 minor then the original also will have; obviously, because. So, because contraction is one of the operations defining the minors right therefore, we can see that this resulting graph also defining K 5 or K 3 3 minor. So, induction can be applied on that by induction assumption we know that it has a planar drawing and in particular the drawing somewhere this v x y will come right v x y will come and you see suppose see the v x y some the edges on v x y will go out of it and where will it go. So, the point is suppose if i had removed this vertex v x y from the graph

It is because we have a 3 connected graph here after removal of v x y we still will have a 2 connected graph and. So, this in a 2 connected graph they any planar plain drawing the each face has its boundaries cycle right. So, simple cycle will come. So, it is therefore, we know that the face in which this particular point is contained that also will be suppose this's the face of that. So, that will also be a cycle you know so

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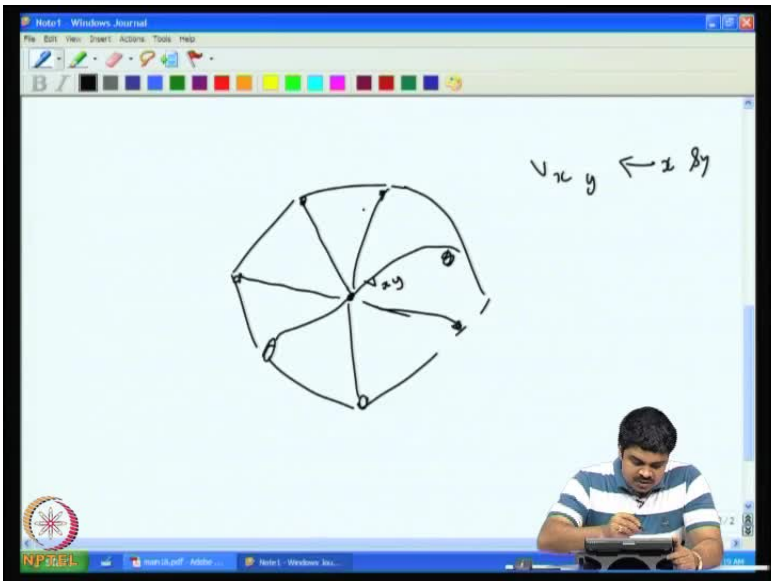

So, if you remove this vertex v i suppose i can i can i will i will remove this vertex v i. So, this is what will see v x y this is what i will see right now we we see that this edges are going somehow to these some points on the boundaries right that only way can go v x y now suppose i had removed. So, v x y removed remember v x y as formed by contracting x and y. So, essentially these neighbors some of them are from x some of them are from y. So, see them might have come mixed up but, let say first we consider the neighbors of x along right

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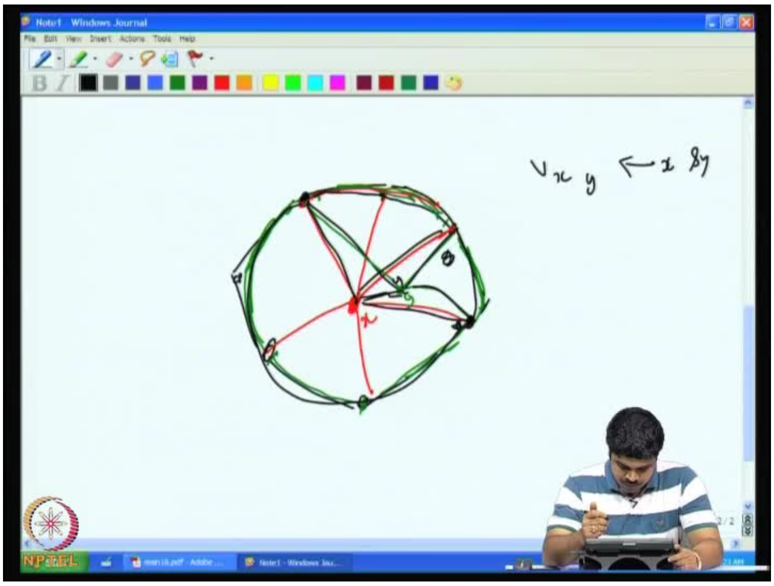

So; that means, some neighbors may be disappear may be suppose this disappears this disappears right. So, we we only return the neighbors of x right suppose these are the neighbors of x. So, i can draw like this i can draw like this. So, this is now instead the points where v x y i just put x and then some other neighbors the the neighbors of x which are not neighbors of y are drawn right.

Now, the point is look at the y neighbors the neighbors of y sorry neighbors of x all the neighbors of x are drawn now i can look at the neighbors of y how can they come. So, suppose. So, there is a region here. So, this this essentially they have several regions on the boundary of this face. So, which is essentially given by the positions of the neighbors of x right one one region another region right another region right another region another region is it possible that we have some y here inside suppose is it possible that v of x i mean each of the regions other then is a not touching not including the the neighbors of x itself is it possible that no y appears in the interior regions anywhere

So; that means, the y are always on the on the positions where x neighbors are placed right the same same vertices right; that means, essentially y will share the neighbors of x but, then at least 3 neighbors should be they y should have at least 3 neighbors right

So, i should have at least 3 neighbors so, it will. So, happen that. So, we can right. So, it y y can be like this. So, it can be some 3 neighbors it can be now what do we see if y has 3 neighbors shared shared neighbors with x. So, see not need not be this consecutive you can explain all the neighbors also here. So, suppose this neighbor of x and this neighbor of x and this neighbor of x is shared by y right and see y x also has.

So, then you can see that the 3 neighbors together will form. So, these 3 neighbors together will from one. So, that say i can i can draw it black. So, this one to. So, because of the 3 neighbors which which has. So, one 2 3 these the triangle here that these 3 neighbors and then this is connected to this this and this and x is also connected to this this and this and between x we have. So, essentially we have a connection. So, essentially we have a K 5 available here right. So, moon more nearly we can draw here

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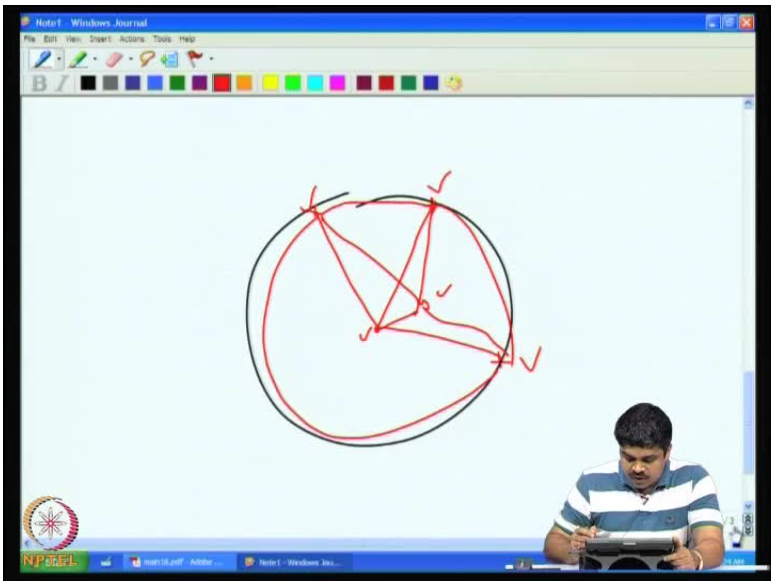

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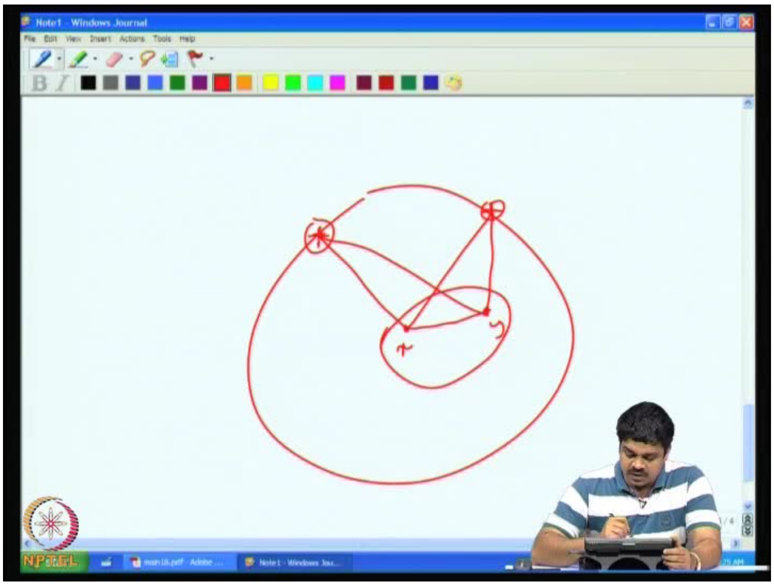

So, it will be for this is the x 3 neighbors now between them. So, we see the triangle one 2 3 then the y may be here y and y is also having these 3 neighbors. So, together we have a K 5 here. So, this triangle this this this and this and this together we have a K 5 minor here. So, it is a contradiction to the assumption that there is no K 5 minor in the graph. So, see of case you can assume that the when i selected between x and y. So, i decided that x as smaller neighbors than y. So, suppose y has only 2 neighbors right. So, i has only 2 neighbors. So, then it will. So, happen that x also will have 2 neighbors maximum and these 2 neighbors if i remove this 2 2 neighbors right see because x should have. So, this is x y at least 3 edges should be going out of x and y.

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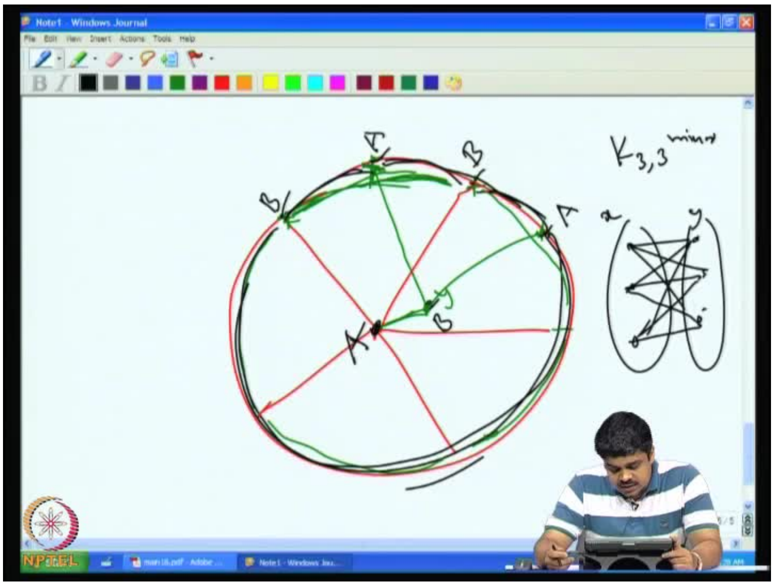

So, because a 3 connected. So, if there the only 2 neighbors for x then y also y then x also only 2 neighbors and then together this removal of this 2 vertices will disconnect this portion from the rest to the graph therefore, it want be 3 connected right. So, that is why we are see saying that they at least 3 neighbors of y and if they all are shared with x then we already got a K 5 minor right now it means that they should be some lets again again draw this thing x in this interior the region of this thing which a way which a mark here this region this region and this region which is not including the in the neighbors themselves the portion of the neighbors themselves they should be at least one (( )) say this here at least the y has i am drawing y is y here.

So, i have at least one neighbor here right now where else can y's neighbor b. So, and claiming that all the neighbors of y has to be in this region there in the same region in the same segment here right. So, y suppose including this we can have we can share this neighbors of x but, it cannot go beyond this thing suppose it goes beyond this thing suppose say y has a neighbor here right somewhere else. So, then what happens is. So, you see this connection is there. So, these 2 are there now you can see that here we have already got a K four K 2 2 right K 2 2 including this vertex this vertex and this vertex and this vertex already defines a K 2 2 like this minor K 2 2 minor here right and then how here is a the third we can add make a K 3 3 by adding this and this on different sides of the K 2 2 right because this on the a side this is on the b side right and these 2 neighbors can be on the a side and these 2 neighbors on b side right

So, this will make a this this this these 3 vertices and then this this and this will be the other 3 vertices. So, this will be x this will be y and these are the 2 neighbors of x and these are the 2 neighbors of y we have this connection right and we have this this also here because that is what this circle the circle provides that connection. So, here to here one edge here to here one edge here to here one edge here to here another edge K four sorry K 2 2 right that K 2 2 and together with this thing will give a K 3 3 minor K 3 3 minor right

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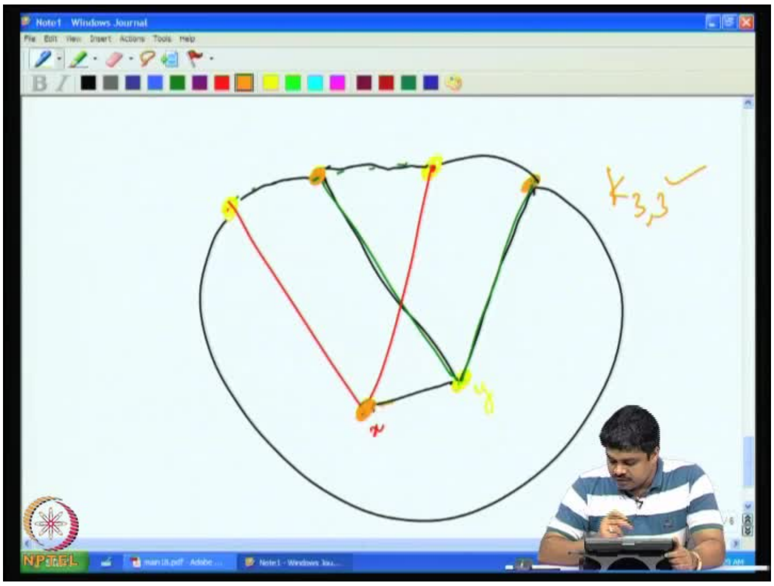

So, this is these tells us that. So, maybe we can draw it a little more clearly. So, i am showing you the K 3 3 minor in case. So, the. So, this as x neighbors this is x x neighbors here suppose in between here. So, y as a neighbor and then suppose y has a neighbor beyond in not in this region if suppose beyond this region then what happens is y i take y. So, this sorry i can i can color it yellow as these 2 and this this is y. So, these 3 as one side and then x sorry these are use orange color (( )) this this and this and other side in here we have the connections. So, here here is a connections sorry is a connection here is the 3 connection to the yellow vertex and then this yellow thing this is there is a connection there is a connection like this here in there is a connection here and then there is a connection here see

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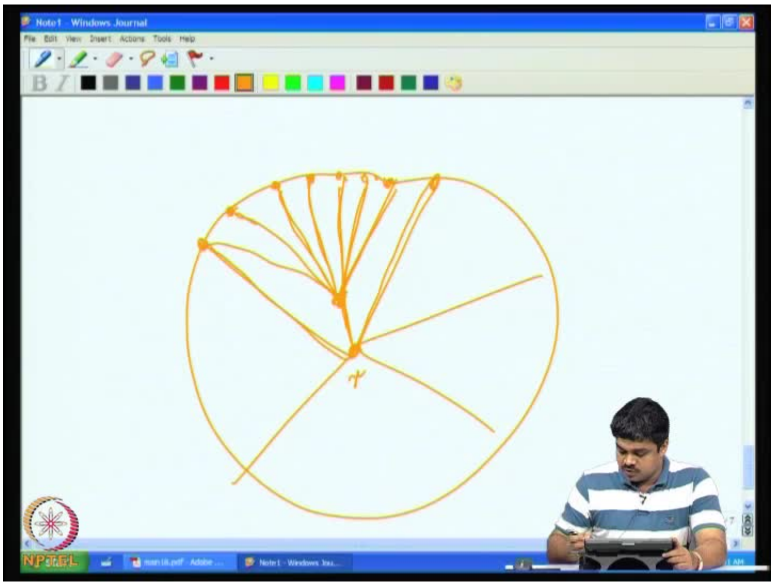

This is. So, that is the every vertices connected to every brown vertices connected now to every brown vertices connected now to every yellow vertex to be brown vertices connected now to every yellow vertex and every yellow vertices connected to this is a K 3 3 is a K 3 3 right. So, what we infer now is that all the neighbors of suppose these is the thing

So, if you for consider x and this this the portion of x and all the neighbors of y should be in this region only. So, it is may have some other things here. So, the all the neighbors including this points through y y neighbors all will come here. So, which means that i can place y somewhere here and then i can necessary i can connect in like this and then. So, this will be a planar drawing of the original graph see the key point is we can argue out that y comes in one of this sectors right

So, inside. So, it will not go in there one be any confusion about how we can place y with respect to the the the graph which contains only x after removal y right we can nicely place it in (( )) segments and all the neighbors are y will be inside this region only

So, that is y we can we can get a drawing of the graph from the smaller drawing. So, it it follows that. So, if there are 2 2 things you have made use of one is there is no K 3 3 minor in the graph then one is a K 5 the graph is K 5 minor free another is 3 connectivity if all these conditions for met then we can we have a argued that we can get a planar drawing of the graph; that means, the graph is planar right

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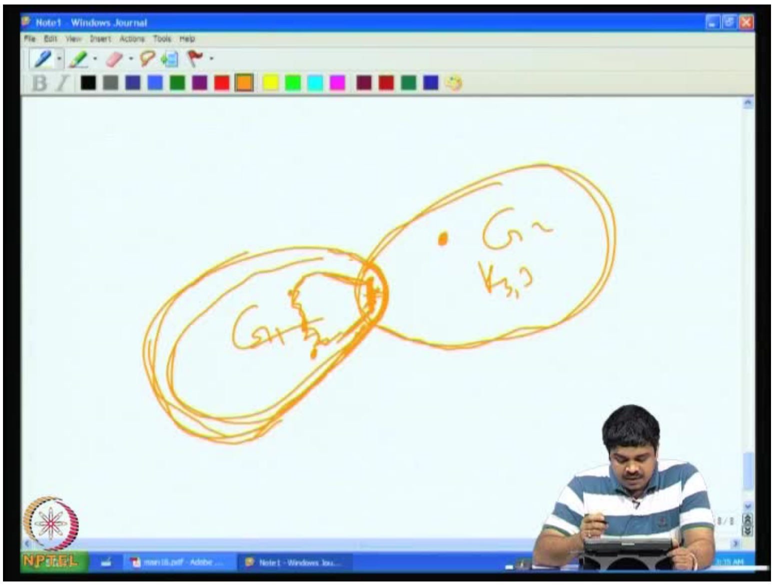

So, the using induction now the the only thing left is to show that suppose it is not 3 connected. So, the connectivity’s only 2 or one or something like that then also we can get a planar drawing provided K 5 or K 3 3 minors not there. So, what we can do is suppose the connectivity’s 2 so; that means, there as 2 there are 2 vertices right. So, when you remove these 2 vertices they will get disconnected into 2 pieces. So, we can see that. So, there are 2 possibilities. So, then here there is an edge or there is no edge. So, whatever it is we can we can assume that there is this edge

We will if it is not there will put it right and then we consider this graph this is g one and then this is g 2 see by putting this edge see see to see that K 5 or K 3 3 minor will not be introduce if it is was not there before edge will leave to the (( )) to verify that. So, the the the why why is it. So, because again suppose the minor is intuitively this is the reason suppose the minor is such a minor is introduced. So, you can you can see that this edge right because is it 2 connected graph even if the edge is not there right the the minor the the minor which got introduced here has to be totally from g one totally from g 2 it is not possible that the minor shares it is brand vertices from g one and g 2. So, a little bit of it is here and little bit of here. So, because it has to be totally from here because of because this K 3 K 3 3 and K 5 or the 3 connected graph the for instance if there is brand vertex here. So, how can how can suppose how can the share it between the 2 sides. So, suppose if there is one vertex here right

And. So, all the other brand vertices are on this side right. So, it has 2 of parts to the other side 3 if there are 3 vertices on this side they should with 3 different paths from here to here right because of the 3 connectivity and that is not possible because there is only 2 vertices in the separator we can only 2 have to disjoint parts. So, we can infer that if K 3 3 or K 5 is a minor of this entire graph the after putting this edge right then it is it is it is completely from this part or completely from this part we can see a K 5 or K 3 3 whichever is in g one or g 2 right

And now even if this edges not there we could of contracted suppose it was there K 3 3 minor K minor in this thing it say that that this edges critical for that but, then this edge can be produced by contracting this entire portion because there is at least 2 edges here and they should be because of the connectivity of this they should be some path and then this will become an edge here right when you contract this entire region into here right. So, therefore, , it will it will. So, happen that so, the we can we can we can produce this K 3 3 minor in the the in the total graph also right

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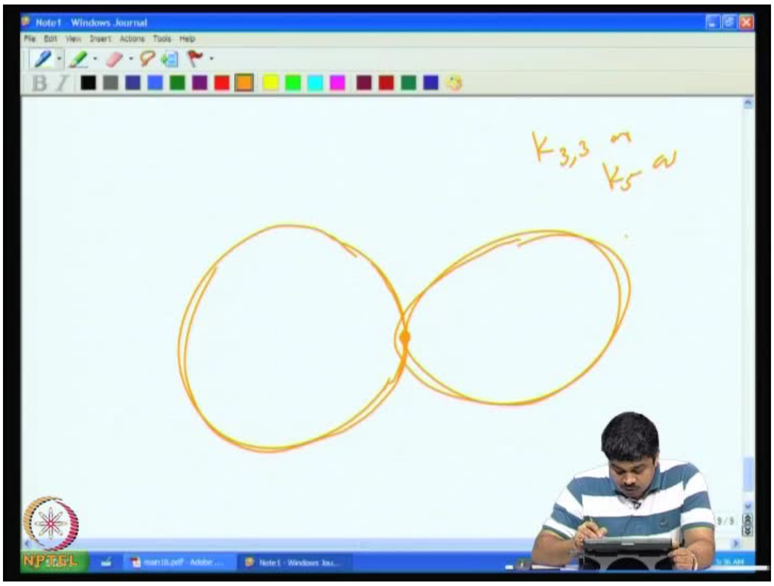

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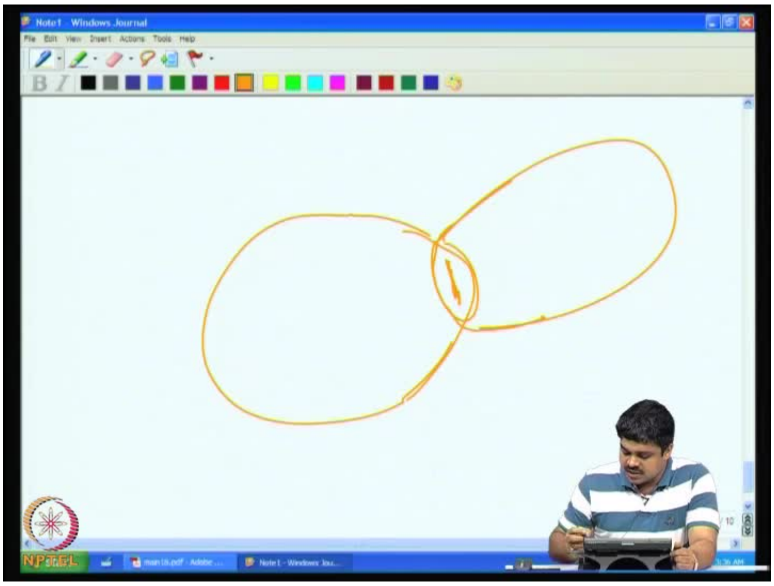

So, in other words if you consider a the connectivity as 2 or for that matter even one even in this case it is like that the argument is correct right. So, the assumption that the entire graph does not contain k 3 3 or k 5 as a minor in it means that. So, this portion alone or this portion alone also does not contain K 3 3 also K 5 minor similarly, if it was the connectivity was 2 is this was the picture then this portion this portion even with an addition of this edge cannot contain K 3 3 or K 5 as a minor because if that minor is there in one of them it should have existed in the bigger graph also that is the reason

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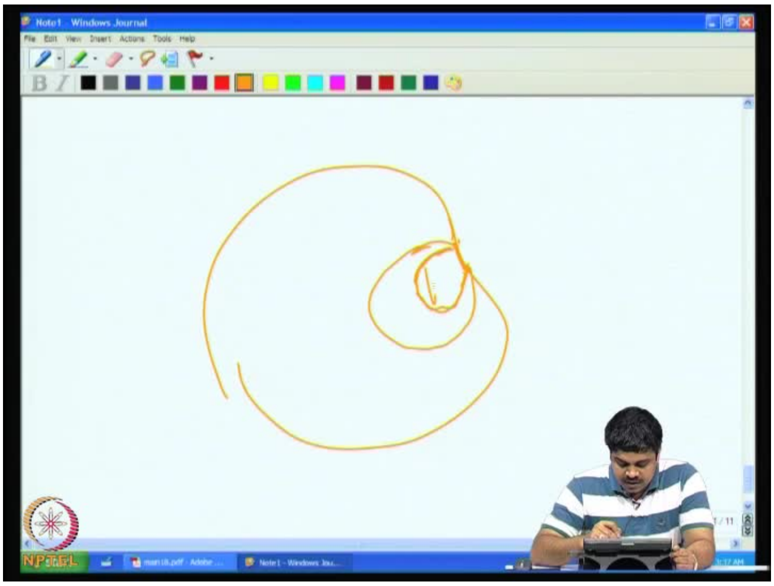

So, then we can see that this smaller portions can be drawn because they are 3 connected and or by induction make and say. So, the smaller pieces can be drawn on the plain and then we we we have to do some scaling and adjusting. So, what what we do is we can draw one of them and identify the edge and make sure that the face is right they consider the face in which that edge is appearing and then we have to make the other parts smaller and bring that edge on the outer face and then draw it

So, i leave it to the student to verify this details but, it is not very difficult. So, or other rigorous statement is available in details book you can read from there we will skip that portion to avoid it tedious details. So, the that point is just imagine to plain drawings for g one and g 2 and think how we can draw a plain drawing for the the complete graph by pasting them together on the correct place because there is that an edge one edge and that and not that the face containing that edge can be made the outer face and we can bring that particular edge in the correct place in after making its smaller the picture making its smaller and then we can paste it in the correct place and get planar drawing that be i leave it to the student if figure out the technical details of that. So, the multiple the more interesting portion was the to provide for 3 connected case right

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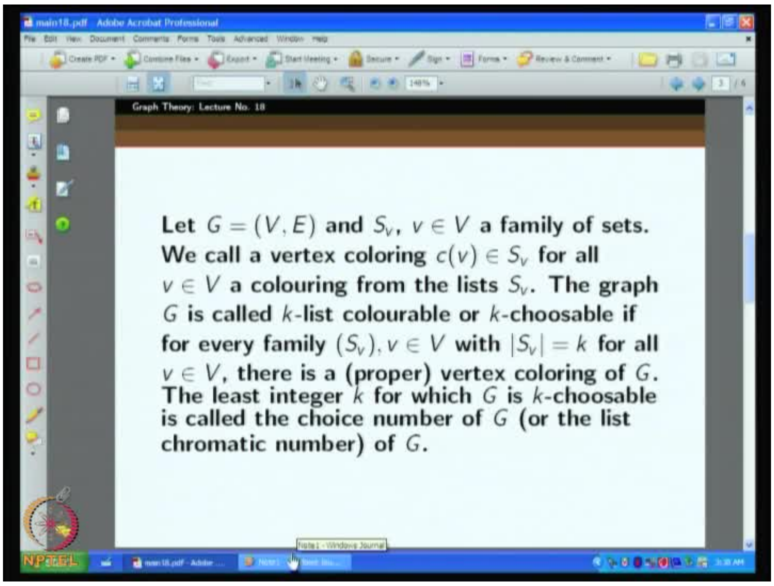

So, now with this thing we will finish the the planar portion we will revisit planar it a little latter but, today my intension is to do different kind of coloring and different kind of coloring called list coloring this is also an important topic in the. So, this is a way we are defining the list coloring. So, we are given a graph and along with the graph for each vertices of the each vertex of the graph we are given a set of colours also; that means, essentially to each vertex we are associating a list our condition is that the vertex should be coloured from using one of the colours from that list on the other words the vertex does not allow any other colour any outside colour. So, it has a list of colours and it is interesting that it will get coloured by only one of the colours from that list. So, the a coloring such that each vertex gets a colour from the list associated with it is called a coloring of the graph from the list you from the list then we will say that a graph g is K list colourable or K choosable.

If our every family. So, as we. So, with the cardinality of the list is equal to K then there is a proper vertex coloring of g from the list see the point here is the the list can be anything but, the only thing we know there is each vertex has a list and it has at least K colours in it the the content of the list may be anything but, just the length of the list are how many colours are there then the list is known K. So, but, then they should be able to say that you respect you to whatever in the list as long as the list sizes the cardinality of list are at least K we can come up with a coloring for each vertex such that the colour is from the list for each vertex and the it is a proper colour and it is a proper coloring in that case we say that the graph is K list colourable see suppose if a graph is K list colorable ah

Then you can. So, fine and another another thing we have we need to understand is. So, suppose this K this smallest value of K for which this happens is a the follows smallest value of K four which this happens is the the list coloring number this chromatic number of the graph or choice number of the graph. So, we also say that the graph is K choosable. So, this the smallest K four which is the graph the g is K choosable is the list chromatic number of the graph list chromatic number of the graph them now the. So, of case to understand it for instance you may see you may say why for instance can i can i suppose can i colour a graph suppose every every every vertex as the the same list same vertices the list of size one then definitely you cannot coloring because then the each vertex says that i have to get this color only right for instance

So, if there is an edge. So, but, on the other hand. So, if a graph is K list colorable if a graph is K list colorable; that means, respective of the list content of the list just the guarantee that each list has at least K colors in it tells us that the graph can be colored with the colors from that list then it also clear that it is a K colorable then usual sense of the coloring why is it. So, because we can as associate with each vertex the the same list namely one 2 3 up to K and; that means, that any vertex can any vertex can be colored with any of the colors from one 2 K right

So, in that case if a coloring happens; that means, it is a K K K coloring only we say that K we can get a coloring of the graph using K colors right. So, because we are not restricting the colours that can be s assignable to a vertex other than saying that it has to be from one to K right

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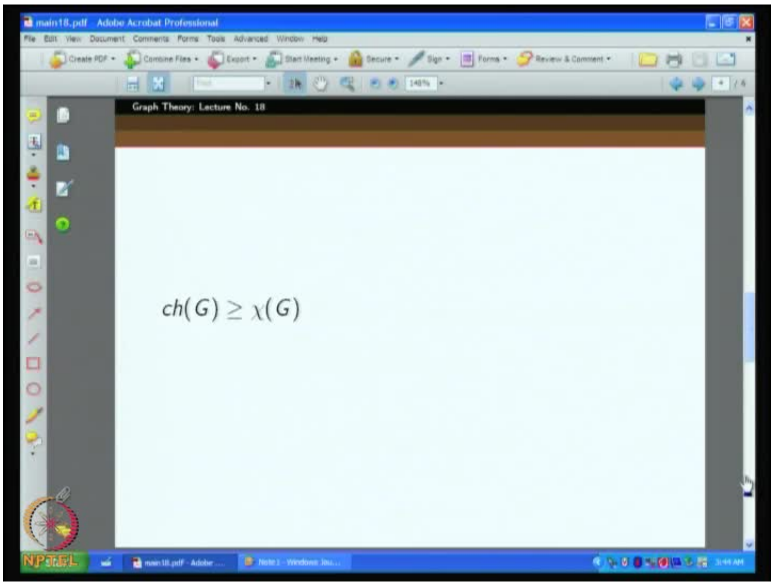

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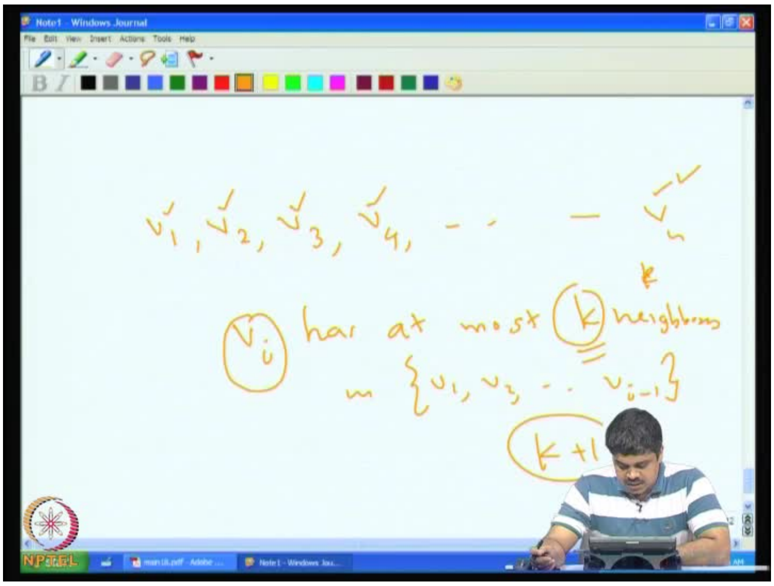

So, therefore, it is very clear that the choice number of the graph g is always greater than or equal to chi of g right because if the choice number is some K then some of chi of g also chi has to be g has to be K or less right. So, but, sometimes choice number can be much greater than chi of g. So, the next our see the and another thing is for instance if you see some of the our earlier bounds like the greedy coloring strategy will it working the case of list coloring also,. So, you can see that suppose you have an ordering of the vertices in such a way that. So, i write the vertices in the order v one v 2 v 3 and v four v n with the property that v i right has at most K neighbors in say v one comma v 2 up to v i minus one suppose such a ordering says we have see that in this case K plus one colours are enough to vertex colour the graph

Why because we will colour v one first and then v 2 first v 2 second and v 3 next and. So, on when whenever i try to colour v i the only the vertices starting from v one and ending at v i minus one has got colours and can be look from v i we see that only at most K neighbors of it are coloured then we can be we can use the K plus one’th available colour for v i and therefore, it can be coloured using K plus one colours this was the greedy strategy and then essentially we told K K plus one as the coloring number right

So, for K was the of the graph now thus same K can be an upper bound for the size of the list so,. So, that we can come up with coloring of the graph right one other words the choice number or the list chromatic number also can be upper bound by the same K how is it. So, because now to each vertex we are associating a list of cardinality at least sorry this K means K plus one i mean the list a list of size K plus one or more and

Now, when i colour starting from v one and then the v 2 and v 3 and one words then i reach v i we see that the the we see that the the the colours which are already used on its neighbors right say essentially because v i will see its coloured all its coloured neighbors are from in the range v one v 2 v 3 up to v i minus one and then we know that only K neighbors are there only K colours are used up but, it is list as at least K plus one they should be one more colour available to it in its list

So, it has to take it from takes from its own list earlier we were using just any available colour now we have to insist that it has to list colour from its own list. So, but, then because K any list has at least K plus one neighbors then you can always kept one more colour and therefore, it is possible. So, the same argument works here altherefore, the general c plus one coloring number still a bound four in particular maximum degree plus one is a bound right

Similarly, brooks theorem can be proved also for this case without much effort therefore, we we see that some of the bounds which working for the vertex chromatic number the is also an upper bound for the list chromatic number

Now, the next point is we have to see a case where the the the for instance planar graphs if you consider it is a four colourable vertex coloring but, then it needs 5 colours for list colour right. So, its list chromatic number is 5 but, how do we show that the planar graphs are 5 list colourable how can i show that if with every vertex if i associate a list of cardinality at least 5 then we can always come up with proper vertex coloring of the given planar graph from the given lists right

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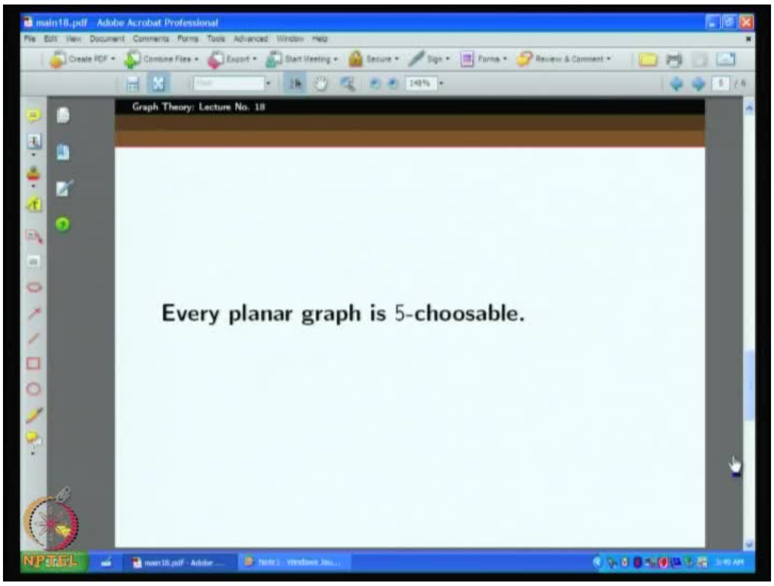

So, have to do this thing so, this is the one extreme. So, this is also a result from messent. So, will show that this every planar graph is 5 choosable. So, the. So, the that key points here is to rather than this an induction proof but, rather than doing the induction directly on the for this statement like means every planar graph is also lets consider a smaller planar graph in assume that it is 5 choosable then i will try to extend the 5 list coloring of this planar graphs to every bigger planar graph this not the strategy rather than doing this thing we will slightly strengthened induction assumption and then will prove the induction for that stronger statement this a is slightly delicates strengthening of the induction hypothesis this is that trick here

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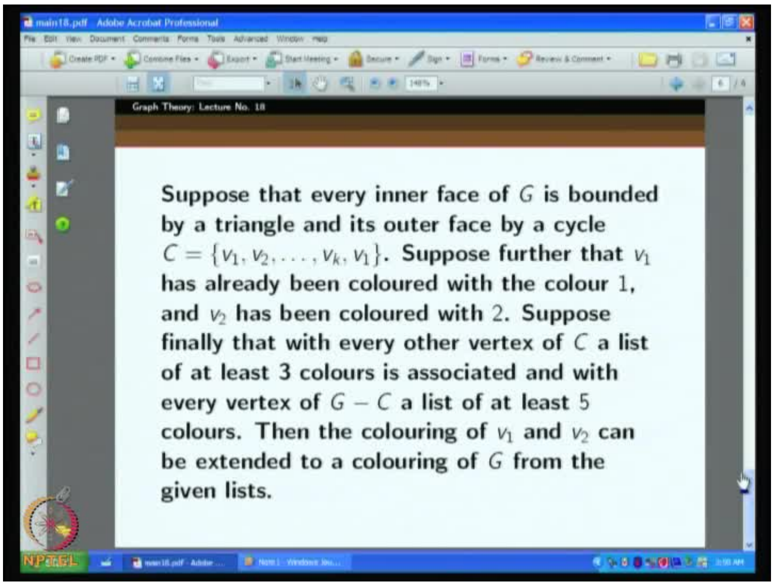

So, how are we going to do this. So, this is less strengthening. So, know suppose that every inner face of g is bounded by a triangle and its outer face by a cycle. So, we consider a plain graph plain drawing and we say that suppose there are several inner faces the all of the inner faces are just triangles and outer face is bounded by a cycle. So, v one v 2 v 3 up to v K and v one it maybe a K length cycle suppose further that v one has already has already been coloured with colour one and v 2 has been coloured with colour 2 suppose finally, that with every other vertex of c a list of at least 3 colours is associated and with every vertex of g minus c a list of at least 5 colours is associated then the coloring of v one and v 2 can be extended to a coloring of g from the given lists this is what we going to prove. So, on other words. So, you consider as special planar drawing

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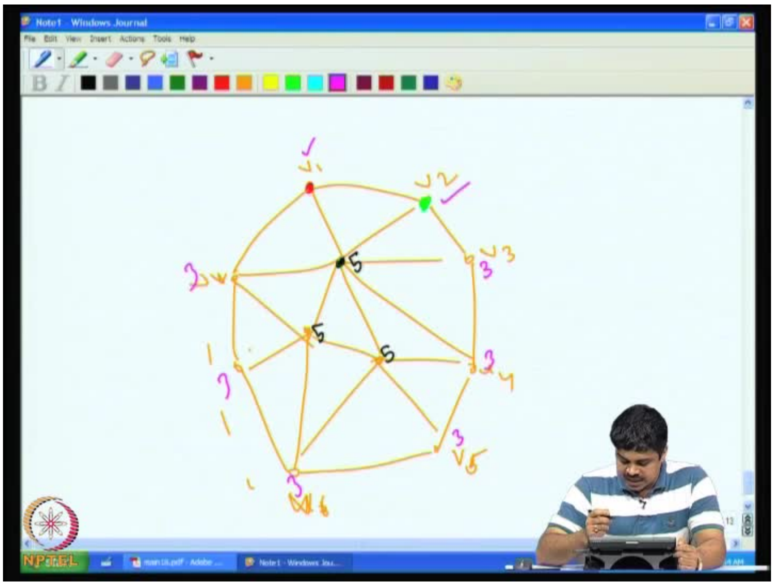

So, the requirement is that our outer face this the only non triangular face which are which were allowing on other interior faces should be see triangles already triangles. So, and. So, this can be number. So, v one v 2 v 3 v four v 5 v six and. So, on up to v K this can be number like this the inner faces are triangles the outer face only is scaling cycle now also we assume that this is coloured this is also coloured these 2 colours are already known to us right now we will say that the lists are there for for suppose a vertex like this for a vertex like this there are 5 colours 5 colours associated in choice 5 lists the lists are of cardinality 5 each lists contains 5 colours in it but, the vertices on the face they contain only say 3 they only 3 3 colours in it they can be any 3

So, 3 only 3 colours right. So, see it is possible that this red and green also maybe part of this v one and v 2 are already coloured this may be part this things but, we do not have a any control at least 3 colours are there right

Now, we say that see we can get a coloring of this planar graph this kind of planar graph from that lists from the lists each vertex will get the colour from their only list and we can retain in that coloring even you even retain the already given colours of v one and v 2 this what the strengthening suppose this is true this statement is true this is some as this is a stronger statement then what we need why is it. So, you may say there why it looks like a special planar graph is not the case because you see if you given a planar graph you can always add edges to it and they because when you add more edges the number the requirement of colours may only go up right

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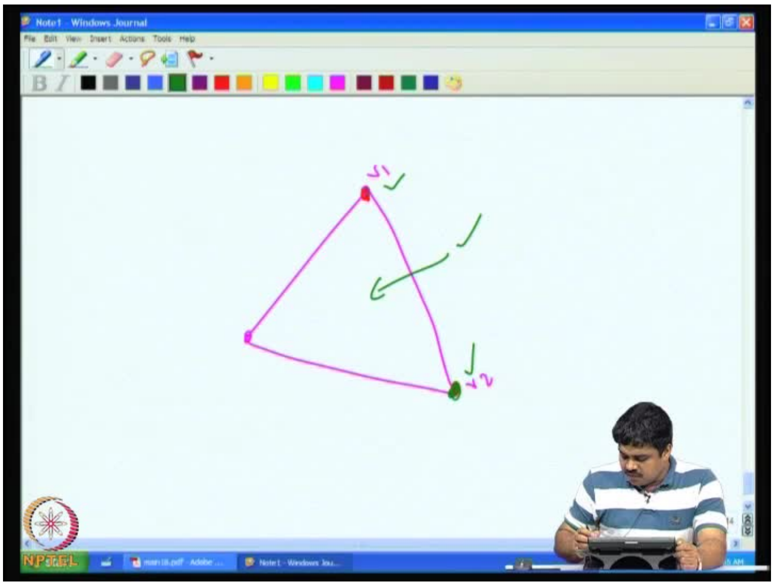

So, if after adding few more edges if you can still colour that graph then the earlier graph can definitely be colours because after removing that edge coloring will not be spoiled. So, the more edges only create trouble. So, what we do we make the planar graph maximal by adding as many edges are possible. So, when the planar graph becomes edge maximal; that means, it is a triangulated planar graph the outer face will become a triangle so, we can say that here right

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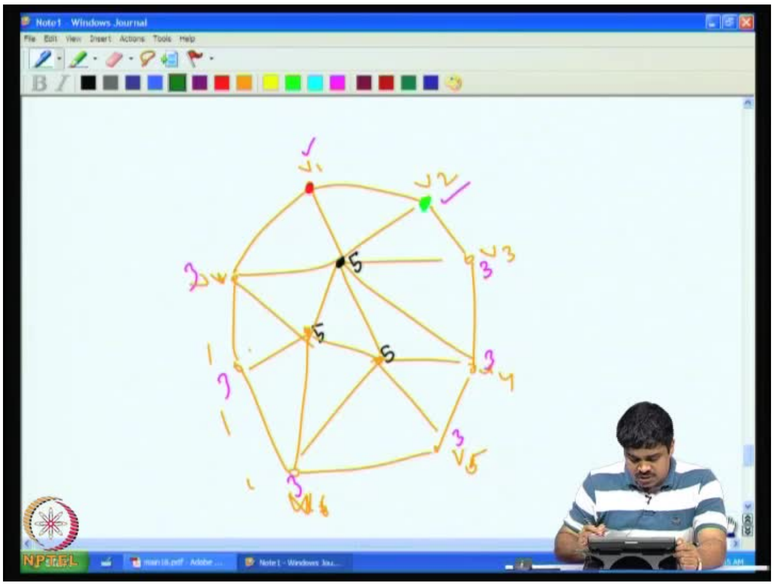

So, already 5 colours are there suppose are 5 colours are given to all of them now let us colour this v one and v 2 with 2 colours say red and green and then you see red green and then here of case this is the all the conditions of this thing this is triangulated graph and then all the conditions of that statement is met this outer face this is already coloured with one thing and this is 5 colours instead of 3 fact all the interior face is have 5 colours and therefore, interior vertices of 5 colours and interior faces are all triangles and here the exterior face the outer faces also triangle that is all therefore, definitely if that statement is true you can extend this coloring v one and v 2’s we has already got colour then everything can get a colour colour from there corresponding this therefore, if these statement is proved then whatever statement we are looking for; that means, a planar graph has 5 choosable is immediate because we just have to consider the maximal planar graph then is because the outer cycle becomes a triangle in that case that is all

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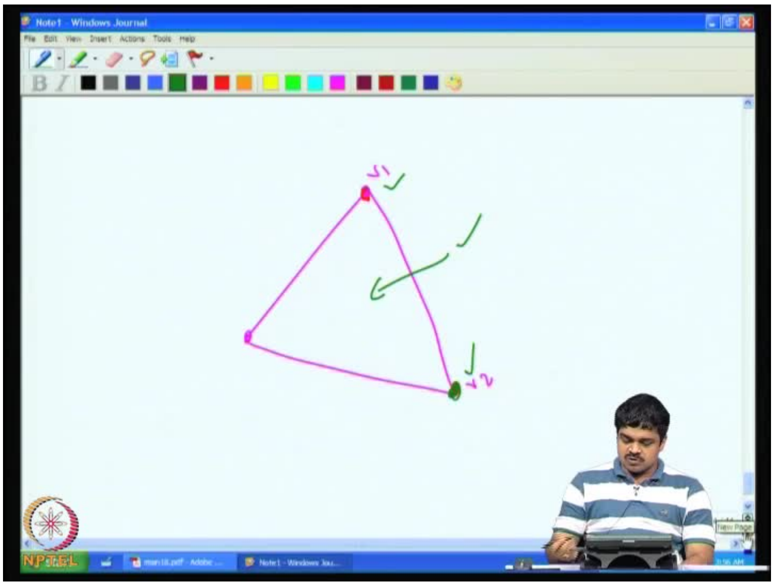

And now we see therefore, we just have to prove this thing. So, essentially we are trying to prove is stronger statement therefore, right. So, because here we have much more things because you know that some other vertices only 3 colours in it in their list so,me the remaining things have 5 colours but, 2 vertices are even precoloured i mean in sense they already got there coloring therefore, it will it will require some more effort will assume that it may require some more effort but,. So, happens that this strengthening helps us to proof it faster because induction hypothesis also in get strengthened right now will let say this is the thing. So, we will consider this outer cycles

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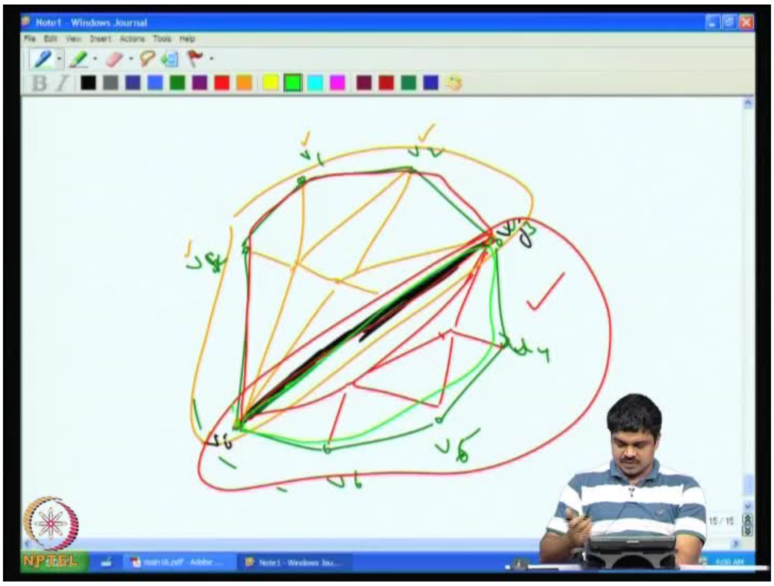

So, outer cycle looks like this. So, this the outer cycle. So, this is v one v 2 v 3 v four v 5 v six and v K right. So, first thing we do is we see that there is any code for the thing is it possible to find some some code say this is the code right code means some vertex to direct edge connecting some v i to v j some v i to v j this can be. So, it is possible to find such an edge right

So, see it not necessary that such a code exist suppose such a code exits; that means, the direct connection (( )) direction connection is not non adjacent vertices on this cycle a connected by a code if such a code exist then what happens says you can see that here there are there are 2 cycles visible right one and this definitely is a smaller number the number of vertices in this induce sub graph will be smaller right here whatever and of case this will be triangle it turned inside right

So,. So, whatever therefore, the all the conditions required for that statement is met because is the each interior face here is triangulated and here all these things of 3 colours associate with that this this can you can give some colours to thing whichever from the list one of the colour without the 3 colours one colour here and then the another colour taken from (( )) and we can get a coloring of this thing by induction hypothesis

Now, once you colour this thing. So, we can consider this portion. So, this portion this portion means here see this 2 vertices already got colours because of this coloring other part and nothing a is shared other then these 2 vertices right these 2 vertices vertices

Now, all the other conditions like interior triangle interior faces are triangle and 2 here also right. So, and here if you go through they also have 3 colours in the list interior face 5 colours only thing is these 2 has already got colours from the coloring of this part this part. So, therefore, , we can apply the induction on this thing on this portion and get it see therefore, if there is a code for the outer cycle then we are easily through the reason is we can that code will help us to see this outer cycle splitting into different cycles one this here and the other this here right and for one of this induce sub graph here this cycle and the interior vertices here of we can apply the induction and get the coloring as we need and then return the colours for these 2 only common vertices and then extend the coloring to the remaining part the green cycle here right

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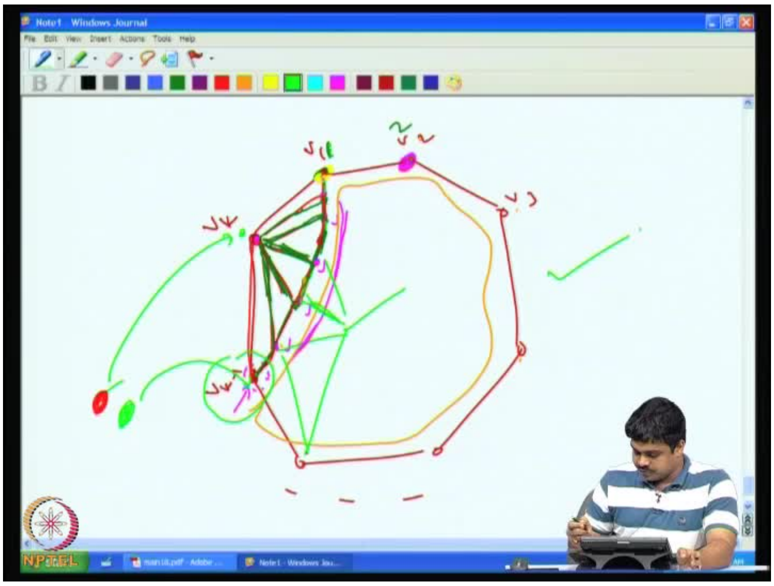

And the interior vertices. So, that cases. So, that cases easy. So, if there is a code then the things becomes easier in this case. So, the therefore, what we are going to do now is to assume that there is no code; that means,. So, this the (( )) this is the cycle the outer cycle this is v one v 2 v 3. So, this the v K. So, v K minus one and. So, on right there is no code; that means, nothing like you can you cannot see any code here in particular if i look at this v K and v K minus one and if you look at the shortest path from v K minus one to v one it will it will contain more vertices right one or more vertices in it it form a code direct on be there

Now, you see that if it is a shortest path right and if you look at the neighbors of. So, this is there is a path here if you look at the neighbors of a v K. So, v K minus one is neighbor one neighbor. So, we get some neighbors here right it'll it will look this and then we can see that. So, the because of this because inner face as at triangles. So, it should right. So, essentially they should be a edge here. So, now, this along this we should get a path like this right the neighbors should along the neighbors they should get a path along the neighbors we should get a path.

It is not even possible that there is there some gap here because i can once i can if i if i if you look at this thing is not possible to have a vertex of this vertex otherwise because how can it be triangulated face right it is not possible. So, should be just including this neighbors we should have a path from v K minus one to v one along with this tuff because that is the all because all the inner faces are triangulated this faces has to be a triangle and this face has to be a triangle then the next has to be a triangle then next has to be a triangle and then finally, it reach here

So, this is the structure we get now what we do it is. So, here see call that by our assumption we have v one and v 2 have colours. So, this is coloured one and 2 suppose. So, maybe you can used 2 colour for this thing. So, this is already precoloured and also this maybe its yellow colour is already there violet colour is already there violet colour is already there now the yah

So, now v one and v 2 already coloured now now what we do is. So, here we will look at the colour list of colours of this there are 3 colours on that right 3 colours now 3 colour out of this 3 colours one of them can be one this colour sorry this yellow but, then there are other 2 colours other than yellow right. So, those 2 colours i will remove from each of this neighbors each of this neighbors except here here i will retain like this because here this has only 3 colours v K minus one only 3 colours we cannot get rid of 2 colours on that because the number colours will go now. So, the v K we will not disturb but, all these case internal vertices they had a 5 colours then therefore, we can get rid of 2 colours from that which are the 2 colours because v K had 3 colours on it

So, other than the yellow colour which is already on the v one we will take out the remaining 2 colours and get rid of all those things there right now what happens is when i the my plan is to consider this this cycle. So, this and entire cycle this cycle this is the route a excluding v K right.

So, now you see that all these v one and v 2 has already got colours v 3 v four all these thing up to v K minus one had 3 colours we have in disturbed them and then these case which had before 5 colours have lost 2 colours on that at most maybe there may not have lost but, then if they had shared some colours with v K they have loss those 2 colours on that right

Now, let say those 2 colours which were removed our red and say green red and green colour were remove from the list. So, they also have because initially they are 5 colours now 3 colours are now you know that by the induction this can be coloured because all these are interior interior faces are triangles and all such such assumption has still true here for this thing and these 2 are precoloured and every every outer face as a 3 colours interior faces 5 colours therefore, it can be because the number of vertices as reduce by one it can be coloured a from the lists right

Now, what we do is you look at the colour by that coloring which colour came for v K minus one see if it is not red or green we can use red or green to colour this thing for the red and green is not anywhere in the neighborhood in this is already coloured one and then this is different from red or green and therefore, this will be valid coloring because if i give red but, in case red or green came suppose green came here then give red to this if red came here then give green to that.

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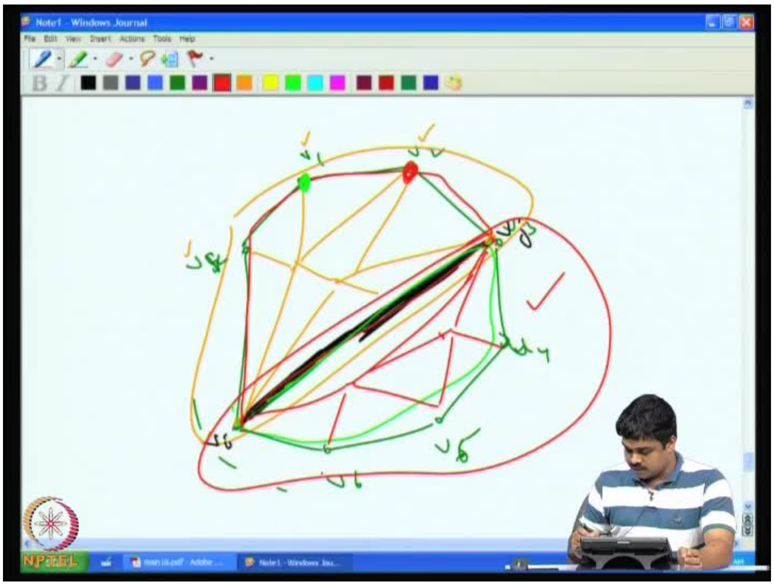

So, it means that we can extend the coloring of v one and v 2 to the entire graph in the previous case when the code is there we remember that the v one and v 2 were precoloured. So, that; that means, this and this is already got colours therefore, we start that we start from this side because we have to extend that first and get colours of this thing and then extend another side. So, i had forgot and tell

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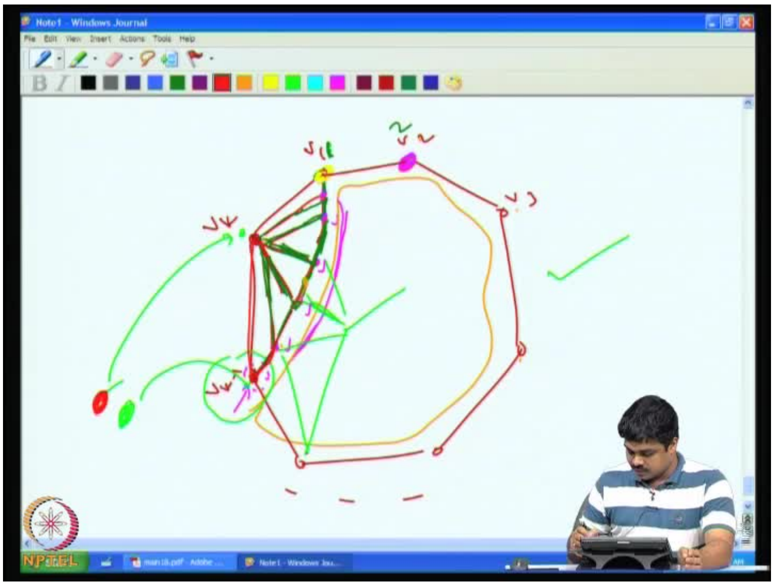

So, therefore, this is the way we prove that planar graphs are 5 choosable and then it also see this also gives a different proof for a the fact that minor graphs are 5 colourable because it the earlier proof for used euler formula and then the the typical the the Kempechi argument we had used. So, the we have use a small different strategy of considering the parts form by 2 different colours and some. So, but, in the last class we had done that but, here is a proof is does not involve any of those things is just induction proof but, here the stealthy in the induction hypothesis and the strengthening the clever strengthening of the induction hypothesis right this is. So, with this we will finish this class and in the next class will consider the list coloring for edges say edge which chromatic index will consider the edge coloring version of the list chromatic number thank you