Graph Theory
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Module No. # 03
Lecture No. # 16
Proof of Vizing's Theorem, Introduction to Planarity

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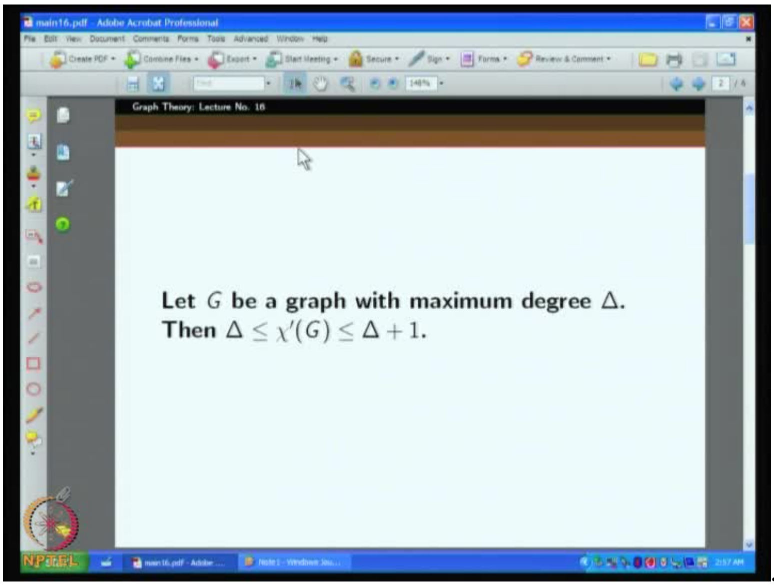

Welcome to the sixteenth lecture of graph theory. So, in the last class we were looking at Vizing's theorem. So, this was the theorem. Let g be a graph with maximum degree delta and then, its chromatic index sky dash of g is in between delta and delta plus 1. There is either equal to the maximum degree or 1 more than it. So, depending on whether delta delta plus 1, we call it a type one graph or type two graph.

So, now we were in the middle of the proof. So, we will look at that once again. But, before getting back to the proof, we were at the the main concept in the proofs, both in the edge coloring and the vertex coloring up. Now, what we have seen was essentially, by considering just two colors, concentrated in two colors and and we will look at, say in the case of vertex coloring, we will look at the vertices color with these two colors, say may be red and blue or red and green. Then, we consider the sub graph formed by those vertices.

So, we can see that because, the entire graph is not colored red and blue. There are other colors also; naturally will not get entire graph. So, it may so happen that it may give us a connected component; some connected components right. Something like this may come right.

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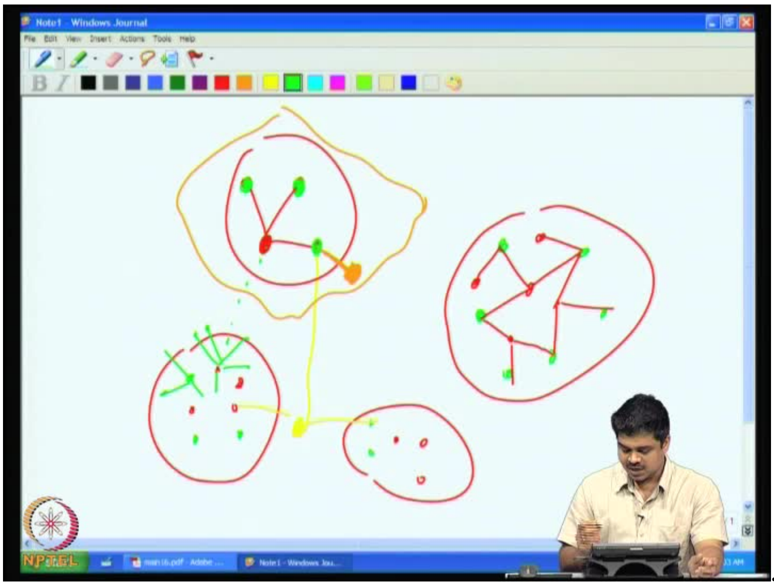

So, right. So then, some structures right. I will just draw like this. This may be the connected components. So of case, this will red will be connected to green, green will be connected to red. How they are connected is we do not know. So, may be some some structures. So, there may be several such structures. So, with red and green, right some connections in between so with red and green and yellow.

So, all the other vertices; may be, if there is a yellow vertices. So, it will be that yellow vertices will not come in any of this thing. It may be connected to this, this, this but, say these are all outside vertices.

So, now the observation that we were making is of that, here it does not matter. For instance, in this connected component of red and blue, red and green, if I change each green vertex into a red and red vertex into a green, is that it would not affect the coloring.

It will still be proper. For instance, I could make this a red and this green. This is actually, interchanging the colors red and green in this component.

Why does not it effect because, other connected components of red and green vertices will not have any edge across this. Therefore, this will not change affect them because, there what do they care. They only want their neighborhood to be different from them. The neighboring vertices should be color different from what color they got, right. Therefore, they do not mind if we change the color of each red vertex to green here, green vertex to red here in this component. Who can the other, the remaining vertices, say for instance; if there is a wrong vertex here, so this vertex may see that there is a color change in in its neighborhood.

For instance, if this was say, so the red change to green but, it does not matter to it because its color is brown. It is neither red nor blue, right. Because therefore, its neighborhoods, whether it is red or green is unimportant to it. So, this change will not affect it that also. Therefore, we can, this connected components, within the connected components of red and blue, we can inter change the red and blue with each other. This was what we made use of in the last proof in the vertex coloring.

So, like many times in the proof of Groggs theorem, so we were studying the connected components formed by certain two colors say green and red and then, containing a green neighbor of the vertex, which you were considering as part of the induction. Then, first we argued the red neighbor also will be part of the same connected component where the green neighborhood comes, right. So, when you consider the red and blue, red and green connected components and then, we argued the that there has to be a path; this neighbor to the other neighbor. So, that was by regress analysis.

But in general, so in that proof, it worked out like that. But, in general, this, in the vertex case of vertex coloring, if we look at the connected component formed by red and green vertices, that can be an arbitrary graph. I mean that, there is no reason to think there. It should have this, it should be path or this should be cycle or it should be of a particular structure. It can be anything right.

So therefore, so, no more help available there. Just that you can exchange colors theorem but, that itself is good enough help. That in many cases, we can make use of that opportunity, that possibility we can manage coloring. But, in the case edge coloring, so, we are, see first of all we should notice that edge coloring is only a special case of vertex coloring. Just that we are the edges, for instance, if we consider the line graph of the graph. What of the line graph? So, each edge of the graphs becomes a vertex. Whenever two edges are adjacent, then they will make the corresponding vertices adjacent. So, that was the line graph. We had studied it earlier.

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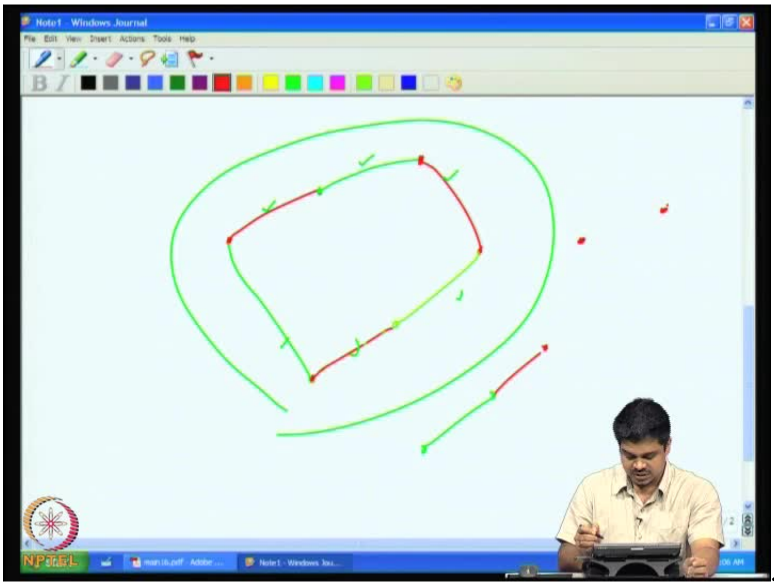

So therefore, it is very easy to see that edge coloring of the graph corresponds to a vertex coloring of its line graph, the proper vertex coloring of its line graph. Therefore, edge coloring is only a special case of this the vertex coloring. So, whatever it is, but if it is edge coloring, for instance if you consider edge coloring, so you can see that, the if you consider the edges corresponding to two colors say red and green, they will not see this connected components, connected sub graphs formed by them. It will not be very complicated because for instance, red and green. So, here this vertex if we consider, there can be only one red edge and one green edge incident on it. So, it is not possible to have another here because, that will not be a proper edge coloring or another red here or red here or it is not possible. So, you can one red here now. So these are all wrong. So this is not possible, this is not possible right.

So therefore, we can have a red edge from here and then, we have a green. That is all. Because, that will, after one red edge, we will see only one green edge. How will it stop? It can abruptly stop. Like, it can go with a stop. Then, I do not see any green edge anymore. So, it can be a path or it can go back and may be complete, a cycle like this. That is also possible right.

This is also these are the only two possible cases. So, you can see that this is completing a cycle. Then, it is a red green, red green, red green and even number of edges are there in such a cycle. Or it can be edges, just a simple path like, yeah, red green path. It starts and some maximal sub graph will be like that right. The connected sub graph will be like that.

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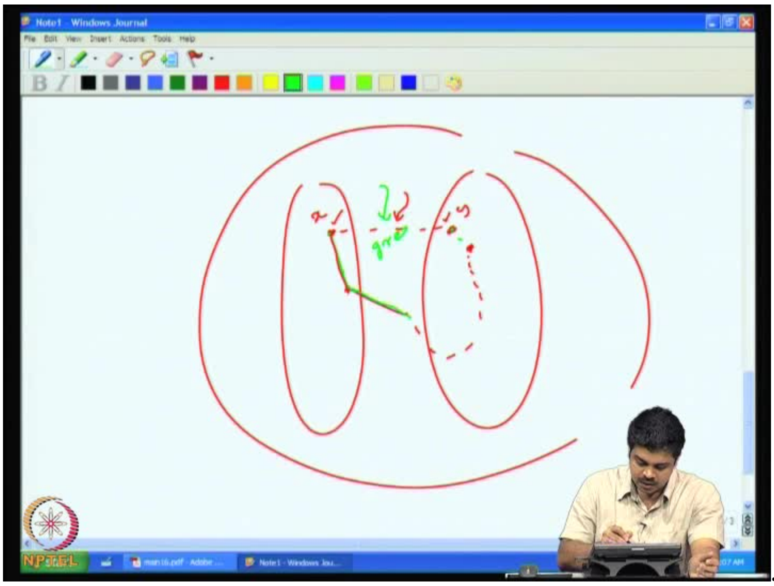

So therefore, in that bipartite graph proof, that means we prove that the edge chromatic index of a bipartite graph is delta. So, we used this property only. We pointed out that if you consider induction, if you did induction, so and we considered some edge and removed it right. This was what we did, x y and then we showed that. See of case, by induction, we have an edge coloring in the remaining graph. Then, so we identified a color which is missing at this point and identified a color which is missing at this point. In fact, if we get a missing color, the common missing color at x and y, then that edge color would have been ok for this edge, because then, there was not conflict at x or y right. We can give that common missing color here.

So, if there is no common missing color. We just took a missing color of x and missing color of y. That means, missing color of y will be present at x, say if green is missing here, green will be present here. Then, suppose red is missing here. So, the red, we can we can keep track of the green red and you know it is a path. We argued that, if it is not coming up to y, so it cannot come up to y because, otherwise will form the odd cycle; it is not end of bipartite graph. On the other hand, if it abruptly stops, it is just a path. Then, we can switch colors on this green and red. Then, here red will come and this will become green and therefore, the green will be missing here. Then, here also green will be missing, so that green color will be available for this. This was the argument of bipartite case.

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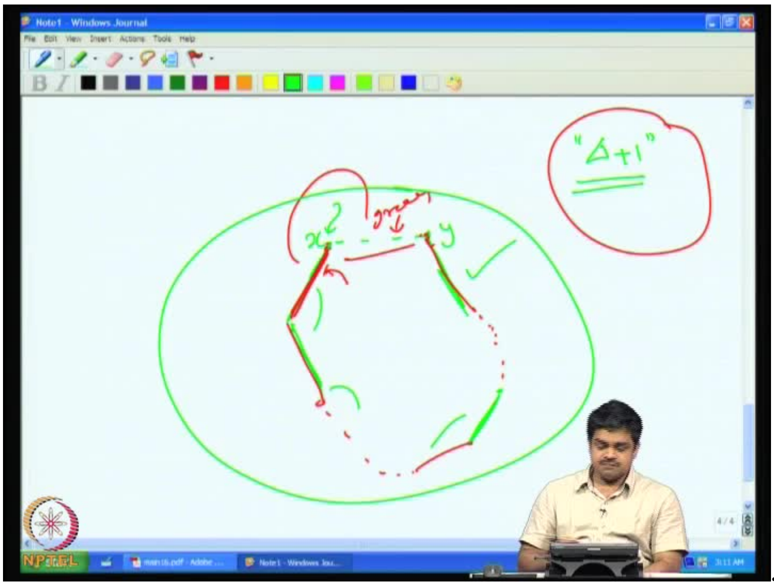

Now, coming back to the Vizing's theorem, we want to do the same kind of trick. So, but this is little more complicated here. Basically, because this is not bipartite graph and the argument that fails is that, see when you consider an edge x y like earlier and you remove it. You consider by induction, an edge coloring which uses only delta plus 1 colors for the remaining graph. Then, if you consider a missing color here with respect to whichever color we consider, so there will be missing color because delta plus 1 colors are available. Then, now the degree has to go down to delta minus 1 here. So, two missing colors will be there. In any other missing color say green is, sorry, red is a missing color here, so and then, green is a missing color here. So, common missing colors are not present. Because, if common missing colors are there at x and y then, naturally that missing color is enough. We can give that color to x y, because would not be any conflict either at x or at y. So therefore, we can assume that the missing color at y will be present at x and the x missing color at x will be present at y. May be, red is missing at x. So, then green is missing at y. This will be there.

And then, if I keep track of the red green missing color, somehow if you could argue that this will abruptly stop somewhere like this. So, and then go somewhere and stop, sorry, so go somewhere and stop. Then, we could have exchanged the colors red and green in this path and then, that would have been enough because here, red will come, here red will come and this will become green and so on. Then, that will that will release the, sorry, that will release the green color and green color would be possible. I mean, for this edge, we can use green color to color x y edge. This was, this could have been possible.

But then, unlike in the bipartite case, we cannot argue that this path cannot reach y. It can reach y. In fact, if we want to argue that ok, indeed suppose for contradiction, assume that we cannot extend the delta plus 1 coloring of this. Underline. So, other by induction, whichever coloring we got for g minus e cannot be extended to g. Then, it means that whichever this, using this missing colors, whichever path we created, it has to, if it is, whatever it starts from x and it should reach at y, right. That is why, when we try to exchange the colors, the missing color, the the the one color appears here but, then the missing color becomes different. But then, now y also loses its missing color and the original missing in fact, the missing color change their role. I mean, x to y they move. That is all right because, the green was missing at y. Then, green will be missing at x. Now, red will be missing at y now, because this will become green after a coloring right.

So therefore, that will not help us here. So, but, one one preliminary observation that we make is that, if you, for inductions sake if you remove one edge and consider green coloring of this smaller graph, it should have this property. If you start from x with a color that is missing at y, and then follow with its missing color at x and so on, that path has to go and end at y. With of case, the color which was missing of x, right. That is a must. This is what we noticed in the last class, first thing. This is the first observation for the proof.

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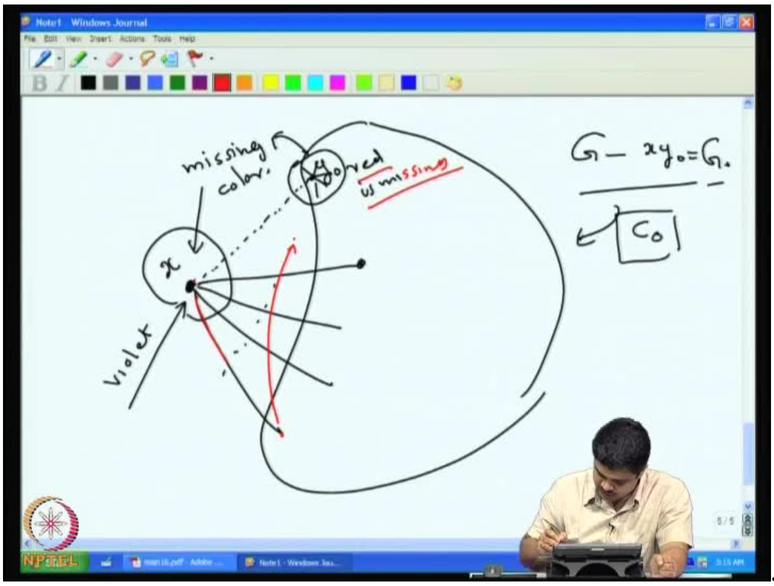

Now, we setup this induction this way. So we picked up an edge, sorry, picked up a vertex, right and then, so we looked at its neighbors. May be, these are the neighbors of that vertex. So, we can call it x and these neighbors, one of them can be called y 0 because, we are going to number them in a particular order.

So, the first thing is to remove this edge, y 0 edge. So, maybe we can put it as, when I remove that edge, I can I can put it as, see this is removed right. So, when you remove, so now, it is g minus x y 0, right. So, this one does have a coloring using delta plus 1 colors. The question is, can we extend the coloring to g. That means, can I somehow find a color for x y 0 also from the, using delta plus 1 colors from the, using colors, right.

Of case, we will look at the the edges, the colors that are used (( )), edges of x edges incident on x. We suppose, we find out a missing color here, missing color here, right and we look at the neighborhood of y and we find out a missing color here. Suppose they were common, there was the missing color y 0 was the same as the missing color at x, then that missing color would have to been enough for coloring this edge, right. But, then that we can infer that color, that is missing at x say suppose, I will say that this color, so this violet color is missing it. This color is missing at x. So, that means none of the edges that is adjacent on x is violet, right. Now, what I say is, so this color violet color is missing at x. Now, you can say the violet color is presented y 0, right. Violet color is present it but, whatever it is, it is clear that with respect to this coloring c 0 c 0 y because, c 0 is the coloring which by induction be picked up for this graph. This will be called g 0. This graph by x y 0 is deleted.

And now, what I what I am going to do is to search for an edge among this thing. For instance, this y 0, at this y 0, there are, there is one is missing color, right. May be, at least one missing color is there, in fact two are there.

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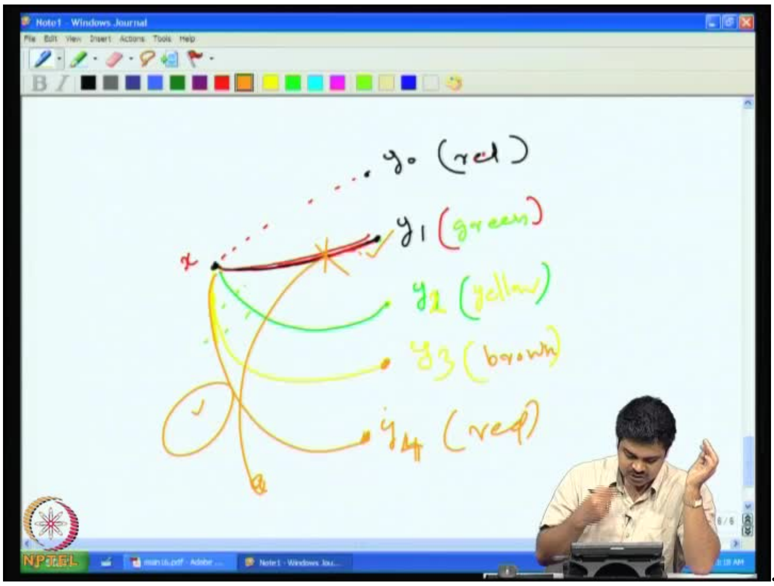

So, let say this is red color missing, red is missing here, red is missing here, red is missing here, red is missing here. So, suppose red is missing here. If red is missing here and then what happens is, so you can, you know that, if red is missing here, red cannot be missing at x. So red, it should be there, somewhere here right, among this. But, so we will pick up the red colored edge and then, let say, we will, most of this to here. In fact, in other words, I will rearrange the neighbors in such a way that, so this is x right. So here, we have the y 0 and missing is red. Red is missing here. Now, I will pick up the the red colored edge here and then, I will call it y 1. You remember this was, this edge was essentially, you note there, this edge was like this. So, this was not present in g 0. But, you see, this is a red colored edge. This is a red colored edge. That is and this is a, red is missing here.

Now, we will look at what is missing here. May be green is missing here. Green is missing here. Now, because green is missing here, so this should, some edge here among the x, which is green. So, there should be some edge which is colored green, right. So let say, this is green colored. I will pickup that edge and then bring it here.

Why should it be there? Because, if green is missing here and green, then green cannot be missing at x; it should be present at x. There should be at least one edge. So, then I will bring that green edge here and call it y 1 and then, will see what is missing here. May be yellow is missing here, yellow is missing here, right and this is, sorry, y 2 and then, we will search for the yellow colored edge here. There should be a yellow colored edge.

So, yellow color edge, if I get here, then I will bring it here and I make it here y 3. Then, I will look for the brown colored, right. It may be brown is missing here. Then, I will look for the brown. So I will keep doing it until I can keep going. When do I get stuck? See the, see as I take more and more edges, the remaining edges will finish of and it may so happen that the missing color, it has to be present among the edges of x. But, it may be already taken. It is possible that here the next time, I may see that the red color is missing here say y 4, red color is missing here but, red color is already taken right.

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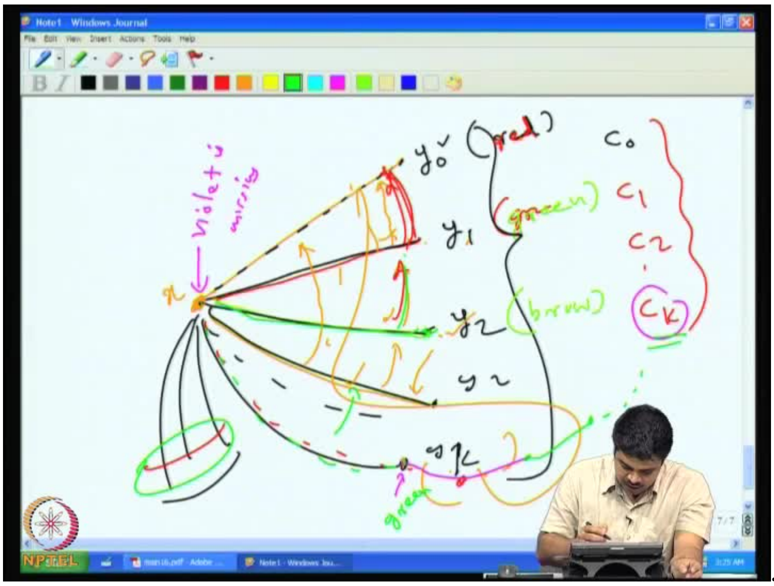

So we we cannot take it again and put it back here. In fact, this is cannot to be moved to here. This has already got a position. So, we will stop there, if such a situation comes. If missing color at this point is already taken, then we will stop there. So, the situation now is like this. So now, what we have done is, we picked up this vertex x and then, we of case, this was the edge in g 0; when we remove we got g 0 and this is y 0. Based on this only we got the coloring that is c 0, the base coloring of which we are starting.

Then, we are arranging the neighborhood of x. Not necessarily all the neighbors but, yes bigger sequences possible, like this. So for instance, this y 1, this is y 2, this y 3 until some y k y k. Then, there can be other edges but, we are not bothered about that, right. We are, we will play with this y k vertex. But, what is the property of this y k neighbors. So, that color with respect c 0, what is missing here? So say, for instance, if red is missing here, then this is red missing here, and then this is red colored, right.

If suppose, now green is missing here; so green is missing here. Now, this is green colored right. Suppose brown is missing here, then this is brown colored like that. So, this is the way we have selected it. What color is missing at y i will be the color of the x y edge. This is the way (( )). What happens at y k is that, y k is that missing color here, is already among this. So, somewhere here. The missing color of y k here is present somewhere here. That is why, we could not take another way because, so it is a missing color and should be present at x. Because, otherwise, what we can do is, the otherwise what we can do is so we will we will move moves, we will put back these edges x 0 and then give the color red to x y 0. Because, any way here, it was missing and then x will not mind because, x is already seeing a red. If you move it here, it is ok and here this will be colored with this coloring.

In fact, the green will move here, because y 1 will not find any problem because, missing here. Say, x also will not find any problem because anyway just moving colors here, red, rotating colors like and then this is will be moved. It will be some kind of, we will rotate with respect to this x. We will be rotating color colors towards x y 0. So, because x y 0 did not have any color before, so that x y 0 will get the color of x y 1 and x y 1will get the color of x y 2 and x y 2 will get the color of x y 3 and so on, right. Therefore, what happens is, x y k will become empty; color less and then, it if it so happens that the missing color here is missing at x also, we could have given that color to. Therefore, it should, if we cannot do this thing and get a coloring for g; it means that, it will never happen, right. Because, the color should be there at x. But then, just that is not in the remaining edges, it is just that we have already seen that color somewhere. So, we will not be able to color the x y k, it should be that. That is why it happens. So, the trick here is a kind of a rotation. For instance, in any of this y 0 y 1 y 2 y k, see you, so the, what you can say is, you can empty, see x y i edge. So in other words, you can derive a coloring from the base coloring of c 0, say for by rotating. Once if you rotate, you will get the coloring c 1, that means x y 1 color will go to x y 0 and this will become empty.

C 2 is obtained by rotating two times, this will go here and this will be going here and this will become empty, that means colorless. Similarly, c k is a coloring obtained by rotating it k times that means y 1 goes to y 0, x y 1 goes to x y 0 and x y 2s color goes to x y 1 and so on. x y k is color goes to x y k y k minus 1 and x y k. This is some kind of derived (( )) but, rotating.

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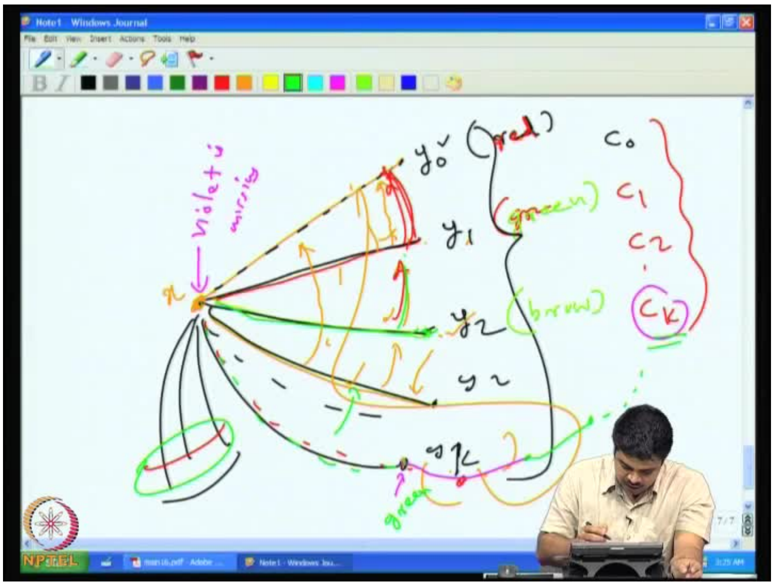

The good thing about these colorings c 0, c 1, c 2, c k, they are coloring so, very related graph. Essentially, we will see some kind of way of deriving this I th coloring from the base c 0 coloring by some rotations, right. By i rotations, we can get it get it back. So, they are all the colorings of very close graphs. Just that one edge is missing. But, very, you know the edges that are missing are, we understand which edges are missing and then, we know how to convert a particular coloring to the other coloring and all, you know by its rotation strategy. This is going to help us, right.

Now, the final step for all these things is a setup. How will you prove it? Now the point is, I will, I will look at y k and then there is a missing color at y k and there is definitely a missing color at x 0. So, we already told that let it be violet color is missing here. It is missing right here.

So what should be violet in y k? y k violet should be there because as I told, because if I had considered coloring c k by rotating and emptying the c y k, the violet is missing at y k also and then, violet color would have been appropriate for that, right. We will rotate it and create my c k, where all the other edges x y 0, x y 1 all of them are colored but, but, just x y k is not colored. There, I would have given the violet color to it, right. Is it not?

So therefore, so, it is, if violet color is missing at x, and then violet color is present here, but, then I can start. So, there is some color which is missing at here. Say, let say the green color was missing at here; suppose green was missing here, green was missing here, then I can start a violet green path here, right. Where should it reach? So, because this green color is missing here and violet color is missing here, as I told before, this should go and enter x and reach x somehow this path, violet green, violet green, violet green path should somehow reach x. When in c k, in c k, when I consider the c k, when I have rotated all the colors and now the empty edges this only, right.

So so, I should somehow enter it. We know that the violet color is not among these, because in that case, this would not have been the last in the sequence. I would have put that violet colored edge also after this. So, violet color is somewhere here, sorry, not violet color, green color is somewhere here mainly, say this is the green colored edge, right. So, somehow the only way to enter the system and reach x is to go through this green edge. That is the only way because, violet is anyway not there. The violet green path has to enter through a green edge.

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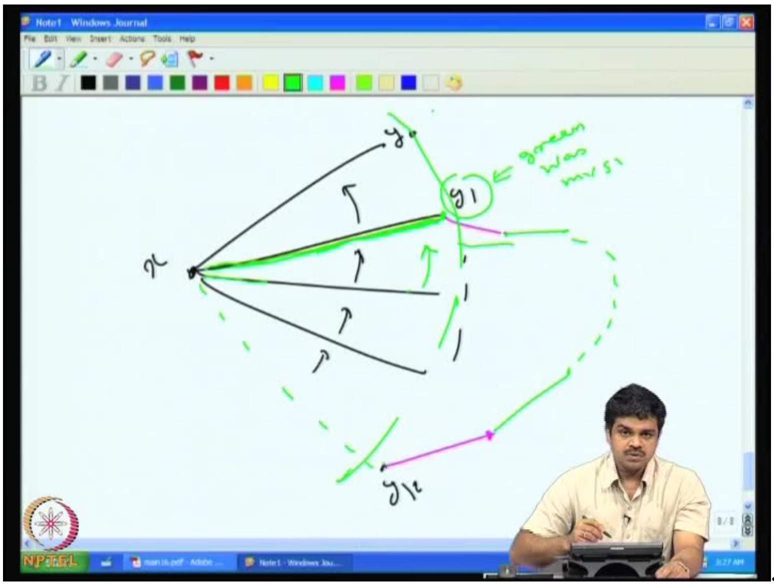

So, it will somehow reach, you see, because a y 2 was this thing. When I created, I have rotated all these things. Now where will, so y 2 had these colors. So let us say, I will try it once again. So this is x, this is my y 0 y 1. So here is the y k. Now, I have rotated the colors. So, therefore, see initially green color was here. But by rotation, this is gone here, right. So, because essentially this was the the green that was missing here, green was missing here. So, that is why the green is available here. So, by rotating then, this it took the place. Then here of case, here we have, you can see that here we have the dotted edge now and here we have the dotted edge that means the missing edge. Now, we trace the violet green path, green path in this and definitely it has to reach here by a violet green path, right. Somehow, it should reach here and it should enter through this thing.

So this is, this entire sequence violet green, violet green is outside the system of (( )) here because, we have this x and its neighborhood, like this edges is incident on this. This is not participating in this part and the only place to participate is this. Once it participates, it reaches x right.

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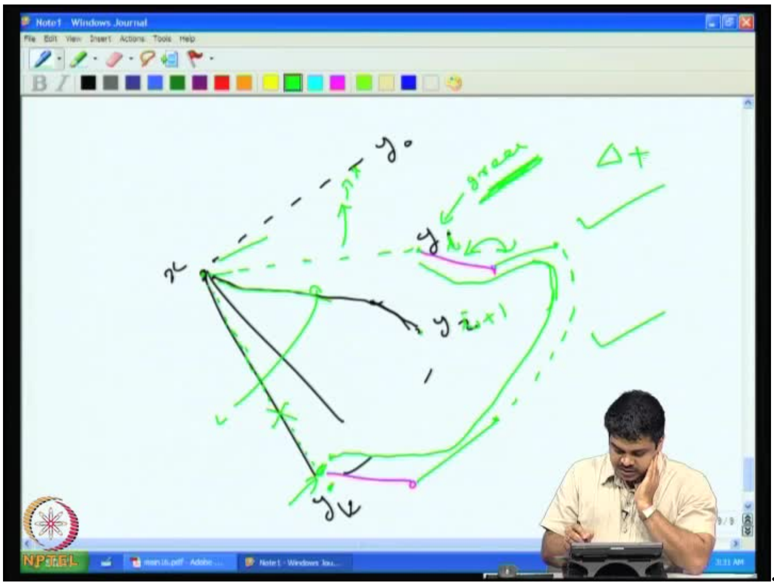

So, therefore, so when I start the violet green path from y k edge, it should reach y 1. For instance, if I had started the violet green path from y 1, it should reach here also, right. Because, it is totally outside and if after deleting this thing, now, what we do is, consider again the original coloring. So this is x, right. So now, our, this is the original coloring. So now, I am going back to the situation where this is the missing color. So, this is the missing color. So, this is y 0 that is my base coloring, of case c 0, so this is y k, right.

So, now you see the green color was here, right. Now green is missing here. That is why green color was there. Now if I considered, when I consider consider the violet, which is miss, which is present here, violet green path from here, you know it is reaching here right, violet. We have already seen that it will enter through a violet green path here. It it is reaching here. But, then the green edges, now here because we rotated back right, by one step.

Violet is missing here; therefore, violet should be missing here. Violet should be present here, we know that violet is present here. Then violet green path, if I trace, what should happen? We know by our earlier argument, this violet green path should enter x somehow. But then, if you trace the violet green path, we know that it has to reach here, all the way to y k, right. There is no other way because, we already seen that there is the violet green connection from y k to y i and there is only one such connection and it will reach here.

But, we know there is no green edge to enter here. This edge is not green right, is it not? The green was missing here, is it not? So therefore, we will not be able to enter here, green was not. So this is a contradiction. This is a contradiction, in fact that while green path should reach x 1, so this is a contradiction. Then therefore, we assume that we will be able to swap colors here. Now, violet and green, and then make violet available for violet will be missing in y also as well as x. Therefore, violet will be, we can give violet to x y x y edge. This is the proof.

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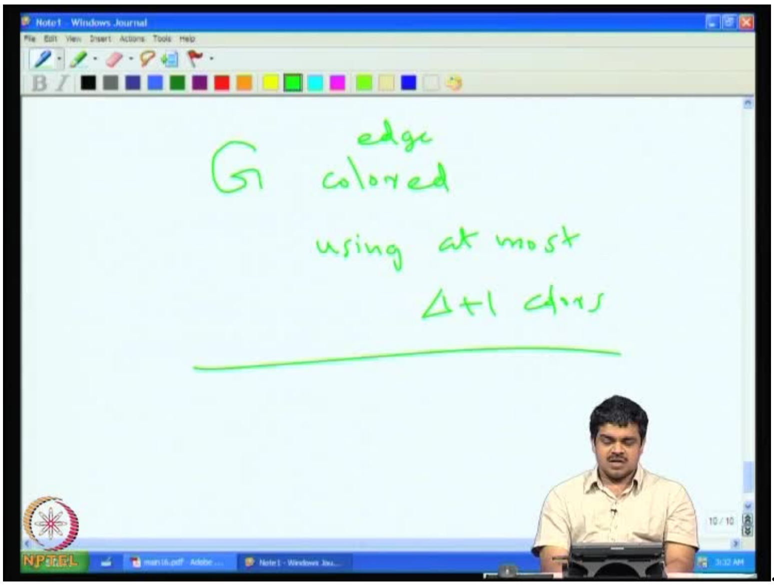

So therefore, we see that this can be colored in delta plus 1 color, induction completes. So what we have now done is, the proof that any graph g can be colored using edge colored using at most delta plus 1 colors. So, we will leave edge coloring here. Then, now we look at some other topics related to coloring. Like for instance, when it comes to coloring, may be one of the most important graph class, special class of graph, is planar graphs. Why is it? Because the most famous problem in graph coloring theory probably in graph theory itself is, the four color problem.

So, it asks about the chromatic number of a planar graph. What is the chromatic number of planar graph? So, the conjunctive was that, the planar graphs can be colored using at most four colors. So, this conjuncture was started in the 1800. So when (()) wanted to color the map, the the countries, the world map with just four colors, such that the neighboring country gets different colors. You can easily see that this will convert to, for instance, each country is considered a vertex, and if we draw a graph such that two vertices are connected. When the corresponding countries are the countries corresponding to those two vertices are neighboring countries, then we will get a planar graph. It is not very difficult to try that out and figure it out that is a planar graph.

So therefore, this problem of, so he asked whether any map can be colored with at most four colors, only four colors. So, this became the famous four color problem and then it took so many years almost a century or two to come up with the proof. So, we are anyway, so we will not do a proof of four color theorem because, here it is quiet complicated and long. So, it is what we are going to do is to see how it can be vertex colored using six colors, five colors etcetera. Four colors, you will leave out because it is not possible to do it in a class.

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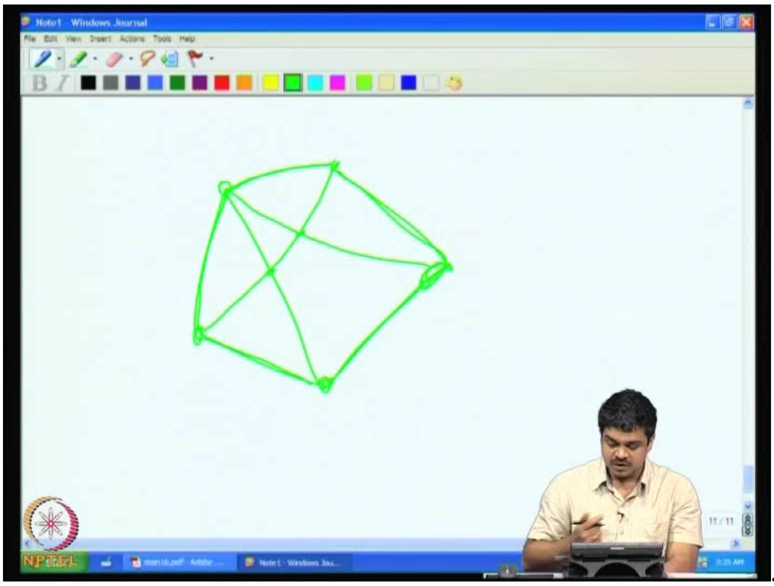

So let let us explain, what are planar graphs? First, (()). So, the graph drawn on the plane in such a way that no two edges intersect, other than the end points is called a plane graph, right. So, we can say this is, right. If I am drawing a graph like this, so now you can see, that the graph is drawn such that if you take two edges, they do not intersect at all other than the end points together, end points intersecting. So may be, you can draw like this. So, this is like that, but then when I draw like this, then this edge is intersecting, this edge right. See, therefore, this is not a planar graph right.

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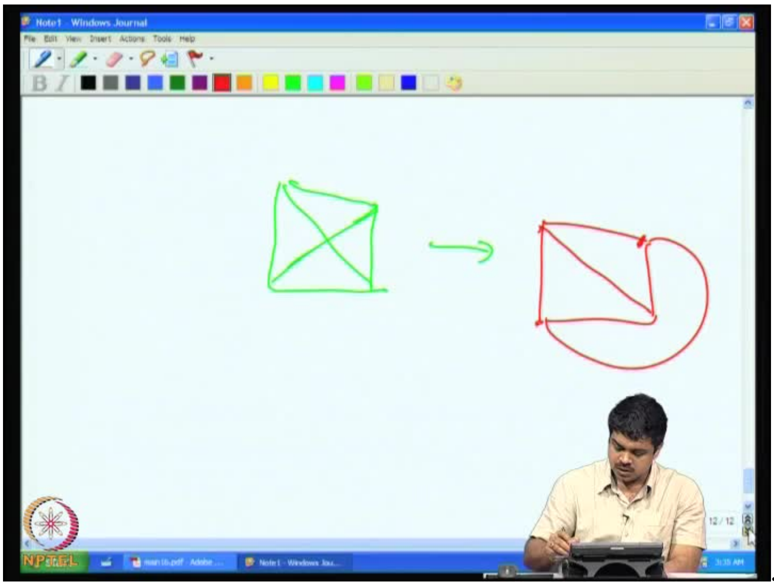

So, the planar graphs are those graphs, which can be drawn on the plane, is a plane graph. That means without the edges intersecting, edges are crossing each other. For example, if you take this graph, so this is not a plane graph. It is a drawing because, we have drawn it only. On the other hand, this is the planar graph because, we can draw it like this. We can draw it so, this is the graph right. We can draw it like, it is possible to draw it like this right.

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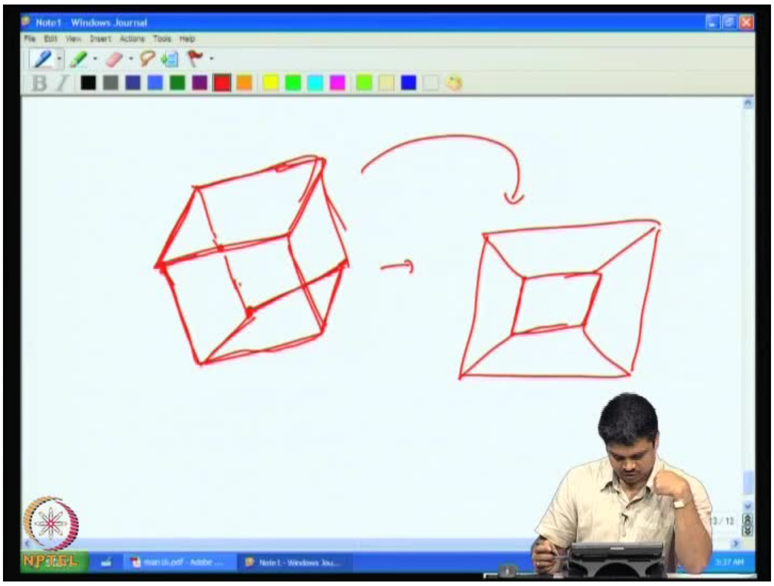

So, now no two edges are crossing each other. Now two edges are intersecting with each other, other than the end points, right. So, for instance, so what about this one? This is a typical non planar looking figure. So, in the sense, that ok, some kind of a cube right. So structure of a cube is drawn here. So, is it possible to draw it on a plane? Is it a planar graph? See now as we have drawn, it is not a plane graph because so, the edges are crossing here like several edges are crossing each other here. So, this edges is crossing here, these two edges are crossing here. So, this is not a plane drawing. But then, it can be drawn on the plane like this. So what we do is, this lower square can be drawn like this. Now, this upper square can be drawn like this and then, see these connections. These connections, these connections, these four connections, we will draw like this.

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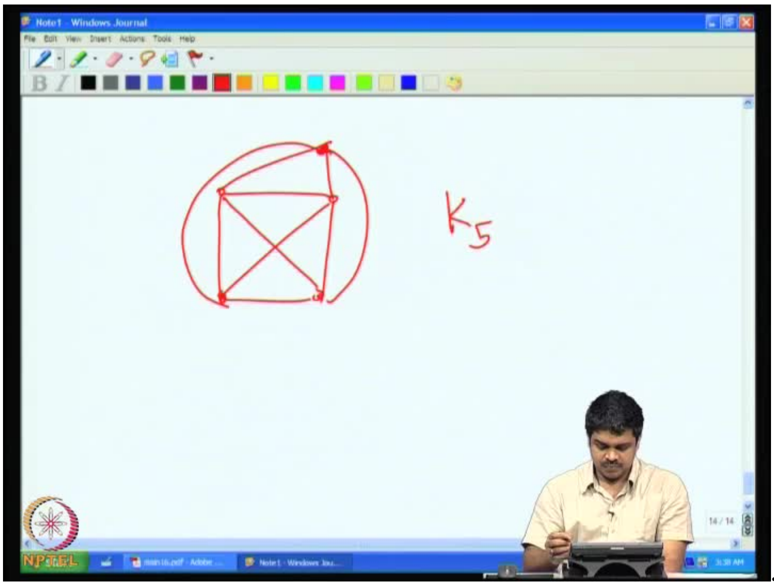

So essentially, this is same. So, we as if, we have just projected it with a little, so this upper faces become like this, right. So that, at one look, some graph may look non planar. But, they can be planar, so like this. So, it will take some effort to figure out whether it is, whether it has a drawing, it has a drawing or not. It has a drawing or not. For instance, you can you can ask whether this graph, so this is what and then, we are here. This is a k 5. So this is a k 5. k 5 means, the complete graph on five nodes. Is it a planar graph?

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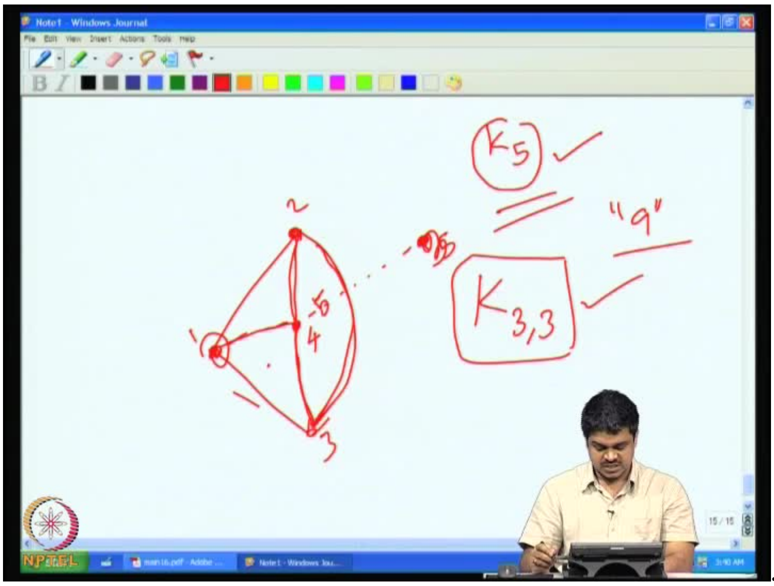

So with, you can try it out whether you can draw it on the plane. But, after some effort, you will see that it cannot be drawn on the plane. So for example, you can argue like this. So you take a triangle. So k, from k 5 you take a triangle. So triangle, anyway, however you draw, it will form a closed region, right. It may, it may be like this, some triangle 1 2 3.

Then, the fourth node has to be either outside the triangle or inside the triangle. Say, if it is outside also not a problem, inside also because, finally, outside and inside you can, with some careful, if you carefully look, you can see that it does not make much difference. So, suppose it is inside right, then 4.

So of case, so this is probably smallest in the number of vertices. This is the smallest graph on the, with the smallest number of vertices which which cannot be drawn on the plane, right. So, there is another graph called k 33 which cannot be drawn on the plane, which I leave it to you to figure out, as to why it cannot be drawn on the plane? This has nine edges; this is the smallest graph, in the essence number of edges which cannot be drawn on the plane.

Now going back to, this is a just an introduction to what kind of graphs or planar graphs. So, my intention is not to give a rigorous treatment of planar graphs. So, I recommend to read the chapter from Reinhard Diestel's graph theory text book, to get a rigorous treatment of this subject. But, to avoid the boring details, I would rather stick to some intuitive explanation about it.

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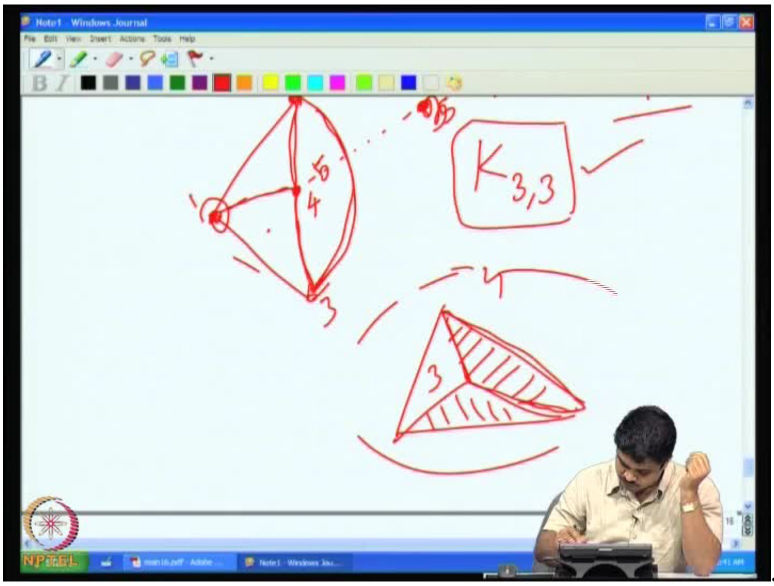

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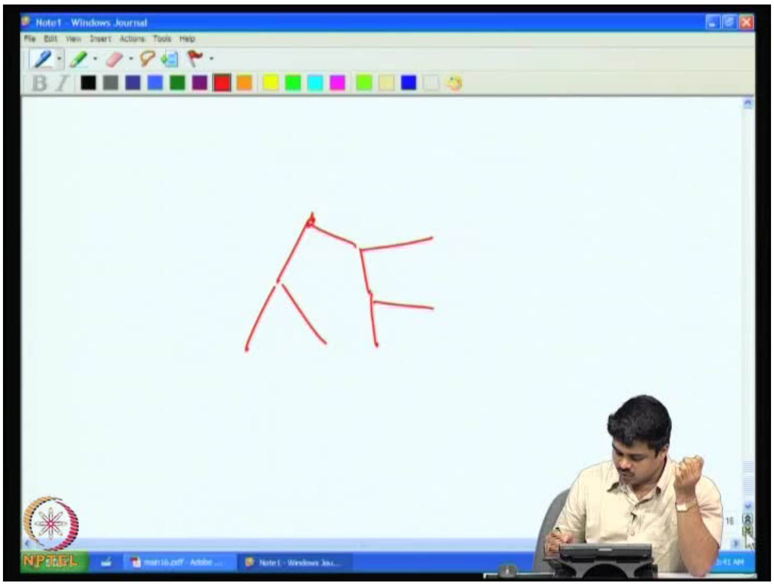

So, so first thing I want to consider is a famous theorem, famous formula regarding planar graphs. So, it is called Euler's formula. So see, in this cases, see in this cases, we have seen, consider a plain graph. See in a drawing like this, this kind of maximal connected regions, right. So, this this kind of (()) are the faces, these are the faces of this thing. The first, second, third, and fourth face. So, these are the faces of this thing. So for instance, if it is a tree, then it will be only one face right. So, this entire one face is connected.

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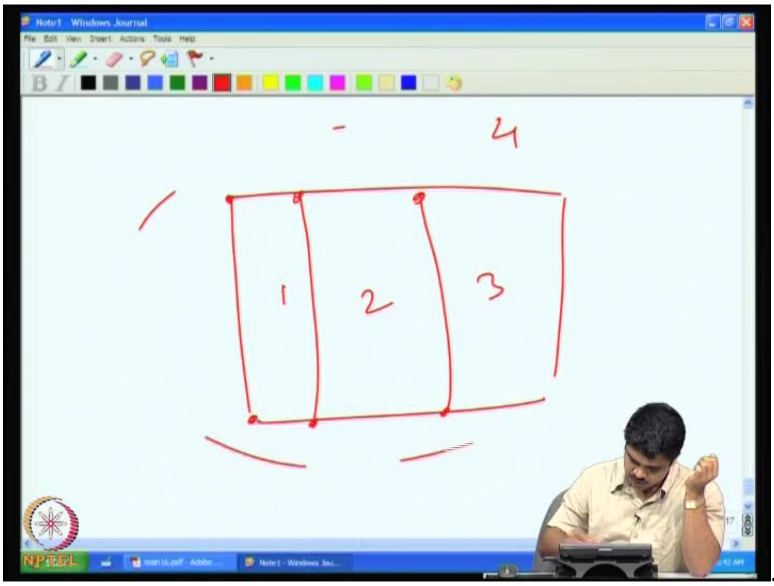

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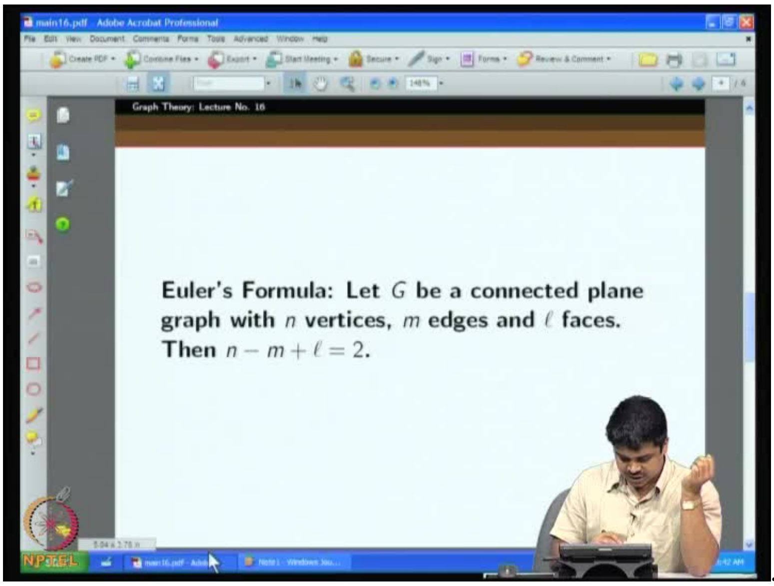

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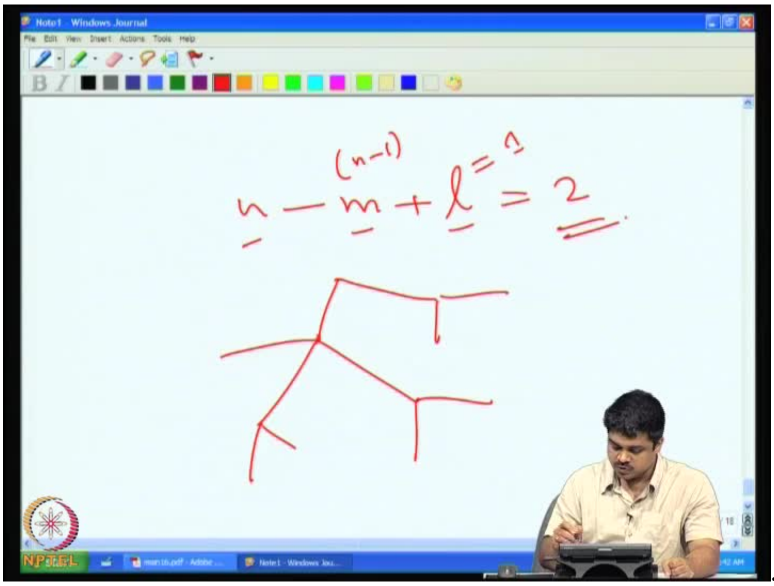

So, see now the euler. So, again another another example. Here, how many faces are there? Therefore, for instance first face, second face, third face, fourth face outside, this is one face right. So, now the euler's formula tries to connect the number of vertices, the number of edges and the number of faces in a planar graph. So, it says n minus m plus l equal to 2, n minus m plus l equal to 2, where m is number of edges, n is number of vertices and l is number of faces.

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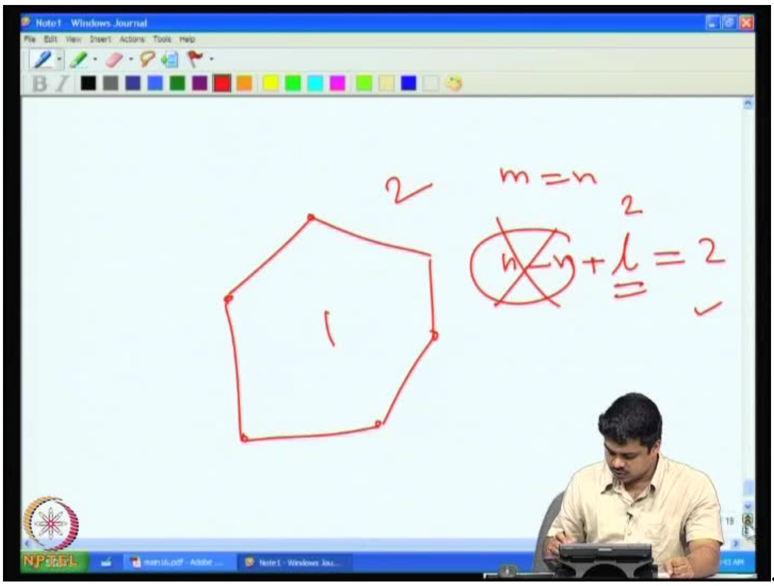

For instance, if you take a tree in a connected planar graph, not in any plain graph, so here number of faces l equal to 1 and here is number of vertices is n. Then, then how many edges? n minus 1. So, n minus 1 n minus 1 is 1 and another one is 2. So for instance, you can take a cycle. Say cycle, there are, number of edges equal to n here. So, there are n vertices and n edges. So, n minus n plus l equal to 2 is what it says. So, what is 2? l, l is number of faces. There is a one face and second face. So, this is 2. So this cancels of, and so here, that is also correct.

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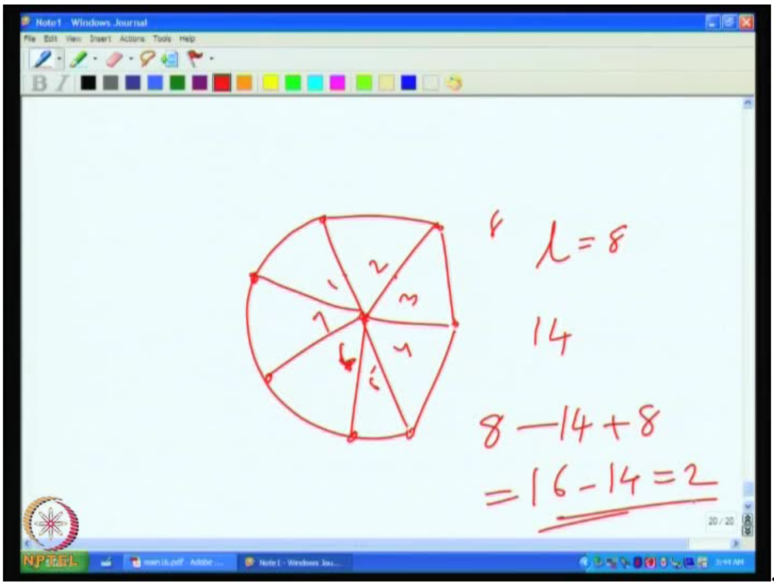

So, this is what euler formula says. We can try some other examples also. For instance, we consider a wheel graph. So, here it is essentially 1 2 3 4 5 6 7 faces here, right. 7 plus 1, 8 faces here, l equal to 8. Now, how many edges are there? 1 2 3 4 5 6 7 edges, 1 2 3 4 5 6 7 edges here and then, similarly 7 edges there, right.

So, we have 14 edges. So, number of vertices is how much? 7 plus 1, 8 minus 14 plus 8, it is 16 minus 14 equal to 2. So, I just wanted to make you familiar with the parameters. I just gave some examples. Now, let us look proof of it. It is very easy, in fact. So you can fix n, that too I am doing a very informal proof. Not that, I do not get into the details of, topological details of why certain things faces are like these and all. So intuitively, I will tell why it is.

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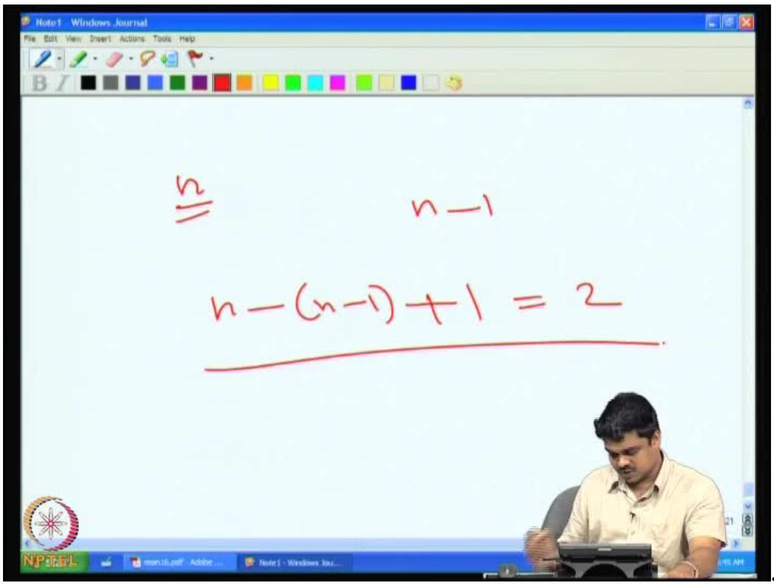

So, suppose we consider n vertices. You fix n vertices. Now this, you can do induction on the number of edges. So for instance, the connected graph, the minimum number of edges possible is n minus 1, right. So, we have already seen it is a tree. So, the number of faces is 1. So, we have n minus n minus 1 plus 1 equal to 2. So that is correct.

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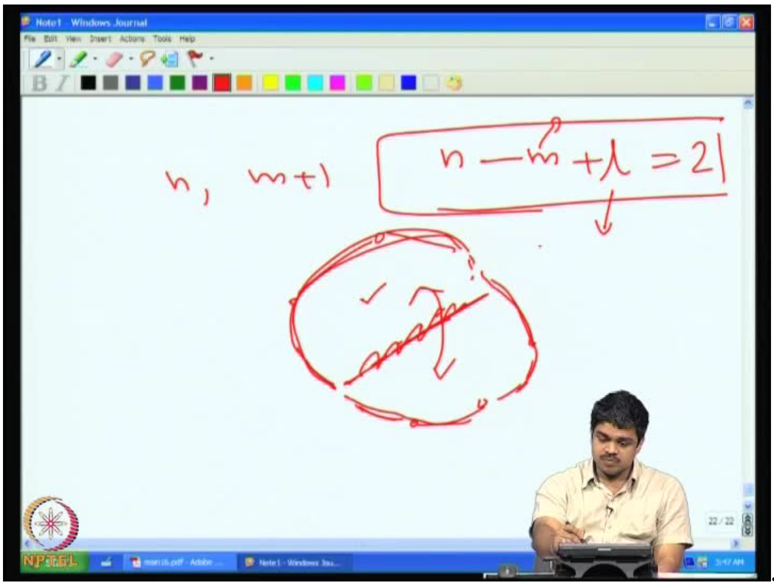

Now, for m edges it is true. Now, if you consider on n vertices m edges, sorry, m plus 1 edges, m edges is verified. So now, what do we do? We see that because there are more than n minus 1 edges. There is a cycle. So, you can remove one edge from such a cycle.

So easily you can see that. See, when you remove one edge, the number of faces, so increases. For instance, if this was the situation before, so if you remove one edge like this, so these regions, these two regions will combine and will become a bigger face, right. That means, the number of faces will decrease by 1. While we have added, sorry, we have reduced the number of edges by 1. So in other words, so for instance, compared to the earlier one, wherever this face, we have, when we added this edge 1, the number of faces increase by 1 and number of edges increase by (()). If you look at the formula, it is n minus m plus l, right.

So, if this increases by 1 and this also increases by 1, they cancel off. So that 2, earlier it was just 2, then it will remain 2. So, that is the induction on the number of edges and the number of faces. So because, as you add, so suppose earlier you had this formula correct n, n is fixed for a smaller number of edges is correct and we are adding one more edge. So, this will tend to go down but, then the number of faces increases to balance it. That is the only thing.

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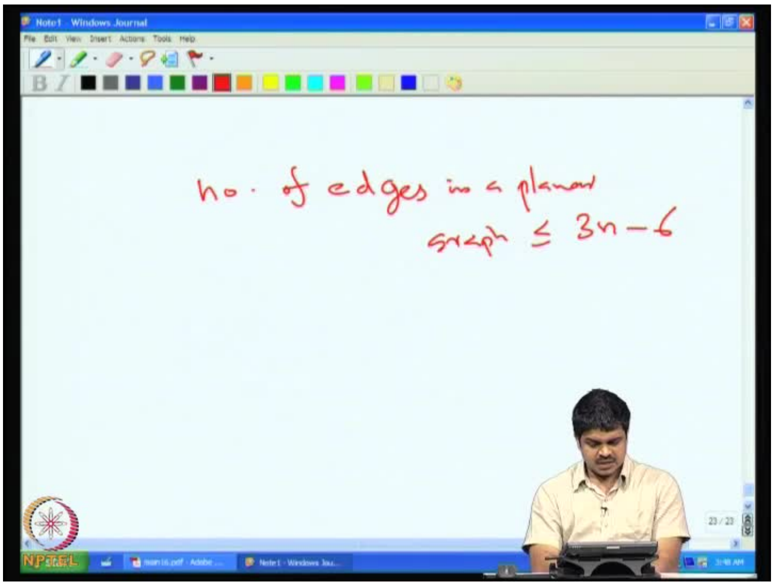

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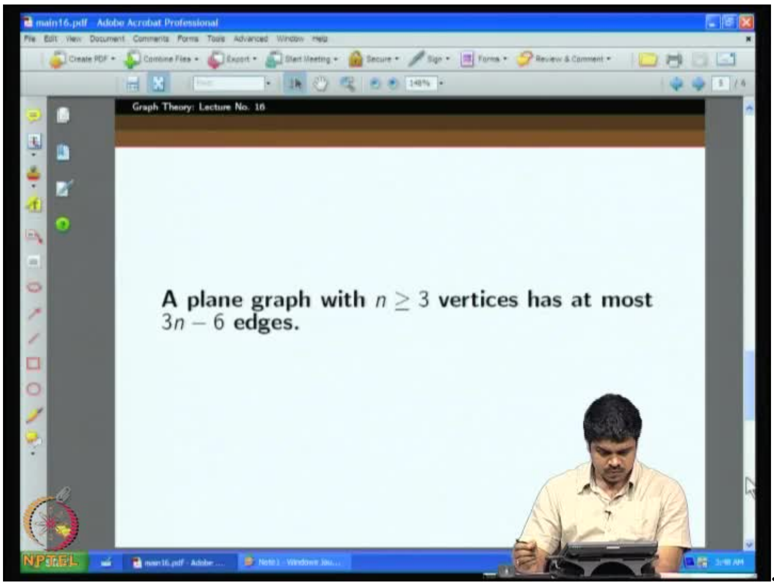

So therefore, the euler formula is correct. Now, we will show that from using the euler formula that the number of edges in a planar graph can at most be the number of edges in a planar graph. In a connected graph, it is at most 3 n minus 6. So of case, so here to do this thing, so let us say, so the number of a plane graph with n greater than or equal to 3 vertices has at most 3 n minus 6 edges. Simple. So this is.

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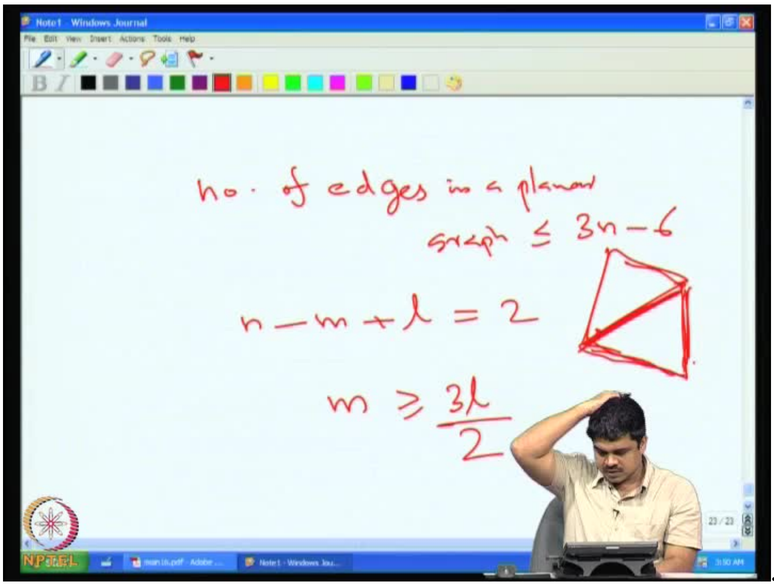

So here what are we going going to do? Here we can see that, see the euler formula tells us n minus m plus l equal to 2. Now each, so we can always, we want to convert, we want to get read of this l. Now you know that every edge is part of so, so for instance, if you take, if you want to get how much is m. So, we can say that each face will contribute at least three edges. So there should be at least three l edges, three l edges right. You may tend to think like that. But then, because the same edge, see for instance, two faces may say that this face gives one edge here, but then this face give same edge here. So, we can tell this thing, right. So, number of edges has to be at least 3 l by 2, right.

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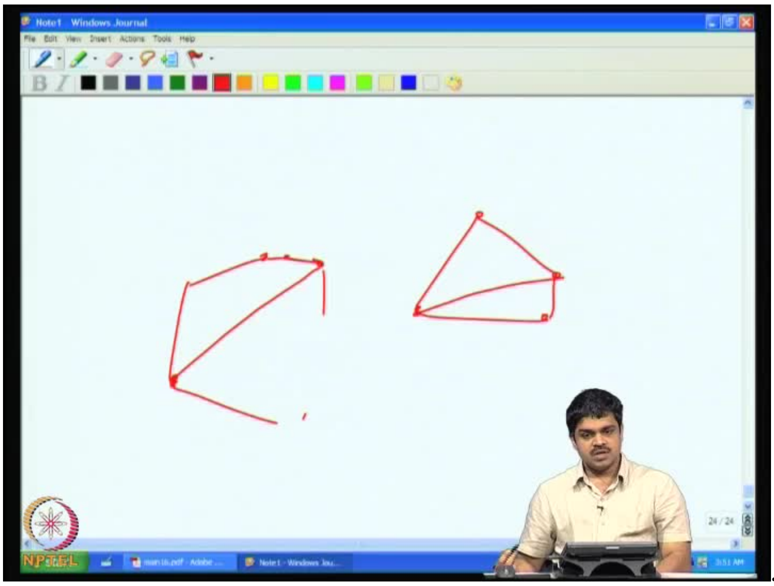

So, right. So we can, m is greater than equal to 3 l by 2. So, what we do is initially correct. We can add edges, so that it becomes maximal right. So, you can easily see that there is no so face which does not give (( )). It becomes triangulated because, if it is not triangulated right, you can see that you can add every faces a triangle, you can see finally, right. If it is not, every face is not triangle, and then you could add one more edge, right. What we are going to do is fix a planar graph. So, let the number of vertices be n. Now, we will add edges to this planar graph in such a way that the number of edges is as big as possible, as much as possible.

So in other words, it becomes edge maximal with the property of planarity right. So, it is very clear that any every face has to become a triangle at this point. Why because, if it is not a triangle, so will see a face like this right. Somewhere, so then you can always add connector to such edges and put one more edge. It is possible to put one more edge without (( )) because inside the face. So therefore, it becomes a triangulated graph and therefore, essentially what we are telling is correct, is is strict here it is.

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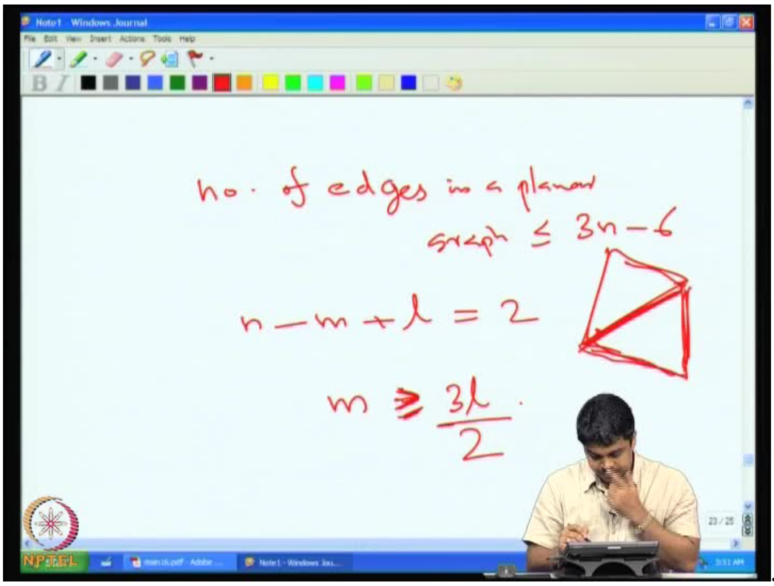

In fact, each face will contribute three edges. It will be equal to 3 l in that case. So when it is triangulated maximal, so m is, I can say that m is equal to 3 l by 2. If I am considering only triangulated graph, I can write m equal to 3 l by 2, right.

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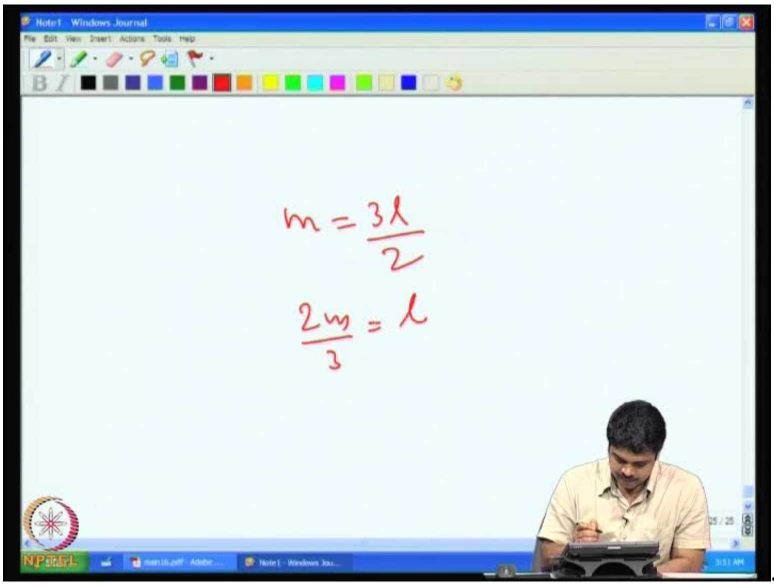

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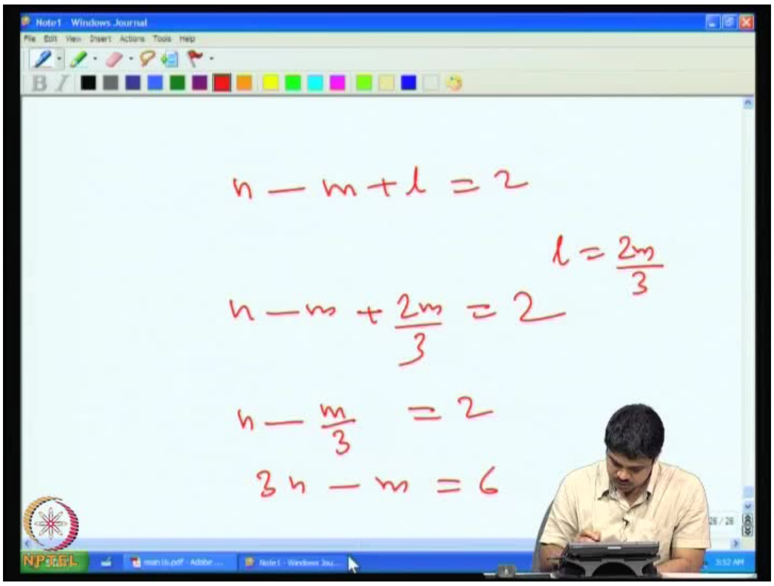

So now, so where l equal to 2 m by 2, is it not. Now, I can substitute in the euler formula n minus m plus l equal 2. So, this l equal to 2 m by 3, l equal to 2 m by 3. So, what happens n minus m plus 2 m by 3 equal to 2, is what it says.

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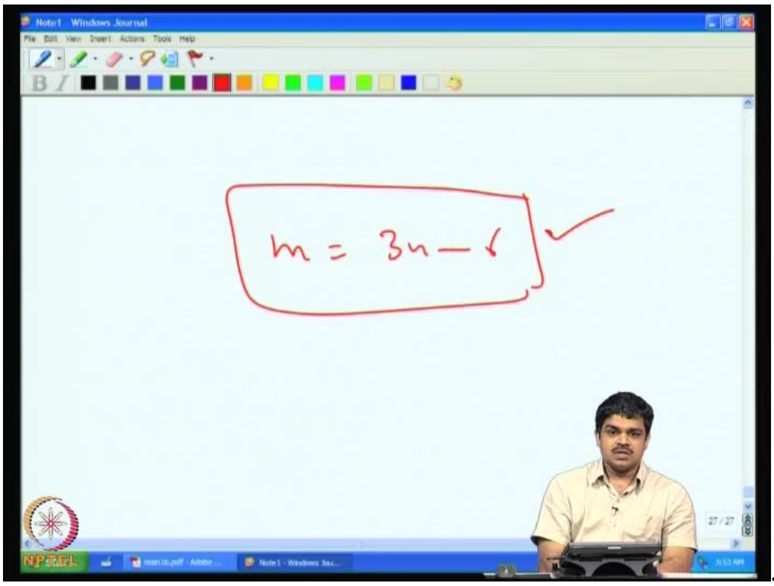

Now, this is essentially, n minus, see m by 3 right equal to 2. Now, multiplying by 3, so we get 3 n minus m equal to 6. So, we get m equal to 3 n minus 6 and this is the case of the maximal planar graphs. So, essentially, when I cannot add any more edge to the planar graph without losing planarity.

So at that case, this is equal to the 3 n minus 6. That means any planar graph which is not maximal will have strictly less than 3 n minus 6 less than equal to. So, we cannot ever make a planar graph more than 3 n minus 6, the number of edges more than 3 n minus 6.

So, we will in the next class, we will see how this observation can be used to get a six coloring of planar graphs, its vertex coloring using using six colors. Thank you.