Graph Theory

Prof. L. Sunil Chandran

Computer Science and Automation

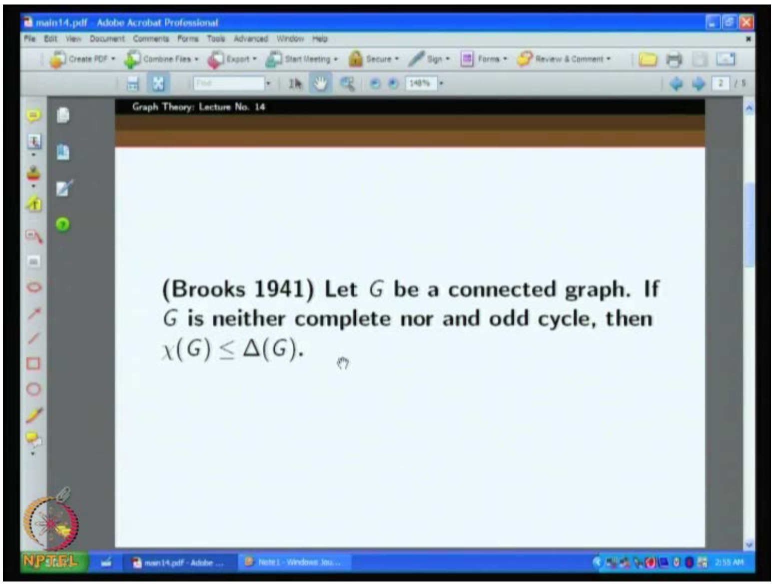
Indian Institute of Science, Bangalore

Lecture No. # 14

More on Vertex Coloring

Welcome to the 14th lecture of graph theory.

(Refer Slide Time: 00:37)

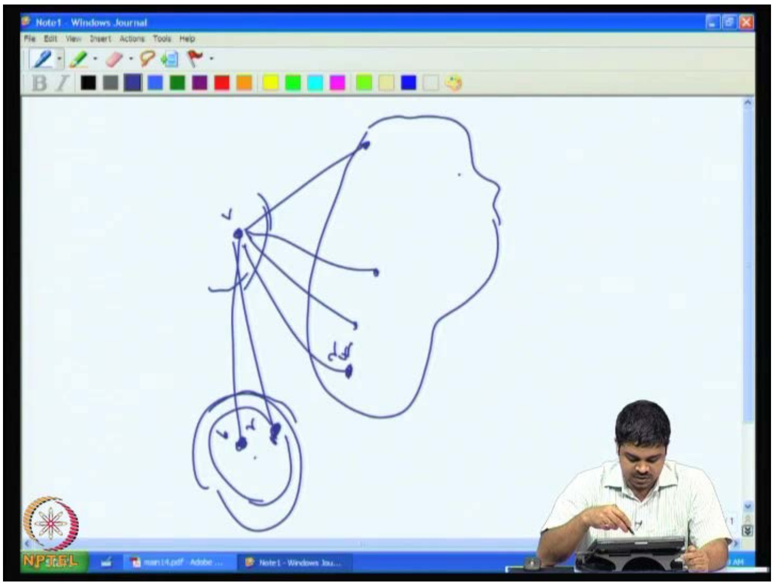


And so, in the last class, so we were discussing Brooks theorem, which tells us that if G be if G is a connected graph, and not complete and not odd cycle. So, G is a connected graph which is not a complete graph or an odd cycle; then the chromatic number is at most delta. The background was that the greedy algorithm, the easy algorithm of coloring which you discussed namely, color 1 vertex; and pickup another vertex, and give it to a color, which is not already used; but among if something is available, among them already used colors, then use it right.

This simple algorithm would definitely color the graph in at most delta plus 1 colors, but the brooks theorem says, even delta plus 1 colors may not be required, many most of the time delta colors will be required; but then there are two exceptions namely, a complete graph, it needs delta plus 1 colors and odd cycle, delta equal 2 there, 3 colors are required; then if a connected graph is different from these two cases, then we can always manage to color the graph with delta colors, this is what brooks theorem says.

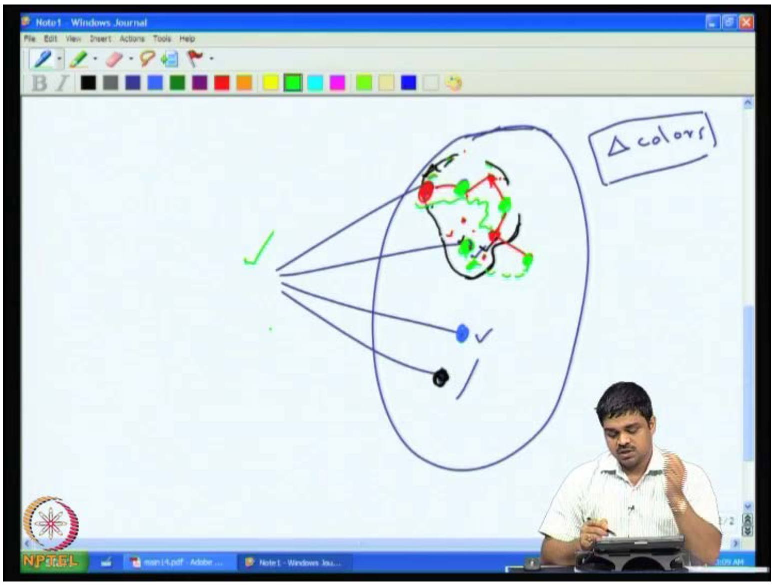
The proof we had gone through the proof, and then reached almost half way through. So, the key ideas for like this initially, we we decided to do an induction for small values of number of nodes, it was easy to verify that only delta colors are required, if it is not a complete graph or an odd cycle; then we noticed that if suppose an n node graph requires delta plus 1 colors, then it should be a regular graph; that means, all the degrees has to be delta; if even 1 node is of degree less than delta, then the same greedy algorithm would have allowed us to color not the same greedy algorithm, but little carefully if you decided the order of picking the vertices, decide a little carefully, we could have colored it in delta colors. So, now the claim that you need delta plus 1 colors tells us that, it is a regular graph, all the vertices are of equal degree and equal to delta.

(Refer Slide Time: 03:25)



Now you picked a one vertex and decided to remove it from the graph. So, this is the vertex V and look, so the neighbors of this thing, so this or this; and then the another observation we made was, so this tail, when you remove this vertex V, with these delta edges, the graph will not get disconnected, it is not possible to have some more than 1 component, because in that case we have a recoloring strategy for the connected components, because here we can always, it is possible that if I if I look at these colors, and then suppose some red color is used here, and the blue color is used here, I can within this thing, I can change blue and red, I have making sure that there is a repetition of red color in the neighborhood. So, if there is a repetition of some color in the neighborhood; that means, almost delta minus 1 colors are used, one more color is left. So, I can extend the coloring of this thing to the this thing.

(Refer Slide Time: 04:38)



So, essentially the idea was, so if you consider the vertex V, all its delta neighbors so if you remove that vertex v, there is only one connected component; and moreover all its delta neighbors should get different colors, because if there is any repetition of color, then the number of colors used is only 1 less and therefore, we can extend the coloring using delta. So, of case we we also argued why this induced sub graph is colorable using delta colors, because of induction hypothesis, we had to take care of two cases; what if this became a complete graph on an odd cycle that we took care right.

So, now the the point now is this is delta colorable, so if even a new repetition, even 1 repetition among the of colors among the neighbors would allow us to use the remaining color to the new vertex, this vertex; and that would essentially mean that we can do the coloring of the entire graph with just 1, so delta colors right; so that would be contradiction. So, we can assume that all these colors are different; so, let us say this is red, this is green, this is blue, and then this is black, so like that.

So now, the next argument we did was, suppose we consider the connected components, so if you consider the induced sub graph on the vertices, which I colored either red or green right, two colors if you fixed red or green, and then if you consider induced sub graph, they will form connected components. So, now the connected component which contains this neighbor and this neighbor, the green neighbor and red neighbor, in that collection should be the same; it is not possible that they belong to two different connected components; suppose it was like if it is so happen that if they have two connected two different connected components; for instance, this was this, and this was this, for instance, this is the red and green, see the red and green thing, so some red and green induced sub graph. So, here we are not interested in other other colors, you only look at the red and green, red and green vertices right.

So, if it so happens that if it is disconnected, then what will you do? In this component, this component I will exchange green with red, and red with green; essentially, all these red color vertices will become green color vertex, now for instance this will become red green, this will become green, and the earlier green ones will become red well, it can do that, because this will not affect the validity of the coloring, because if you look at any say red vertex, its neighbors will see that see so the change is only from the red to the green, if it is a yellow neighbor how does it a matter, because anyway as far as its sees a different color, it is okay for it; the red neighbor the green neighbors sees a change, but then it has change this color to green. So, therefore, if you exchange the color of red and green, the overall coloring of the graph will remain same.

So now, the effect is that when you did this, so this became green, so this is already green right. So, there are two greens on the neighborhood of this vertex v, so two greens means there is a repetition; in other words red was released, red is not used by any other neighbors of these things, now red can be given to this thing right, so red can be given to this thing; So, this is the effect; so therefore, we extended the coloring. So, what what can we infer? So, this will not happen, so we can infer that so if you look at the red green, some the induce sub graph due to the red green vertices, this our two neighbors namely the red neighbor the red neighbor and the green neighbor, they should come in the same connected component should be something like this, they should come in the same connected component, it is not possible for them to be into two different connected components. So, it can be something like this right. So, it is not possible for them to be into 2 different connected components.

Now, we wanted to study that connected… the next step is to study this connected component of red and green right. So, when I say red and green, it is any two colors, in fact, I am picking red and green for the purpose of explaining, so it can be blue and green also, so these they any two color, if we take they connected component with respect to that two color, the color classes, if you if you look at the connected form components formed by the only those vertices, which are colored blue and green; one of the connected components will contain both these vertices, then which are the neighbors of this new vertex v, which had colored red and green right. So, so this is the situation.

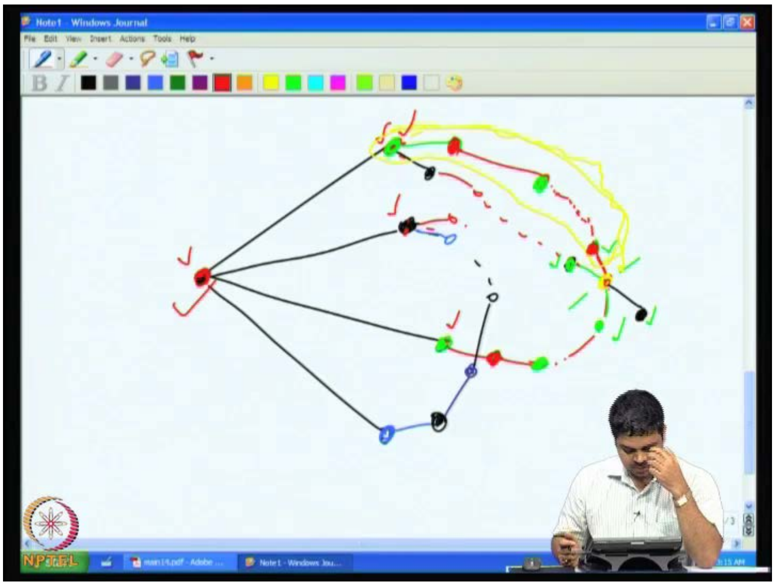
So, now let us see how this connected component will look like; now because it is fully connected. So, the first question was how many… So, if you look at this red neighbor, how many green neighbors can it have? Can it have a red neighbor now? Then the red vertex cannot have an any other red neighbor, because it is a proper coloring; now it can have green neighbors, but it may have several green neighbors that is the first feeling, we get; but then we argue that is not possible to have more than one green neighbor, because if you look at its degree, already one edge is going back to our vertex v.

So, now only delta minus 1 neighbors are there for it, which are already colored in the graph; and now two of them are colored green, means there are two extra colors; one of which is red, which is already given to this thing, but one more color is there, I could have changed into that color right. So, say may be I use, we use this blue to indicate that right, the the the one extra color which is available. So, if such a color is available, so I give it to that, so what is the good thing? So, the good thing is that you have released that red; red is not anywhere else right. So, the red will go to, again red will go to here right.

Now, we again say that suppose it has only one neighbor, it does not have two neighbors right; it is only one neighbor, and then we argued that this one neighbor can have only the same kind of argument, this one green neighbor, if you look at its red neighbors, can it have several red neighbors? It cannot have any green neighbor, but can it have several red neighbors, but we argued that is not possible, because if it has more than I mean, one red neighbors, so this one, two, three red neighbor will come, one dark and two new; so total three red neighbors means, so out of the delta neighbors, three are same; so, only delta minus two colors are used. So, out of which one is definitely green; so, one more is there may be blue, so that blue, I can give it here. So, which essentially means that here I can do an exchange, because now the red can be made to green without any problem right, because it is a single term component in the red and green connected sub graph. So, therefore, we could have exchange the color of these two green, and release the red right; this is the point.

Now, so now, coming back to this, so we have, so we have continue this argument to say that now this will go to red, and then this will go to a… then this will go to green green, and then this will go to a red right, then this will go to a green like that. So, it is not possible to have more than one new neighbor any time, so it will form a path only, it will not be any complicated graph; and it has to finally reach here right. At any point if you see a, see more than one new neighbor a of that same color green, then that means, the three neighbors, it will one back and then two forward. So, three of them has the same color, and then that means, one free color is available other than the color, which is already there on the vertex. So, we can change it. So, that would allow me to consider this component which, so that means, it is component is not containing the other vertex, these neighbors; if I can do a red green interchange in this component up to here, which will make this green here at this point, so green, green will allow me to release the red, this was the thing.

(Refer Slide Time: 14:24)



So finally, so the argument up to now, told us that so if you consider a red, green; for instance if I red, if you considered a red and a green neighbor here, so this is green neighbor here, so the if you if you explore the red green path, there will be a red green path, and then it will come all the way here, and then it will be like this, it will be like a like a path you know, it will be like a path like a path, (( )) a path.

(No audio from 15:00 to 15:17)

Now, you can see that the argument that we have given for red and green is valid for any pair of colors, I could have taken say the black and blue or I could have taken red and blue for instance this could have been like right, so it could have been black and blue right, in which case you will see a path going from here like this, black like this, and then slowly reaching here, so with the blue here right; or it could have been red and black, so then it will happen to be like this right red and black, so something like this; so finally, so it will come like this; so you can see this kind of paths between any pair of colors in the neighborhood right, so two neighbors if you take, you see a string here.

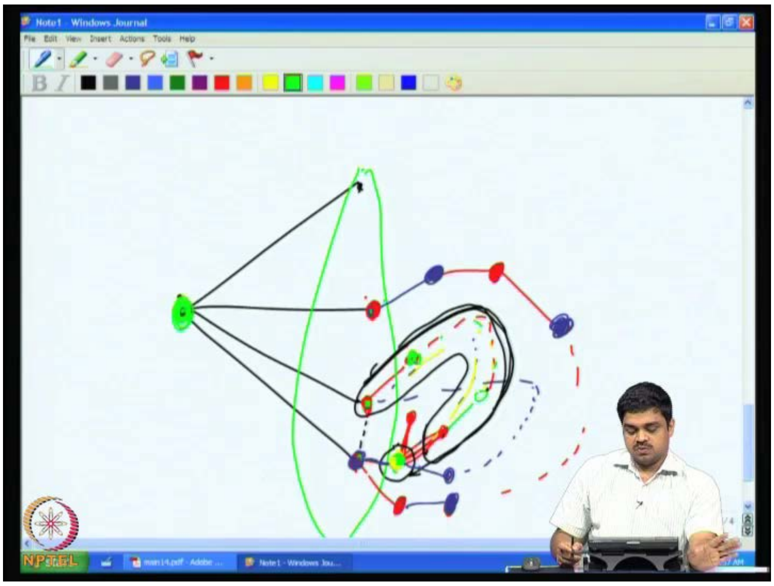
Now the next question is so you see several path here is can this paths intersect? Of case, you say that there intersecting here; this path is intersecting; at the beginning and ending they are they are like, they are intersecting actually, they are red black is intersecting here, this red black path is intersecting with the red green path in the beginning node here; similarly, this red black path is intersecting the red blue path in this node. So, naturally there are several intersecting points this, this; so so, but then these are only end points right these are only end points right, but is it possible for two such paths to intersect on a in say in between point that means, interior point. So that definitely for instance, if you consider a red green path, it cannot intersect with say black blue path, because the at the vertex suppose it is intersecting at certain vertex, interior vertex, so what will be the color of that? So, red blue path will say there it has to be either red or blue sorry black blue path will say it has to be either black or blue, well red green path path will say that it has to be either red or green.

So, essentially its not possible to have a interior vertex, which is common to such paths; but on other hand what if it is a red black path and a red green path; now they can be a common vertex probably, because what if, see because it can intersected a red vertex right; for instance its possible for red to say this path, see this red, a green path to come come like this, and then at this vertex, it may be a red vertex here. So, what will be the previous vertex? Here it will be a green vertex sorry this is this will be black vertex, so red, I am telling red black vertex; and the here it will go via black vertex out, and here in the other path, we will see this red vertex is in fact, immediate neighbors are green right it is green, green.

Now, the question is here this black vertex is repeated twice, the green vertex is also repeated twice, so now, total out of delta neighbors, how many colors are used on the delta neighbors; black is used twice; green is used twice. So, total of delta minus 2 colors are used, and this red is one of the remaining two colors; and then one more is there, so that color we can always you say let us let us say it is yellow, so then what will happen? So, it means that this path is broken, for instance if you follow this red green path, here it is broken, so this path does not go all the way to here. Now this is a connected component of red green by itself, this this one. So, we can exchange this green and red in this path, this will become green, this will become red, and this will become green, and this will become red, and so on.

So, if I do that what will happen? Here is a green, here is also a green, so green is repeating on the neighborhood of the vertex V, so red is released, and then red can be given here right; is it not? So, this is what will happen. So, we can we can assume that these two different paths in this collection of and choose to paths right sorry sorry d delta choose to paths, any any pair of colors we have a path, so they never intersect, so they never intersect; even if they share a common color, it is it is not going to intersect.

(Refer Slide Time: 20:24)



So, this is one good observation, which they will make use of in the… And to conclude the proof, now again, consider this situation now again, so this this vertex, and look at its neighbors, so one of them is colored red, say one of them is colored green right. So, now, first we have to find two vertices in the neighborhood, such that they are non adjacent right. So, is it possible that every pair is adjacent? But if every pair is adjacent, this is already a clique, because there are delta neighbors, this is a delta clique, and along with this thing this is a delta plus 1 clique. So, the entire graph is going to be a delta plus 1 clique right. So, so we do not have anything to prove in that case. So, the of if there is some other vertex outside this delta plus 1, degree will be more than the delta right that will be a contradiction, so to the assumption that the degree was delta.

Now, so let us say this is the this is the two vertices, which are non adjacent; this is the two vertices which are non adjacent; now this is let us assume that this is green color, this is red color. So, first we consider the red green path, so this is the red green path red, red, so this will come all the way to like this right, this is the red green path.

So, what we are interested in is now, so this look at this vertex, so this is this is crucial vertices, this is the second vertex, this is the green color right. So, this vertex is definitely available, because in this path however, short at is it should have at least three vertices including these two vertices, there should be one more vertex, this is therefore, these vertex exist. Now, we can consider another vertex say this blue vertex here, this blue vertex between this red vertex and blue vertex, we have the connected component right, so for instance, if you consider the red green connected component, it is going to be a path like this right. So, here this is a red, so this is sorry this is blue, this is red. So, there will be a path like this. So from here, it will be… And this is we know that this is the this is the this is the full connected components involving these two vertices and only red blue vertices. So, this there would not be anything going this, we had already argued that nothing like this will be there right. So, this is not possible right.

So, it is just this is a full connected component involved in the tube; now our plan is to interchange the colors red and blue, along this path, along this path, so along this path, we will interchange the red and blue colors, what will happen? This will become red, this will become blue, this will become red, this will become blue and finally, this will become blue, this will become red, and this will become blue. See, this will not help us much, because here the earlier argument will not work that, we cannot say that there is a repetition of colors on the neighborhood, because the earlier it was red blue, this was blue, red blue now, in this red is here, blue is here. So, that is not going to help us, but so what is going to help us is the fact that the if you consider these vertices, none of these vertices are affected, because this path is gone not going to intersect the green red path right the original green red path will be intact up to this point, this entire green red path will be up to here, it will be safe. So, it would not be changed at all because of the blue red interchange.

So, it so happens in particular that this vertex will still be green, this vertex will still be green, because nothing has affected this; but on the other hand, if you consider the blue green path now, what will happen? This is blue, so the blue green path has to contain the second the nearest vertex which is green in the in it right. So, this will essentially contain the… So, it will it will go like this right the blue will come here, and it should go somewhere to somehow reach here right its it should.

So, essentially what was happened is there is a sharing of a colors sorry sharing vertices by the blue green path and the green red path now; blue green path and the red green path now, namely this vertex; this was a contradiction; why is it a contradiction? Because now if I look from this vertex, it sees a right it so we have this thing that, because now they should be one more red going out of it right, because it has a finally, reach the red color vertex, somehow it has to be reach here sorry this is the red color vertex now; right it has to reach here somehow. So, finally, there should be a red going out of this somehow, so it should see two red in its neighborhood like this, and then also two blues right. So, totally delta minus 2 colors only are used, and one of them is definitely green.

Another will be there may be it is a yellow one. So, this yellow one will allow us to break this path, this entire path, here I can interchange red and green on this on this path, on this path means this path to here this path, I will be able to here, here I will be able to interchange the red with green, and in that interchange this green will become red, this green will become red, red, this will become green and like that. So, here is a repetition of colors green and red and red; in the process we have released the green color, and then that green color can be used here right. So, which will which will allows extend to that that will be the contradiction. So, finally, this gives us the final contradiction; that means we have shown that whatever it is, it has to yellow us to release a color from this thing. So that that hence the proof.

To repeat the main ideas, so the proof we wanted to show that the entire graph with maximum degree delta can be colored with delta colors, if it is not a complete graph, one delta plus 1 vertices or an odd cycle I mean, for delta greater than or equal to 3. So, odd cycle cases not there right. So, the first we observed that it has to be regular, so that means, and also then we observed that if you remove any vertex, the remaining has to be connected, then more over all the delta neighbors should get different, different colors all the delta colors should be used up.

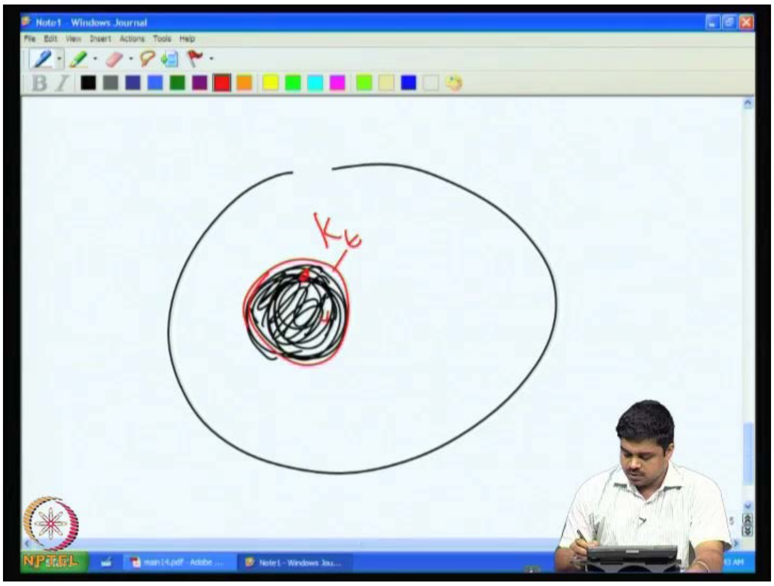
Then we explored between any pair of colors, we considered the connected components form by the in the induced sub graph of those two colors alone, the two vertices of the two colors alone, and this connected component which contain these two vertices, which are on the neighborhood of v, should be in the same should be same; that means, they cannot be part of two different connected components, in which case we would have exchanged the colors red and green in that particular, one of the connected component, and we would have got a repetition of colors in the neighborhood of v. So, therefore, there should be in the same connected component.

After that we told, we will explore what is the structure of a connected component, which contains these two red and green neighbors of V, and we show that it has to be a simple path starting at the red neighbor of V and ending at the blue neighbor of v, red should be a simple path red, green, red, green, red, green path; and this argument is true for any pair of colors, out of the delta colors, we have delta choose to such pairs for any pair of colors this is true. So, therefore, then next thing we did is, whether the paths can intersect or not? We show that the paths can only intersect at the end points, it can never intersected in interior points; and then we did this final trick namely we found out a non adjacent pair, say we call it red and green be the let it let red and green be the colors given to this non adjacent pair.

Now, we located this second vertex in the red green path, starting from the red vertex, and then after that this color is going to be… So, let say it is some another color, so it is it is, so it is color is green, red red green path right and now we located another color say blue, and then we interchanged the blue and red in the red blue path right in the red blue path, which means that the original red green path was unaffected. So, in particular our vertex which we noticed earlier was still green. So, therefore, the that earlier red green paths should contain that vertex, and also our new blue green path will also contain that vertex, because now blue is the earlier the blue neighbor of V is now the earlier red neighbor. So, that is this green neighbor as in the it is adjacent. So, therefore, the path the blue green path should contain the vertex. So, the blue green path, and a red green path is sorry a green red path is containing the same containing common vertex, so which is the contradiction, so by our this is this is how we conclude at the proof.

So, now this finishes the proof. So, now the question, the next question we will address is this; so we can see that the chromatic number is able to affect some parameter; for instance here it says that if the chromatic number is high, the maximum degree has to be high. So, similarly what other things can be tell about for instance, some what is it that causes chromatic number to be high. So, for instance, if we cannot color a graph with few number of colors, small number of colors; what can be the reason? So, the initial guess for anybody probably would be that so, if there is a clique complete sub graph in the graph, original graph, then you cannot color it small number of vertices.

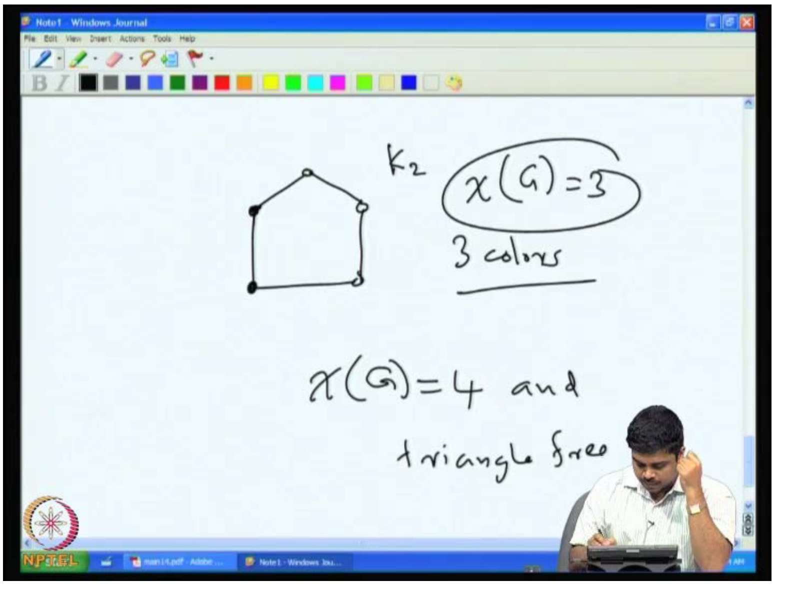
(Refer Slide Time: 32:38)



For instance, if G is this graph right, and then if there is something here, which is complete, so this is the complete sub graph you know, this complete sub graph itself will require so many colors, suppose if there are there is k t here, and t colors will be required in this itself. So, how can we color that the entire graph with less than this. So, it is very natural initial guess to think that probably the complete sub graphs in a given graphs is the reason for the number of colors to be high. It is true that if there is a complete sub graph, large complete sub graph, the large number of colors are required as if k t is there, in fact, t number of colors are required at least; but then is it actually the only reason why the chromatic number is high.

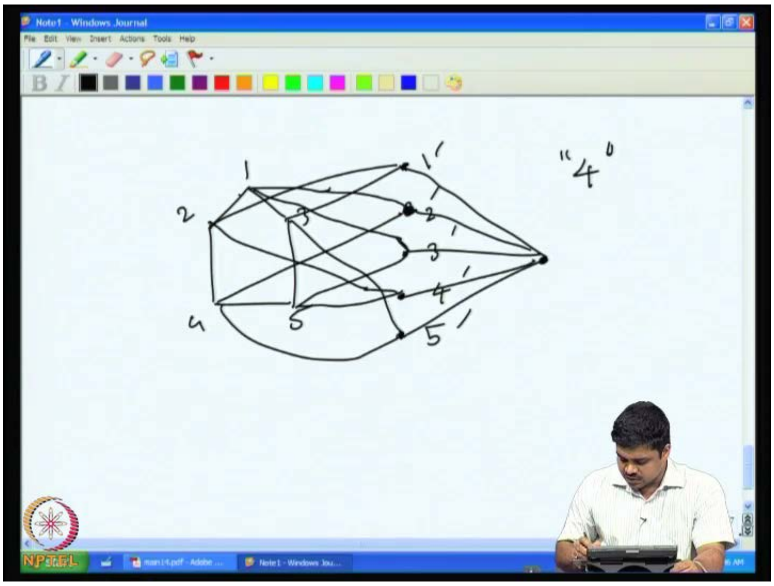
For instance is it possible that we do not have any complete sub graph, for instance see any graph there will be a k 2, that means, an edge will be there, if it is not as it collection of isolated vertices. So, let us say we do not have a k 3 in the graph, k 3 means the triangle, suppose the triangle is not there, is it possible that then the chromatic number will be low. So, that is what our initial intuition told the complete sub graphs, the presence of complete sub graphs may be causing the chromatic number to be high. Now, we are asking this question is it possible that so there is a graph with no k 3's in it, no triangles in it, but still the chromatic number is very high. So, is it possible to construct such a graph; it surprisingly turns out that it is possible to construct such graph, so also.

(Refer Slide Time: 34:06)



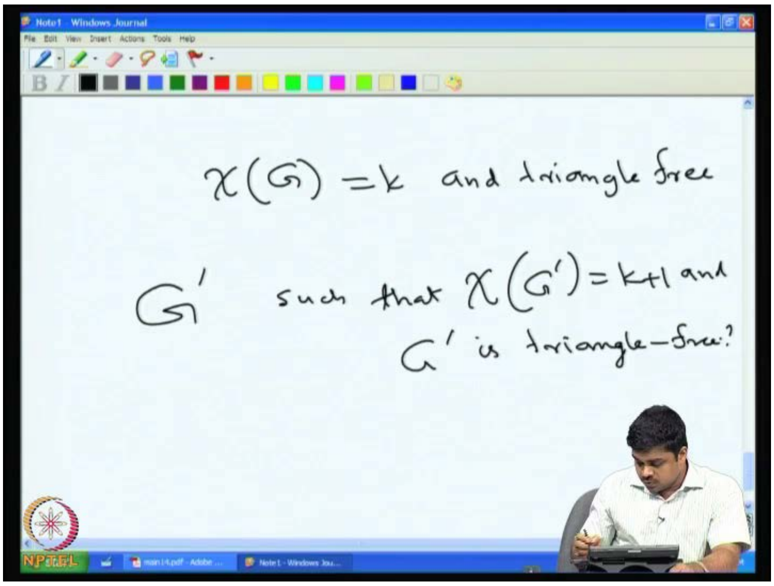
So, let us see the in this example, so this is let us say for 1vertex right it is true. So, then let us say this is a… So, this is a 2 colorable k 2 right this is no triangle, but then it needs 3, 2 colors; now the third one see you can see the this pentagon, so this is the pentagon. So, this requires three colors, this is this chromatic number is 3, G is equal to 3 here, but it does not have a triangle right. So, triangle free graph without… So, which requires three colors right, so this is the first example; now how do I get a triangle free graph which requires four colors. So, I am looking for a graph with chi of G equal to 4 and triangle free, and triangle free. So, this is the way we constructed.

(Refer Slide Time: 35:10)



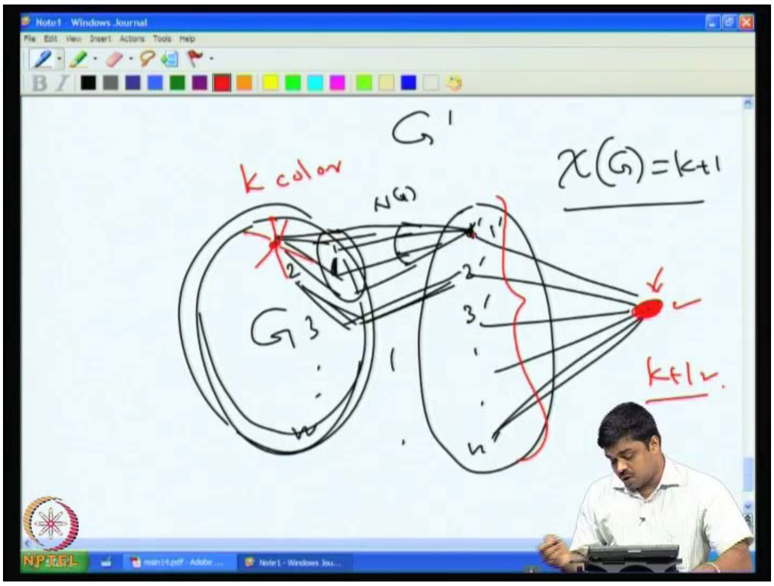
So, we take the pentagon, so this is 1, 2, 3, 4, 5, so this is 1 dash, 2 dash, this is 3 dash, this is 4 dash, and this is 5 dash; now this I will connect 1 dash to all the neighbors of 1; that means, here and here; and then 2 will be connected to all the neighbors of 2, that means, here and here; and then 3 will be connected to all the neighbors of 3 like this; and 4 will be connected to all the neighbors of 4, 2 and 5; 5 will be connected to all the neighbors of 3. So, this is and then we will introduce a new node here, and everything will be connected to this, so how many nodes here, so here 5 plus 5 plus 1 - 11 11 nodes. So, the it is easy to see that here we do not have any triangle, and also we can show that it requires 4 colors.

(Refer Slide Time: 36:36)



So, in general suppose we get a graph with chi of G equal to k and triangle free; if we know that if we know that G is of chromatic number k, and that is triangle free; then how do you find a G dash such that chi of G dash is equal to k plus 1, and G dash is triangle free, to this is the we are repeating the pentagon kind of a examples

(Refer Slide Time: 37:20)

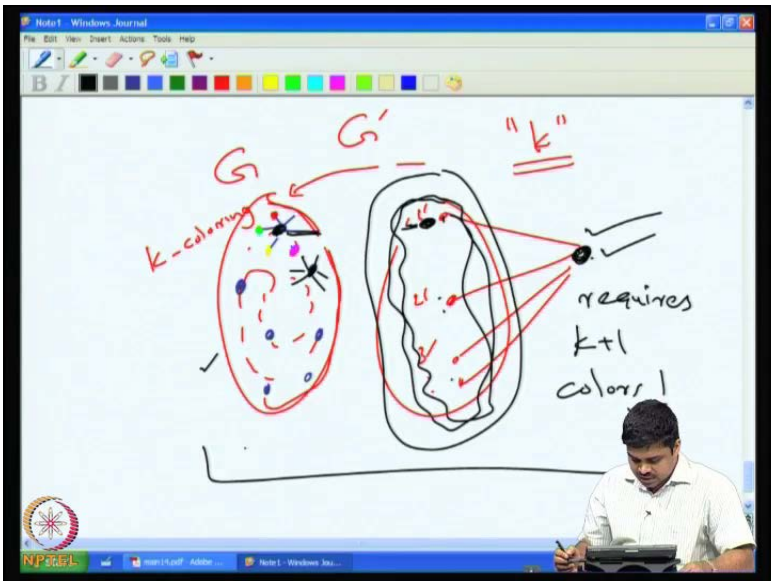


What we do is we just take G here to construct G dash; what we do is we take a G and then let say this is 1, 2, 3 up to n these are the number of vertices, and we just introduced 1 dash 2 dash independence set 3 dash up to n dash. So, these are copy of these vertices in some sense, but then no edges here in this set right. So, now, this 1 will be connected to all the neighbors of this neighborhood will be n of 1 right, the neighbors of 1 dash will be essentially the neighbors of one here, so here so here it is a same neighbors of whichever the neighbors 1 is connected, the same thing it will be connected; for instance 2 is connected to this, 2 dash will be connected; between 1 and 1 dash, there is no edge; between 2 and 2 dash, there is no edge; but the for instance if x is a neighbor of 1 x would be a neighbor of 1 dash also here right this is the way it is constructed.

Now, you will introduce after this connections, you will introduce a new vertex and connect it to that; clearly this is not going to be a triangle sorry this graph is not going to have any triangle; first of all there is no triangle involving this vertex, because this is an independent set; and then if at all there is a triangle, which involves say 1 dash, then you can see there is this neighborhood, there is an edge, and then this instead of 1 dash, if we use 1 right here; then here itself there should be a triangle right. So, which will be a contradiction to the graph that G was originally a triangle a free graph; G was a triangle free graph right. So, whatever triangle we get in G dash involving someone dash, so the it should be a triangle in G involving 1, so therefore, it will be a contradiction. So, this is the reason why it is triangle free.

Now, the next question is why why is it key chromatic; why is it chromatic number k plus 1; I have ensured that the chromatic number has increased. First it is easy to see that it is it can be colored with k plus 1 colors, because here we have only k colors right; now what will be the color of 1 dash, same color as one; because one its neighbors one's neighbor is a same as 1 dash neighbors and therefore, if 1 has no conflict with this neighbors, 1 dash can share the same color, because 1 and within 1 and 1 and 1 dash we do not have any edge, we can 1 dash can share the color of 1; similarly 2 dash can share the color of 2, same color; then 3 3 dash can share take the color of 3 and so on, because their neighborhood is same, so they want to be any conflict them, and between then, there is no edge. Now, you see that you can always use a new color for this last vertex and that we will make it k plus 1.

(Refer Slide Time: 40:25)



k plus 1 colorable, but then how do I show that it cannot be colored in less colors; suppose it can be colored, suppose G can be colored in less number of colors right. So, this is 1 dash, 2 dash, 3 dash etcetera; then so, see suppose its k, color colorable using k colors; definitely it cannot be colored in less number of color; now you see if you take the coloring in this part alone there, this is G right, this is G, this part alone the coloring is essentially a k coloring of G, but this k is its chromatic number, we cannot color with less number of colors. So, you can consider any color class; for instance you can consider the blue color class here. So, these are the blue color class here; blue color vertices here.

Now what can I tell about the neighborhood of these vertices of case, their neighborhood no other blue vertex will come in the neighborhood; now I can say that at least one among them should be such that it is neighborhood contains all the other colors, other than blue all the other colors, say if you using red, green, yellow, violet all the possible, all the k colors, k minus 1 other colors we used here should appear in the neighborhood; and with this blue it is all the k colors right at least one one vertex should, I am not saying that every blue vertex should be like that, at least one vertex should be like that. Why is it so?

Otherwise what we can do is, suppose some color is missing here in this thing, suppose say we have black color missing here; I could have simply made it black right made black, no conflict, and this here, for everything if it, if something is missing, I could have in the neighborhood some color is missing, I could have converted that vertex to that color, moved that that vertex to that color, and because these are independent right they want this change of colors will not affect them each other. So, that would have been a valid coloring.

So, with one less color, because we are replacing each blue colored vertex with some other color right. So, therefore, from the remaining set of colors, so it is so happens that the the the that it is not… It is it should be … Because it is not possible to color with k minus 1 colors, it should be the k is there, at least 1 vertex should resist this attempt to reduce a number of colors used; that means, there should be at least 1 vertex, blue vertex such that in its neighborhood, all the remaining k minus 1 colors are appeared.

Now if this is true for blue color, it is true for say yellow color also, it is true for red color also, whichever colors is used in the k coloring, this is true; for instance they should be at least 1 vertex of that color, such that in its neighborhood all the all the vertices are colored with the remaining; suppose this is colored, this is such a black vertex, so definitely this is 1 dash. So, it is of case, 1 dash no connections here; but then this, its corresponding vertex here we so, we also its connected its neighborhood, all the k minus 1 remaining colors are used up, all the k minus 1 remaining colors are used up in its neighborhood. So, it has to reuse the same color as this one, namely the black right. So, of case no on this connection is not there.

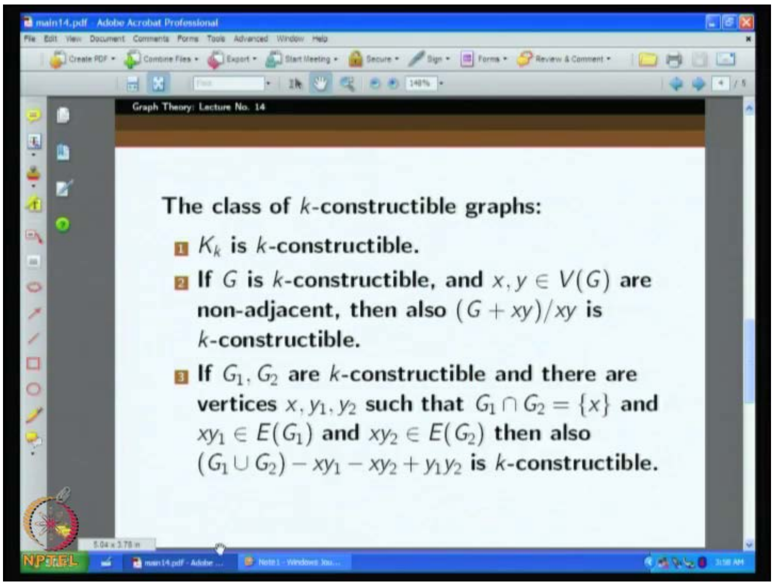
So, similarly the blue color what has special vertex (( )) the neighborhood is using all the remaining colors, its partner in the this side should definitely have have to use the same color as it, because in the neighborhood, because its neighborhood is same and all the other colors are appeared in the neighborhood, it will have to use the same color. So, it is so happens that all the k colors will appear in this place, in this set, in this in this group also; and now this k has to get the new the new vertex, the final vertex has to get a new color, because all the available k colors are appeared in its neighborhood. So, it is should get k plus 1, there is no other way. So, it requires k plus 1 colors. So, that is what we can infer.

So, so this is interesting constructions called my Seals Kane construction, so which tells us that there are triangle free graphs, which requires arbitrarily large number of colors. So, in other words you give me any any number k, then I can construct a triangle free graph with whose chromatic number is at least k, this is what it says. Now there are other graph construction is also which will, which can show the same thing; but there is an interesting result that for instance, so so what about say one direction is seen triangle free, suppose if we do not even allow a 4 cycle, 5 cycle, suppose the girth of the graph means, girth means the shortest cycle of the graph has to be greater than k; will it probably allow us to infer that the chromatic number will not be arbitrarily large; for because for instance if you look from one node, if the girth is large, so it will look like a tree in the neighborhood; is it not? At least up to girth by two distances, it will grow like a tree.

So, is it possible that so because the trees are two colorable; it locally it looks like is it its only very small number of colors are enough to color such a graph, but then, but is it possible that such graph can also be of arbitrarily high chromatic number. So, this result from interesting, for every integer k, there exist a graph G with the girth greater than k, and chromatic number greater than k; you can you can get both this parameters girth and chromatic number, which intuitively looks contradictory, but both can go high, arbitrarily high; any value of k if you put, you can get a graph may be it is big graph, but you can get a graph with both the value, both the parameters having value greater than k.

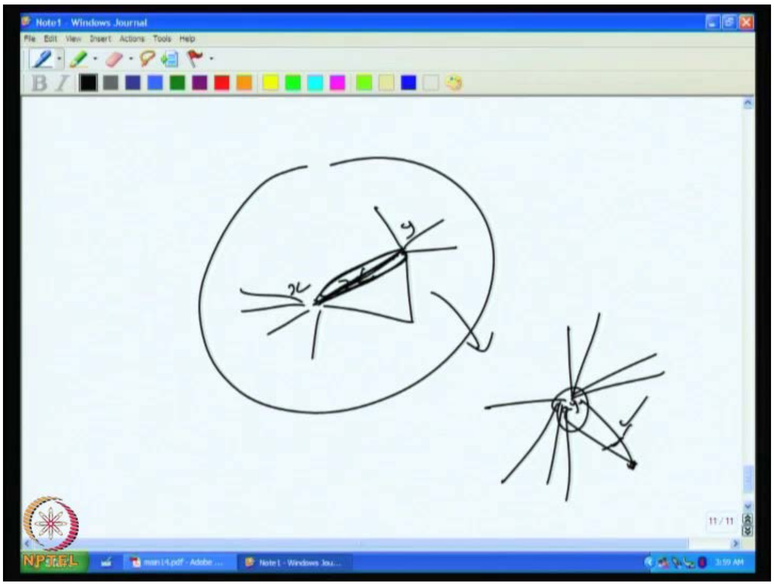
So now, so the final final thing today is something about k constructible graphs, so what is a k constructible graph? So, this like in the connectivity k; as we told, so all 3 connected graphs can be constructed in this way; so like that is it possible that for all k chromatic graphs, the graphs which there is some such constructions of case, so we cannot hope for a construction for all k chromatic simple construction, but what we will say is that if you if you look at a k chromatic graph, a graph which requires at least k colors to color, then they should not be some structure in it, some sub structure in it, which will require so many colors sorry which which can be produced in certain ways that is that is this essential idea of k constructible graphs.

(Refer Slide Time: 48:34)



So, what is the k constructible graph? So, a class of the class of k-constructible graphs are defined like this; so the complete graph K k is k constructible, so completely graph K k is k constructible; and then if G is k constructible, and x, y is element of V (G), x, y two vertices in V (G) are non adjacent; then this non adjacent vertex vertices can be kind of identified together, they can be contracted. So, what it is says is G plus xy contract, then contract that x, so so we can we can look look at it like this.

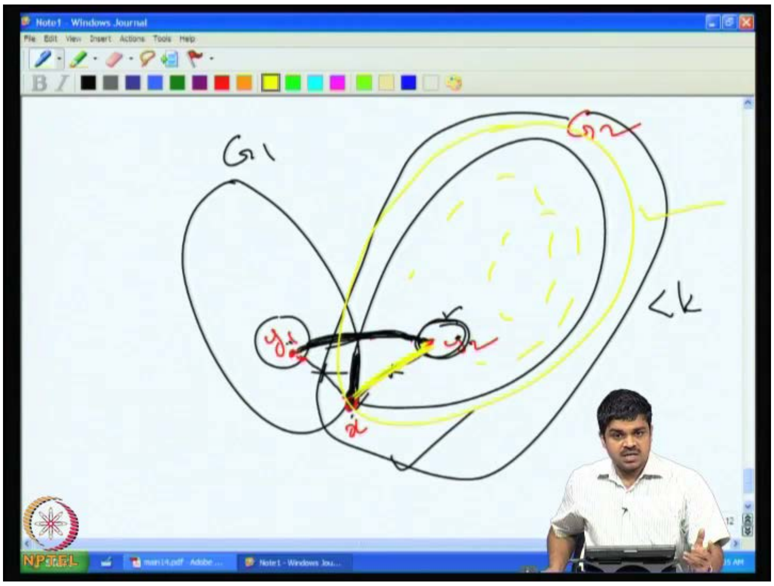
(Refer Slide Time: 49:09)



Suppose if this is a graph. So, you have and this is a k-constructible graph, and then suppose you find out x and y, so that is there is no edge between them right, those are the some neighbors, so now we can just pull them together. So, the thinking you put this edge and contract or otherwise we just collapse them together right. So, that is what it says; so then what will happen? So, you will get a vertex, so together here, with all the neighbors some common neighbors may be there, but you can always drop the multiple edges, if you do not like. So, if you want a simple graph right, so then if you do this operation, then it will still be k-constructible.

And suppose you have given 2 k-constructible graph G 1 and G 2; and there are there is a common vertex for them, x just one common vertex, and one common vertex x, and y y 1 and y 2 are such that x y 1 is an edge of the first graph, and x y 2 is an edge of the other graph; then G 1 union G 2 minus x y 1 minus x y 2 plus y 1 y 2 is k constructible. So, it looks like this.

(Refer Slide Time: 50:35)



So, suppose you get; so, if this is a graph G 1 you say k-constructible graph and there is this vertex x, and this is another graph this is the only vertex, which is common, so this is x, let us say x; and this is G 2, this is also a k-constructible graph; now we have a y 1 here, this is this edge is present here, and then there is a y 2 here, this edge is present here and in G 2; now and of case, this edge is not present right, because this is y 1 and y 2 are not part of the same, this edge is not present. So, when what what we say is we put this edge and remove these edges, so this graph is also k-constructible together; so, it is also k-constructible. So, so this the graph class of graph, which can be constructed by these operations by repeated application of these operations are the k-constructible graphs.

Now we can easily see that so any k constructible graph is k colorable right, so which is chromatic number is k, which requires k colors the color; so, the reason it is by an induction, because the first complete graph K k requires k colors, and now suppose any smaller so the so if you are doing it by an operations - sequence of operations, at earlier step we just assume that it is k chromatic. So so now, if you consider the second operation, namely this operation, so you can… So here this operation, you can easily see that is it possible that the chromatic number decreases? So, if the chromatic number decreases by this kind of contraction operations, when the non-adjacent vertices are contracted; then why you can always go back, and use the whatever color is there for that contradicted vertex, you can use it for both vertices, any way they are non-adjacent; so, then the same color will work both of for both of them in the original one, because the neighborhood any way is different, the colors on the neighborhood is different; is it not?

So, therefore, we can we can infer that by the second operation is not possible to decrease the chromatic number, if you do that; if you decrease it, then the original also has. Similarly, you can consider the other operation G 1, G 2, the here also, if you say suppose you decrease this operation here also, so when you here it is only k colorable, for instance you drop these edges remember this two edges are dropped, and this was put right.

Now, you know, because of this edge, this color and this color is different right; one of them should get a color different from that of x, for this entire graph right. So, this color and this color is different, then you can put for this edgem, suppose with (( )); and now look at this graph alone, with so so this entire graph as a k less than k coloring, now x got a color with respect to that, y 1 got a color with respect to that, and y 2 got a color with respect to that, it is possible that y 1 and x have the same color, but if not then it is not possible to have y 2 and x 2 have the same color, because y 1 and y 2 have different colors only one of them can be the same as x.

Now, in that case it is a clearly x and y 2 suppose they have suppose they have different colors; then even if you put this edge, then we do not have any problem right, because they have different colors already; and other colors whatever was there with respect to. So, we have only using k colors for these things it is a contradiction, because we told that these requires G 2 and G 1 required more than k color. So, we can infer that so by this operational also, it is not possible for the number of colors required to reduce right. So, this k constructible graph are definitely k chromatic, I mean they need the chromatic number is greater than equal to k, but now we will say that any graph, whose chromatic number is greater than k is considered, they should be some sub graph in it, such that its its it contains a K k constructible sub graph; so to we will we will do the proof in the next class, so this class we will conclude this class with this much. Thank you.