Graph Theory

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Lecture No. # 11

More on Connectivity: K- linkedness

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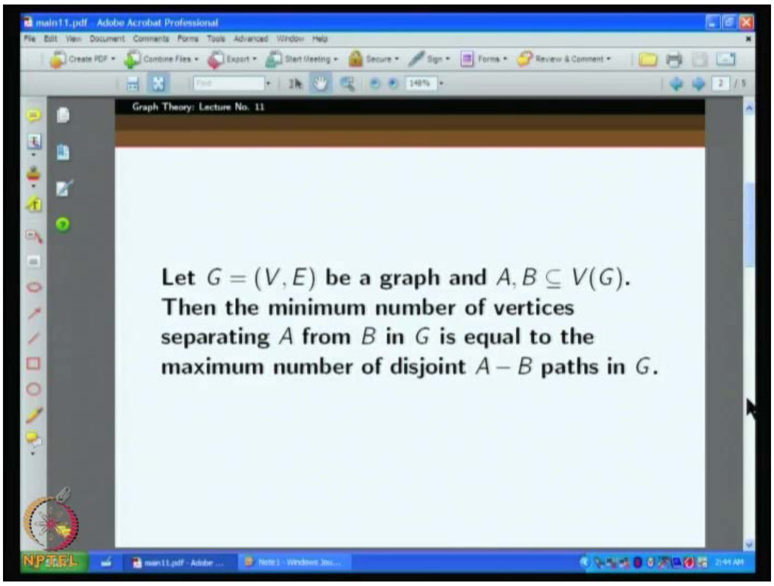

Welcome to the 11th lecture of graph theory; in the last class, we were discussing Menger's theorem in this generalized setup. So, we considered a graph and two subsets A and B of the vertex set of G; so, Menger's theorem states, the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A - B paths in G; we had discussed what is meant by a collection of vertices to separate A from B, which means that, when you remove those collection of vertices, then they will not be any more paths from A to B from the remaining vertices of A and G, A to the remaining vertices of G; it is possible that we may remove vertices from A or B or both.

Maximum number of disjoint A - B paths; A - B path means a path which starts in A and ends in B, but this path will take exactly one vertex from B A, and exactly one vertex from B, it means the starting vertex of the path will be in A, no other vertex of the path will be in A, and similarly the ending vertex of the path will be in B, no other vertex of this path will be in B, such paths are A - B path; we are interested in disjoint A B paths, so they should not have any common vertices in this paths; how many is the question; this maximum number between in a given pair A B of subset of f G, this is equal to the minimum number of vertices separating A from G. So, today we are going to do the proof of this statement.

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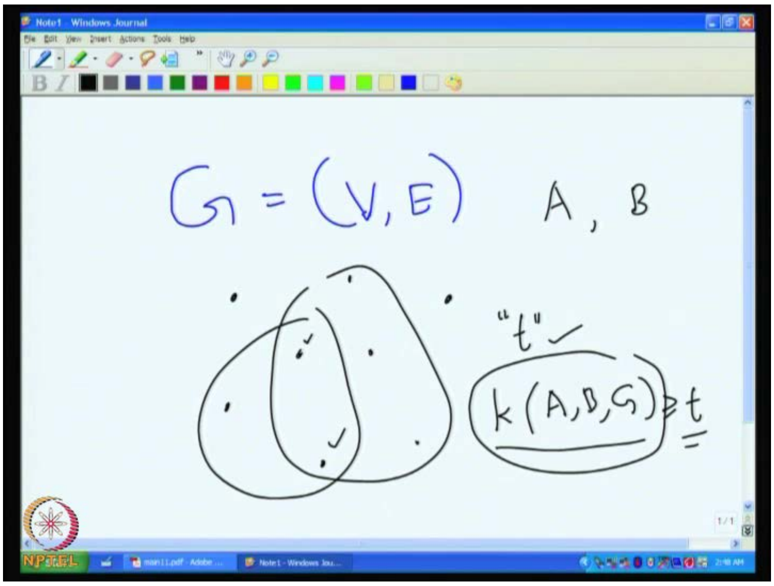

So, we are going to prove this by induction; induction on number of edges in the graph. So, you can consider an n node graph G, so G equal to (V,E). So, the n node graphs, in the n node graph, so suppose there are no edges that means, 0 edges, we our induction is and the number of edges, so there are no edges; then the minimum number of vertices to separate, and also your A your A and B are given both the subsets of B; is the… Before before getting in to that we should not is that, if there are suppose some t number of disjoint A - B paths.

Now, to separate A from B, we need to remove at least one vertex from each of this path, otherwise in the path from which, no vertex is removed will remain in the graph. So, an A B path will remain, so A will not get separated from B by removing whatever we did right. So, therefore, at least t vertices we will have to remove, if you want to separate A from B. So, if we write K comma (A, B, G) to denote the number of vertices required to separate A from B in G, then this has to be definitely greater than equal to t, the number of disjoint A B paths, maximum number of disjoint A B paths; the question is can it be strictly less? That means, can we show that if K is the number of vertices that is to be removed in order to separate A from B, then, do we always have t disjoint A B paths in the graph.

Now, if am given A and B in a graph without any edges, then it is very clear that to separate A from B, we just have to remove the vertices, which have common to A and B; that means, for instance A can be like this, B can be like this, then if you want to separate this vertex and this vertex is to be removed, and this are the ones to be removed, this are the only ones to be removed.

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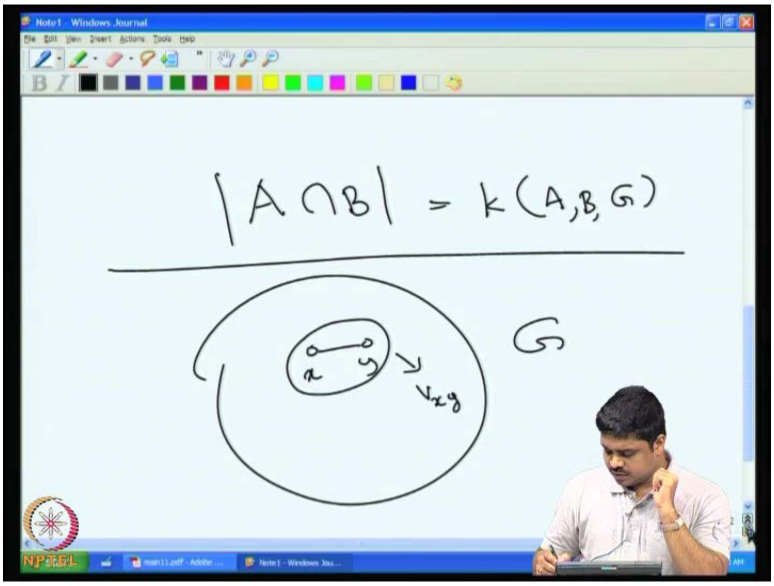

So, in this case the minimum number of vertices that is to be removed from the graph so that a get separated from B is equal to A intersection B. So of case, corresponding to each vertex in A intersection B, we have one A - B path, because that is a trivial A V A B path having just one node, it is starts in A and ends in B, because both vertices, this vertex belongs to both A and B. So, therefore, we do get so many A - B paths also; so in this case, when there are no edges, the statement is trivial. So, we can assume that there are some edges, so we started of the induction now; now there are some edges, it pick up some edge x y, this is x y; in the graph, this is G; and how plane is to contract this vertex to get a new vertex V y; what is the advantage? We have, by doing this thing, we have reduced the number of edges; and for all graphs with smaller number of edges, we are assuming the theorem.

So, the first question that comes to your mind. So, I am going to use induction on this smaller graph with a smaller number of edges; for on the same sets A and B, but the first question that comes to mind is, will this A and B remain same? Of case, the A and B need not remain same, because we have destroyed two vertices x and y and created a new vertex V x y.

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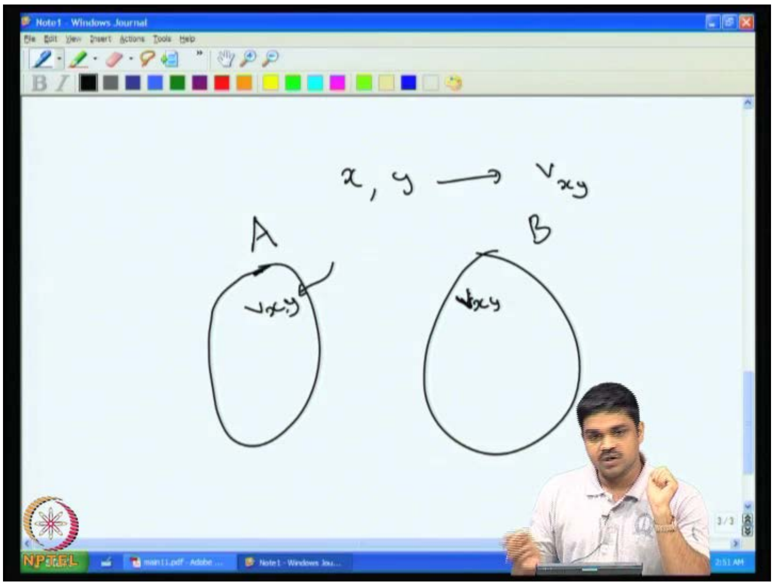

x and y was converted to a contracted vertex V x y . So, if A and B contained x or y or both, then the set is going to change, A and B are going to change. So, what we will do is suppose a say for a set A, it did not contain x or y, than it is as such, we will we will not have any change for A, but if it contained x, then we will add V x y do it; similarly if it contained y, we will add V x y do it; if it contained both, then also we will add V x y do it. Similarly for B, suppose we did not have x or y, then B is as such it would not be effected by this contraction operation, but if x or y was present in B, then we will put V x y in it, also if both x and y were present, then also will put V x y in it. So, what we should notice is, it is possible that V x y may be added to both A and B, it is not just that if I add V x y to A, then I cannot add V x y to B, it is possible that we may add V x y to both of them.

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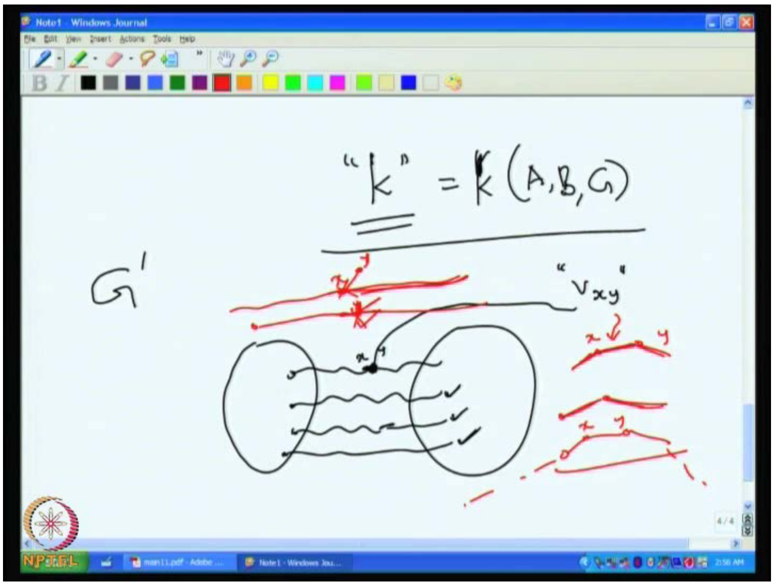

Now, the idea is to ask this question first. So now, is it possible that see we have this graph G, A the two sets A and B, and we know that minimum number vertices to be removed from the graph so that A gets separated from B is exactly K that is the number we are we are we are this the letter we are using to denote the number of vertices required to separate A from B right K of A,B,G sorry, so this is the letter. Now, we also assume that there are no K disjoint A - B paths in G; if there are already K disjoint A - B paths, we do not have prove anything. So, we assume that the number of the maximum number of disjoint A- B paths in G is strictly less than K.

Now, the first question is suppose I do the contraction operation, and I get a smaller graph say G dash, is it possible that in G dash, suddenly we have K disjoint A - B paths; A and B as defined just about it may get modified, but as above as I described above; is it possible that we end up with K disjoint A - B paths and G dash, so suppose it happens; suppose it happens, then why cannot we use this same A - B paths in G also that is a question. So, if this A - B paths were without that new vertex V x y, the contracted vertex V x y, definitely those vertex, those paths will be available in G also, we could we could definitely use them and therefore, that will be contradicting the assumption that there are no K, we do not have K A - B paths in G.

So, if at all G dash has K A - B paths, they should the newly created vertex V x y by contracting x and y should participated in this paths, but it cannot participated in every path, it can only participate in one path, because this is disjoint A - B paths, should be at least one of this path should contain V x y; then why do not I try to convert it to A may be corresponding A - B path in G; for instance all the other K minus 1 A - B paths can be used as such and this V x y, we can replace by the original x y, what will happen? So, this path is it possible, it is possible suppose path is there this V x y, it is possible that x and y, when I separate it, put it, it may be look like this, then of case, x takes paths in the path and it is okay; it may be just that these two edges both this two edges, which have a practicing may might have come from y, then also we can put y and we get the corresponding path or it may be possible that one this edge came from x, one of the edge came from x and the other edge came from y.

So, by putting back x y, we do have the A - B path again, just that we are long hitting the path by one more edge, because we will this if this was the path before you may make it like this being x and y this is possible right; this is the that is disjoint A - B paths. So, what we can understand is, in G dash that means, after contracting though edge x y, the resulting graph if you if we have K A - B paths, then the same K B paths are may be with some little modification will be available in G also, the original graph also, the instead of V x y we may have to use x or y or x y right. So, therefore, it is not possible to have G dash to have K A - B paths right, G dash also will not have K A - B paths.

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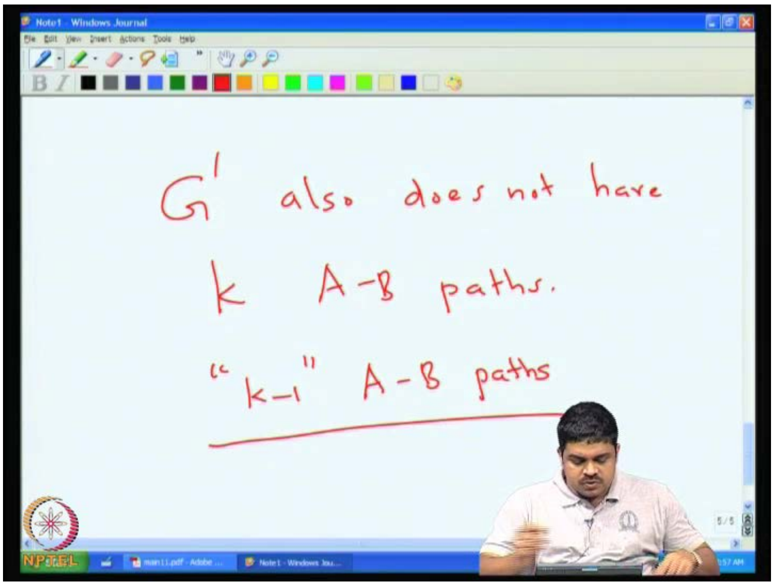

So, we can say that K dash does not have… So, G dash also does not have K A - B paths; otherwise G also will have which will be a contradiction. Now, if G dash does not have K A - B paths it is suppose it has K minus 1 A - B paths; if it it has K minus 1 A - B paths. So, this K minus 1 A - B paths means that there are by induction hypotheses, because we know that for all smaller graphs, the induction was true the the statement was true; that means, K minus 1 A - B paths means the number of vertices to separate A from B there has to be K minus 1 right. So, it has to be K minus 1 right.

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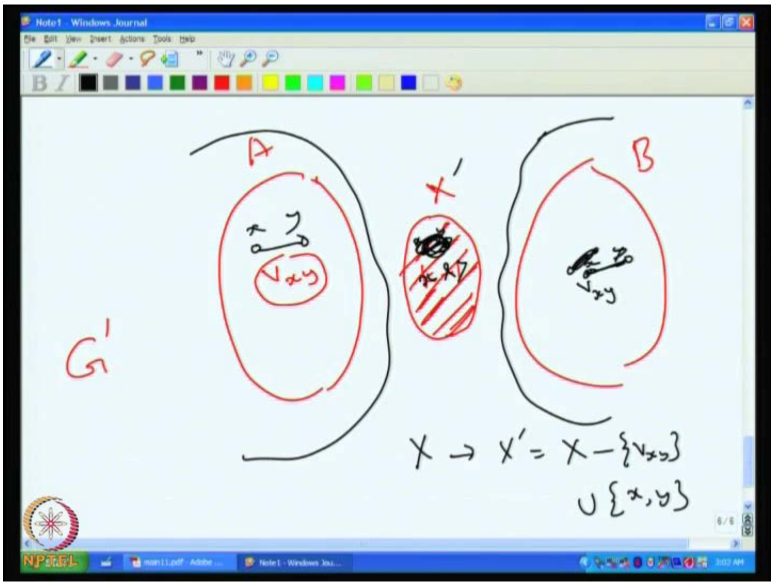

So, this is K minus 1; is it not? So, so you can separate A from B using K minus 1 some K minus 1 set y. So, say let us call it this set as X, some set X will separate A from B, and it will contain only K minus 1 vertices right; you may ask why it is not K there, because if it is K there, induction has to be true there, so so it would mean that they will be K K A - B paths. So, you can only have K minus 1 A - B paths; you know, now, you should ask if there is K minus 1 sized separator of A and B in G dash, where will be V x y have a V x y, is it possible that V x y is totally inside A of case, not; because if V x y is totally here, I could of replaced V x y with the original edge x y here, and this x will again separate A from B right. So, it is very clear, because it is does not change anything, because I would not be when you remove it, you cannot introduce any new x A - B paths; similarly x and y both of them cannot sorry V x y cannot be here also right, because I would have replaced x y V x y with this edge x y, and then how when I remove X, they would not be any A - B paths left.

So, they they would not be A - B paths, so x should be separator of A and B to contradict the assumption that there is a contradict the assumption that there is a, the separate the number of vertices, minimum number of vertices separate A from B is K. So, you cannot you could not have separate A from B using just K minus 1 vertices. So, what do we infer? We mean that the contracted vertex V x y, it cannot be in A, in this side sorry or in this side not A or B, I mean whatever is separated, now X is separated by this things. So, it is not possible to have V x y outside X, so you you should have x y sorry it is not A, so whatever.

So, it is because if V x y outside X, then we could have put it back that edge we could a put it back, without affecting this thing, because whatever we remove here is the same, and then this would not be capable of creating an A - B path here. So, V x y has to be a name in the separator, so x and y has to be in separator. So, what we do is so, this is not V x y, so V x y could be replaced by original x y, what will happen? So, the X will get converted to separator X dash norm, which is essentially X minus V x y, union x,y right. So, instead of V x y, I am putting x and y in this thing, so we will get we will call it as X dash. So, this X dash is a separator for A and B in G also; is it not? X dash is separator for A and B in G also. So, because if you remove X dash, then it is it is like removing x in the smaller thing right only thing is there are one more vertex now; so it is a K separator not K minus 1 separator right.

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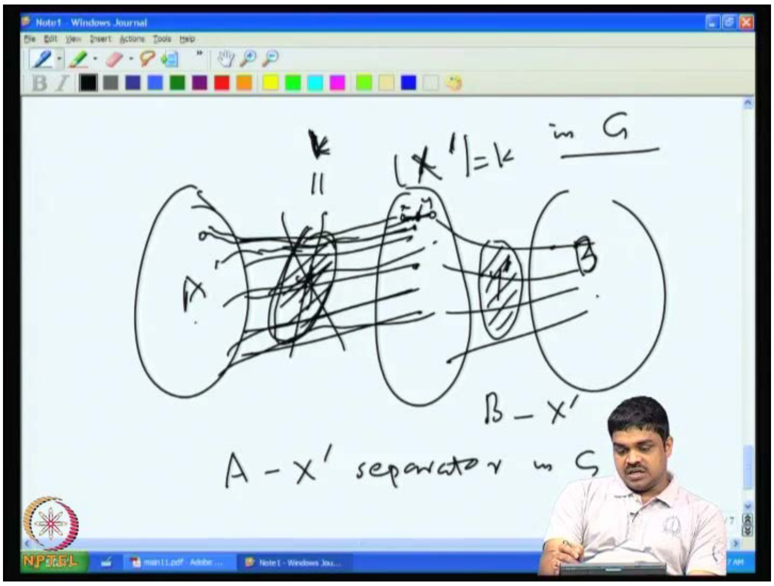

So, now let us consider this X dash separator here, this is the X dash separator; now we have A somewhere here, B somewhere here. So, A this A separated from B by X dash in G, in G itself; now we consider this A X dash separator in G; what we can do is we can remove this x y edge from the graph, let x and y both are in X dash. So, therefore, the induction hypotheses halts here, right in a smaller graph, but then see what it would suppose there is a some separator of y, which separates A from X dash . So, what can we tell about the cardinality of y; can it be greater than K? No, it cannot be grater then K, because X dash itself is a separator A X dash, separator of A and X dash. So, X dash is only cardinality k, so therefore, it cannot be it cannot be grater then K, it should be at most k, but can it be less then k; but if it is less then K, then what will happen is, so in every A X dash path is separated a cut over is passing throw this y right that is why y can separate A from X dash.

Now, also we know that every A to B path is passing through x dash. So, now any A to B path in G has to through pass through X dash, that path is also passing through y, because y is an A X dash separator. So, when you remove y from the graph A will get cut from X dash, so it means that A will get cut from B also, separated from B also, because all the paths, which starts in A and go to B should pass throw pass through X, all the path that is starts from A and reaches X dash has to pass through y. So, all the paths that is starts from A and reaches B has to pass through y.

So, when y is gone a separated from B. So, y has to be of the cardinality k. So, otherwise we will have a smaller cardinality separator of A and B in G, which is not possible K is the minimum number of vertices required to separate it by assumption. So, therefore, we can definitely assume that the cardinality of any A X dash separator is equal to y, is equal to K. Now what about B, suppose if I consider a separator y dash of B - X dash, the same argument is true here; its cardinality has to be K; why? Because it cannot be more than K, because X dash itself is a separator, X dash is of cardinality only K, and now every B A path has to pass through X dash, and then every B X dash pass path has to pass through y; that means, every B A path has to pass through y. So, essentially sorry y dash; so which means that y dash has to be of cardinality K.

Now, because we can apply induction hypotheses here for G dash, because we have removed this 1 edge that is why x y edge we can remove and argue; then that means, there are so many A X dash paths also right K A X dash path also, coming from A to X dash and coming from B to X dash, K dash B - X dash disjoint paths are available; similarly K A X dash disjoint paths are available that is by induction, but you may ask me suppose if I put back this edge, is it possible that number of vertices required to separate A from X dash will change? So, it is not possible to change, because this y edges added inside X dash that is not going to change anything, because if it is separated by y, now putting in edge between x and y, how does it change the situation, because if X was in A, then if X was not gone and it is not separated. So, we can assume that x and y, if it is still there, then it is not part of A right.

So, otherwise putting a connection between x and y is not going to provide a path from x to A X dash to A therefore, whether the edge X dash x y is there or not why will remain as an A X dash is separator, and y dash will remain as A B y B X dash separator. So, it means that we can we can we can say that even with the this new edge x y; that means, original graph G we have K edge disjoint sorry K disjoint A X dash paths in G, and similarly K disjoint B X dash paths in G. So, we will get the picture like this.

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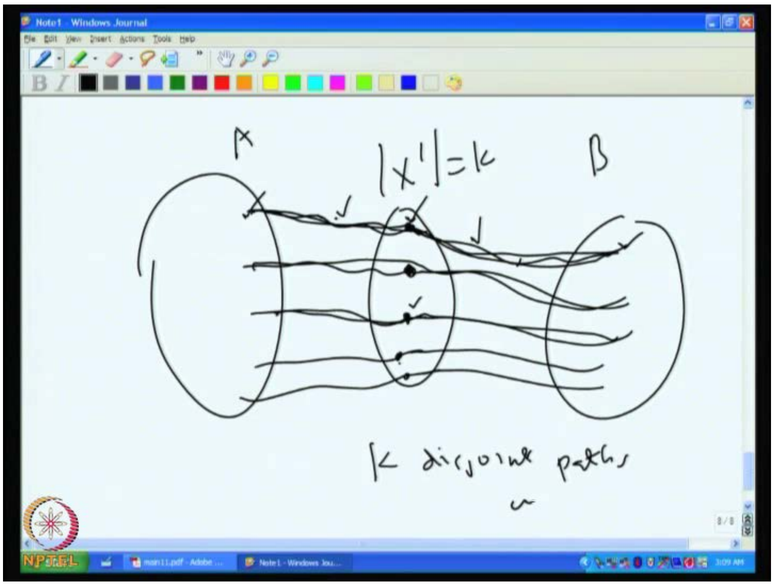

So, this is X dash, this is A, so we will get some paths like this K of them, so because there are K; similarly here this is B, so here here we will get K disjoint paths, you remember x cardinality of X dash is equal to K therefore, the n vertices are just sufficient to fill all the vertex take up, it takes up the n vertices of this A B A X dash paths takes up all the vertices of X dash, something starts form here and say 1 vertex and another vertex, another vertex, another vertex that is nothing left here; similarly this B X dash paths also come and take up this vertices, so there should be 1, for each 1 vertex should be A X dash paths B X dash path like that. So, they will they will merge like this; right for every vertex, there will be one coming from the A side, one coming from B side, so I can start from A, reach this vertex and go to B right. So, these are disjoint any way; is it not? So, so you should see, so it is not that this portion will take something from this portion, in that case we would have got a A - B path, which does not go through x dash, so they cannot take anything form that, because here there is the first vertex in A, here the first vertex in A X dash, the similarly there is one here, if something goes and takes here then could a short circuited and got a path between A and B, which does not go throw x dash; that means, X dash does not separate A and B.

So, we can paste the paths together, and we get K disjoint A B paths in G right has been added, that is what we wanted to prove, we wanted initially we started at proving that if X dash is a separator of A and B of cardinality K, then we should have K disjoint A - B paths in G. So, we ended up proving that. So, we in the last class we had seen in the applications, how this generalized version of Menger's theorem can give me give us back the original Menger's theorem that whenever two vertices are considered the number of vertices other than A and B required to be removed from the graph so that A is separate from B; this number is a same as the number of vertex disjoint, internally vertex disjoint A B paths in G, such things we had already seen in the last class.

Now, the only thing probably which remains regarding Menger's theorem is to look at it is edge version, so to look at the edge version of the Menger's theorem, the see one easy thing to do; so rather than proving it separately, to is to consider it to derive it from the vertex version of the Menger's theorem in as follows.

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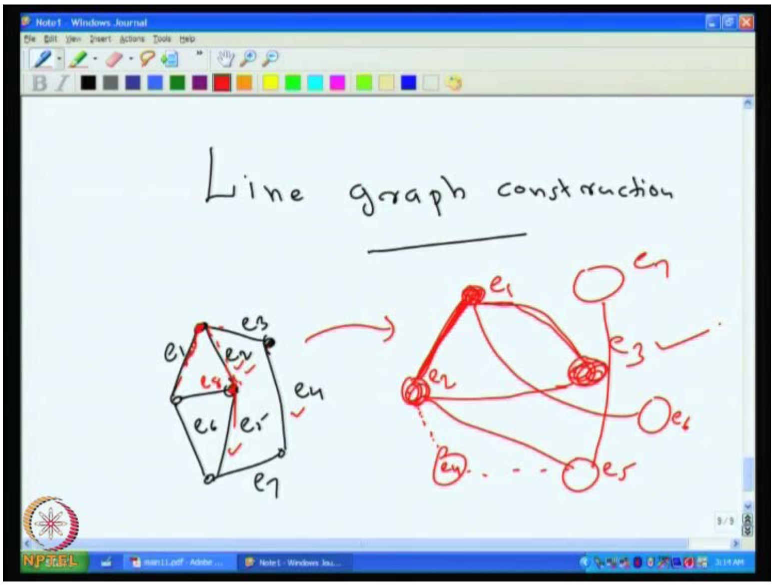

So, let us just talk about a construction called Line graph construction; Line graph construction what is a Line graph construction? So, for instance you can consider this graph, so small graph here, so right here is say e 1, e 2, e 3, e 4, e 5, e 6, e 7, so this 7 edges are there here; now I can construct graph on 7 vertices, so this is e 1, this is e 2, this is e 3, this is e 4, this is e 5, e 6 and e 7. Now e 1 - e 2 is a connection; e 1- e 3 there is a connection; e 2 - e 3 there is a connection; and e 1 - e 6 there is a connection. So, what I am going to do is here, whenever this e 1, 2 edges e 1 and e 2 are adjacent, then I am putting a connection between them. So, here this is e 1, this edge e 1 is connected to this edge e 2, so e 1 therefore, I am this is the edge e 1, this is the edge e 2, then put am putting in edge between them right; and similarly you consider e 3, so e 3 and e 1 is adjacent, because there adjacent incident on the same vertex, so I put a edge between them.

Similarly, if I consider this is e 8 sorry e 6 and e 1 are adjacent; what about e 2 and e 5? e 2 and e 5 are adjacent, because they are sharing an edge here. So, e 2 and e 5 are adjacent, we put an edge between them e 2 and e 4 are not adjacent, because e 2 is here, it is both endpoints are different from the both endpoints of e 4. So, there is no vertex, no connection between them right. So, this is e 2 and e 4 are not adjacent; now e 4 and e 5 are also not adjacent right, this edges is not there, because this are not sharing any endpoints, but on the other hand is e 5 and e 7 are adjacent this way. So, if I construct a graph like this from this graph, then it is called a Line graph.

So, here in the Line graph the vertex set is the set of edges of this, and then two vertices here will be connected if that that two vertices here are in fact, two edges of this graph and whenever these two edges share an end point in this graph, we will connect them together with an edge, this is the Line graph right construction. So, what is the good about Line graph construction?

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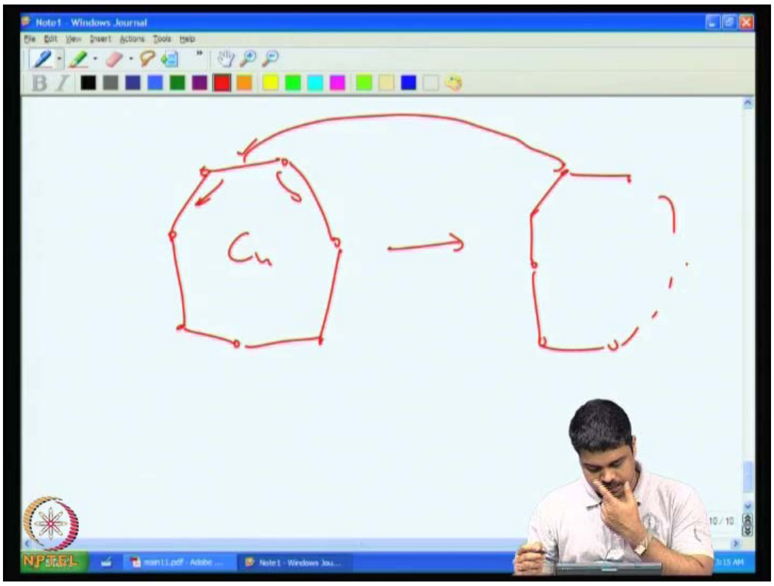

So, see to understand Line graph, we can construct some simple examples for instance this is the C n, the cycle; the what will be the Line graph of this C n? Of case, that will be C n itself, put some, next thing we can see that the same cycle will come right because the edge will this edge will become a vertex that is only thing, but then the two vertices will be adjacent to this. So, the same cycle will come here.

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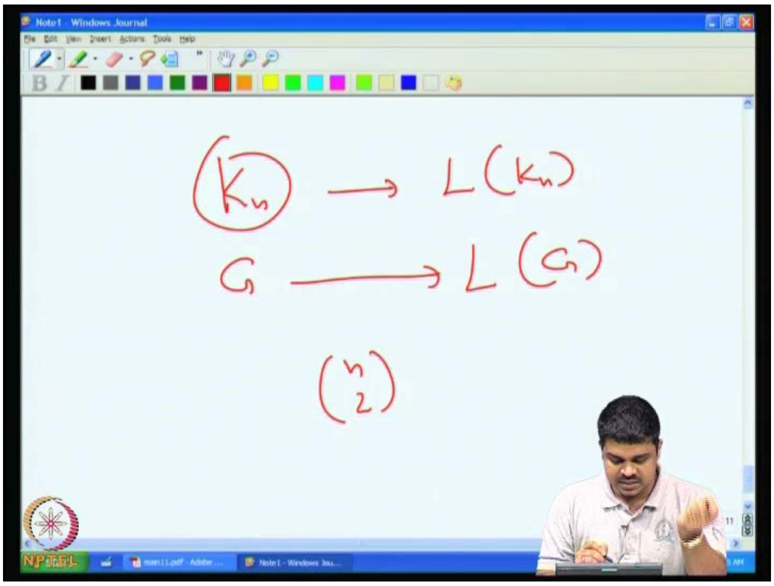

So, for instance if we consider a complete graph K n, and we call it the Line graph of K n, the Line graph of G. So, given in a G, we will write Line graph of G right. So, in the Line graph of K n, the number of vertices will be n choose 2, because there are n choose 2 edges in K n, and then whenever two edges are adjacent, then we will have to connect them together right.

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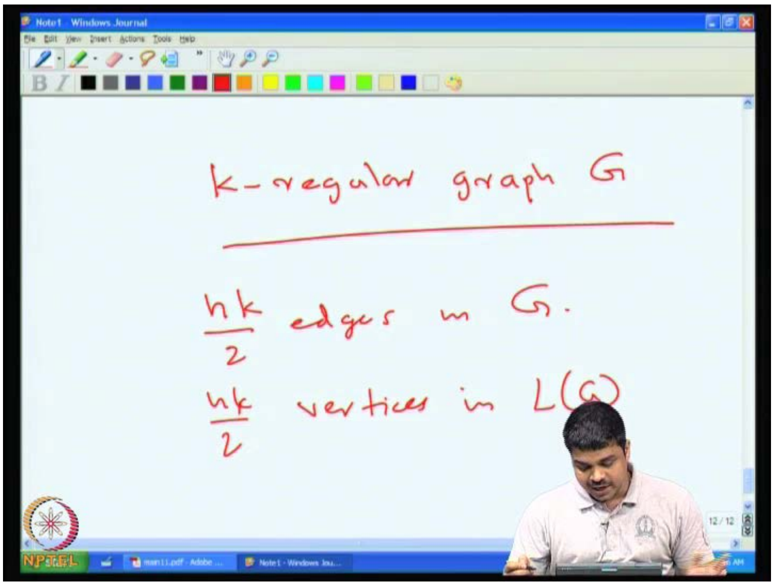

So, so for instance we can consider a K regular graph G, K regular graph G K regular graph G. So, how many edges are there in the graph G, there are n in to K by 2 edges in

G. So, in G, so n k by 2 vertices in the in the Line graph of G L of G right; now, you may ask, so what is L of G a regular graph? Yes, it is regular graph what should be the degree of each vertex in L of G.

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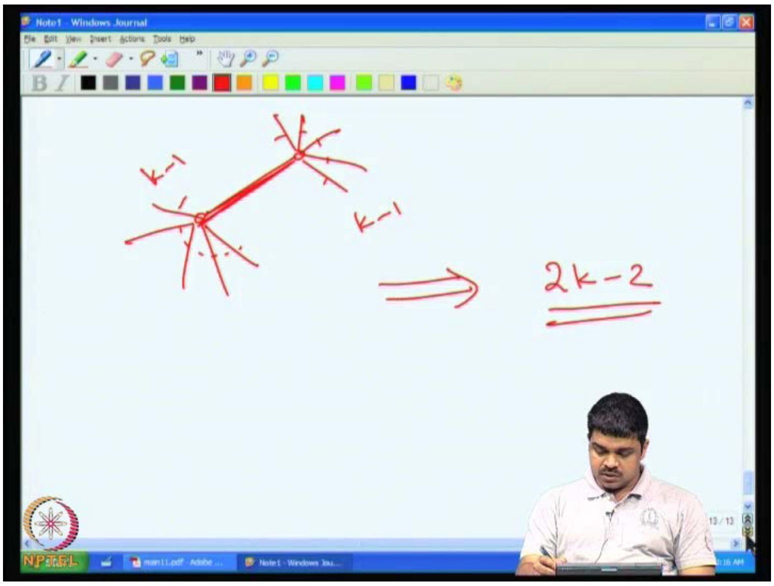

So, because you if you look at G the this is an edge in G, this is going to be a vertex in other other Line graph of G; now the neighbors of it is going to be this edges and this edges right this edges, so how many of them of there? Because it is a K regular graph, there are K minus 1 here, there are K minus 1 here, total degree will be 2 k minus 2 right. So, this is the the the so, the this is a typical situation in a Line graph.

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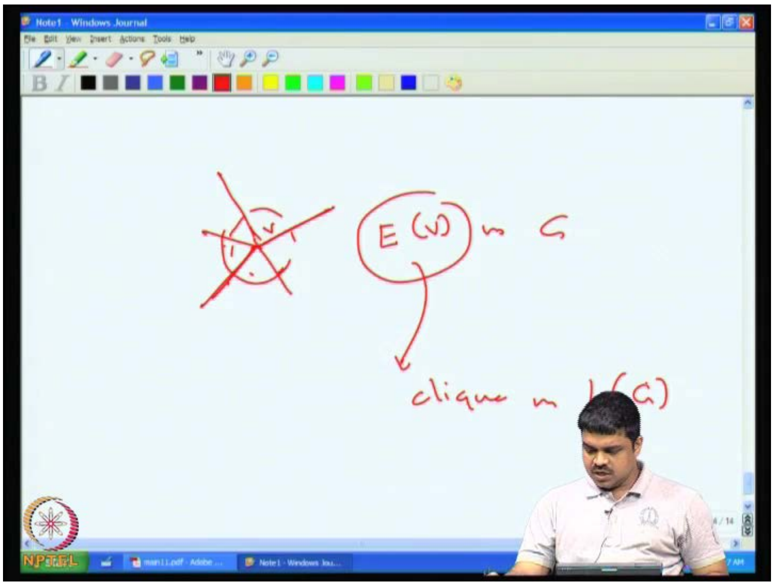

So for instance, if you look at a vertex V and the edges incident on E of V let say; vertex incident on the vertex V in G in G; what will happen to this in the Line graph, this will become because this are all this vertices, if he see they will all be a adjacent each other in the Line graph right so that will become a clique clique in L of G right; they will form a clique in L of G, so this I told all this things to just make you familiar with the concept of Line graph, why do we… So, when do we do call a graph is a Line graph? So, we call graph a Line graph when we can find some other graph h so that G is the Line graph of h. So, in other words we if we know that G can be represented as the Line graph of some other graph then we can call this graph a Line graph right.

So, usually we will say this is the graph and here is the Line graph of it, then sometimes will simply say this graph is a Line graph, because we just want to say that this graph there is some other other graph such that this graph can be represented as the Line graph of it, so that is the notion of Line graph; now how does the Line graph helps us to do the edge version of the Menger's theorem, so let us look at the edge version of the Menger's theorem.

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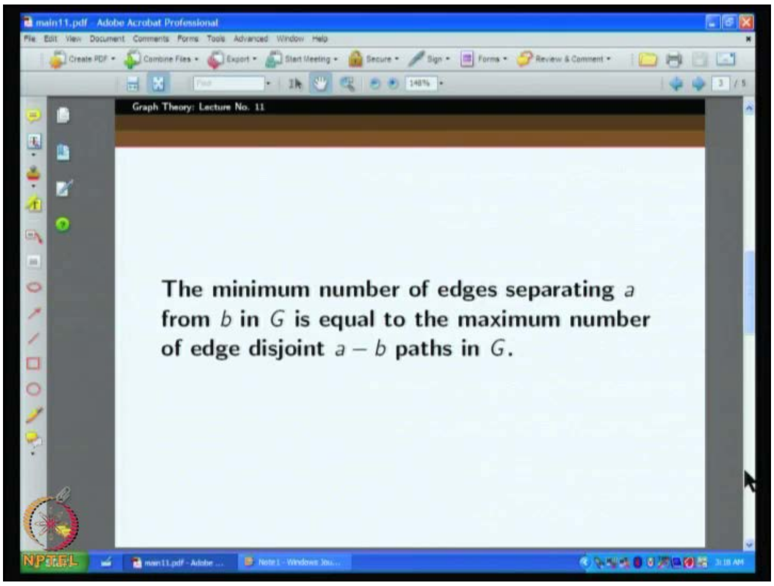

So, this the minimum number of edges separating a from b 2 a and b are two vertices in G; suppose a the minimum number of the edges separating A from B in G is equal to the maximum number of edge disjoint a - b paths in G; that means, you are given two vertices, so consider the minimum number of edges that is to be removed from the graph, so that a and b are in two different components that results from this removable of edges right.

Now, you also ask how many maximum number of… How many edge disjoint paths between a and b can be formed out in G; this both this numbers are same. So, how do we do this thing? So, to do this thing, so we should we should use the Line graph concept, we would consider the Line graph of G, and then all the edges of G will become vertices there; now for we are given two vertices a and b right. So, the incident vertices on a have become a collection of sorry incident edges on a would have become a collection of vertices in the Line graph, this collection of vertices are collected and let us called a say that it is E f a and similarly collection of edges, which are incident on b would become a collection of vertices in the Line graph. So, we will collect that vertices.

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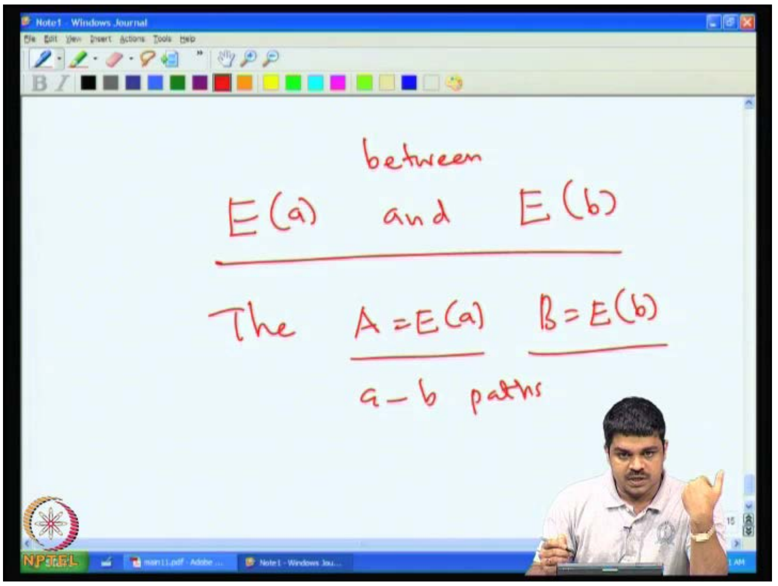

So, now between E of a and E of b between E of a and E of b, we consider the separator a equal to E of a B equal to E of b the number of vertices required to separate A from B is essentially equal to the number of edges that need to be removed from the original graph, so that a get separated from B in the original graph; similarly then number of disjoint a - b paths in this Line graph will be the vertex version here right. So, in the it will be equal to the number of edge disjoint a - b paths in the original graph a - b paths in the original graph.

So, therefore, just apply the Menger's theorem here, you will get edge version of the Line graph here. So, you will just look at the statement; the statement is just is the minimum number of edges separating a from b in G is equal to the maximum number of edge disjoint a - b paths in G, maximum number of edge disjoint a - b paths in G.

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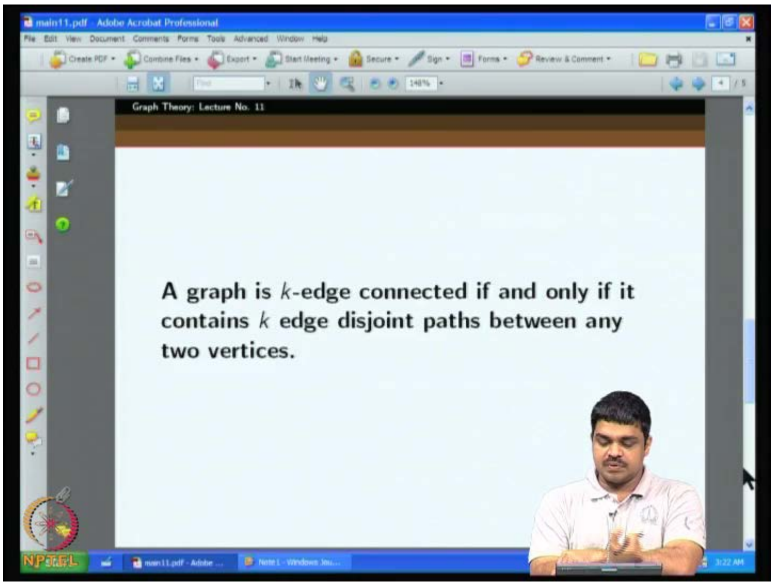

Now a graph is K edge connected if and only if it contains K edge disjoint paths between any two vertices graph is K edge connected if and only if it contains K edge disjoint paths between any two vertices. So, what is a what do you mean by K edge connected, we have already discussed it; it means to disconnect the graph, we need to remove at least K edges right. So, it happens only if there is for given any two vertices, they should be K edges disjoint paths that is that is that the same as the graph B in edge connected right, this is an immediate consequence of what which is so before; So, so we therefore, any way it is just have to think about it what it to understand it is, and now we close with this Menger's theorem, we will we will look out lightly difference notion of connectivity called a linkedness; linkedness what what do you mean by linkedness?

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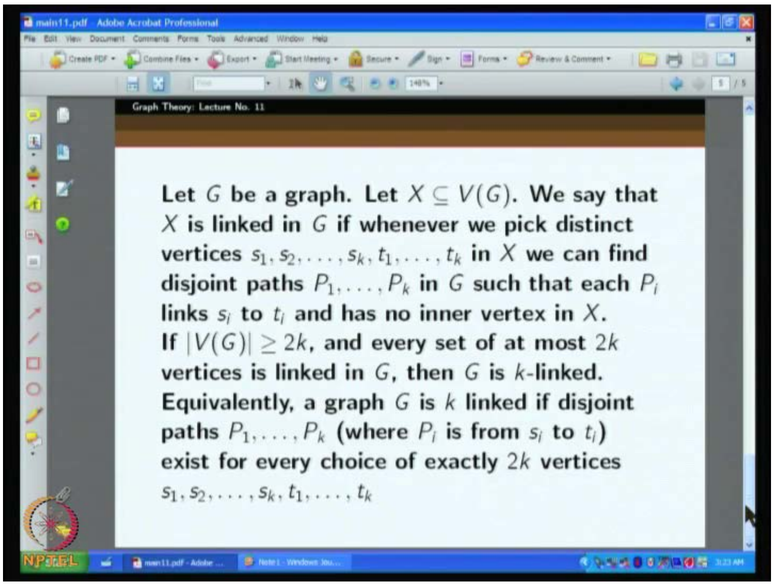

So, here let G be a graph; then let X a sub set of V of G, we say that X is linked in G, if whenever we pick distinct vertices s 1, s 2, s k and t 1 t 2 t K in X we can find disjoint paths P 1 to P K in G such that each P i links s i to t i and has no inner vertex in X. So, in other words if V of G is greater than equal to and also what is linkedness by V of G is greater than equal to 2 k, suppose there are 2 k vertices in the graph; every set of at most 2 k vertices is linked in G, every set of at most 2 k vertices is linked in G, then G is K linked, then G is K linked. Equivalently the graph G is K linked if disjoint paths p 1 to p k where p i is from s i to t i exist for every choice of exactly 2 k vertices. So, this is this are all Equivalent notions. So, let us try to explain this what. So, the final thing is much easier to understand.

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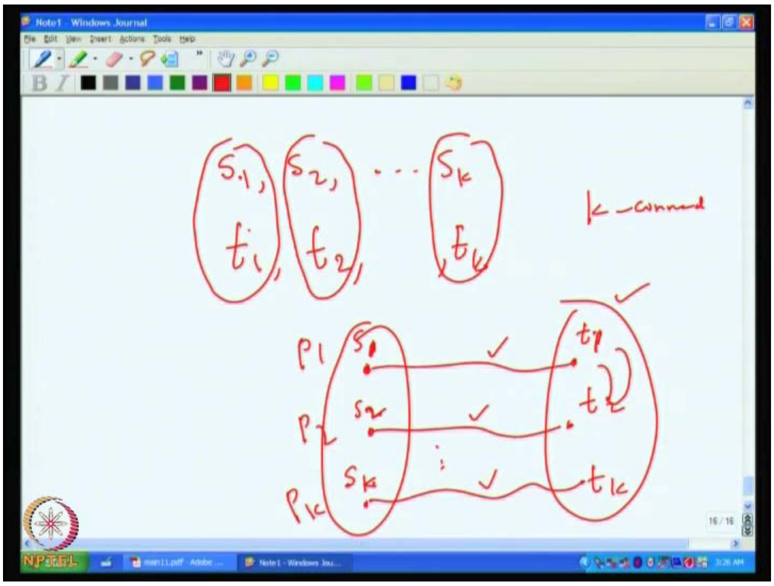

So, here we are seeing you give me whatever way you like you number s 1 s 2 up to s k some K vertices say and so this can be called sources vertices, none other disjoint different vertices t 1 t 2 t k as a sink vertices. Now the somehow this s 1 and t 1 is one pair right sourcing pair, this another sourcing pair, this is another sourcing pair. Now we should be able to find a path p 1 between s 1 and t 1 t 1 and a path p 2 s p 2 in s 2 and t 2 and path p k between s k and t k right. So, this path should be disjoint, this should not share any vertices, even the end point should not be shared; and see there the key thing is of case, this paths some paths do exist between this k set and this k set, by just connected graph is k connected, we know that between these two sets we do have k disjoint paths, but it is not enough here, here we are also the enemies also low to number the vertex s 1, s 2 up to s k that t 1, t 2. So, we may find these paths then we may say that I want to want one, so this is t 1 now, and this is t 2. So, in interchanging the role of t 1 and t 2, and then ask for new paths new set of paths.

So, where in other words, you fix this s 1, s 2 up to s k here in one order, the it is possible for the this t(s) to appear any of the permutation, in any of the order, this t 1, t 2, t k or may t k may take the roll of t 1 or anything take the roll of t 1, any permutation is possible here; we should be able to match them in such a way that this is match to this, this is match to this end right; that means, we should find. So, this should be so many paths different, so many paths. So, the question is how good should be the connectivity for this to be happen.

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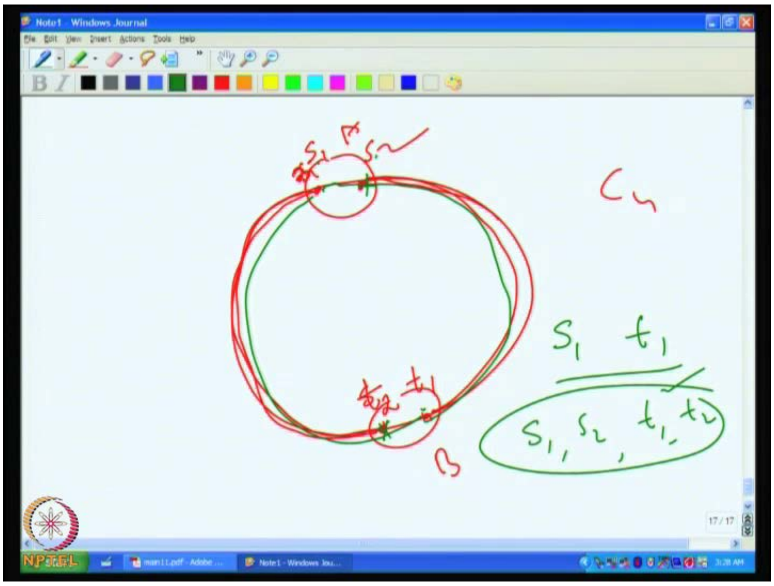

So, of case the k connectivity is not enough, we can simple simply take some examples, for instance see this cycle, we know that C n is to connected right any any two vertices if we take you can find two paths. So, over any so two paths are available or if we get two vertices I can show 1, so these two sets, this is A, this is B; I can indeed find one path here, one path here right, but what if I say this x 1 sorry this is s 1 and this is s 2, and this is such I name this in a wrong way; that means, I will said I will call this t 2, and this as t 1, then from s 1 to t 1, if I want to find a path, I either have to come like this right, come like this, in which case I will already take up t 2. So, that is not allowed, because t 2 s 2 paths should contain t 2, and then that intersection is not allowed or if I try to come this way, then s 2 will be already inside this that is also not allowed.

So, in this case, if I take s 1 here and s 2 here, and t 2 here and t 1 here, then I am struck, I can have only, I cannot have paths from s 1 to t 1 and s 2 to t 2, which do not intersect with each other right, so that is a problem. So we here we can say that if I give s 1 and t 1, then I can find the path, but if I if you give s 1, s 2, t1, t 2 then I cannot I cannot guarantee it depends on right, how s 1, s 2, s 3 are arranged. So, the K linkedness are demands that in a graph, if you are given s 1, s 2, s k in whichever vertices set or whichever naming it is and t 1, t 2, t k different set of vertices form s 1 s 2 s k and whichever is the naming right irrespective of that we should be able to find from s 1 to t 1 and s 2 to t 2, s 2 to t 3, s k to t k disjoint paths, disjoint paths p k; p 1, p 2, p 3, p k right this paths should be available; otherwise it is not K linked. So, in the cycle though it is two connected, it is not two linked.

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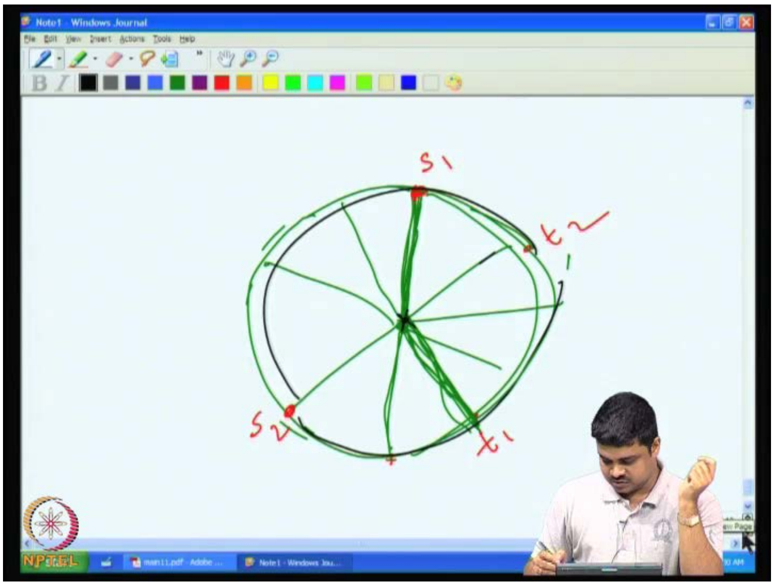

So, you can try if even this for instance you would (( )) this take this wheel structure which is three connector in fact. So, right if we take any two vertices you can get three parts here. For instance, if we take a vertex here and vertex here, so this is a wheel of case. So, right. So, you cannot you can always go through the centre and also you can go through this and this, there are three paths. So, but it is not even two link. So, the question is of case is it three linked? Of case, not it is not 3 linked, it is not even 2 linked, because you can you can take this s 1 and s 2, write s 1 s 2, and then we can as usual we can take our t 1 and t 2 here, what will we do to here?

So, of case we can we can try to make use of the central vertex, but it does not help because s 1 to t 1 if if you use this right, we use this, of case the centre is not available for the other pair any more right. Now, the other pair t 1, t 2, if it if I am trying to go like this s 1 is blocking it, if I am going to like this t 1 is blocking it; so both ways I cannot go, neither is this possible nor is this possible, and nor is this possible, because is vertex is already K converted. So, is already taken by other path. So, so here we can say that even with three connectivity it is not happening right. In general, so I can say that right…

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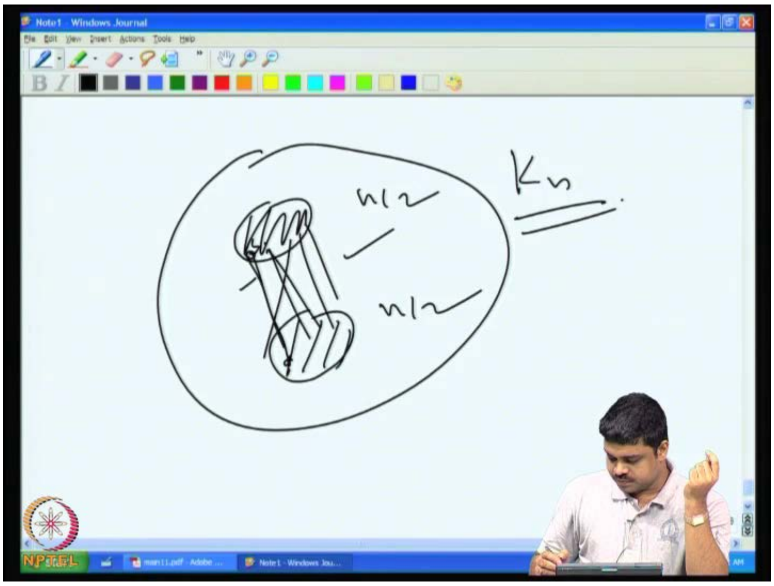

So, but on the other hand there are in some cases, the K linkedness may be very easy in some cases, for instance we may complete graph K n you can say that see any K set right any n by 2 set if you take another n by 2 set is available. So, we do that between any of them a path right. So, you just have to (( )) which s I which t I we can always take give the paths required. So, two links paths at two vertex paths right; an edges, edges can be given. So, they are n by 2 linked.

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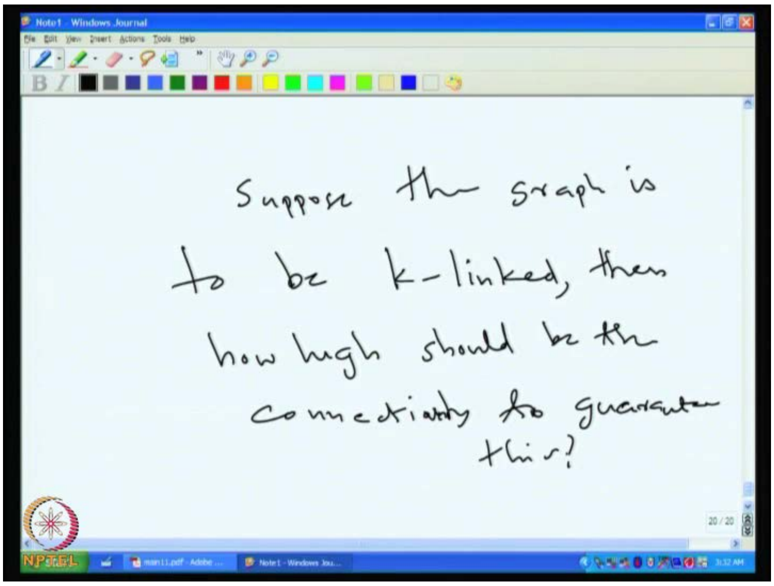

But (( )) on on the other hand, the question in the relevant question is suppose I want the graph to be the graph suppose the graph is to be be K linked right K linked, then how much how high should be the connectivity to ensure this? should be the connectivity how how should I keep the connectivity connectivity to ensure this to guarantee this? This is the question? How high should I keep the connectivity it? So, we have seen that I the connectivity if the connectivity is just K it may not happen as in the case of the the the as in the case of the cycle C n or in the case of V it was one more, but still it was not happening. So, for instance up to where?

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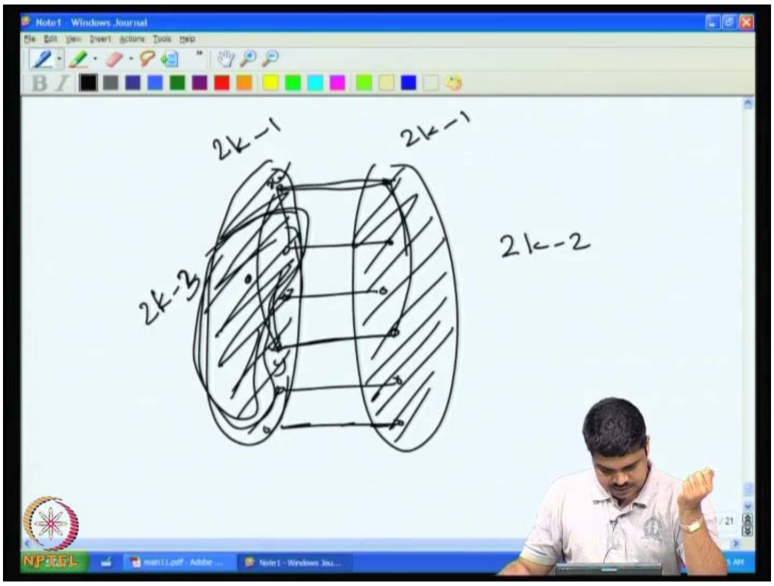

For instance you can see this example, this simple example, here we have a 2 k minus 1 connected graph. So, you take a clique of 2 k minus 1 here, 2 k minus 1 clique, this is the clique, and the 2 k minus 1 clique here right. So, 2 k minus 1, and then you the corresponding vertices you just connect like this matching just match them here right.

Now, what is the connectivity of this thing? For instance if you want to big inspect the connection between these two vertices right. So, we have how many paths between them? So, each of the intermediate vertices will give one path like this and one direct path is there right. So, total we have 2 k minus 2 plus 1 path is that. Because always possible to go to this side and come also 2 k minus 1 paths are there right. So, we have to… So, these two vertices 2 K minus 2, see other than other than see how many vertices are to be removed from the graph. So, that this two vertices get this disconnected that K also we can ask right.

So, you if you remove if you leave you 1 vertex left here will be paths, so will have to remove all the 2 k minus right 2 k minus 2 vertices here. How many how many vertices are removed? So, this is 2 k minus 2 vertices sorry 2 k minus 3 vertices here, and you have to remove at least 1 more from here right is 2 k minus 2 connected. This graph cannot be K linked.

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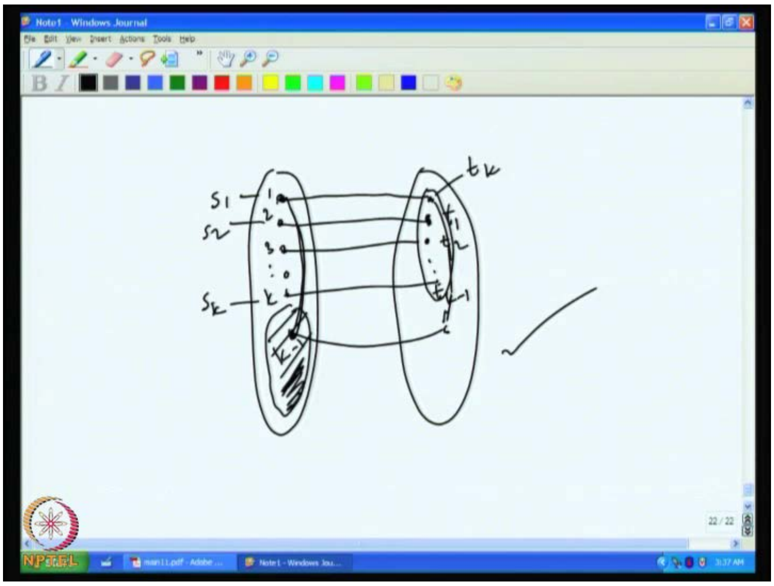

Why because say this simpler game in here, because if you look at say vertices k vertices here, 2, 3; 1, 2, 3 k vertices here, and then you just say that this is my s 1, s 1 this is my s 2, and this is my s k. Now, here I would rather say this is my the second vertex here is my t 1, and the third vertex is t 2 and this is t K minus 1 and this is t k; that means, just rotated the destinations; the direct connection is not there between s 1 and t 1, this is s 1 is connected to t k array, the t 2 is connected to t 1, and 3 is connected to… k is connected to k minus 1 and so on.

Now, we have only K minus 1 vertices left here right. So, you can see that if I want to travel from here to here for any any any pair here to here, we cannot use it is direct edge, because it will hit up another destination vertices so that is not allowed. So, you have to the only other way to go to this class, this group, and then go to this side and then jump into the side right. So, every time for every pair, I have to use 1 from here right, but there are only K minus 1 vertices here. So finally, what will happen is I will not be able to for 1 pair, I will not be able to get this resources available. So, K minus 1 will finish for K minus 1, so 1 will not be able to 1 pair not be able to connect between to each other that is if 1. So, that that is the the the the the that is the reason why this is not K linked. So, we have showed as in other words. So, even with the some connectivity like 2 k minus 2, we are unable to K linked it.

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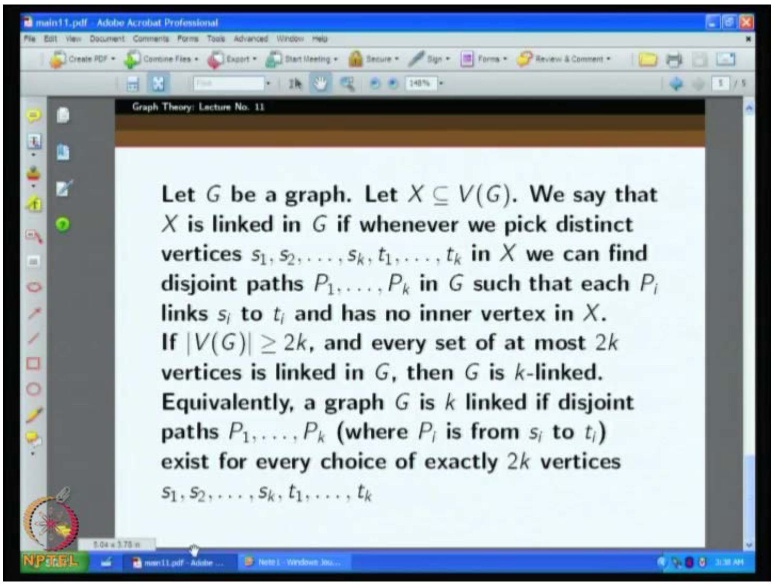

But it will be it is shown that shown that we want show it in this class that if the connectivity is 2 k with the assumption that the average degree is also a little by there are the number of edges is a little more than just implied by the connectivity, then it is indeed, K linked that that is prove proven by Thomas and Volan, that proof will not do in this class, but we will do some we will we will do a little lighter version of that of case, we want, we will just show the if we keep the connectivity is sufficiently large, then it is possible to K link the graphs to ensure that guarantee that the graph is K linked.

But otherwise so the in fact, thus the statement can be improved as I just mentioned to say that if the graph is to connected, and then the average degree is a little higher 8 K something, then it can be it can be shown there it is K linked. So, that will be done in the next class. So, next class we will do a proof of this statement that if the connectivity is sufficiently large, then the graph has to be necessarily K linked. So, to summarize the today's class what we did is, we did proof of Menger's theorem for the generalized version, and then showed that through the Line graph construction, the edge version of the Menger's theorem can be derived from the vertex version, and then we discussed about this slightly stronger version of connectivity called K linkedness, and that will be discussed in the proof of this particular statement, which I just mentioned will be discussed in the next class. Thank you.