Combinatorics
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Lecture - 9
Combinatorial identities - Part (2)
Permutations of multisets - Part (1)

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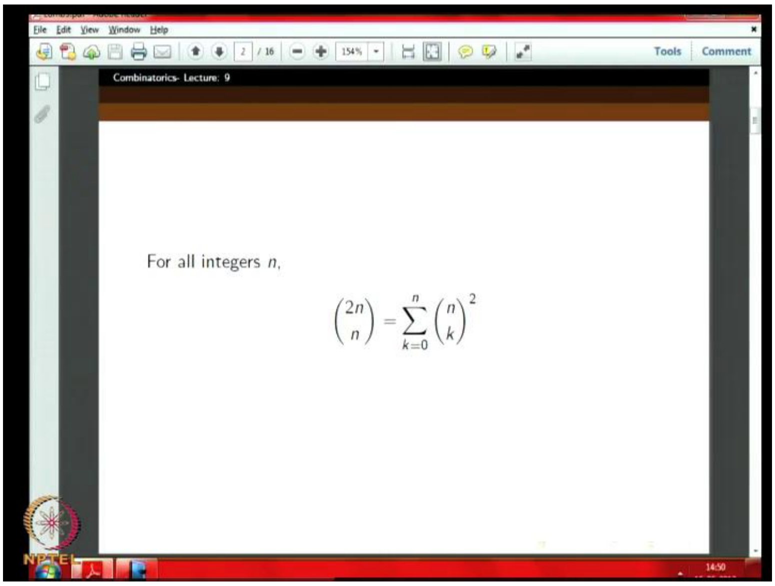

Welcome to the 9th lecture of combinatorics. So, yesterday we stopped with a one problem, so this was a combinatorial linearity 2 n choose n is equal to sigma k equal to 0 to n, n choose k whole square right, what does it maybe in the Pascal's triangle.

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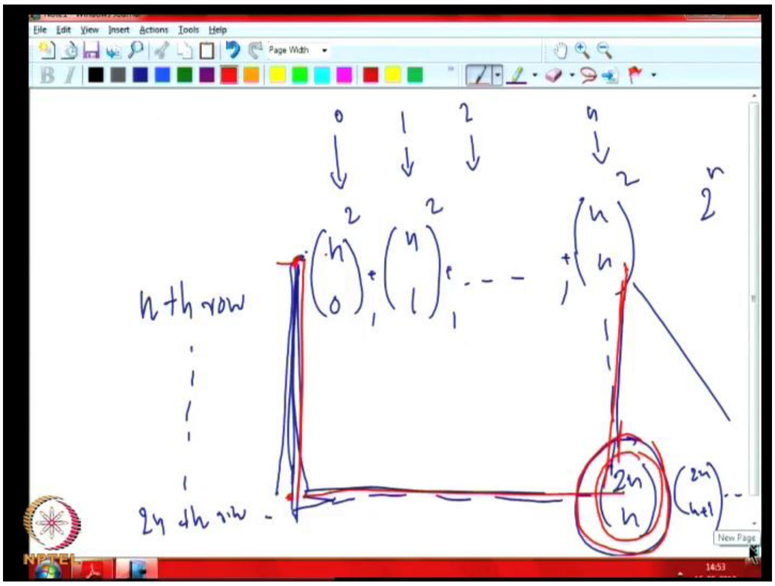

So, you can you are saying that if you take the nth sorry, if you take the nth row right this is n choose 0, so n choose 1 and then up to n choose n right, you square each of this terms right up to then I add them together what you get? So, you remember if you are adding without squaring you would be getting 2 raise to n, now you are saying that if you square and add right if you square and add you will get 2 n choose n. So, where in 2 n choose n this is 2 nth row if you go nth row right, this is 0th column, this is 1 column number 1, 1 column, second column and this is the nth column right.

So, in the nth column in the 2 nth row you will see 2 n choose n right, here there will be more numbers 2 n choose n plus 1 and so on; because we know it is increasing like this right. So, it is something like a yeah you can imagine that this is n to 2 n right it is an n, n units here downward and this was n unit and if you draw a line here and you need downwards in the square this corner right this number will come.

So, the point is here squaring the first a line of first line of this square if you consider; that means, when I the square starts with the nth row and ends with the 2 nth row then up to n columns only you are taking n rows here n column. Say 0 to n means n plus 1 columns here. Similarly, nth row included to n plus 1 columns here, this square here you will reach 2 n choose n. So, if you square up the first rows the members of the first row add up and you will get 2 n choose n this is what it says. This is from the Pascal's Pascal's triangle. So, that is what we can see to remember this right and then to prove prove this.

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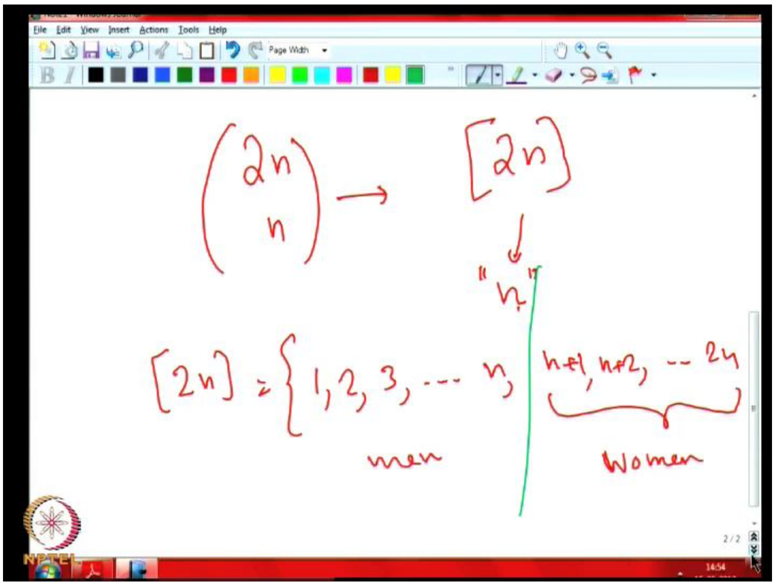

So, we want to show that 2 n choose n is that summation, what is 2 n choose n, 2 n choose n means out of 2 n things say numbers 1 to 2 n you are selecting n things how many ways you can select the n subsets of 2 n right. So, let us say this is 2 n is written like this say 1, 2, 3 up to n and then n plus 1, n plus 2 up to 2 n right this is one group let us say these are woman and these are the men, men and woman. So, we are saying that to select an n set from this, we could have a selected 0 men and n women that is one possible possible way.

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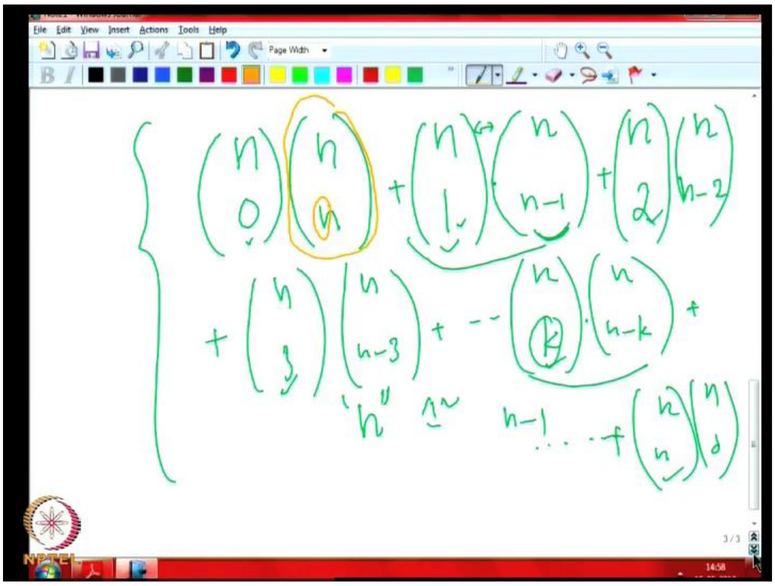

That means, that is you can do it in n choose 0 into n choose n ways that means 0 men are selected no man is selected and all n woman right. Or you could have selected 1 man that can be done in n choose one ways because there are n men and out of that one should be selected n choose 1 ways. And then we can multiply by the possible ways to select the women means n minus women, because you need n total right; so 1 man is selected now, now we need n minus 1 women right. So, total number women is n from that n women we can select n choose n minus 1 possible ways of selecting n minus n women are there now for each possible selection of the man you can multiply with each possible selection of the women. That will give all possible possible ways to select an n set n elements set n member set consisting a 1 man and n minus 1 women.

Next you could have selected n n people where are out of them 2 are men and n minus 2 are females right. So, 2 men can be selected out of the n available man in n choose 2 is and for each such selection you can select n minus 2 women out of the n women available in n choose n minus two possible ways. Similarly, this can be done with see n choose three right 3 men are selected and then n choose n minus 3 and so on. And for k men, when k men are selected this will be the general term n choose k into n choose n minus k possible ways you can select.

And finally, you can select all men; that means, n men and no women right this, so this is this will give you the total possible ways to get n people out of the 2 n available people. Because when I say first n are men and first the remaining n are women the any n subset you make out of this thing should contain some women’s and men. The number of men can be either 0 or can be 1 can be 2 it can be 3 it can be n n possible. For each such selection we have to get the remaining say for n choose k, k men are selected we have to get the remaining n minus k females.

So n choose n minus k possible these are the total possible ways of, so why I am multiplying this thing. So, because k men are selected then for each such k selection of men. So, you have n minus k possible n choose n minus k possible selection of n minus k women out of available and women that is why this is, then that is by multiplication rule we get each of this term. And then by addition rule, we because these all are disjoints sets right.

Because you know a collection a subset consisting of 0 men and n women that is that kind of subsets are not counting the subsets where 1 men is there and n minus 1 female is there.

So, this are disjoints things, so we can add up right by addition principle we will but then still this is not what we want we wanted n c o square plus n n choose 1 square plus sorry n n choose 0 square plus n choose 1 square plus n choose 2 square and so on. So, that we will get if you observe that this term is actually n choose 0 for instance here I can put 0 instead of n why because I can use the identity the symmetry identity.

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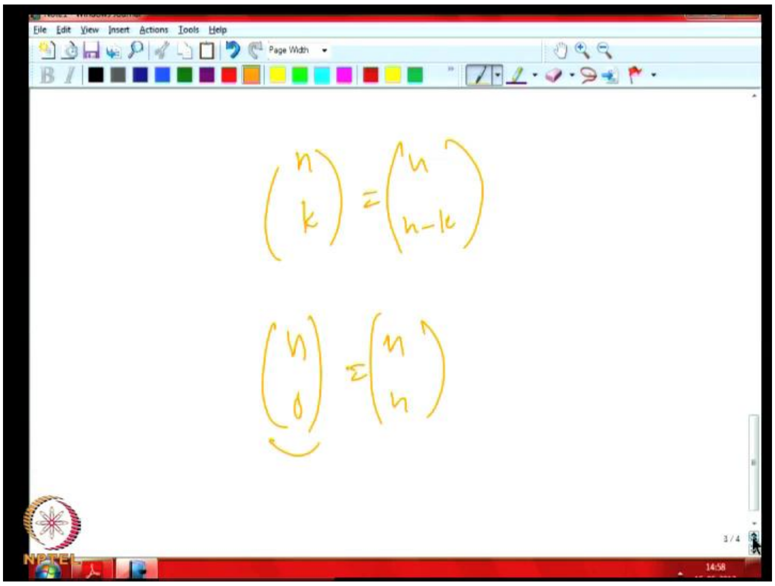

That means, I know that n choose k equal to n choose n minus k right and principle n choose 0 is equal to n choose n choose n, so choose n can be replaced by n choose 0 right.

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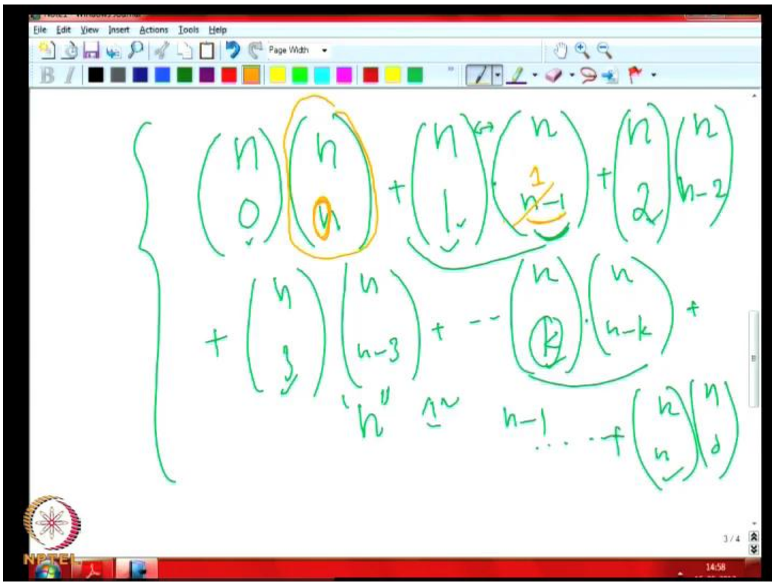

So, here I can replace n by 0 here similarly, n choose n minus 1 here can be replaced by 1 right instead of these thing.

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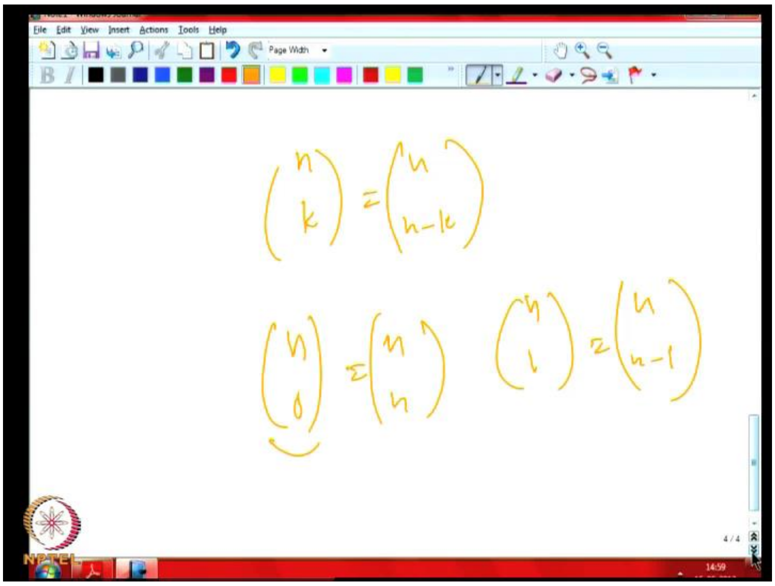

Because n choose 1 is equal to n choose n minus 1 right.

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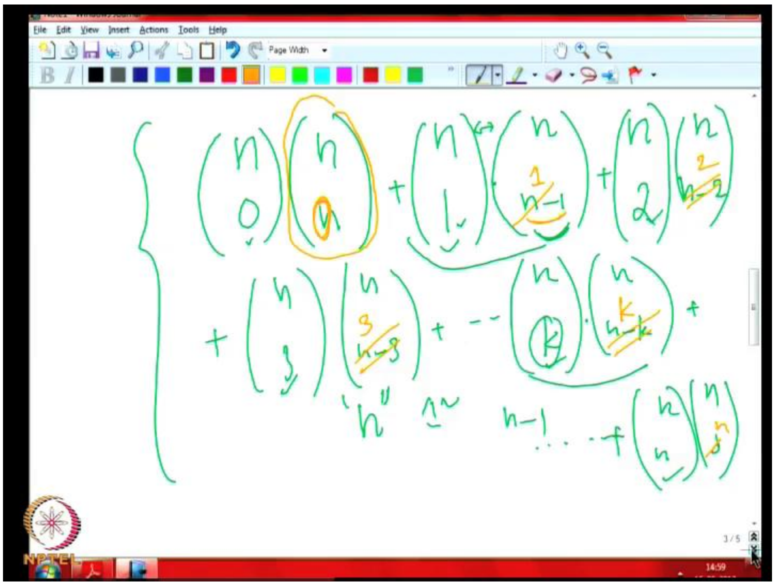

So, similarly, each term we can replace here I can put 2. So, here I can put 3, here within that I can put k and here it will be n itself. So, what is this n choose 0 into n choose 0 n choose 1 into n choose 1, n choose 2 into n choose 2 like that.

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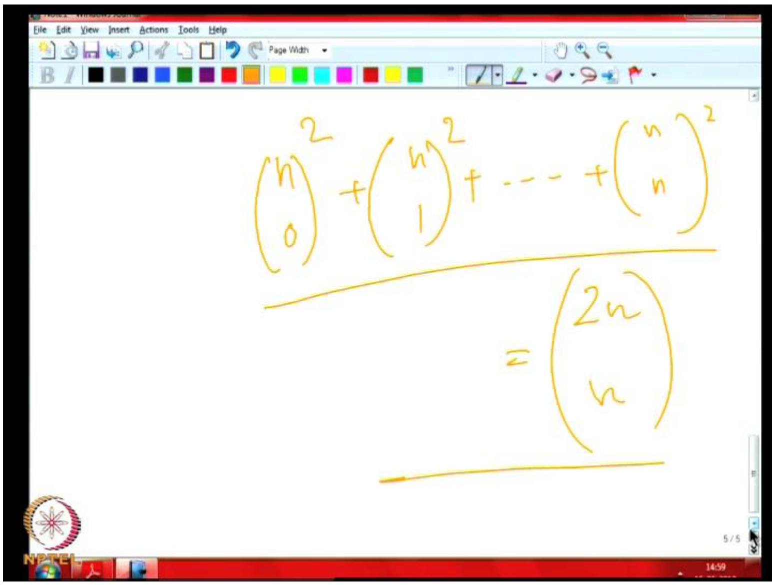

So this is essentially n choose 0 square plus n choose 1 square plus like that n choose n square. And we know that this is the total possible ways to select n people out of 2 n available people, this is a combinatorial proof of that identity.

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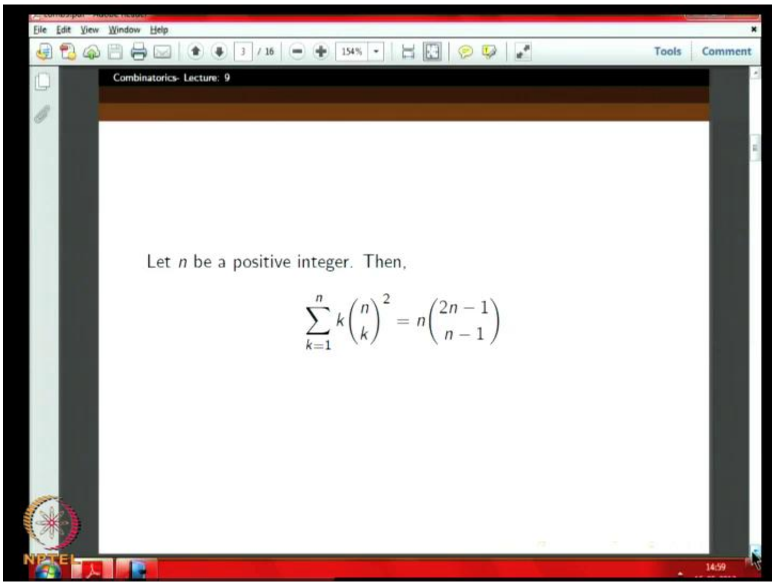

Now, the next identity we want to prove is this one. So, similar 10 be a positive integer of case, k less than equal to n and non negative integer of case.

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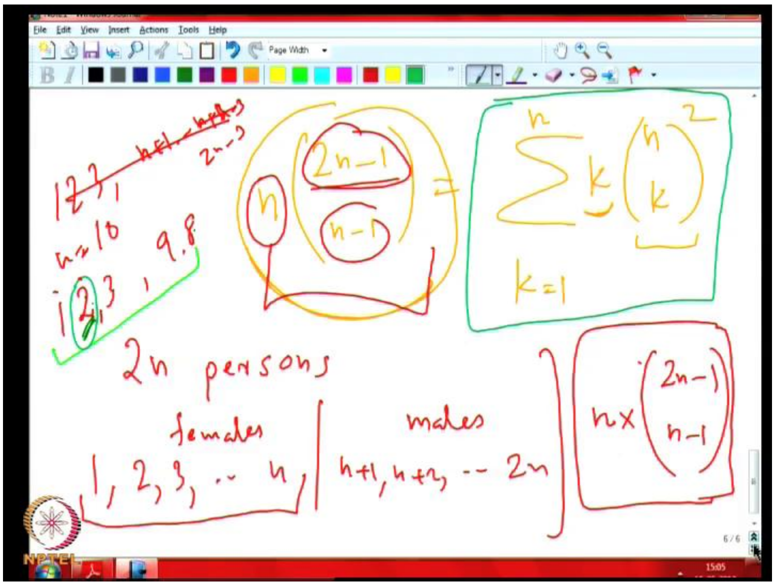

So now, T equal to 1 to n, so this is the identity says yeah this identity says n into 2 n minus 1 n minus 1 is equal to k equal to 1, 2 n right k into n choose k whole square k into n choose k whole square. So, how do you prove this things it says looks this, here we have n choose k whole square an multiplier is there into k n choose k whole square into k. So, equal to 1 to n and here it is n into 2 n minus 1 choose n n minus 1.

So, we first interpreted in the following way we can try to interpret the following way. So, again we a have 2 n people 2 n persons are available say out of which a say 1, 2, 3, the first 10 are females and the remaining n, n plus 1 n plus 2 up 2 n are males, these are females and this is males. Now, I want to form a committees the subsets of size n as usual but this time I also want to see because this is the subset of interpreter as committees and a we want a chair person for the committee each committee needs a chair person. The only condition is that the chair person has to be a woman, so how many ways you can do this thing.

So, this number will give the answer how does it give the answer. So, you can first select the chair person in how many ways. So, there are n women any of them can be the chair person. So, now you select it an n choose one n way n ways right n ways you can select, now once you have selected the committee person chair person I have to just select the remaining members of the committee. The remaining members of the committee can be all men, all women, some men, some women, no problem women. So, therefore we just have to select the remaining n minus 1 person out of the remaining n minus 1 people 2 n minus 1 people.

So, from the remaining 2 minus n 1 people you have to select the n minus 1 remaining people. So, first in n ways we selected a the chair chair person and then for each possible selection we can, we have remaining two n minus 1 people available out of that ican choose the remaining n minus 1 committee members in this many ways, 2 n minus 1 choose n minus 1 ways. So, this number gives you the possible committees along with the chair person, it is a it is just that what we counting is not just the committee it is the committee with the specific chair person.

For instance it can be the committee can be say 1, 2, 3 and say n plus 1 up to n minus 3 or, so right sorry n n plus n minus 3, or so 2 n minus 3 or so right. That means for instance if there are n equal to 10 k’s a the committee can be 1, 2, 5 people are recruited 1, 2, 3 say 9, 8 this can be the committee right. But this in one case chair person can be 1 and another case for the same committee members the chair person can be say a 2 but these two will be counted different.

For instance 1, 2, 3 a committee comprising of person number 1, person number 2, person number 3, person number 9 and 8 with the chair person being 1 will be counted differently. I mean it he will think that it is a different committee, when the committee members are same but the chair person is say instead of 1 the chair person is say 2 right, so that will be different committee.

So, therefore, we are actually not counting just a that subsets, we are also counting the subset with the chair right, who is the chair also I mean, if the chair is the different we are counting it differently right. So, that that is this number, and then we can count it in a different way to get the same number and show that this is this will correspond to that how will you do that.

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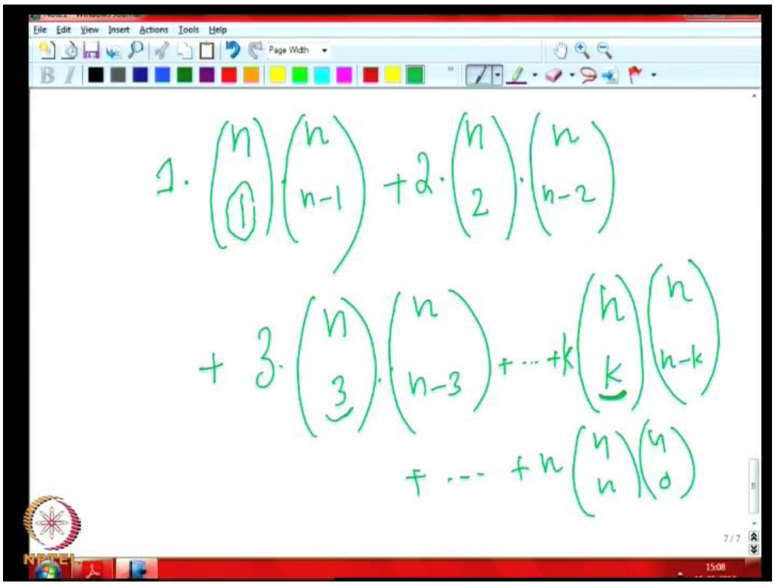

See, we can do it this way first select the women and the committee, one thing went slightly different citation from the earlier one. Here earlier one we could have selected a no females and all males but here it is not allowed, because at least one female should be there in committee; because one chair person has to be a women the chair person has to be a woman every committee should have a chair person. So, for at least that chair person will be there as a female so therefore, we should have at least one woman.

So, we may select 1 woman right out of nth available woman in n choose 1 possible ways and then we may select the remaining men; that means, n choose n minus 1 ways we can select n minus 1 men out of n men. So but then one because there are there is only one women the committee chair can be decided on only in one way right 1 right is it no; because there is only 1 women that women has to be the chair or you can have 2 females in the committee.

So, that 2 females can be selected in n choose 2 is and the remaining committee members are all men say. So, we can select that n minus 2 people from n available men in n choose n minus 2 ways, so this are possible at this are the possibilities. But now to decide the chair person we have two possibilities because there are 2 women here similarly, you could have 3 women in the committee. So, that means a n choose 3 possible ways to select the female members on the committee. And the remaining n minus 3 members has to be selected from the males that can be done in n choose minus 3 ways like multiply them together to get the all possible combinations.

And now because there are 3 females, so any of them can be the chair person, so have to multiply by 3. So, like that in the k case in the general case when we could have selected k female members, from the n possible it is in n choose k ways and the remaining n minus k committee members should be male; so we can select them in n choose n k minus n ways. Now, because there are k females here we can make any of them as the chair. So, there are k possibilities.To decide the chair. So, k into n choose k into n choose n minus k and finally, when you reach we could have selected all females n choose n right and no male because there is no problem in that and then but then there are n possible ways to select the chair this is in it.

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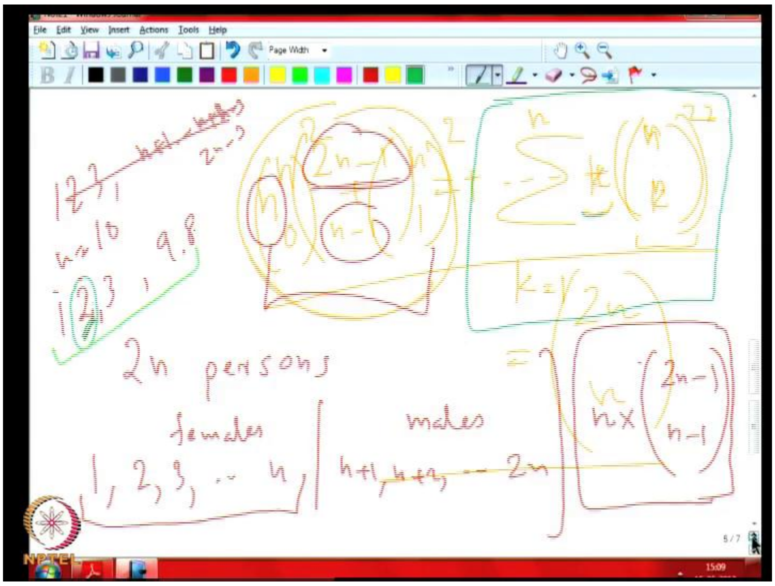

And again as in the previous case how will you get the form because we have to show that this is same as this k you see almost we have this k equal to 1 to n k into n choose k square but we have n choose k into n choose n minus k that we know how to do right.

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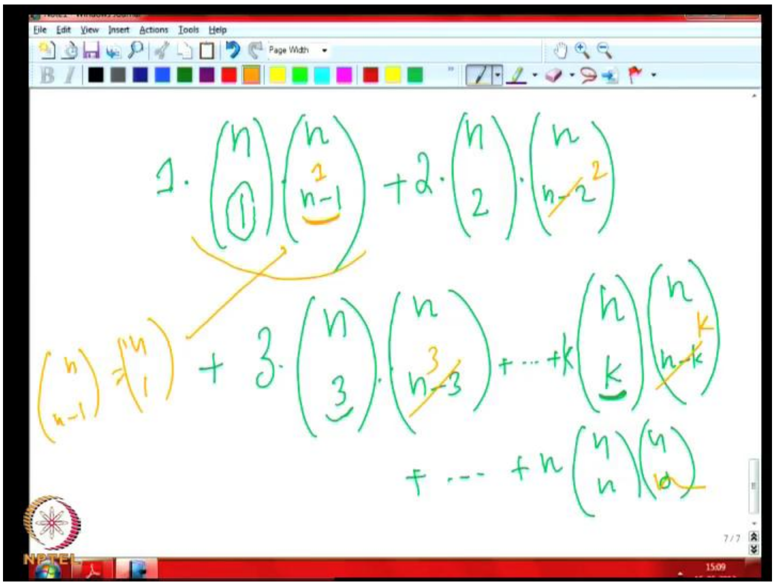

Because yeah here we have to n choose 1 is here, so this should be replaced by 1, why because n choose n minus 1 is equal to n choose 1 by symmetry identity right. So, that is that is how we can replace n choose n minus 1 by 1. Similarly, this n choose n minus 2 can be replaced by n choose 2 this is n choose n minus 3 should be replaced by n choose 3, n choose n minus k should be replaced by k like that. So, here we can put n right, so like we did it in the previous case this term n choose n and choose n minus 1 can be read as an n choose 1 into n choose 1 that is n choose 1 square. Similarly, n choose 2 into n choose n minus 2 is n choose 2 minus into n choose 2 it self, that means n choose 2 square and so on.

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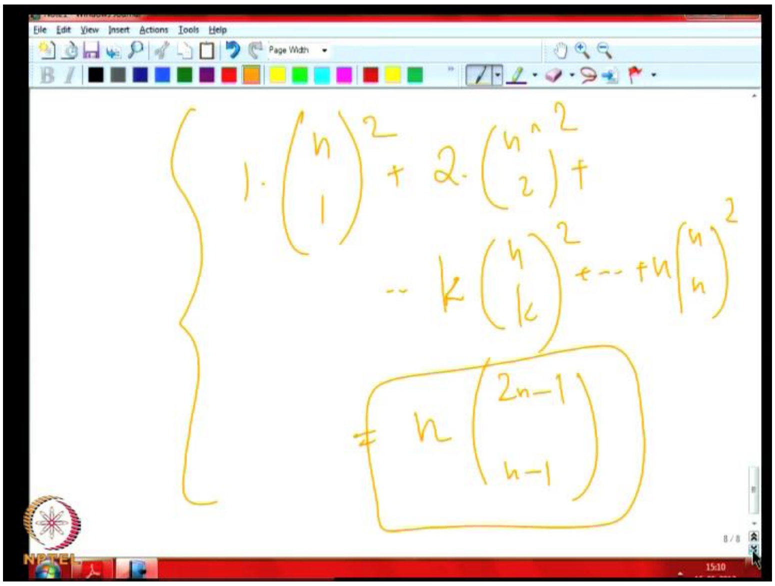

So, finally, this turns out to be 1 into n choose 1 square plus 2 into n choose 2 square plus finally, k into k into n choose k square and n into n choose n square, so this counts that, n into 2 n minus 1 choose n minus 1 the same this.

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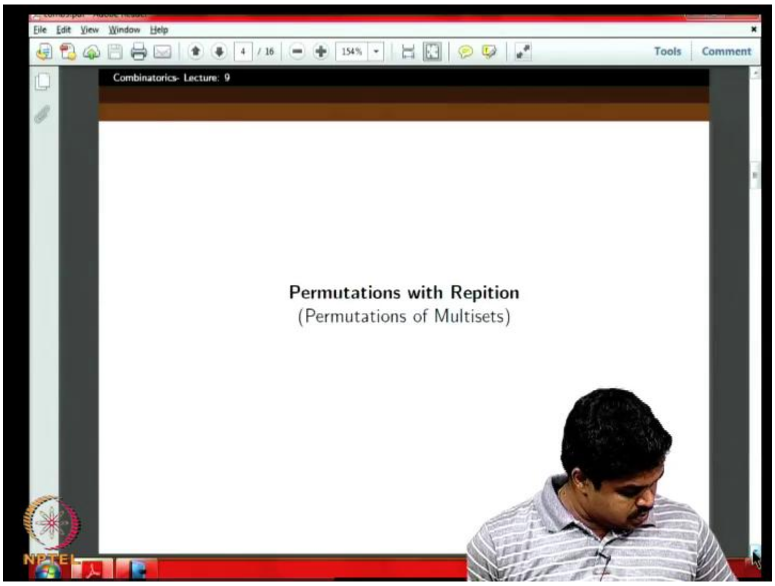

So, by this two examples we have illustrated some reasonably sophisticated counting arguments I mean initially that symmetric identity and that addition formula may proved where quite simple very simple. So, these were reasonably more sophisticated counting. So, many times you can see that yeah, so the this kind of identity see say this summation, suppose if you see the summation somewhere you can replace it with a more need formula like n into n choose n minus 1. So, this will give a better idea of what it is right. So, let us say some many times becomes useful that way, so not that we have to remember all those things but we should be able to do some such manipulations. So, now I will move on to a new topic permutations with repetition, so in other words will say the permutation of multi sets.

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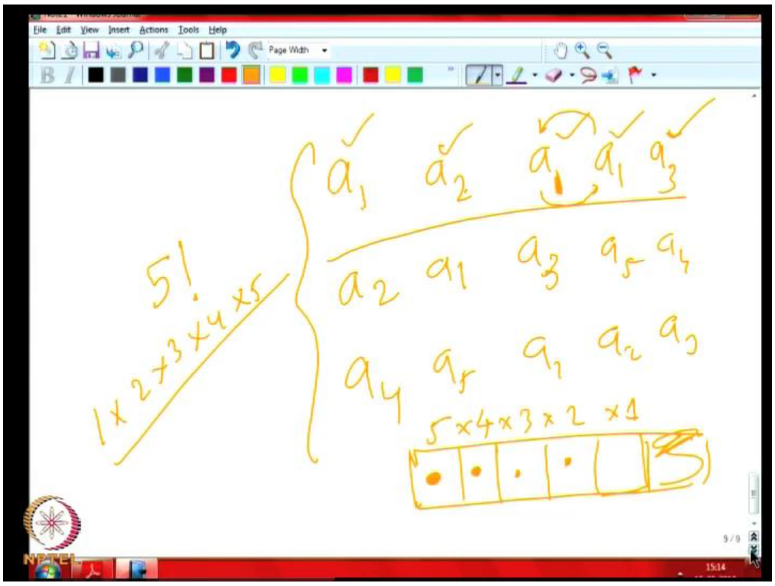

So, that could be better illustrating that means see the point is we have a certain we have seen what a permutation is, for instance say there are these objects a 1, a 2, a 3, a 4, a 5 in how many ways you can permute permute it. So, we have seen that the permutations are say you could have started. So, these are all permutations a 2, a 1, a 3, a 5, a 4 all ways of arranging it right a 4, a 5, a 1, a 2, a 3 right how many ways there are we have seen that there are five factorial ways of doing this right, which is 1 into 2 into 3 into 4 into 5. We have argued that this is because see if you want to fill the positions right five positions. So, this any of these a 1, a 2, a 3, a 4, a 5 can come here there are five choice choices to fill it and then the second one.

Once you have used one thing one of the a 1, a 2, a 3, a 4, a 5 to fill this thing then only four choices are left here. And once here fill these two there, only three choices left here, and then there are only two choices left here, and then there are only one choice left here and so on right. So, can have this three only five five positions are required right. So, that is why got 5 factorial but somehow we are assuming here that these are all different things a 1, a 2, a 3, a 4, a 5.

Suppose they are not different suppose we have a 1, a 2 and this a instead of a 3 we have a way again we have a 1 and this is again a 1 right and this was an a 3 right that means we have an a 1 an a 2 and 2 a 1's and a 3. So, will it be again 5 factorial definitely not because when I earlier a 1, a 2, this this was one permutation, suppose when I put it here and this here it is a different permutation but now because they are all a 1's if I take one here and one here. It is not going to change anything because it will be the same right. So, how will you count how many permutations are there, this is what we are interested in now.

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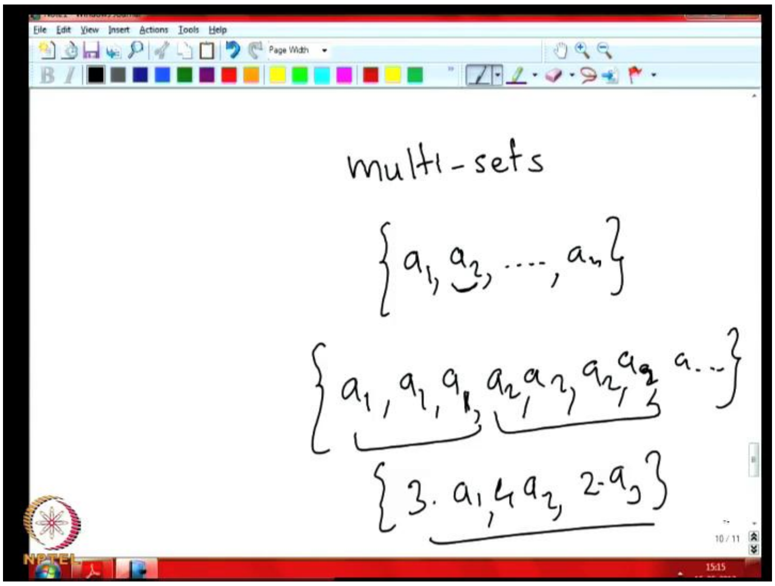

Now, so let us say we will formulate the problem using this word using the form this this concept of multi sets multi sets, what is a multi set? We know that a set is a collection of objects right say something like this but in a set we do not count we do not put the same object more than once we assume that each object comes only ones there in the set. In a multi set we allow copies of the things we can have a 1, a 1, a 1 say then a 2, a 2, a 2 right a 2, so like that. So, we also allow repetitions in multi sets right assume that the same object is present more than one times. So, one neat way of writing it is for instance this is 3 a 1 we can write it as 3 a 1's and here 4 a 2's and say 2 a 3's and so on this is one possible way of writing it on.

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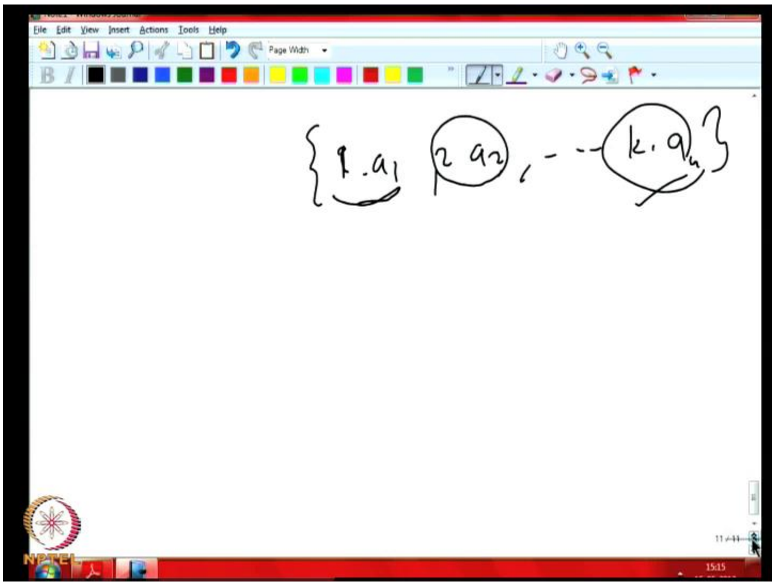

When it is a finite set we can just write that 1 a 1 right, 2 a 2 something and then say some k a n right. So, that means 1 time a 1 has appeared 1 times, a 2 has appeared 2 times, a n has appeared k times and so on right. So, this is going to be our notation for multi sets now we are interested in this kind of things as i mentioned earlier. So, we are interested in the permutations of multi sets.

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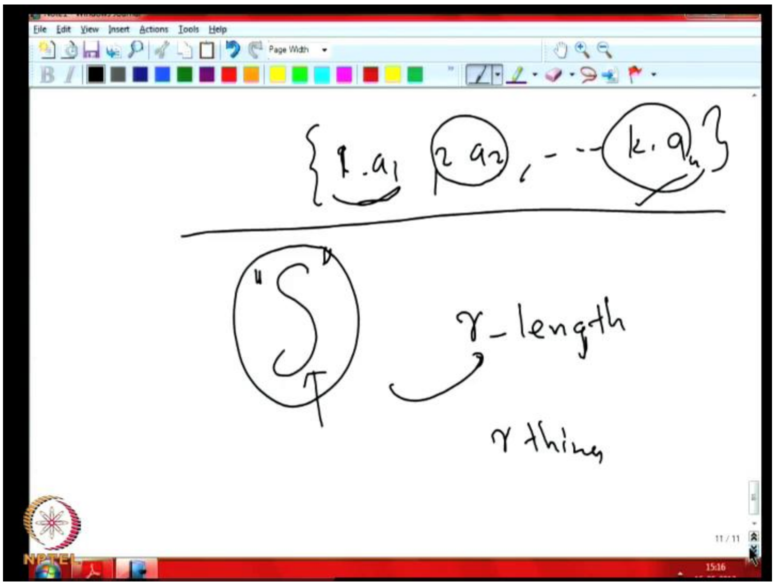

So, the questions can be two different thing for instance I can ask for like there is some multi set S and I want an r length permutation of this thing like we were asking for whether n objects we want to have a we want to take r of them and order them. We have to get an ordered list of r things from them given things, like that from the multi set if I want to get a r things r things ordered list of r things from this multi set of S. So, we will we will not answer it completely here but will answer some special cases.

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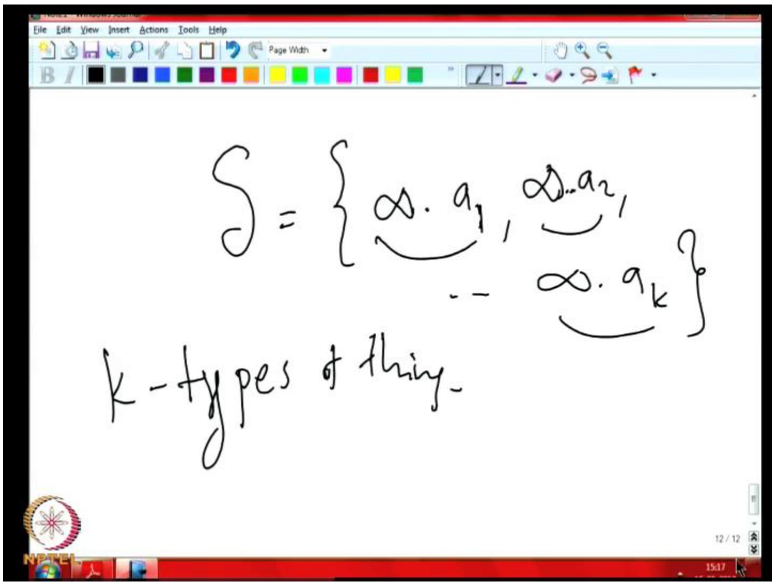

One special case we can easily argue is this case when S is equal to say infinity times say a 1 infinity times a 2 right infinity time. And say infinity times a k there are k things and an each of them having infinite copies right in the multi set this is an infinite set. But the only there are k types of things only finite, k is finite S is finite number k types of things but of each type we have infinite copies right.

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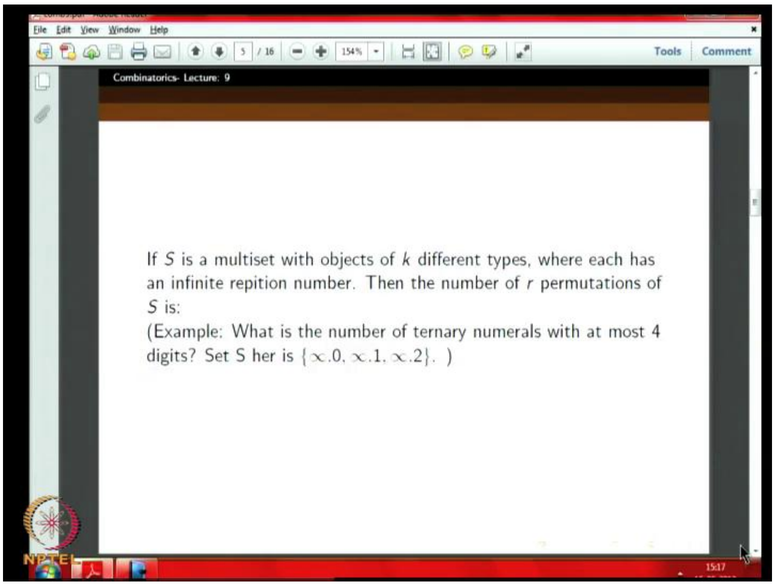

Then we can answer that question easily, so if I say some multi set with objects of k different types, where each has an infinite reputation number. That is a word we use, repetition number how many times the particular object repeats. So, see you say it is of a certain type then the type repeats how many how many times that is a repetition number of that type then the number of r permutation of S is how much.

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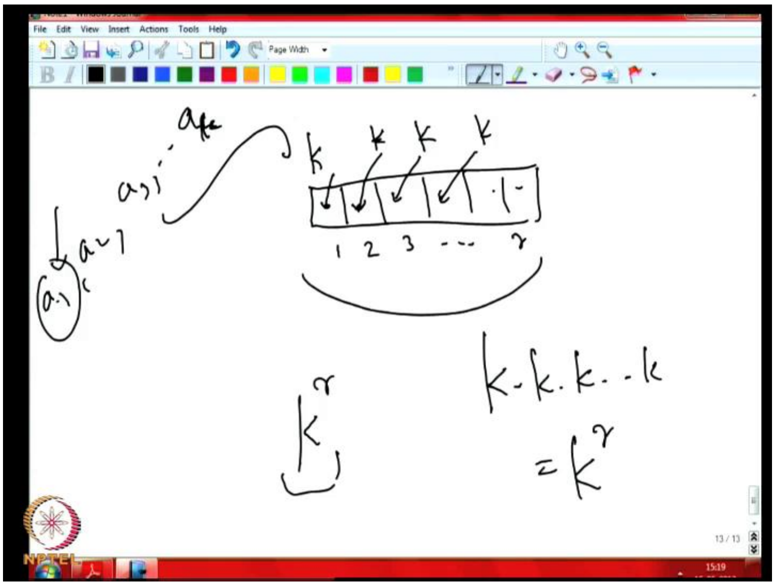

So, this is easily answered, because you know you just have to think of filling our positions our positions this 1st position, 2nd position, 3rd position, r positions. The 1st position can be filled in k different ways, because there are k types of things right you can place it any of the say a 1 or say a 2 or a 3 or a k any of them can be put here right. And a everything is available, because each one has infinite copies and here second one, 2nd position can also be filled in k different ways. Why because we whether fill it with a 1 or not still we have a 1 copies of a 1 left we can use it here.

And similarly, this also can be filled this position also can be filled in k in different ways this also can be filled in different ways there are r positions, absolutely k into k into k into say r times right that is k raise to r possible ways to fill it. So, therefore the number of when the multi set is such that there are k different types of things in it, say first type is say a 1 and then a 2 a 3 a k represent the k different types and each of this a I has infinite number of copies in it. Then we can make r combinations sorry r permutations like r length permutation from this multiset then the things are taken from the multi set we want an r length ordered list right. So, this this can be done in k to the power r it is quite easy right so right this can be done in k.

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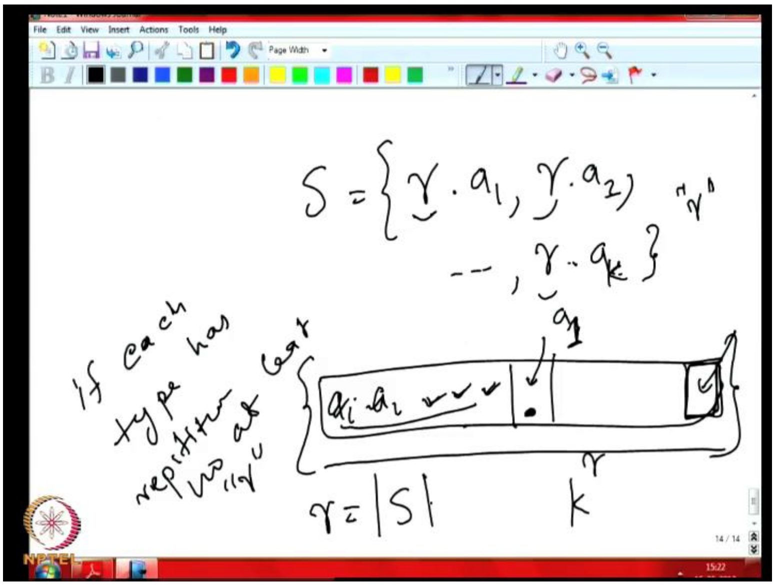

So, then yeah once you look at this proof you see that that exemption that the multi set S is of that this form; that means, in fine infinity times a 1 infinity times a 2 right a 2 and all the way to infinity times a k was not really necessary. Because what we bother is what we are bothered is when I tried to fill this thing you might have fill something something,

something here yeah sure. So, after filling it then I when I reach here it should, so happen, it should not happen that.

So, I want to fill some a 1 here that a 1 is finished of when I seen finite copies are there it will never finish of that was only intention of saying there are infinite copies right. But but this we can ensure even if we say that there are r copies right. So, if we just assume that if each type has repetition number repetition number at least r. That would be sufficient right because you know even if i use a i everywhere even when I reach at the last place I still have a a i left. So, there nothing will run out of stock right. So, therefore I can still say that the number of possible ways to maker permutations r length ordered list out of this multi set is k 2 the power r right because there are each question can be filled in k different possible ways. So, instead of this infinity we can actually write yeah r r a 1 right r a 2 yeah r this is this is this much is enough.

Anything more than r are not necessarily r you can write r plus 1 r anything more than r right then we can say. But if it is less than r then things are going to be more difficult right if it is less than r then things are going to be more difficult we will we will consider this problem in detail later but as of now we will give some, some special cases and probably we will try to solve it. But yeah, so first let us say this we are the first special cases this suppose we are only interested in making r permutations where r is equal to the size of S itself, S as being a multi set you have to, so count everything right with repetition.

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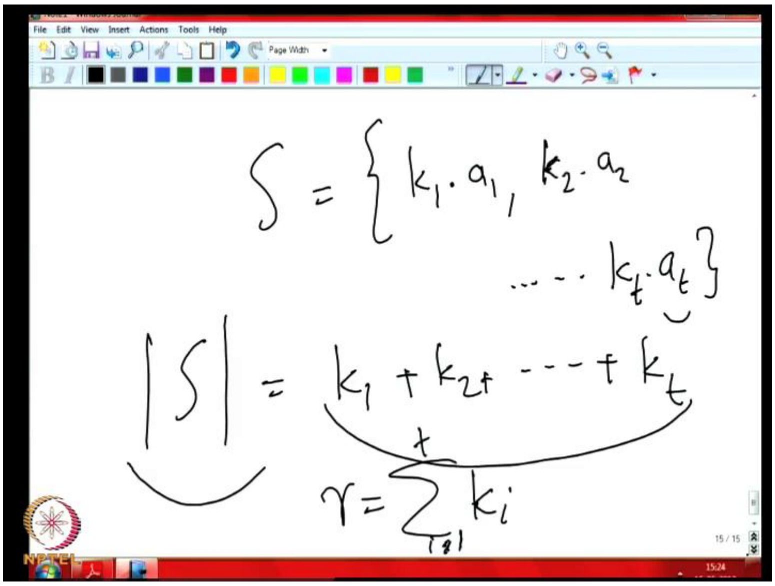

So, suppose s is equal to say this one k 1 times a 1 and then k 2 times a 2 and say k t times a t right this is this multi set not this right this is this multi set. That means, the total the size of s are adding together all the repetitions this is k 1 plus k 2 plus k t right. And we are interested in making permutations of S; that means, everything from S should be ordered right it should be written on the list right or we are looking for r permutations r equal to this sigma k i's this right i equal to 1 to t right.

Now, how many ways we can do, so the we are as I told we are only looking at the special case now namely our multi set is now fine at neither the repetition numbers are invite in fine no a i no type is infinitely available. And the number of types is just t here, in fact, it is finite so is a finite set, multi set and then we are only looking at ordered list of size equal to the size of a set.

That means, we want to permute everything not like a we want we do not want to select l few of them and then want to make all the order possibilities right. So, these are the two restrictions we have now our question is how many ways we can do this thing. So this can be answered like we will give two different ways to answer these things. So, one ways way is suppose your type is a 1 right.

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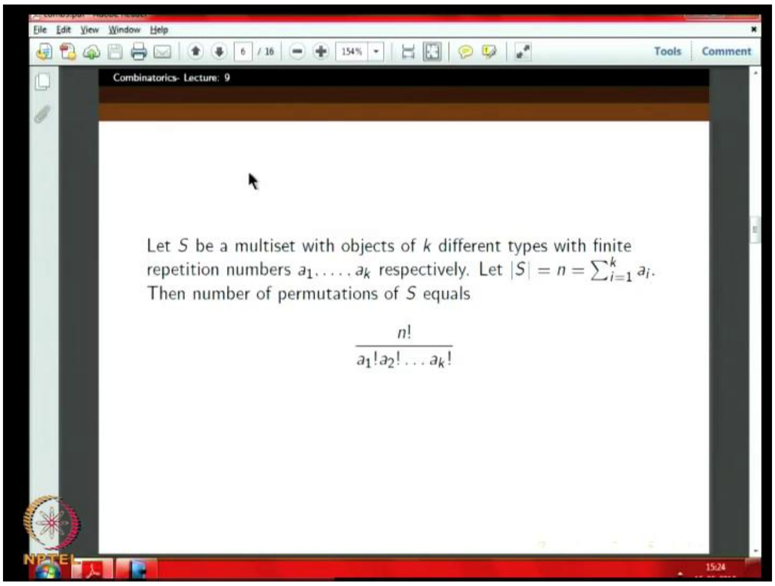

Sorry now yeah here I have a yeah used a 1 to represent the number of fine we will we will do one thing things.

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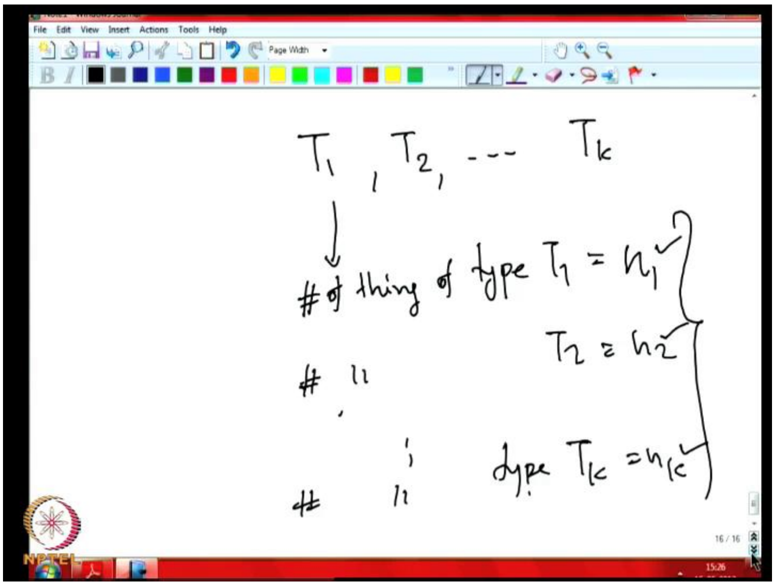

So, we will for the types we will write T 1, T 2, T k there are k different types and the number of things of type T 1 will be a 1 say like that, number of or we can just use n 1 probably that is that is need a. So, a number of things of type T 2 is equal to n 2 and so on right. Number of things of type T k is equal to n k there are k different types and each type as set in number n 1 times it comes n 2 times these are the repetition numbers n 1, n 2, n k.

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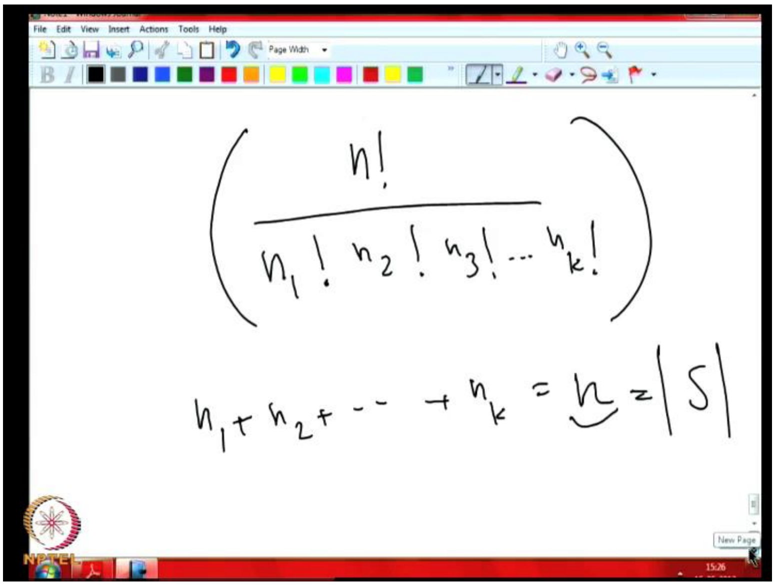

Now we want to show that the total number of permutations of the multi set S is n factorial divided by n 1 factorial into n 2 factorial into n 3 factorial into up to n k factorial, they call this n 1 plus n 2 plus n k is equal to n. Because if you add up the repetition number sorry repetition number of each of this things each of the each of the types then you will get the total number this is essentially the cardinality of the multi set right. So, you want to show that the total number of permutations you can make it these how will you do this thing, so one argument is like this one possible.

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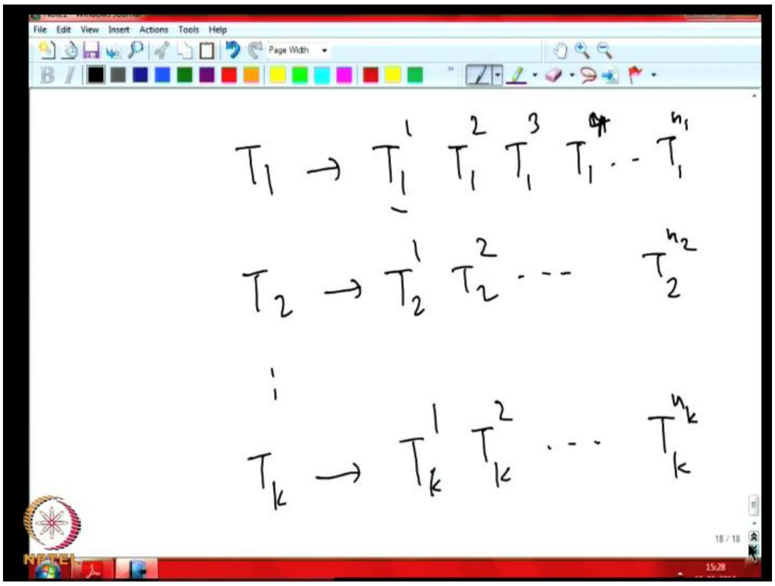

So, will take set t one is the first type right, so we will just write the objects of T 1 as say T 1 1, T 1 2, T 1 3, T 1 4 and up to T 1 n 1, it is like the type we took and then we numbered them separately. So, we made them different somehow we at least we can identify this of same type but see we have numbered them. So, we can identify which is which from that. Similarly, t type of T 2, we wrote like T 2 1, T 2 2 and then there are how many of them T 2 n 2. And similarly, the k th type T k we can write as T k 1, T k 2 and T k n k, this many ways we can do it right. And this now it look at least it looks all different it is just ask how many ways you can permute this set. Now, I mean the multi set as become a set of n things, because that now nothing repeats, because these things we forcefully made them look different right.

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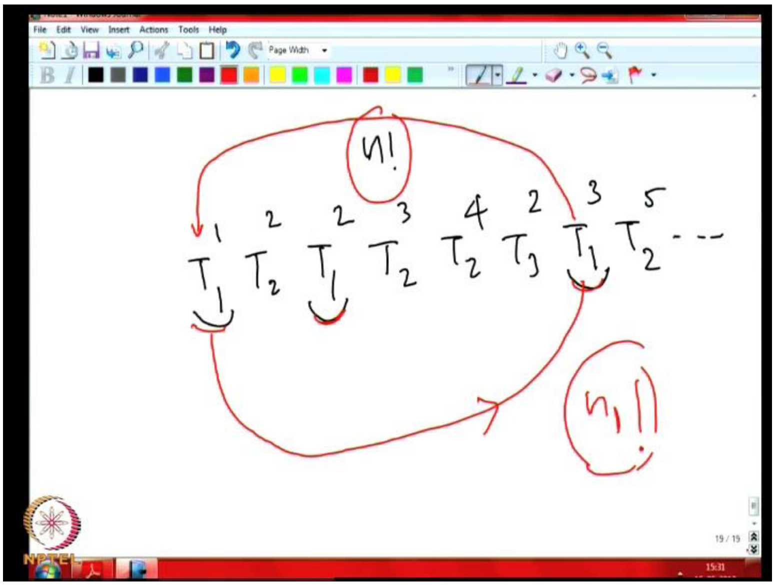

So, therefore, that is only n factorial right n factorial, so for instance that permutation may look like T 1 1 say T 2 2 and again T 1 2 and then T 2 3, T 2 1, t 3 a 2 and T 1 3 T 2 sorry T 2 this is four say like that T 2 5 and so on. It may look like this but then we know that these are all same types T 1 t t this are all same types like T 1 1, T 2 T 1 2 T 1 3 all are same types right. Now, suppose we had a taken this and put it here and this and this here sorry not this one so I suppose we had taken this and put it here, and this we brought here.

Suppose as far as permutation is concerned this a different permutation but we know that actually they are not different because this T 1 1 and T 1 3; they are one and the same except that we have numbered them. They there is no way of distinguishing them other than looking at our number we forcefully gave that number. Maybe somebody else cannot outsider cannot see that number right we have secret number we have say in a very small letter we have using we just marked it.

You see somebody else outsider will not even know that there are this things are numbered. So, we have we are seeing this is what. Suppose if I took it here and took it here yeah I will recognized this and this are different, but then outsider cannot recognize because there are all the same types. So, then essentially this T 1 T 1 type T 1 there are n 1 times we are seeing it in many places right in n 1 factorial ways fixing of all other things I could have a permuted them in n 1 factorial ways. Each of this n 1 factorial permutations of this T 1's right T 1 1, T 1 2, T 1 3, T 1 n 1 would have contributed at different count and an extra count to this n factorial this count right; the total because in the total number of permutation we have counted right.

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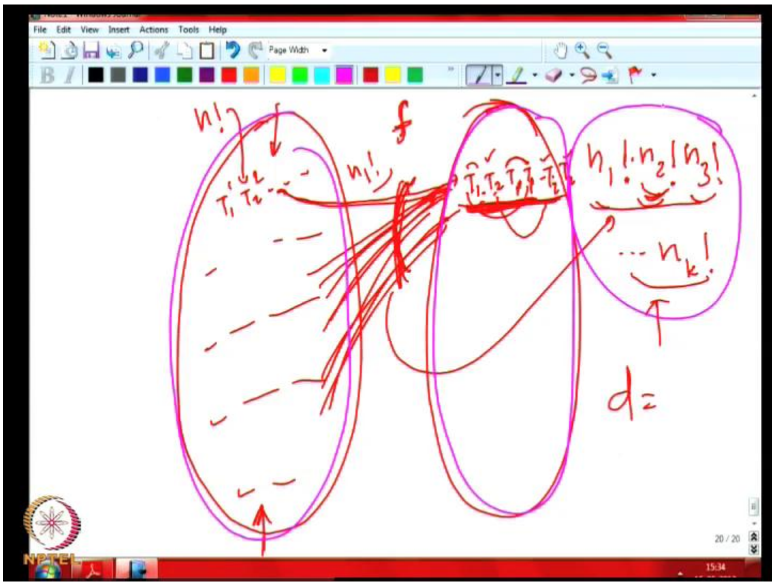

So, we now we can draw to apply the division principle we can do one thing. So, we can list all the permutation all the n factorial permutations we created like this T 1 1, T 2 2 like that right.

So, this permutations of multi set we can write here on this side and this side we can write the actual permutations where instance T 1 T 2 say T 1 T 1 like this and here it, it may have this number also T 1, T 2, t 3 like this but in here it is not there. So, there is no way of distinguishing this differences but you know any of this n factorial permutations of this T 1's I mean here fixing the other things that we know that that corresponding to that same stuff here we can put an arrow into that like this, which means a that all this things are mapped into this by a function right.

So, we are defining a function from here to here right. So, each of this things actually correspond to this; that means, so when you remove our number special numbers that we have given that will fall onto this. But not only this n 1 factorial permutations of that first type actually we could have for any fixing of this first type we could also have permuted. The second type in n 2 factorial ways right and for any fixing of the first type and any fixing of the second type, we could have permuted in all possible ways that third type right.

Then finally, the k th type we could have permuted, so this many things will actually correspond to the same stuff this is this number, same stuff here right. So, for in for instance what I am telling is here this three things I could have one here one here, we could have arrange them among themselves in any ways. Similarly, this T 2's right in the with without changing the positions related to others within themselves. So, this portion, this portion, this portion they could have shuffled themselves in any ways that is n n 2 factorial ways.

Similarly, for the third type we could have done the same thing right n 3 factorial. So, total n 1 factorial into n 2 factorial into n 3 factorial into n k factorial ways of shuffling was there for any given from ordered listing of those things from the multi set right. Because we just imagine that this as a ten type of things there n nn k things of k-th type and they all are different then they could have change them into this things all those things will be listed here in this n factorial.

But, now as nice what is nice, about this function this function means this is mapped to this thing what, what it corresponds to this right and this is a d 2 1 function where d is equal to this number right. As we can apply the division principle and say that total number of things here on this side right is the number of things available here divided by this number t right.

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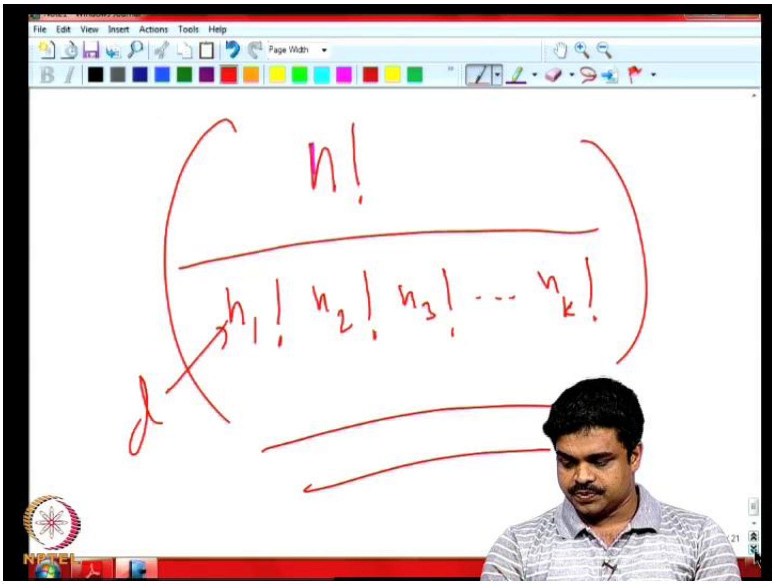

How much is that that is n factorial sorry that is n factorial divided by n 1 factorial into n 2 factorial into n 3 factorial into n k factorial right this we will get. So, we have applied the division principle here what we did is we considered the actual permutations. So, the multi set; that means, when the types are for a particular type there is relative arrangement among themselves does not matter, in fact, right. So, then we just imagine that each type can be within each type the members can be distinguished at least for us we can distinguish it.

And then we know every for instance type i there are n i of them then once you fix a permutation that n i factorial ways we can arrange themselves among their own positions without affecting the other types right. So, each gives rise to n i factorial of them but then this can be done for each of those types, so therefore, actual one permutation or the multi set will give rise to n 1 factorial and n 2 factorial into n 3 factorial n k factorial possible different ways of permuting them the permutations if we really assume that they are all really distinct right.

So, what the total count is n factorial because if we assume all of them are distinct there is only n factorial because there are only n factorial permutations. Now this corresponds to a d 2 1 function where d is equal to this n 1 factorial into n 2 factorial right into n 3 factorial and n k factorial and therefore, we can apply the division principle. And say the total number of a permutations of this multi set is n factorial divided by n 1 factorial into n 2 factorial into n 3 factorial into up to n k factorial right this is what we have done. Of case, this is one way of looking at it, so here we cheated by imagining that they are all distinct and then we use the reduction principle to come back to the correct number. Another way of doing it is rather straight more straight way of doing this thing is, so we know the multi set as n things we need to order them in n positions right.

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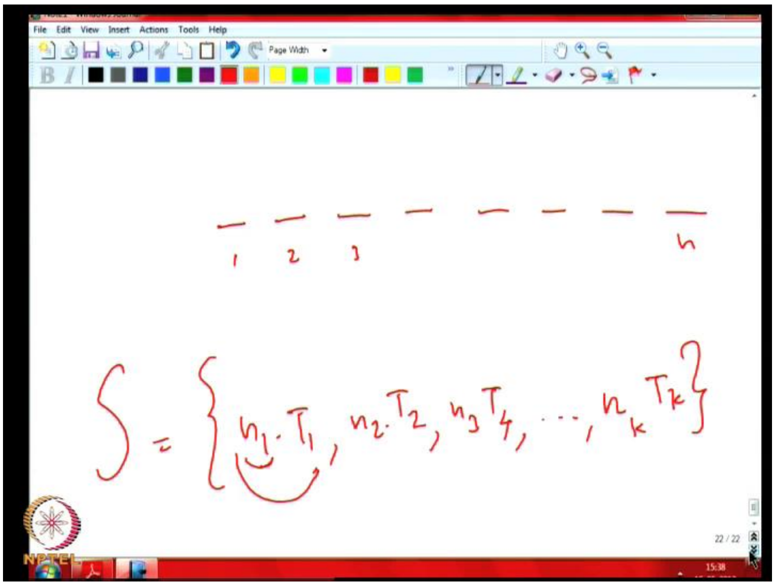

So, we just place mark the number this is first position, second position, third position n positions. Now, the multi set is S, this S has n one things of type T 1, n 2 things of type T 2 and n 3 things of type t 4 and and n k things of type T k right. Now the question is a right, so we can think that this should be placed here. So, this n 1 things of type T 1 has to be placed here but there n positions, we just have to fix the positions for them right. So, we select a out of n positions available.

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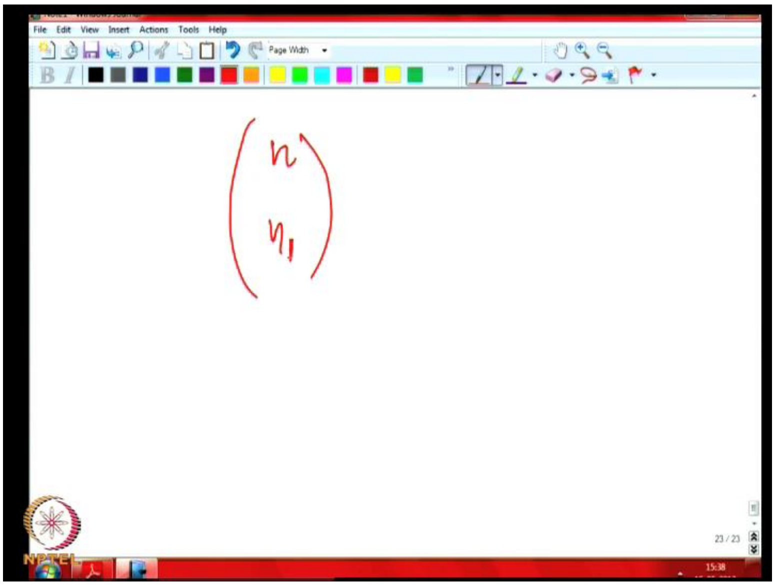

We select n 1 positions for these things that can be done in n choose n 1 ways n choose n one ways right is it not n choose n 1 ways.

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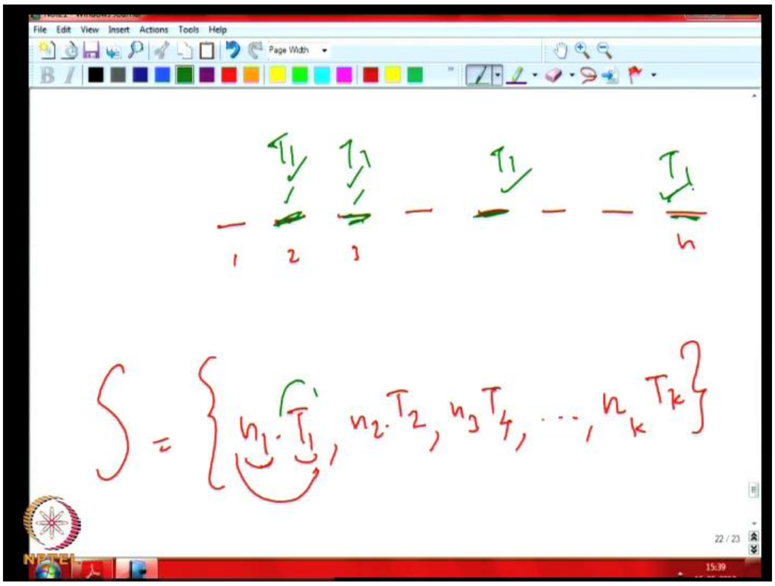

because there are n positions available, we just have to decide which positions we should put that things of type T 1 and there n 1 of them, we just select n 1 out of n positions right. Maybe it is selections can be something like I select this, I select this, I select this, something like this. And I place the T 1 things here, here, here, here there is only one way of placing for each selection or we just have to find out two positions.

If we are actually these things are different we will also have to the the worry which of which thing has gone here and which of type T 1 but among the type T 1 which has gone here. But we know now we are assuming that they are indistinguishable, so we just place them T 1 type things. So, 1 T 1 will go here 1 T 1 will go here one T 1 will go here 1 T 1 will go here like that now now that we have used n 1 positions from this n positions we have only see like we have marked this this this color. So, they are gone right there already T 1 type T 1 things are sitting there.

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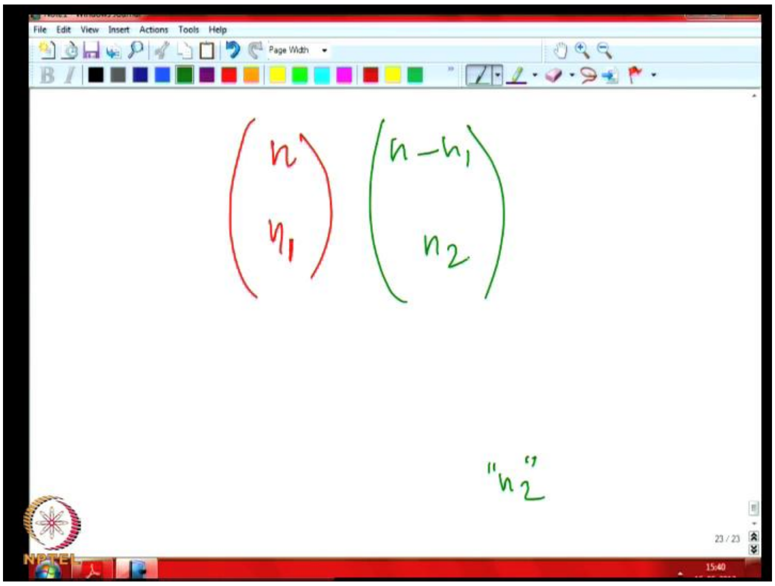

Now, the remaining positions we can we know right how many are there the remaining positions we have n minus n 1 positions are there. Out of this n minus n 1 positions, we have to find the positions to place that type two things T T 2 type things, T 2 type things there are n 2 of them right. So, we just have to select n 2 position select n 2 positions out of this thing right.

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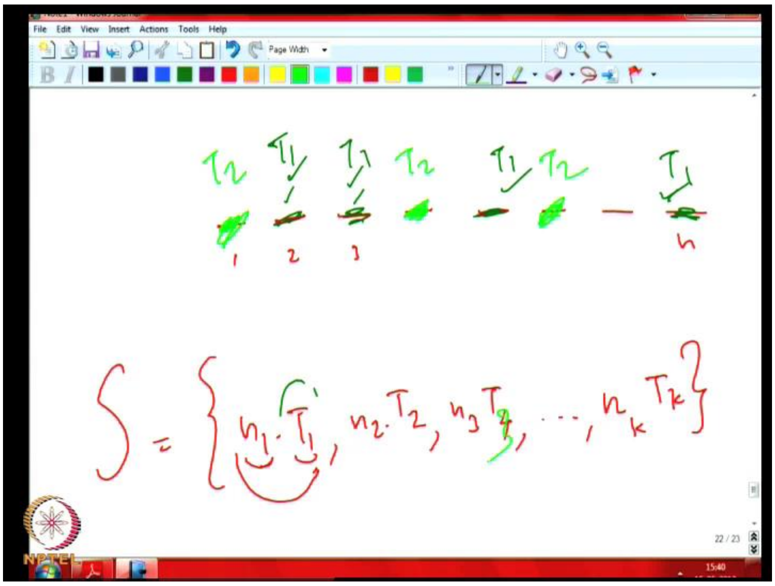

So, we just maybe this way, so we select say some n 2 positions out of this we place them that T 2s are placed there right. There is they are indistinguishable things we just need to select the positions to place them correct number n 2 of them that is it. So, we can place the T 2 things here. And now that we have placed n 2 there n 2 T 2 things there, t type T 2 things there and they have to worry about type it is t 3, n 3 are there like but then are how many positions are left.

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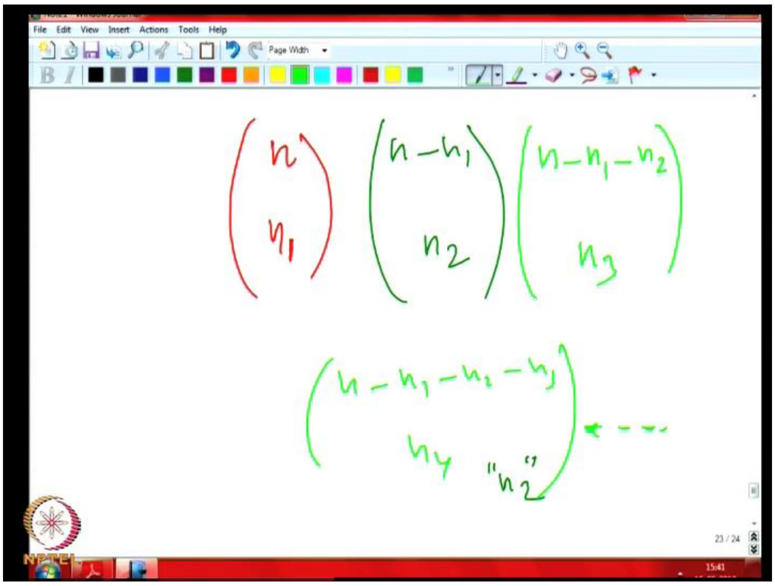

Initially n positions were there, now n 1 was already used up by type T 1 things and then n 2 positions of the remaining are already gone for type T 2 things. Now, this many positions are available and from this thing we have select n 3 things for type 3 things right, so type 3 things and then type 4 things. So, we have to select see from the remaining n minus n 1 minus n 2 minus n 3 positions we have to select n 4 and so on right like this like this it go. So, we multiply the possibilities then we will get.

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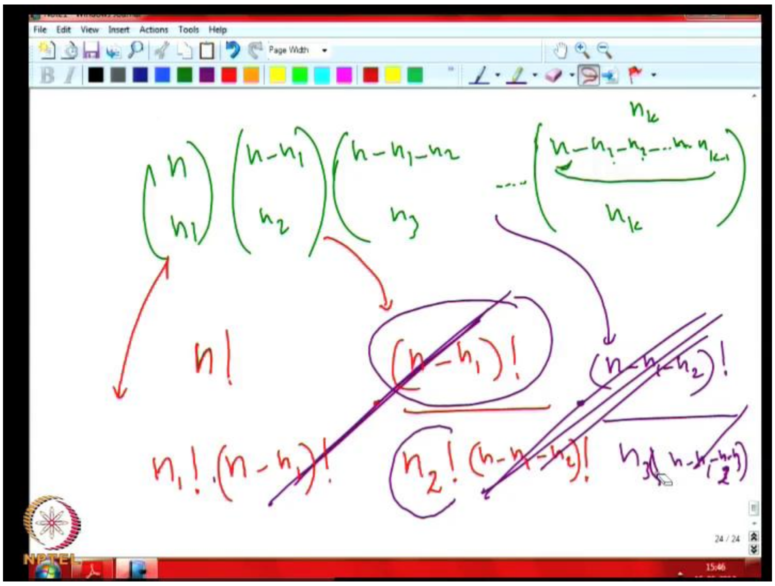

How will the formula look like n choose n 1 into n minus n 1 choose n 2 into n minus n 1 minus n 2 choose n 3. And finally, we the last term will be n minus n 1 minus n 2 minus n, so like that, so we will we will keep on n minus n k n k minus 1, so n k. So, this ways because the last once you fixed all this things last a because this is actually what this will be nth k. Because we this n s the sum of n 1 plus n 2 plus n k minus 1 this will be like n k choose n k that will be just one one way of, because just n k positions will be left will put the last type here that is what.

Now but what is it complicated formula this is indeed a complicated formula. So, we have to simplify it. So, let us try we know this n choose n 1 n 1 that is n. So, maybe we can use a different color n factorial divided by n 1 factorial sorry divided by n 1 factorial into n minus n 1 factorial is this into this one what is this, this is this. This is n minus n 1 factorial divided by because up it is n minus n 1, now n 2 factorial in to n minus n 1 minus n 2 factorial. Good thing is that see this the second term here, is the same as this thing here right, so they will cancel like this.

Similarly, the same thing will happen in the next term right because here is n minus n 1 minus n 2 factorial divided by n 3 factorial into n minus n 1 minus n 2 minus n 3 factorial. So, I am just not writing, so this will cancel off. So, like that it will cancel like this, like this, so it will and the next will go. So, what will be left this n 1 factorial will be left below see, this factorial will not get cancelled and this will get, this this will not get cancelled. So, I will mark it with blue. So, this yeah so yeah, so this is yeah. So, this when when we put this thing this will, this will go away the upstairs the the numerator we have n factorial in the denominator we have in the denominator we have in the denominator we have a this n 1 factorial n 2 factorial and then n t n 3 factorial that will be left and a let us see.

So, so we will get this formula. So, it is here instead of a 1 factorial a 2 factorial we can write n 1 factorial n 2 factorial n k factorial, so before because any way the time is almost over. So, I will just mention one more thing here see, so it is summarize you are considering multi sets and then we are considering the permutations of that, so we started with this questions suppose as a say the multi set. So, there are certain k different types there but each type can be finite or infinite.

Now how many are permutations ordered list of r can be made from this multi set S this was the question right, so the one example for the this a question for instance see suppose we want to create turnery num numerals with at most four digits right. So, four digit means it a at most four digits; that means, we can also initially if it is a 3 digit number we can say that a it starts with 0 0 1 2 0 let say turnery number with that three digits. But we can just say that i always use four digits just that when it is three digit number I will use put a zero in the beginning, that means the first position can be filled with a either 0 or 1 up to 0 1 or 2.

The second position also can be filled with 0 1 or 2, third position also can be filled with 0 1 or 2 right. So, this is one example of this this kind of a setting. So, there are here we have a infinite numbers of zeros available right, because you know zeros does not run out of stock. Similarly, infinite numbers once available, infinite numbers of twos available can fill them because just because you filled them in one place it is not true thatin another place we cannot fill it right, so this is one example of that.

So, four positions we can use four into sorry 3 into 3 into 3 into 3; that means, 3 raise to 4 see this is an example where infinite repetition number is infinite right. After that we consider the situation where all the number of types is finite and repetition numbers are also finite, then we consider this special case, where r is actually the size of the multi set itself. That means we want r permutations where r is equal to the some of the repetition numbers right of all the types. Say you considered repetition numbers to be say n 1 n 2 n 3 n k because there are k types each ith type is n irepetition type is n i.

Now sigma n i is equal to r now we want to permute all the things in the multi set that is we were saying. And then we saw two proofs to show that this repetition number is actually n factorial divided by n 1 factorial into n 2 factorial in to yeah. So, n k factorial then the numerator we have n factorial denominator we have the product of all those n i factorials right, so in next class we continue with some examples.