Combinatorics
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Lecture - 8
Bijective Proofs – Part (3)
Properties of Binomial Coefficients
Combinatorial Identities – Part (1)

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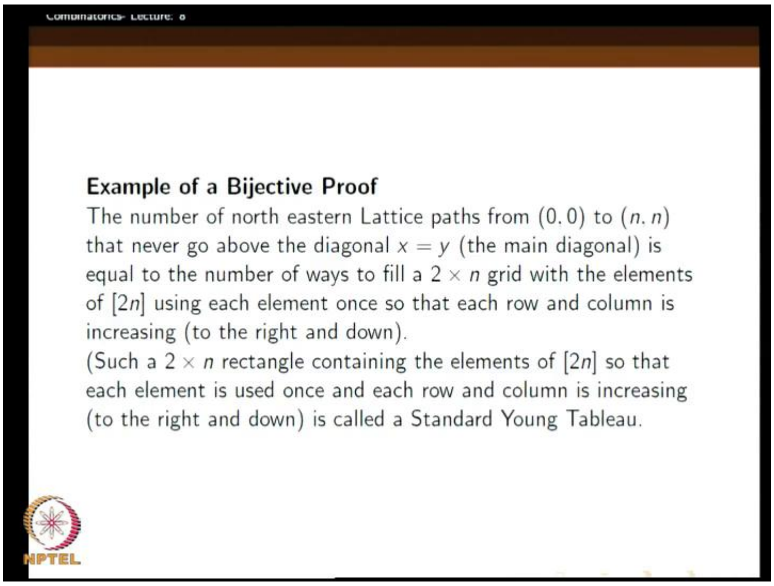

Welcome to the 8th lecture of combinatorics. In the last class, we stopped with an example of a bijective proof, we were considering the number of north eastern lattice paths from 0, 0 to n n that never go above the diagonal x equal to y the main diagonal. And the other thing we were considering was the number of ways to fill a 2 by n grid with the elements of 1, 2, up to 2 n; 1, 2, 3 up to 2 n using each element once exactly once. So, that each row and column is increasing; row is also increasing, column is also increasing, so this is this is such kind of rectangles are 2 by n rectangles are is called by the stand, this yeah this kind of rectangles are called the standard young tableau.

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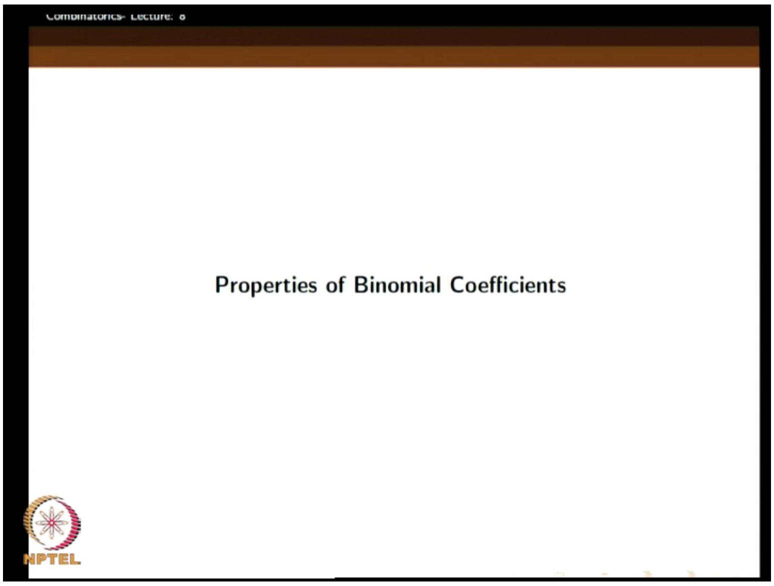

We are seeing that this 2 by n young tableaus. So, the number of such tableaus equal to the number of north eastern paths right in the diagonal.

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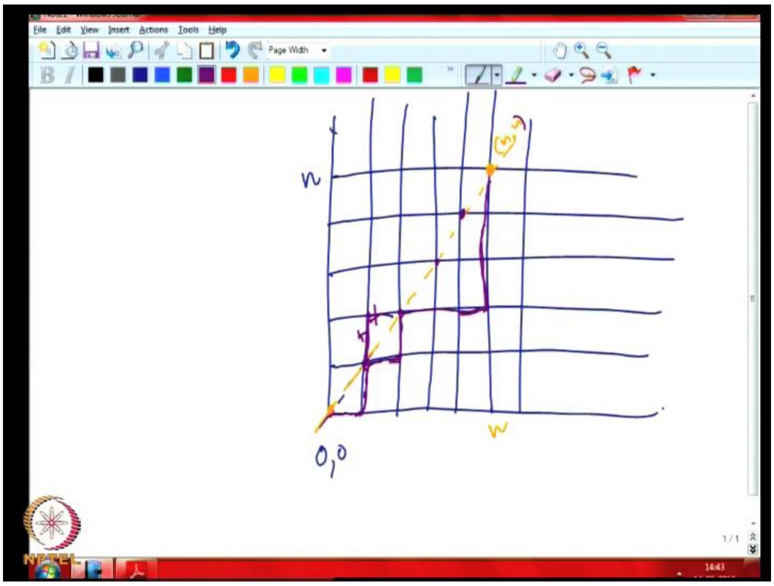

So, let us do the proof once again. So, we just want to set up a bijection between these two things, one side we have the number of north eastern lattice paths. So, something like this; this is 0, 0 something like this right. So, let let us say, this is n and here, so this is n, so this is the main diagonal, so you can use a different pen right, so this is the main diagonal. So, let us say this is m by n this is the m by n point this 1 right. So, now what we are interested in is, so you remember what is north eastern path?

So, we are driving from here to here these are the roads, we can either go straight maybe we can go like this or we can go like this, so we can go like this. And so either we can go upward I mean north direction or in the east direction; north means straight right towards right. So, this is how many such paths are there to reach here is the question right, but the now the extra condition is that we should never cross above the main diagonal I mean you can touch this points, but they should never go beyond this may be we should no go here like this right we are not allowed to take this kind of things.

So, now we we are actually not counting these paths here, we will do it later the using catalan numbers but that will come later right as of now our intention is just to set up a bijection show a bijection between these kind of objects. The objects are see one of the objects is a path starting say something say, this will be one such object right which we are interested in several such objects you can see, set up a bijection between these objects.

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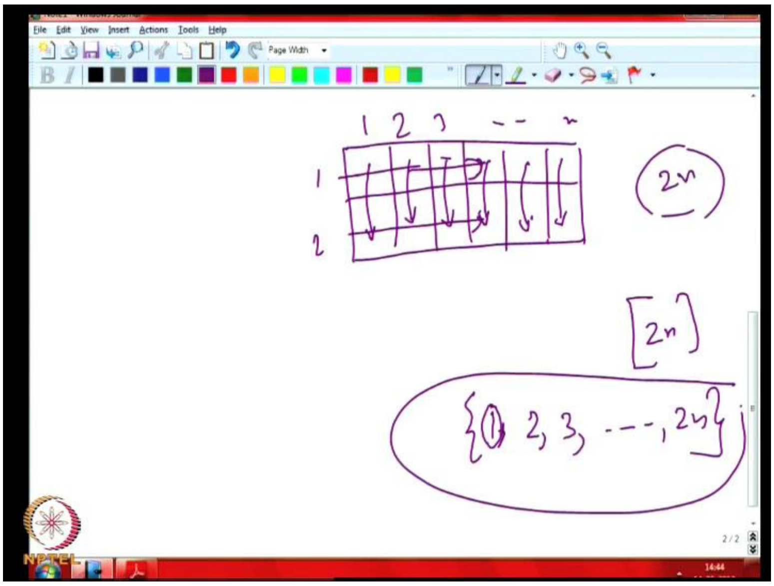

And the standard young tableau of the 2 by n type so this 2 by n means is 2 columns are there sorry 2 rows are there and then n columns are there 1, 2, 3 up to n and this is 1, 2 right. So, there are actually how many squares here 2n squares are there right. So, we have to fill these squares from this set 2 n right; that means, 2 n means 1, 2, 3, 4 up to 2 n right.

Now of case, which means that every number here has to be taken exactly once and put some where right, because every every one of these 2 n things should go there we should not repeat right. And but the only constraint this that when you read after filling in when you read from this direction; that means, in a row when you go from left to right it should increase.

Similarly, in second row also similarly, in every column when if you come from top to bottom it should increase, there only two elements here this should be this should be smaller than this that is what it says right. So, we have claiming that this number is exactly the number of ways you can fill it using this 2 n in such a way that our constrain is satisfied is exactly equal to the number of ways we can drive from 0 0to n n without ever crossing the main diagonal always keeping below the main diagonal right. So, how do we show this thing, so the it is we just need to show a bijection. So, we will show first, we will show that. Suppose you have a route here driving route here, now how to construct a young tableau for that they will we will say that there is for that for each of them I can construct a unique a tableau, so that it is a function right.

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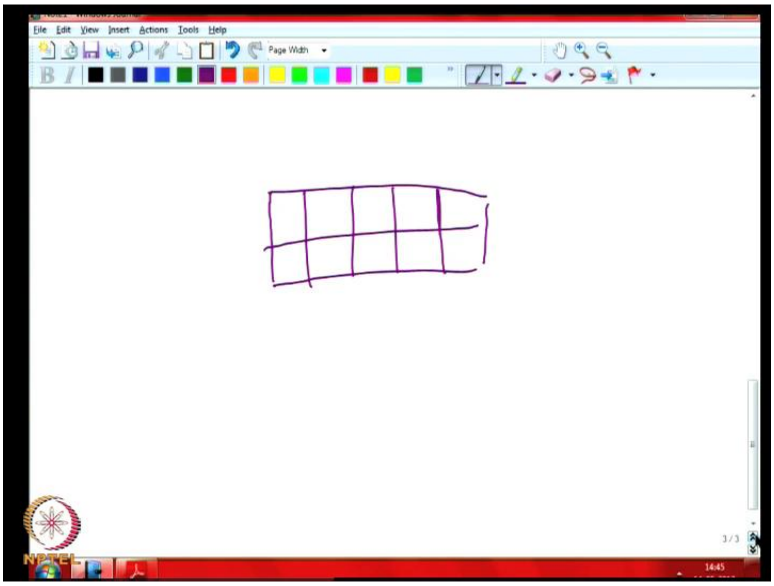

So, this is for instance what I do is I will show filling in this right.

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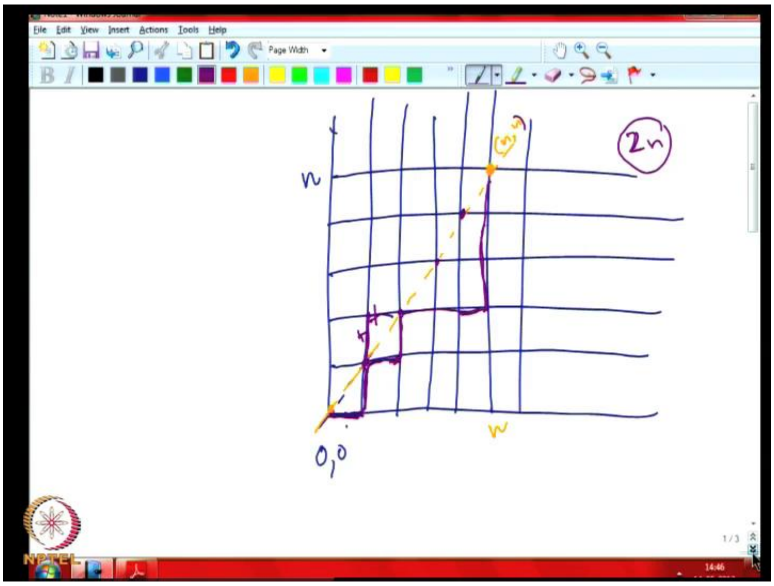

So, it is like see for instance here first step is east step right therefore, whenever I take eastern step; that means, rightward that step number this is the first step, this is the step this is the second step, this is the third step, the fourth step, how many steps will be there total 2 n steps will be there right. The step numbers if it was an eastward step then we will fill it in the upper row otherwise we will fill it in the lower and we will fill it in the order and which it come.

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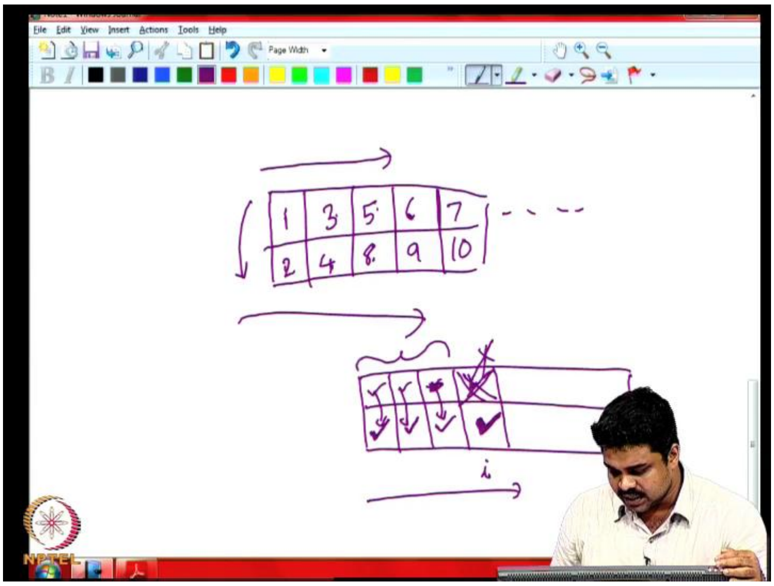

For instance here the first step right that will go you write here 1 the first step is in the east may be in east and then here this is upward I mean a northward step that is second step, second step then because it is a northward step it will come here. And then here the third step is eastward, so that will come again here and what about the fourth step, fourth step is again northward that will come here because it is northward. And what about the fifth step fifth step fifth fifth step sixth step, seventh step. Fifth, sixth, seventh all of them are eastward, so we will write it fifth, sixth, seventh.

And then now eighth step eighth ninth and tenth are northward, so we will write like this like that we can keep on filling. So, see this in this case we reached n was is equal to 10 if it n was equal to 5 therefore, we reached in ten steps to the point n comma n. So, otherwise we keep filling like this, so we claim that looking at that route we we have filled the tableau so it is indeed satisfying the constrains of case we have been filling this step numbers.

So, we have taken numbers from 1 to 2 n, so a because 2 n squares are there all the squares are filled because we considered 2 n numbers. And then of case we were considering this step number in the increasing order therefore, and we are filling here from left to right. Therefore, the rows definitely increase from left to right and the only thing we have to check is that, the columns also increase from from down see downward sorry down to bottom in this direction right, here 1, 2, 3, 4, 5, 8, 6, 9 see it is increasing.

So, why why am I sure that it will always increase, suppose it does not increase right at some point of time for instance it, so happened that the first time. So, for instance I can locate the first column where it is not increasing may be this is the first column, here it is increasing here it is increasing, here it is increasing, here it is not increasing that means here we have a number, which is bigger than this number. So, we can stop at the point we filled it. That means, we filled it, filled this, filled this, filled this may be I do not know whether this is filled or not but we have filled this, this and this right. Upper row we do not know, because we have in the lower row we have reached up to here right, upper row we are sure that this is not filled right, whether this is filled or not we are not sure.

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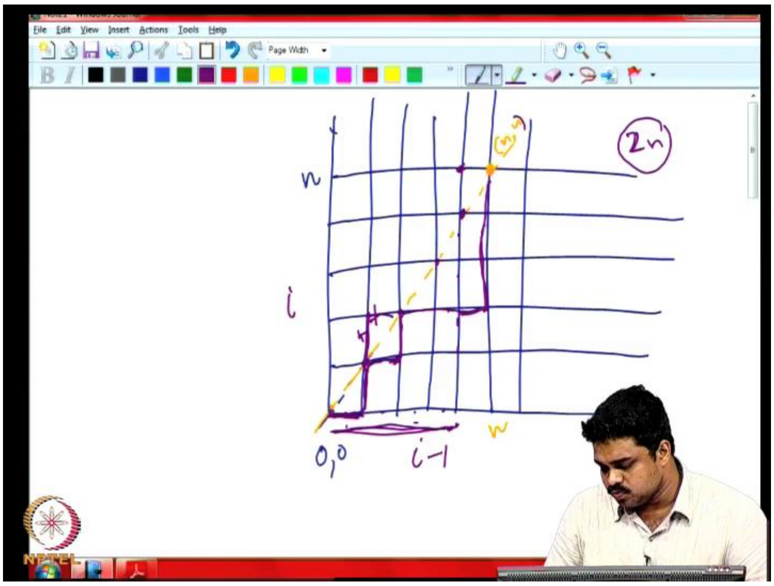

But, now because I have filled the lower row at least this many times this is the ith row; that means, i times I have filled something means, i steps were taken in the northward direction that is what. But on the other hand this square is not filled at all right; that means, the maximum number of steps taken by that time, towards the east is strictly less than i, i minus 1 maximum way it can be even smaller, but i minus 1.

But, then you can see that if you are taken gone only i minus 1 steps, say 1, 2, 3, 4 steps eastward but i i steps; that means, 4 plus 1 5 say one more step northward; that means, you will be above the main diagonal right. 1, 2, 3, 4, 5 you will be above, because if you have if you are here maximum if you want to keep below the main diagonal, it means that you can take maximum by the time four steps eastward, see if you have gone i steps here; that means, you can only take i steps northward also.

But, you have seen that we have already taken i steps downward so only i minus 1 steps are taken eastward. So, you it means that you are above the main diagonal that is why it will never happen, because you know that our path was always below the main diagonal right. The path maintains the property that at any point the number of any steps taken to the east is at least as the number of steps taken to the north right.

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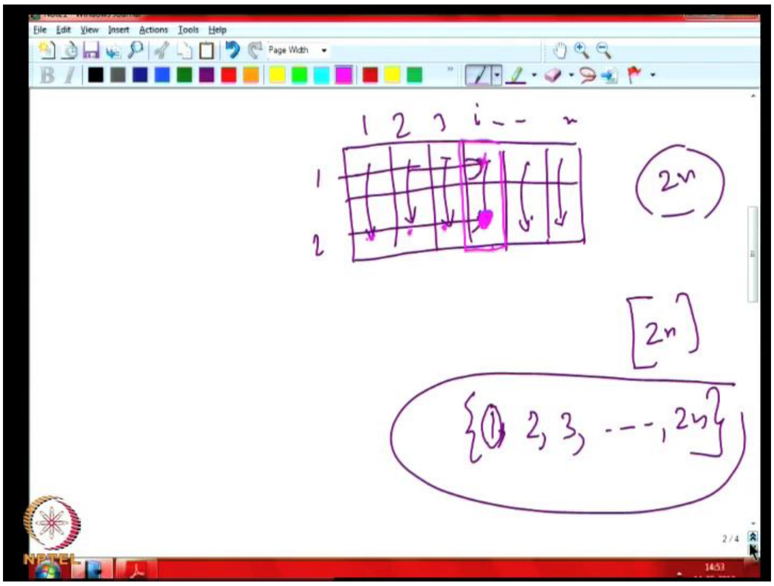

So, therefore, it is not possible that the ith, say ith column right gets filled in the below part the in this this gets filled before this that can never happen. Because if this gets filled before this then; that means, the number of steps taken in the north direction will be more than the number of steps taken in the east direction, that is what it will happen right. So, we have indeed got young tableau for each of valid young tableau for from each path and it is very clear that each path will give rise to a different table. Because you know the for instance if two tables are same; that means, the specific steps taken to the east are same in both the tables right; that means, the paths are also same right. So, the but the paths were different therefore, the tables will also be different for different paths.

So, what we have shown now is that for each path we have a table and for two different paths we do not have the same table right that is whatever that is what we have shown. So, therefore, right each path we have a unique table and then so if we if we consider two different tables that means one will wonder whether is it possible that there are some tables for which there was no paths that also we have to verify.

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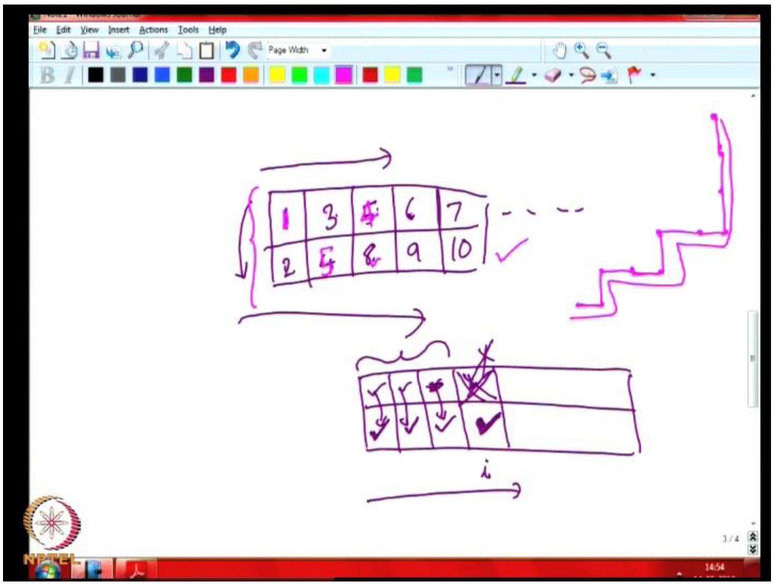

But that will definitely will not happen because once you get a table you can look at the entries in the table so you can imitate the path by this is i I look at this step number this is 1. So, wherever is a 1, so I will put that step in the east direction, second step where is the second step. So, if it is here then see for instance here 1, so I will write it 2 I see, so it should be like this 3, so that it means in the east direction. Suppose 3 I see here I would have gone upward out of case that you will not see it that is suppose 3 it is in the eastward.

So, then if I look for 4, 4 is here if four if 4 was here and 5 was here then definitely I would have taken 3 4 because in the east direction and 5 is coming in next like this right and then 6 I am seeing here. So, I will here, 7 I am seeing here then 8 in the north right 9 in the north direction 10 in the north direction like that I can construct a path of case, because of the property that the whenever I am at a point. So, for instance I have already taken more eastern steps than more or equal eastern step than the northern steps, I will be always below that main diagonal; that is why there is a one to one correspondence between these two right.

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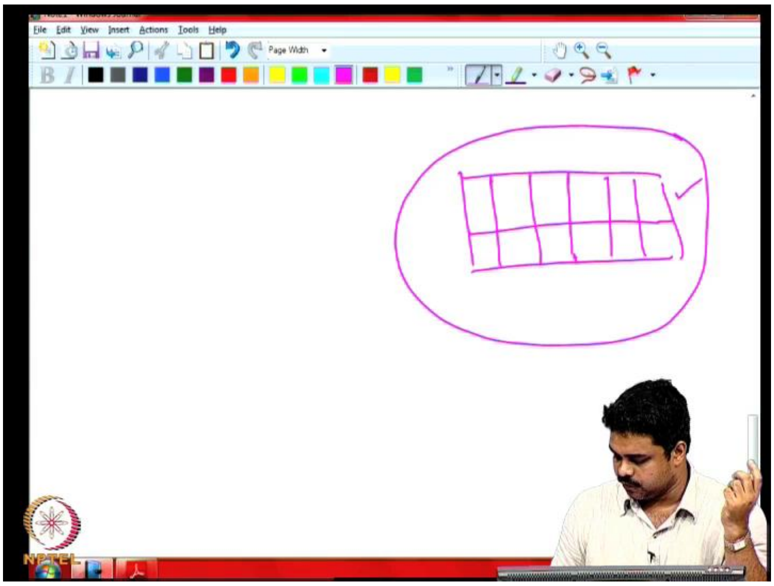

So, of case, so our point was just to illustrate the bijective proofs means here for instance the number of paths starting from 0 0 and reaching n n, which always keeps below the main diagonal. It at first look it looks very different from the other object we can consider namely matrix 2 by n matrix at a block, where we have filling in numbers in the squares.

And we have put just two rules means along rows the numbers should increase and along the columns downward the numbers should increase right and this is very this is a very natural restriction. That means, usually want the rows to increase columns to increase that not a very unusual thing it is not a cooked up rule right. So, but how many such how many ways of filling are there filling the table are here. So, at one look it may look it may seem these two problems are very different, but then by setting up the bijection we have shown that both the problems are 1 on the same the count will be same right.

So, this is this kind of arguments are very common in combinatorics because in many cases one type of thing may be easier to count than the other type. That is why we set up the bijection and then we connect the other maybe it is already known how to count one type of object and then by setting it of the bijection from the other object to this 1. So, we already count we can count the other type of objects also right. Otherwise, thinking directly about it may be a little more complicated of case one has to keep this kind of ideas in mind.

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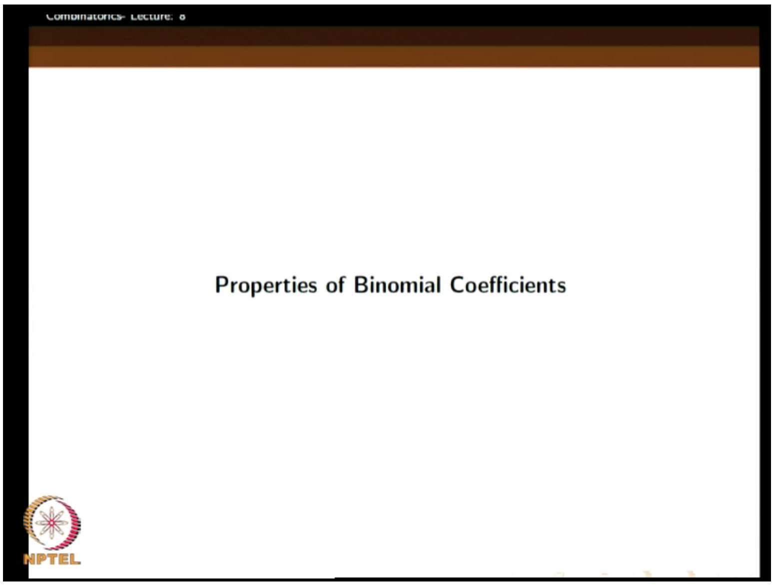

Now, now we will consider, so we have learnt of already we have learned that binomial coefficients those are called n choose r. This choose means we are choosing see if from that subset interpretation of binomial coefficients from below word is coming there are n element set from that we are choosing an r element subset. So, that is why n choose r how many ways we can choose r element subsets from n elements set.

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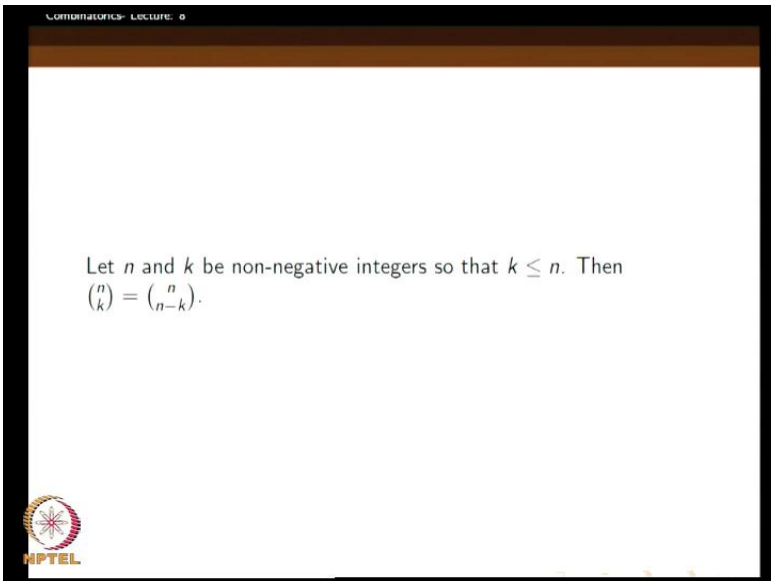

Now, we will study certain properties of this binomial coefficients the most important ones the most famous ones we will consider they are really elementary let n and k be non negative integers, so that n less than equal to k. So, there we are only considering positive integers n then k are positive integers here and k is also sorry 0 it k can also be this 0 can also be 0 of case. So, we are assuming k is less than equal to n then we want to show that n choose k equal to n choose n minus k.

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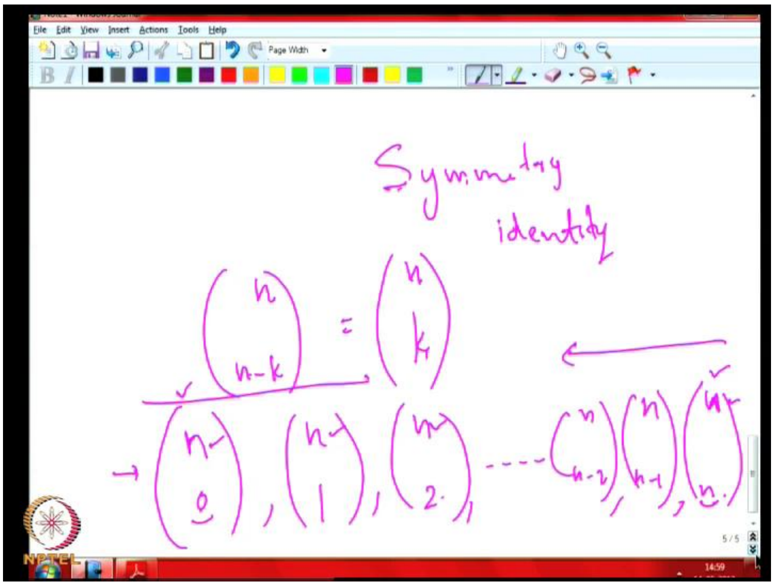

This identity this is called symmetry identity symmetry, why is it called a symmetry identity because n choose n minus k is equal to n choose k some some symmetries is clear about it. For instance if we consider the combinatorial coefficients n choose 0 on choose 1 and n choose 2 and like that up to n choose n right. So, we look at, so this n is same here n is same here now 0 and n, so we read from this side or this side this this this kind of symmetrically positioned from both ends right. If we look from the left and right this n choose 0 corresponds to n to sense.

So, me how right n choose 1 will correspond to n choose n minus 1 right and similarly, n choose 2 will correspond to n choose n minus 2 position wise at least right reading from here or reading from here right. So, that we n choose k correspond to this n choose n minus k because this is the k plus 1 from here this will be the k plus 1 n minus n n choose n minus k will be the k plus 1 th from here; so they are equal is what it says.

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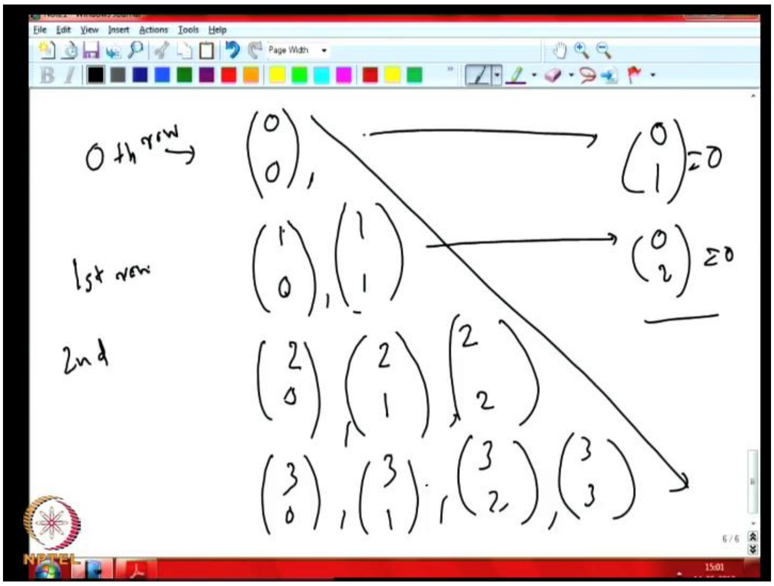

So, may be it to get this picture a little clearer we can introduce one pictorial way of writing or may be tabular way of writing these combinatorial coefficients, which is known as Pascal's triangle. So, it is like this, so we first write n choose 0 in the first run then we write n choose 0 and n sorry. So, not n choose 0, so we can start with. So, 0 choose 0 right the 0 th row and we can write 1 choose 0 and 1 choose 1. So, we can say that this is the 0th row and this is the first row right. So, and the second row will look like 2 choose 0 and 2 choose 1, 2 choose 2 like this of case we have to put comma, if you want to separate them. This will be 3 choose 0, 3 choose 1, 3 choose 2 and 3 choose 3, this is a table, so this we will keep on the the length of the row it will be increasing right.

So, not that 0 choose 0 is 1 by definition is and then we do not have a 0 choose 1 because 0 choose 1 will be equal to 0 we do not write it 0 choose 2 will be equal to 0 and so on these are all 0es. In fact, here 1 choose 0 and this is 1 choose 1 and 1 choose 2 will be 0. Or so 1 choose 2, 1 choose 3 all of them will be this this will be 0 therefore, there would not be anything beyond this this; on the other hand this is 1 choose 0 is 1 and 1 choose 1 is also 1.

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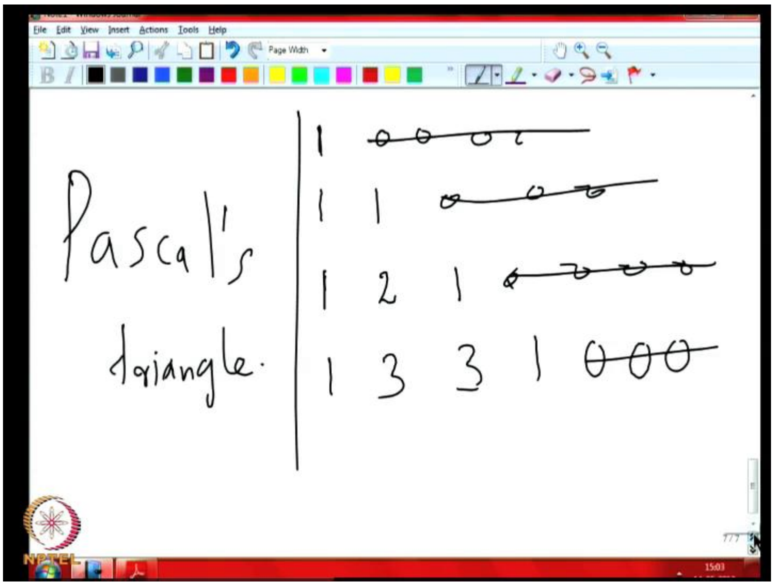

So, we can maybe we can, so write the numbers here 0 choose 0 is 1, so the rest will be all zeroes of case right therefore, we do not write it and then 1 choose 0 is 1 1 choose 1 is 1 and then going back here 2 choose 0 2 choose 1 2 choose 2. This is 2 choose 0 is 1 2 choose 2 is 2 and 2 choose 2 choose 1 is 2 and 2 choose 2 is 1 and the rest are all 0's here also there are 0's here also there are 0's, because 2 choose 3 onwards it will be zeroes we do not write it right.

So, similarly, 3 choose 1 3 choose 0 is 1 3 choose 1 is 3, 3 choose 2 is 3 and 3 choose 3 is 1 3 choose 4 is 0 and 3 choose 5 is 0 and like that. So, we would not write it, so this is the Pascal's triangle this is called Pascal's triangle because Pascal had written a treatise on this binomial coefficients may be the first one in Europe.

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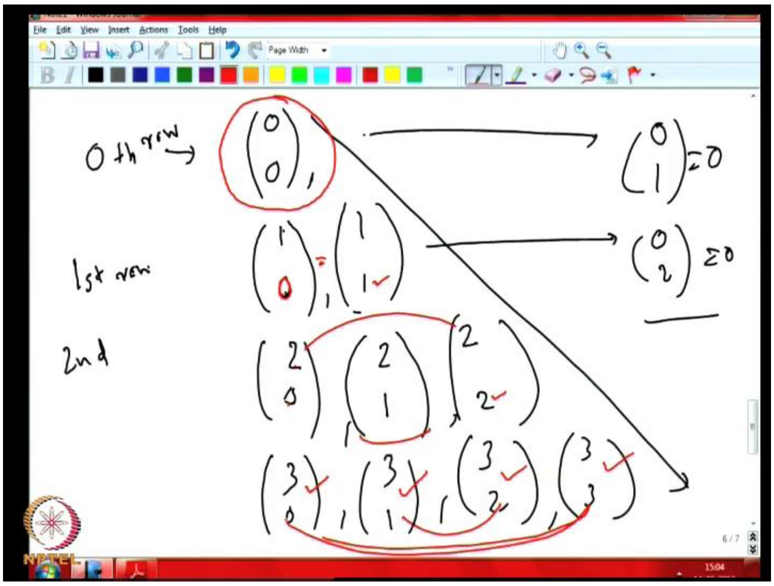

So, about binomial coefficients, so it is a it is known Pascal’s triangle and of case binomial coefficients were known in the east before Pascal so of case we can read it in the history of mathematics all these things. So, then these are this is what we were telling is we were talking about the symmetry identity. So, here see there is nothing see, there is no nothing to talk about symmetry here 0 choose 0 right. So, but here we have 1 0 choose 1. So, if you take k equal to 0 here and 1 minus 0 will be 1 right this and this are equal is what it says.

And then here 2 choose 0 put k equal to 0, then that ill correspond to 2 choose 2 minus 0 that is 2, then this and this are same in between, so 2 choose 1 is equal to 2 choose 1, 2 choose 1 is equal to 2 minus 1 is 1 here 3 choose 0 will be equal to 3 choose 3 and 3 choose 1 is equal to 3 choose 2 like that right this and this is equal this and this is equal like this you pair right.

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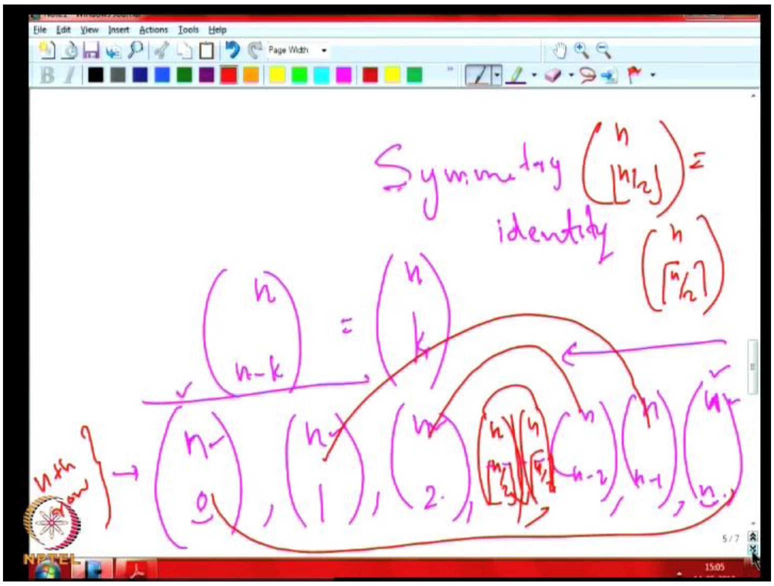

So, now for instance this is the n plus 1 this is the n plus 1th row right, because we started with the 0th row sorry sorry this is the nth row n plus 1-th row in the sense that we will say we named it as 0th row the first row. So, this the nth row nth row of the Pascal's triangle. So, here this will b equal to this this will be equal to this and this will be equal to this and so on like that we can pair.

So, if n is an odd number the how many entries will be here because it is start with n choose 0 and end with n choose n there are total n plus 1 entries the even number of entries that will pair of properly. If n was a the middle the middle 1 will be n choose n by 2 floor and n choose n by 2 seal so both they are equal right, they will get paired of n choose n by 2 floor will be equal to n choose n by two see not I am assuming n is an odd number before this floor and seal both are integers 1 apart right. So, they are going to pair of.

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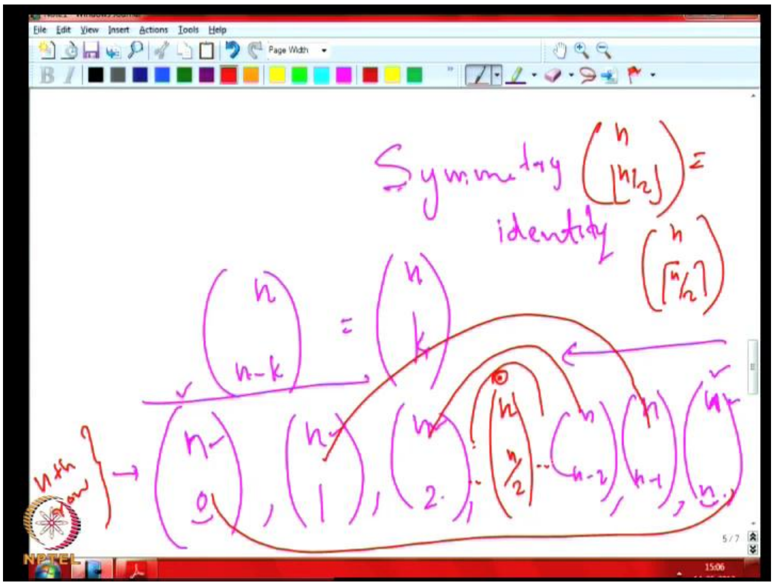

While if it was an even number then what will happen is there are exactly an odd number of things it will pair. So, in the middle n choose n by 2 that will pair itself, so because n minus n by 2 is actually, n by two right n choose n by two will pair it with itself right. While here n choose n by 2 minus 1 will pair to n choose n by 2 plus 1 right, so like this.

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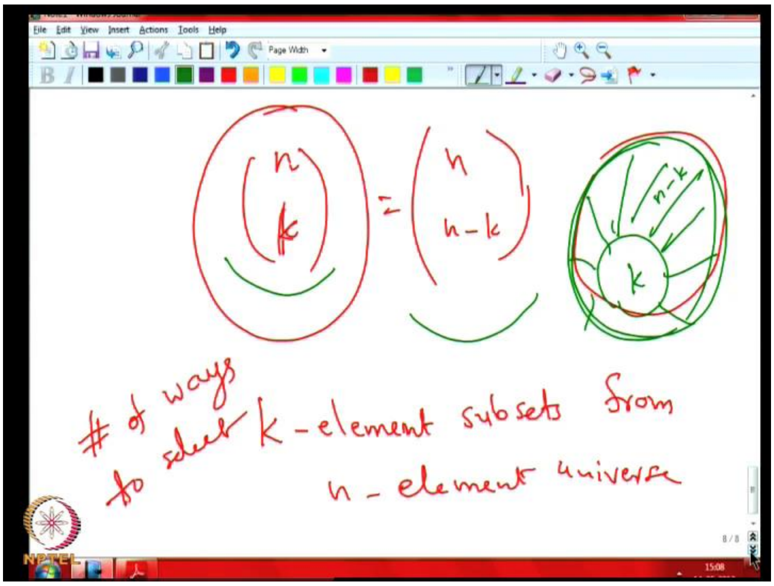

So, so we need a proof of this and this very elementary, so this is what we want to show n choose k equal to n choose n minus k right. So, we can say that the number of ways because this is the number of ways of selecting a k subset, k element subsets from n element universe right number of ways to select k elements subsets from n elements universe. So, but the point is for each such selection suppose I this an n element set this an n element set this is n element set is selected a k element set from here.

So, what is the remaining thing, the remaining thing the number of remaining thing is actually n minus k. So, when you select a k things from an n n set, so n minus k things will remain. So, when you remove when you take k things we have left n minus k is back or it is as good as saying that I have selected the remaining n minus k things it is like I removed this k things and picked up the remaining thing right.

So, you select as you select k things you are actually selecting the remaining n minus k things also, in some way you can imagine like that way, because the process of selecting k things is equivalent to the process of selecting n minus k things. So, therefore the number of ways of selecting k things will be equal to the number of ways of selecting n minus k things that is.

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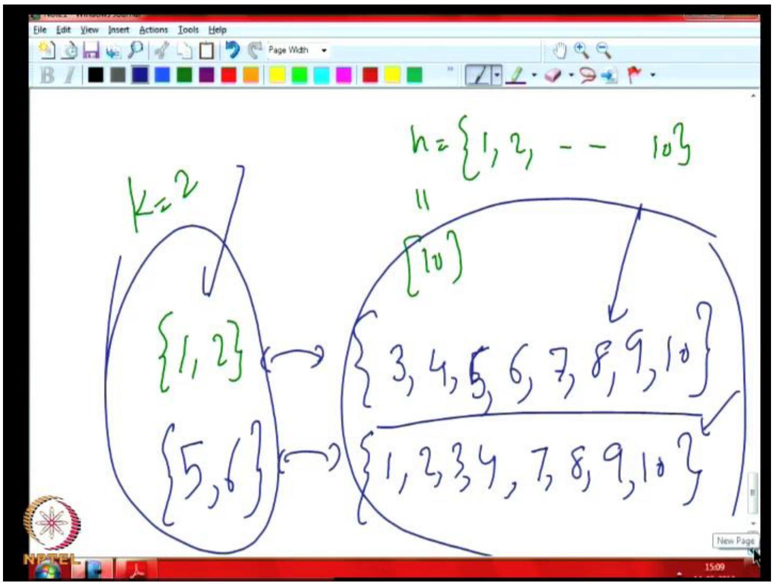

So, so if you want an example for instance put n equal to say 1, 2 up to 10 right that is 10, now a suppose put k equal to 2. So, you can select say a two elements set 1, 2 from this but then you have it is like equivalently you have selected equivalently you have selected the remaining set. So, this is the remaining set 6, 7, 8, 9, 10 this is an 8th element set right similarly, you can take any to element sets say 5, 6 here. So, then it is equivalent to saying that you have selected a 1, 2, 3, 4, 5, 6 gone. So, 7, 8, 9, 10 right the remaining 8 things you have selected like this.

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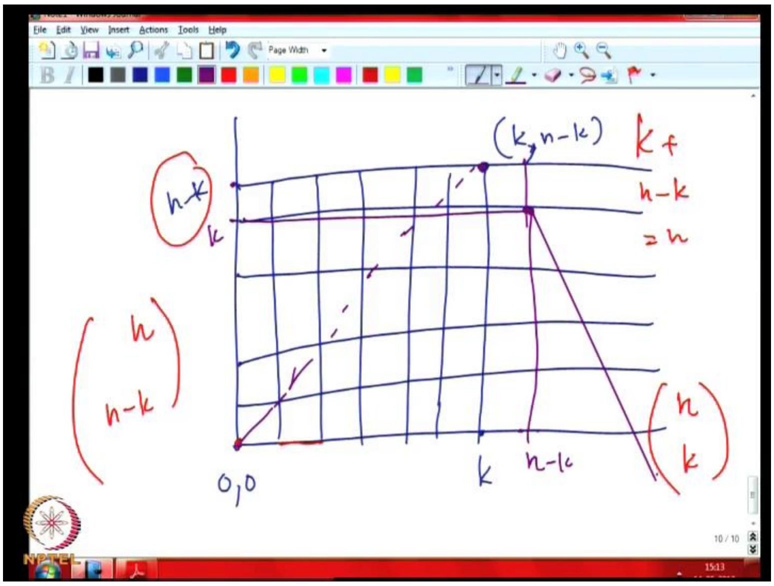

So, the number of ways of selecting this two sets here is equivalent or, so is equal to the number of so selecting these 8 sets that is what it says. Therefore, n choose k is going to be equal to n choose n minus k, so if you want to see it in the north eastern lattice. So, it is we have seen that this n choose k, what was this n choose k, n choose k was like we want to reach from 0 0 to k say this lattice point this point was n minus k right this point k comma n minus k this is where we wanted to reach right. So, I can draw this lattice point right.

So, we wanted to reach from here to here because total k plus n minus k equal to n steps will be required to reach from 0 to 0 to this point and of case it is a like out of this n steps how many steps will be eastern. How many ways you can select the eastern steps eastward steps right east east steps eastward step right that is what we were thinking or mean the steps towards the right. So, that is that will be definitely n choose k right, that we have already seen this argument.

Now, when you want to argue this in a different way one way to think is right you could have argued that this is also to the number of ways to reach here that can also be counted by asking how many ways you can select the northward steps. Because once you select the northward steps automatically the eastward steps will get filled. For instance see here our argument earlier was that we select the second, for instance first step, second step, third step as eastward right and the first k steps are eastward automatically the remaining n minus k steps will be northward.

So, we do not have to worry about the northward steps at all you just have to specify eastward steps right the steps towards the right that is only we as (( )), so that we can take out the n steps in k ways what we told. Or we could have argued the other way also see we we could have selected this steps toward the north that has to be n minus k right. So, we can select n minus k steps n choose n minus k ways towards the north and the eastward comes automatically be fixed, if you want to be slightly different. So, let us say we can also think like this, so as a now right.

So, this is the main diagonal so of case I have not drawn it properly but you can say, we can take a point this is k n minus k we could have also taken another point which is n minus k k. So, for instance here it is approximately right let us say n minus k, so n minus k comes here and then the k comes here something like this right, suppose k comes here right. So, it is a in some sense like it is a number of ways to reach here right that paths if you reflect above the main diagonal that will be away to reach here right here. So, this we have to draw better one it will be clearer. So, the point as may be you can draw it and it is not worth redrawing the figure and wasting the time.

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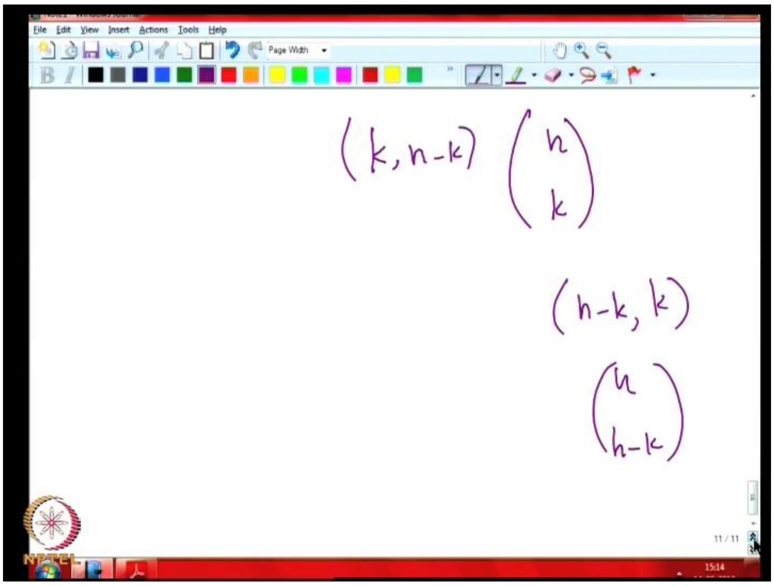

So, the point is you can you can say that to reach n k minus k the number of paths is n choose k right. But to reach n minus k k should be the same number of paths because of the symmetry and this will be n choose n minus k right, even the formula can be written right that is what it says; anyway, so that I will leave it to you to figure out.

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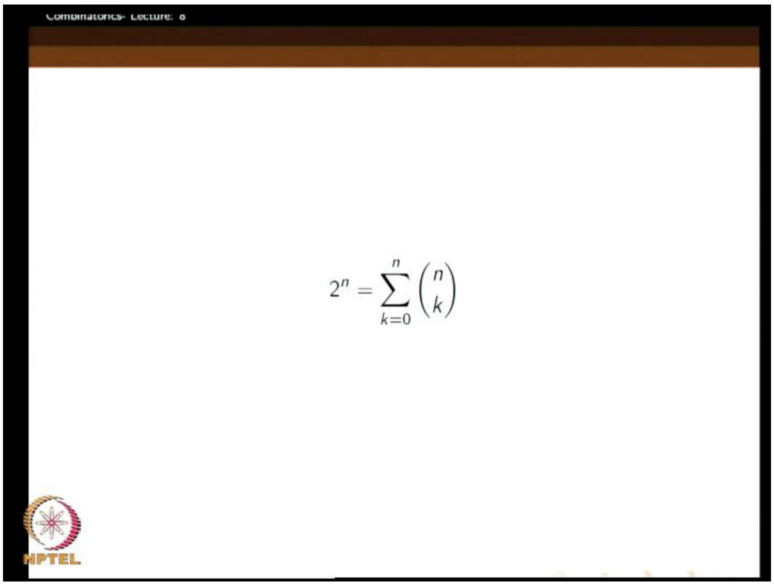

And now the next identity we will consider if 2 raise to n equal to k equal to 0 to n n choose k. So, again going back to the Pascal time, so this is what we want to prove right.

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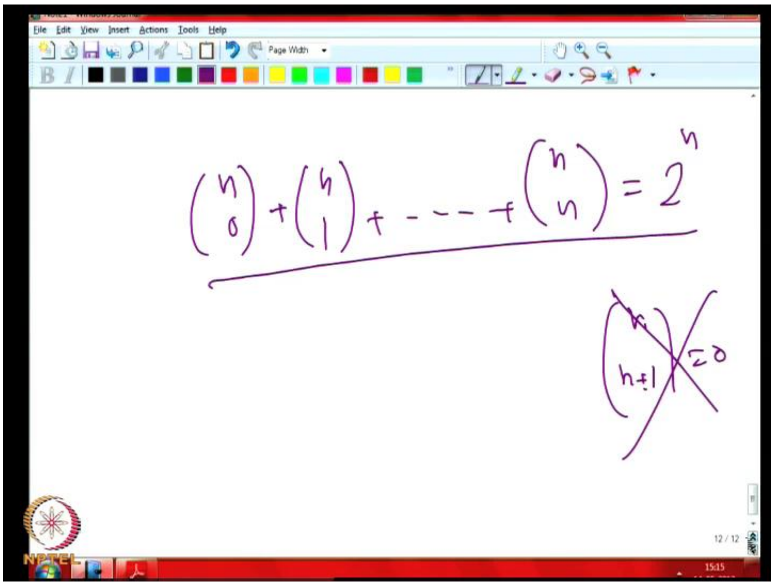

So, n choose 0 plus n choose 1 plus if you add these things up to n choose n what we will get is 2 to the power n that is what it says, note that, so this is starting from here n choose n is the biggest non zero number which we can I mean for instance n choose n plus 1 onward will be zeroes right. So, we do not have to so if we consider all these things it will become 2 raise to n in other words from the Pascal's triangle.

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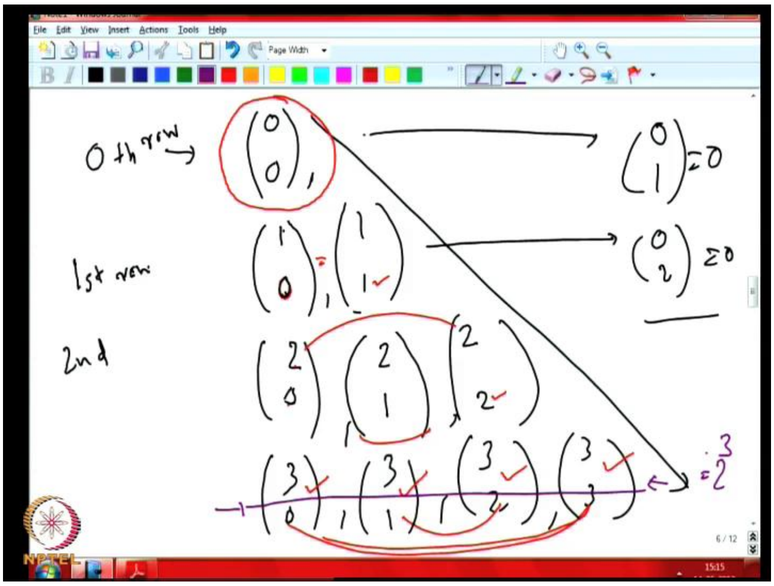

if you want to see this here it says if you add up these numbers say in a row in a certain row this will add up to here it will be 2 raise to 3 right.

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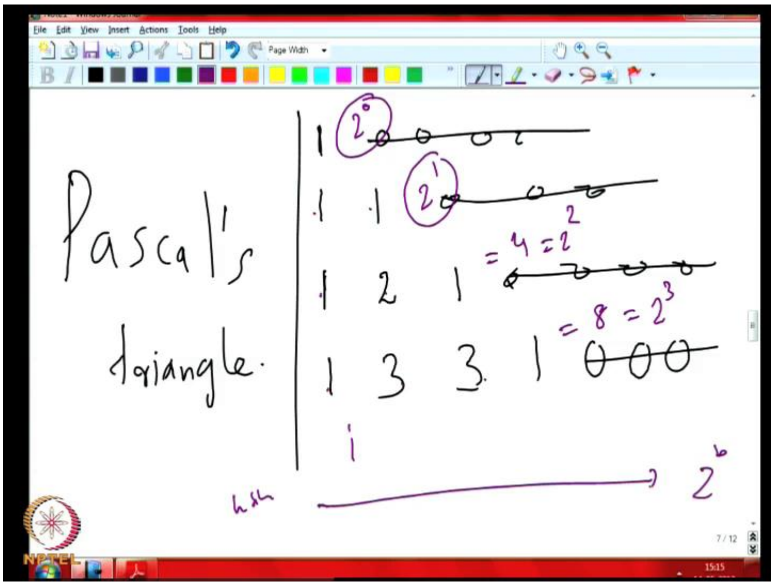

So, if we want to see the next page 1 say 2 raise to 0 this is 0th row right, so here 1 plus 1 2, 2 raise to 1. So, this is the first row 2 plus 1 plus 1, 4 it is 2 to the power 2, second row right 3 plus 3 6 plus 2 8 that is 2 to the power 3, third row sum sub to that. Similarly, n-th row if you sum up n-th row will sum up to 2 raise to n if you add up like this that is what the statement says. Of case this is not new to us we have already seen the proof of this using the fact that actually n choose k is the number of k element subsets, and this will this is counting the total.

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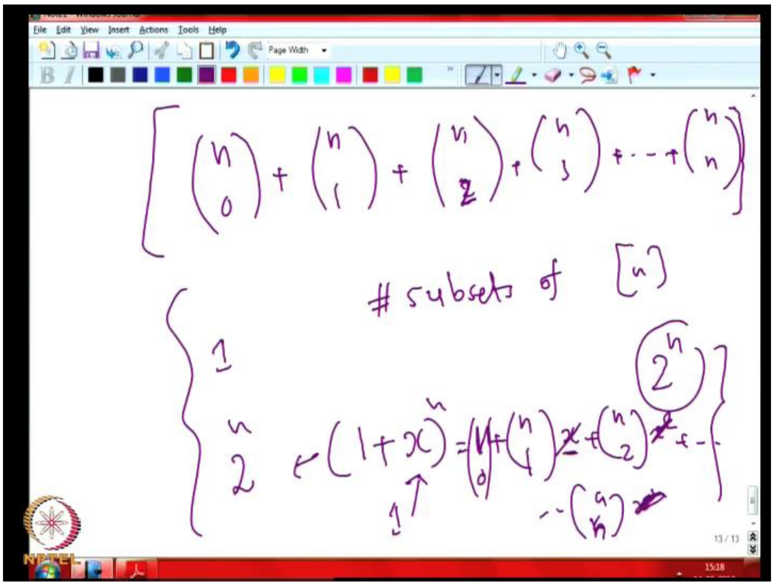

For instance what is it counting n choose 0 correspond to the number of empty sets that is only 1 of case n choose 1 correspond to the number of singleton sets right n choose 2 correspond to the number of two elements sets right. So, two elements sets n choose 3 corresponds to the number of 3 elements set and finally, n choose n corresponds to the n elements subsets. If you add these things up we are actually counting the number of subsets of n right.

And earlier we had shown that by that string binary string argument I mean we set up a bijection between the number of subsets of say n elements set and the n string n length binary string. And told that this is 2 raise to n right here, we are claiming that that is indeed. So, we another proof we had shown for this thing is by noting that 1 plus x raise to n is equal to 1 plus n choose 1 x plus n choose 2 x square this was the binomial theorem n choose n x raise to n choose n x raise to n.

And we put x equal to 1 and then this side became 2 raise to n right and these because x became 1 here right all these things will become 1. So, therefore, this is n this is n choose 0 sorry n choose 0 plus n choose 1 plus n choose 2 plus n choose n these 2 proofs you have already seen right but in the context. So, we can also remember this identity saying that the Pascal's if you take the Pascal's triangle and add up the elements the numbers in the same row right. If you sum up the rows it will always be 2 to the power something 2 to the power that row number that is what it says.

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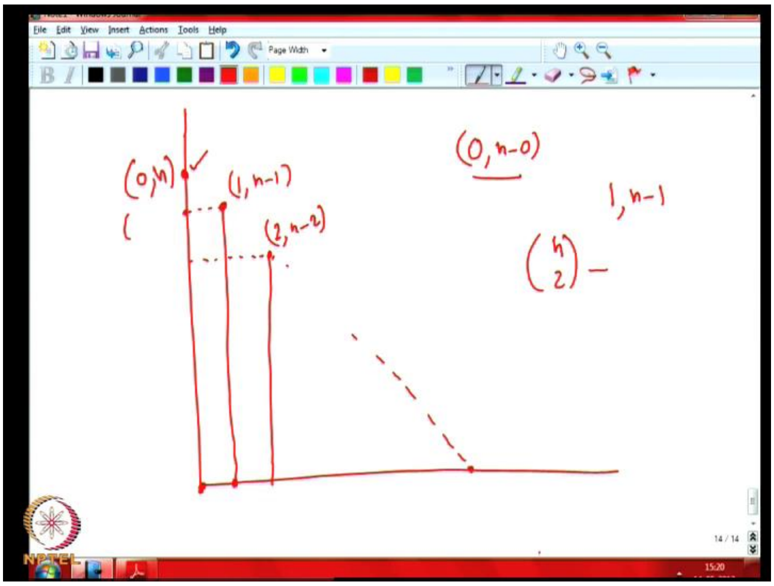

So, next one, so of case you can try to get an interpretation of this using our lattice path arguments also, so for instance you can interpret it like this. So, what will be n choose 0 right. So, that is so we are considering the number of ways to reach 0 and n minus 0 right. So, 0 n minus 0 would be this point right this 1 0 n right 0 n minus 0 is this thing and then what is n choose 1 n choose 1 will be the number of lattice paths I mean north eastern paths to reach from 0 0 to 1 to n minus 1; that means, say this is n minus 1 this being n minus 1, so where here right; so this is 1 n minus 1 right, so right.

Now, the next one will be n choose 2 that will correspond to the number of lattice paths to reach from 0 0 to this 2 n minus 2 right and so on right. And we are actually summing up the number of paths to reach here here here any of these two things right. So, or finally, n choose n this is we can see interpretation in terms of paths, so that is just to visualise and then what.

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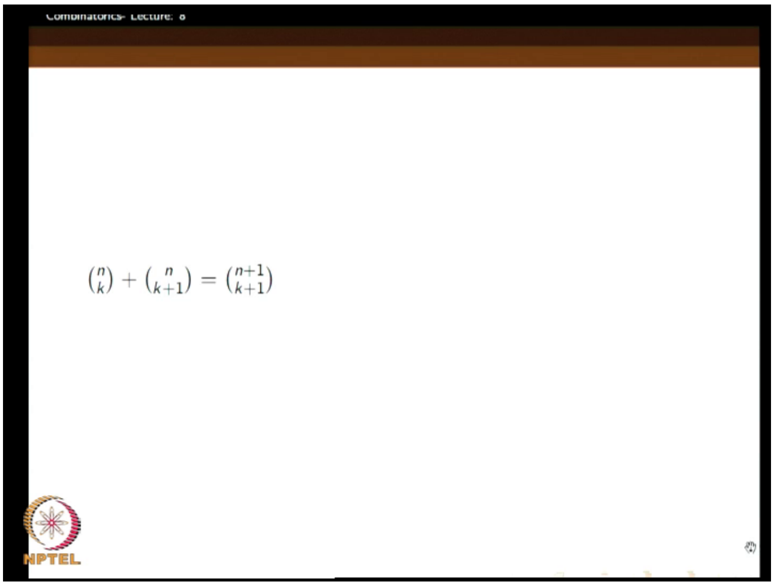

The next question the next identity we want to consider is n choose k plus n choose k plus 1 is equal to n plus 1 choose k plus 1 so this is 1.

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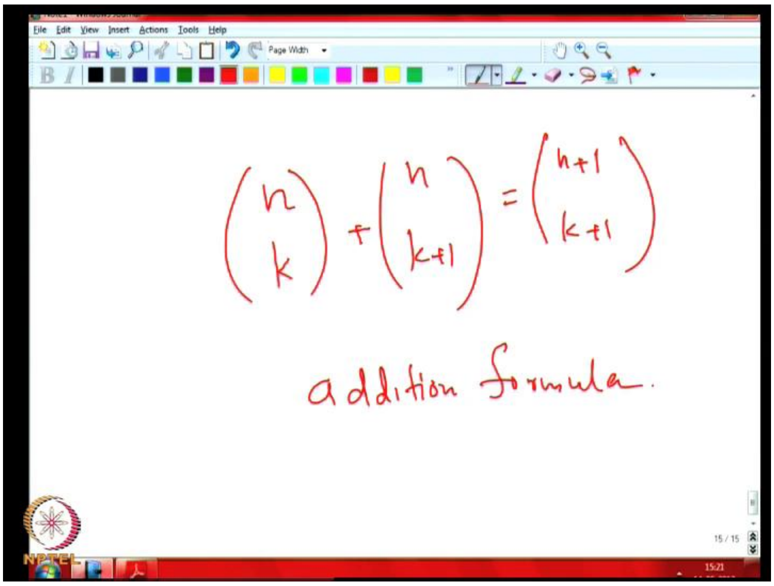

So, an important identity the it is called addition formula addition formula see this is for instance in the Pascal’s triangle.

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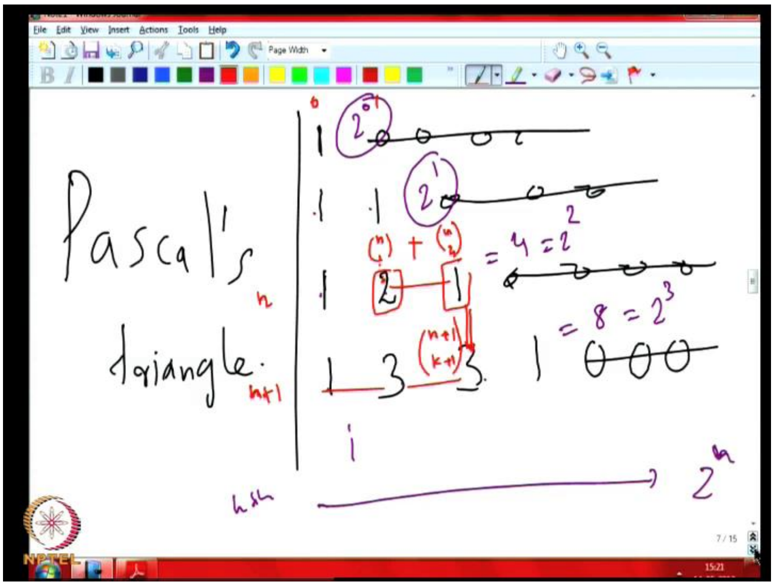

So, what it says is here say this is nth row suppose this is nth row this is n plus 1th row right. So, let us say n k is equal to this 1, k is equal to 2 k equal to sorry this is 0th row this is first row k equal to 1 right. So, n choose 1 here this number n choose 2 will be here right; now we say that I suppose add this 1, n choose 1 and n choose 2 then what do we get we will get n plus 1; that means, in this row choose k plus 1 namely just below this thing k plus 1 right.

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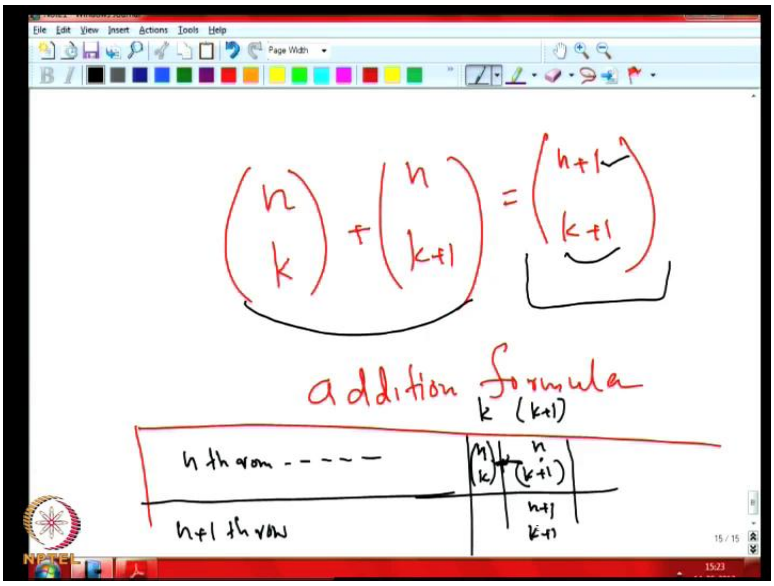

So, this and this if we add you will get this is, what it says right or maybe you can in the Pascal's triangle here in the Pascal's triangle. Suppose I will draw like this suppose here suppose here this is the nth row say we we consider the kth column here kth column here in the kth column we will see n choose k right. And this k plus 1th column we will consider see n choose k plus 1. Now, this addition formula says to add these two things here this and this and what it says it declares that the answer is going to be in the next row; that means, n plus 1th row. So, this is the nth sorry n plus 1th row and in which column k plus 1th column means this column just below this thing n plus 1 k plus 1 right, if you add this things together we can get this thing that is what it says.

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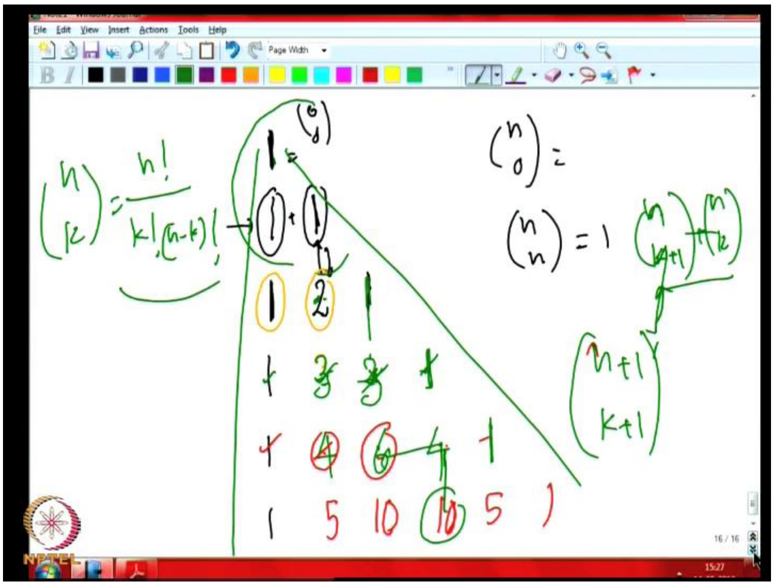

For instance that will help us to create the Pascal's triangle very easily for instance you can you can write like was here 0 to 0 and here the next 1 also we need; that means, 1 choose 0 is 1 and 1 choose 1 equal to 1 right. And then on the third one onwards we can use this formula we always start with 1 because always any n for any n n choose 0 is going to be 1, so this is always like this right.

And the second sorry this is the first row onward first column onwards we can create like this see this 1 plus 1 will be 2 here, this and this if you to get this thing what you should do is you go the previous row and take just above thing this 1 and this 1 and add them together. And then we can write here 2 and of case this is what should be this this is always the last 1 is always 1 we do not want to worry about it because that is n choose n right for any n it is always 1.

So now, next number, so here 1. So, what we do is we take this and this and add them together and then write it here 3 3 and then to get this number what do we do we just take the this number and this number 2 plus 1 is equal to 3 and here it is 1 because. And to get this this number we add this and this together that is 1 plus 3 is equal to 4, to get this number we add 3 and 3, this 1 and this 1 right that is 6. To get this number we add this and this that is 4, 3 plus 1 4, and here is 1 right.

The next one 4, 1 plus 4 this 1 and this 1 is added we will get 5 here and now this one and this one is added 4 plus 6 10 we will get and then 6 plus 4 10, 4 plus 1 5 and 1, what are we. We are just applying that formula n choose to get a the n plus 1th row to get then plus 1th rows k plus 1th entry we are just taking the just previous row that means nth row and k plus 1 entry; and this previous one even n choose k k-th entry right. So, just to get this one took this one and this one right that is what this is just just above this and this previous one that is being added like that we can create the Pascal’s triangle.

So, this is the basic recurrence relation for the binomial coefficients once you know the initial values namely way how you start this ones and something this ones and say this one’s this we are not creating using this formula and then this and this we already wrote. Beyond that we can just produce the n pair table using this right, so we can write down all the formula that is very easy that way. So, you do not have to use the explicit formula for n choose to calculate see n factorial by k factorial into n minus k factorial.

So, not that this is of case here we have to for instance if n is it is too big we have to generate all this numbers. While of case here we have to do the multiplication but as such for small numbers this may be easy to right write down also at least we can generate them easily. So, if you know the previous one if we can easily generate this numbers this is the way the Pascal’s triangle can be generated using that addition formula.

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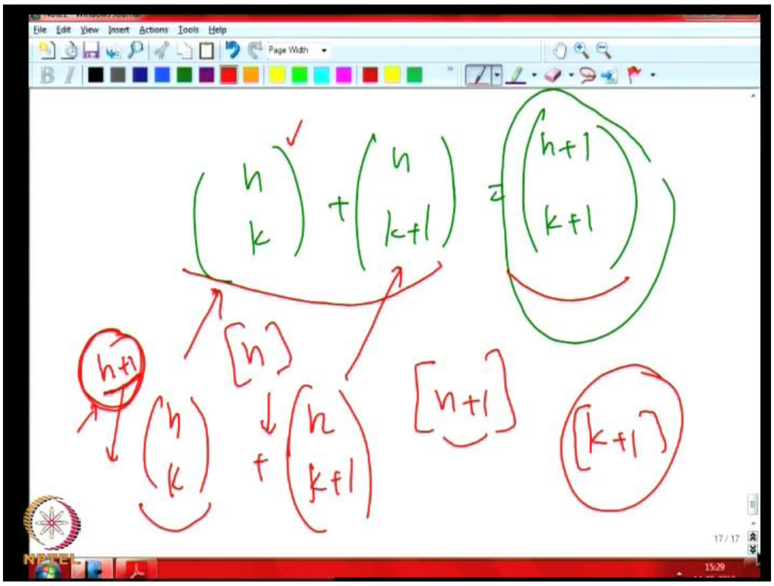

Now coming back to the addition formula namely n choose k plus n choose k plus 1 is equal to n plus 1 choose k plus 1 here n and k both are non negative numbers and then k is smaller than n and how do we prove this thing. So, we just see the interpretation of for fourth of these things. So, what is this, this is like selecting from n plus 1 elementary universe we want to select k plus 1 subsets the best thing to do is to remove the largest case say n plus 1 itself and keeping keep that apart right.

Now, there are two different ways of selecting k plus n 1 elements sets, so from the remaining n elements that means this universe n. So, we can select k plus 1 things all the k plus 1 things can be selected that way; that means, with never putting this n plus 1th element in it that that can be done in n choose k plus 1 ways right this is with this term right. And another way of doing is add this n plus 1 I mean first add him and then we still have n elements left out of that we can select the k things that is n choose k's right.

So, this collection always contains this n plus 1th guy n element right n plus 1 element and we are we are saying that we have grouped the possible subsets k plus 1 elements subsets of n plus 1 into two groups those subsets, which contain n plus 1. And those subsets which does not contain n plus 1 the number of subset which does not contain n plus 1 is essentially. The number of k plus 1 elements subset of n that is n plus 1 is removed right that is we we are not going to take that is this n choose k plus 1. The number of subsets which contain n plus 1 because n plus 1 is already there now we only have to have k elements subsets from the remaining n elements that is n choose k ways right. So, these two things if you add you will get it this is what is the simple thing.

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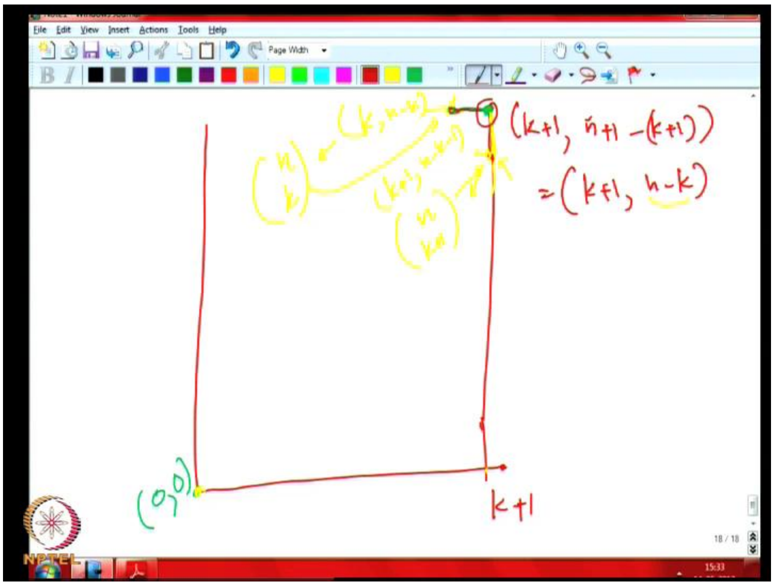

And if you want to as usual if you want to look at the north eastern lattice as you want to say it says for instance you know what is this n plus 1 choose k plus 1. So, say some k plus 1 is here right, so this is k plus 1 this is the point we have to reach n plus 1 minus k plus 1 right that is k plus 1 comma n minus k this is the point right we want to reach from 0 0 right by north eastern lattice paths. So, of case, so we k we have to take k plus 1 eastward steps and the remaining n minus k northward steps right this is the number of ways to reach here that way here. But to reach here from the picture we can see that there are only two ways to reach here either you reach here and then move here or you reach here and then move to here right.

So, therefore, to get the total number of ways to reach here we can count the number of ways to reach here and add it to the number of ways to reach here right. So, the number of ways to reach here is easily calculated by if you know the what is this point, this point is because this was n minus k right. And because right this will be this point will because this is n minus k this will be n minus k minus 1 comma k plus 1 sorry n minus k minus 1 comma k plus 1 right. So, the number of ways is to reach this point is we add these things together. So, that will be n choose sorry this I wrote it wrongly this is this point is k plus 1 comma n minus k minus 1 right. So, the number of ways to reach is we this is this is we just add total number of steps is this plus this that is n, choose the total number of eastwards 2 appears that is k plus 1 right.

This is the total number of ways to reach from here to here right n choose k and to reach here we need to know the point here this n minus k sorry k this point is k because this is k plus 1th row. So, this is k plus 1, so this is k comma n minus k right. So, to reach here this is we add them together so the total number of steps is n out of that k steps is to be select selected in the eastward direction that is n choose k here to reach here, we have n choose k ways. To reach here we have n choose k plus 1 ways right, to reach here we have n choose k ways to reach here we have n choose k plus 1 ways.

So, we add the number of ways to reach here, we add to the number of ways to reach here and then we get the number of ways to reach to this point right, because to reach here we have either this way or this way. And if you have already reached here then there is no way of reaching here if you have already reached here. Then we have no way of reaching here because these are disjoint sets that is just that addition principle, we are using because we have cut these into two disjoint sets.

So, this is only a combinatorial interpretation or what sorry another interpretation or what we have explained earlier. So, that selecting so putting them into two categories and and counting two different subset I mean the what we want to count is the cardinality of a particular set. We are just splitting that set into two disjoint sets and then separately counting this ends and adding them together right and so on.

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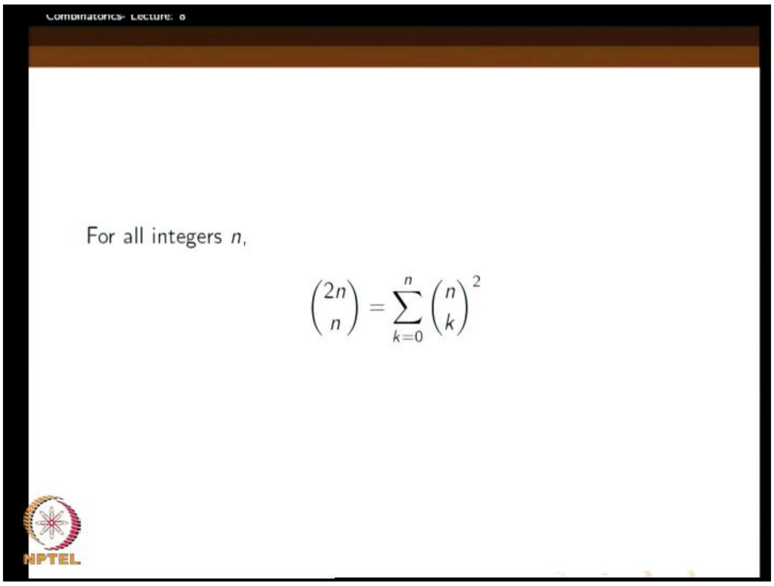

And that is what about the addition formula and next one is we have already described what Pascal's triangle is. And now for fun sake let us consider a few slightly more complicated identities which in a try to remember, but these are not as important as the earlier two identities namely the symmetry identity and the addition formula. So, this let us look at this 1, 2, n choose n is equal to k equal to 0 to n n choose k square how do we prove this thing it is just for practice so.

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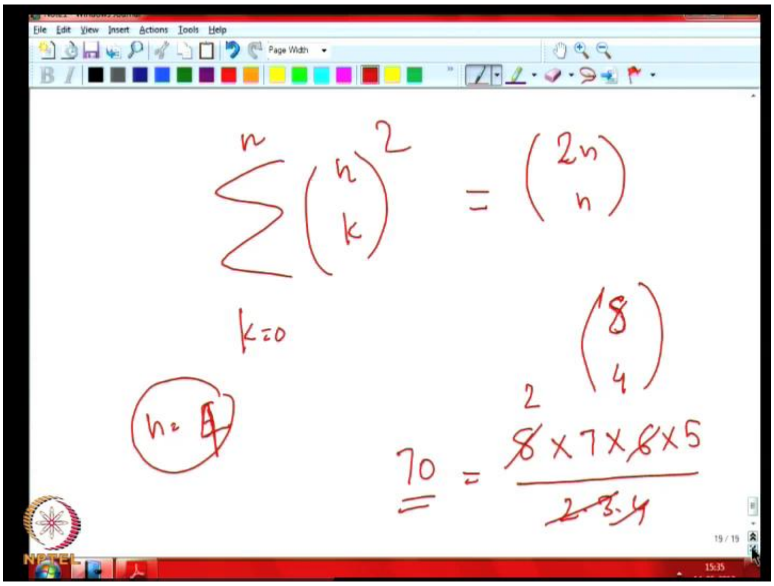

So, we are seeing that if you sum up from k equal to 0 to n this n choose k squares right we will get 2 n choose n. So, do you want to verify it for some small values. So, fine for instance n equal to 5 let us see, let be too big I do not know. So, we can may be a we can take a little smaller, n equal to 4 let us say. So, this 8 and 2 n choose n is 8 choose 4 right that will be 8 into 7 into 6 into 5 divided by 4 factorial 2 into 3 into 4 right. So, this will be 70 if I have not made any mistake.

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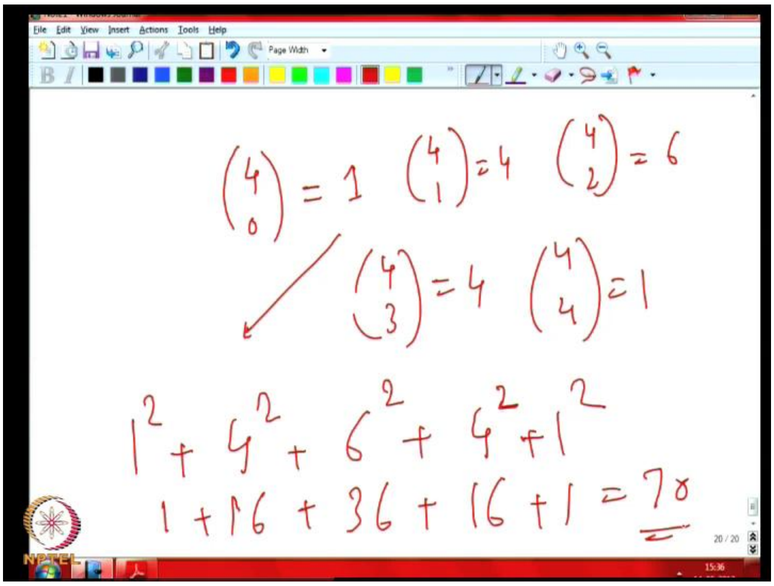

So, but this formula what does it say 4 choose 0 square 0 is equal to 1 4 choose 1 is equal to 4 and 4 choose 2 is equal to 6 and 4 choose three equal to 4 and 4 choose 4 equal to 1, now square it. So, the square of this 1 square plus 4 square plus 6 square plus 4 square plus 1 square, this is 1 plus thirty six sorry this is 16 plus 36 plus 16 plus 1. So, that is 70, 32 plus 36 68 plus 2, 70.

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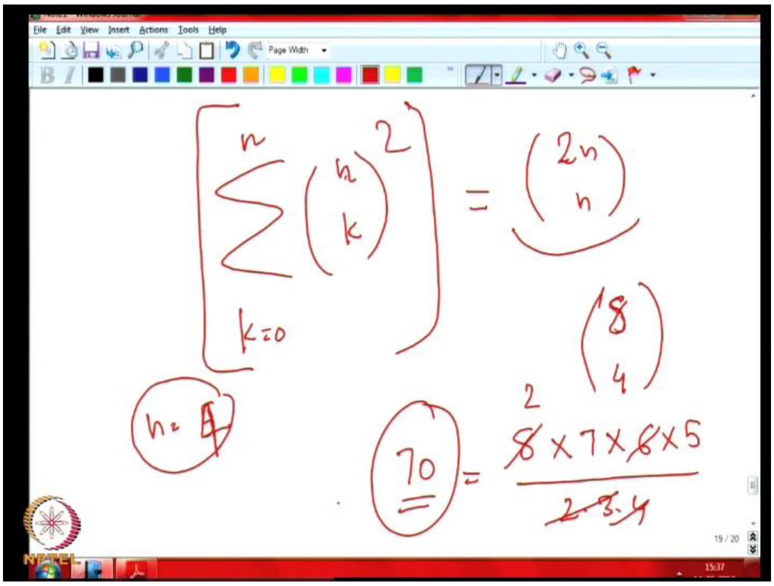

So, it is the same as what we got earlier that 70 right so of case this this can be useful sometimes for instance is at least you have seen that to write it down each of this n choose k terms and square and sum. Of case if you knew that it is just 2 n choose n you in one go we can get the answer right. So, let us say let us try to prove it, so suppose this a combinatorial proof I mean that is what we want to we we do not want to manipulate, and get which is this is kind of combinatorial arguments we have practising here.

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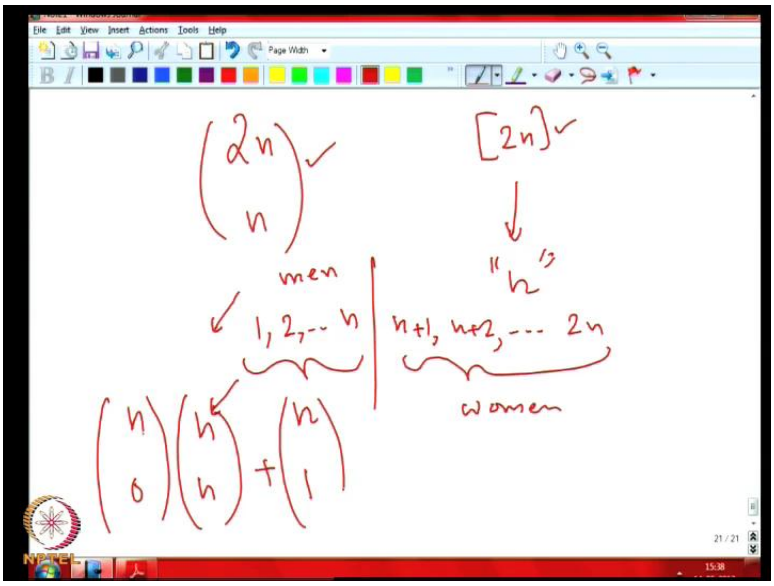

So, the first side says 2 n choose n that means from this set 2 n we want to select an n element set how many ways you can do that right. The other side says you can do it in the following way first group these 2 n things as say 1, 2, n. So, the first part n plus 1 n plus 2 up to 2 n, the second part you can think that these are men these are women say women. Because to this 2 n were people and they were n men and n women we just with this only ask for this 10 people to be selected out of this 2 n we do not care about men and women. But, we can do it this way we can select zero men first out of this n men right and then all because the zero men we select that means all n people we select should be women right out of n we select n women or otherwise we can select 1 man. So, maybe we will complete it tomorrow because time is over.