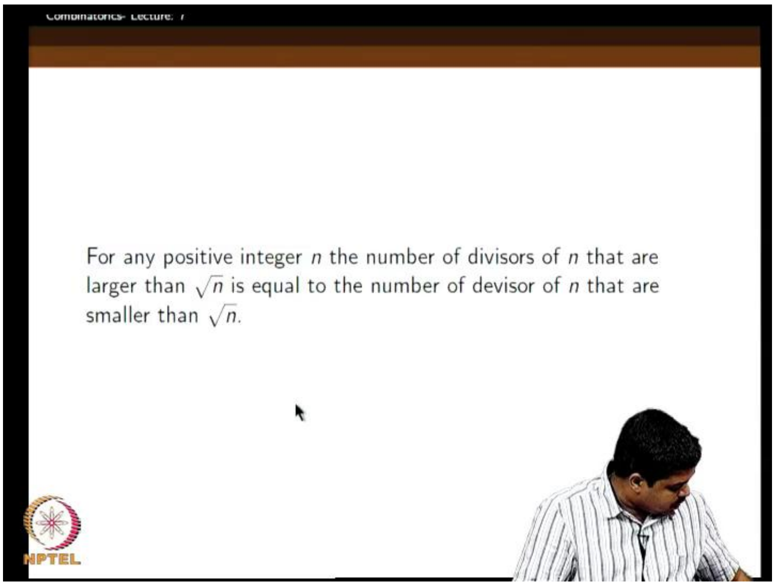
Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian institute of Science, Bangalore

Lecture - 7
Bijective proofs - Part (2)

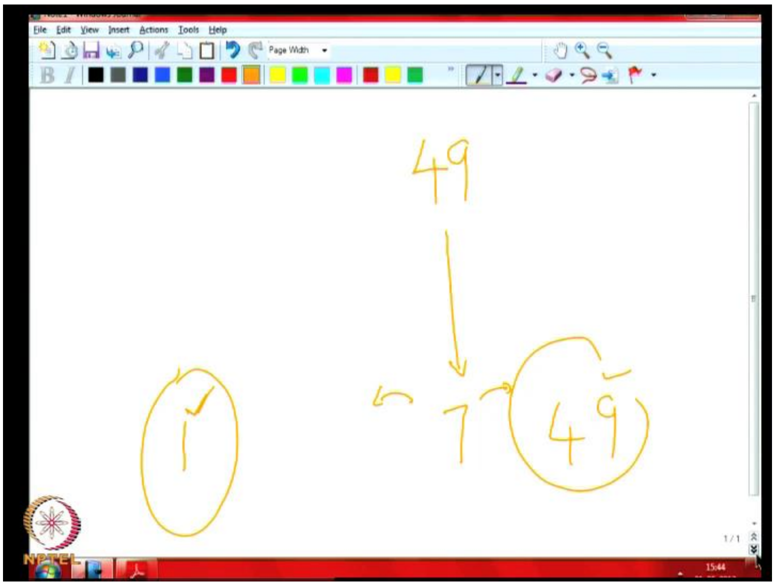
Welcome to the 7 th lecture of combinatorics. Today, we will continue with bijective proofs, right. So, this is like as we have seen, it is a special case of the division principal, where our function between the two sets is a bijection. Bijection means a one to one correspondence, right. So, if it is a function from T to S, then every member of S has exactly one pre-image. It is not possible that two different members of T get mapped to the same member in S, right. So, and also every member in S has a pre-image (( )) bijection. So, the point is if you can establish a bijective function, bijection between the two sets, then both sets should have the same cardinality. Whichever set is easier to count, we will count and then we can infer that the other set also has the same cardinality.

So, in the last class, we started with an example where we counted the number of subsets of n element set. We noted that there is a bijection between the set, namely the family of set of subsets of n element set and the n length binary strings, and it was easier to count n length binary strings, it was 2 raise to n. Therefore, we also inferred that the number of subsets of mn element set is 2 raise to n, right.

(Refer Slide Time: 02:08)



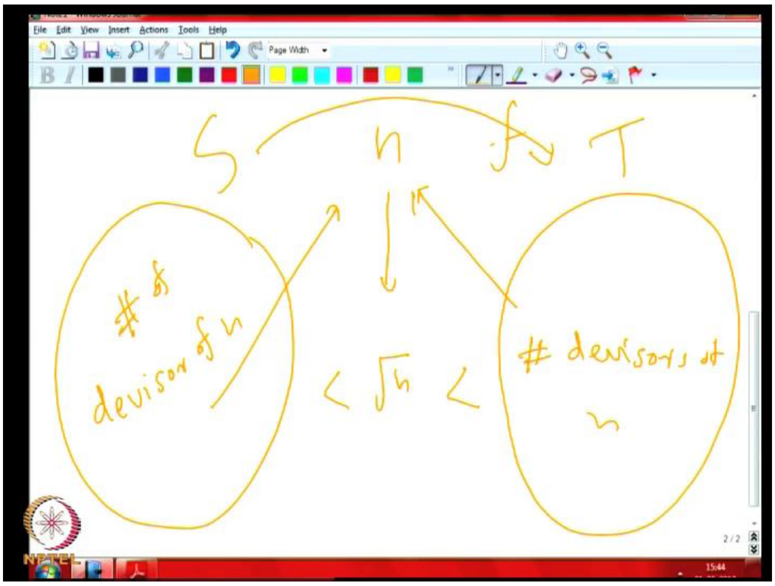
(Refer Slide Time: 02:31)



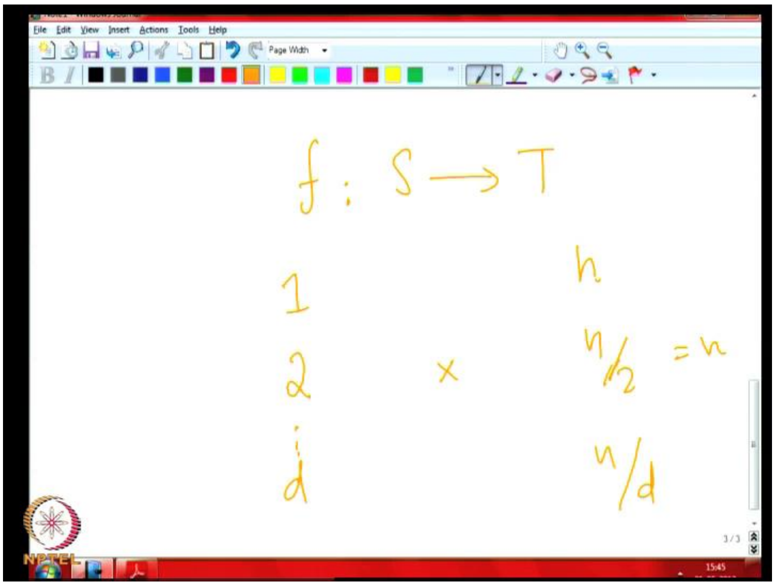
The next question we considered was this. For any positive integer n, the number of divisors of n that are larger than root n is equal to the number of divisors of n that are smaller than root n. We considered some examples in the last class and so it is for small numbers, it is indeed holding. For instance, I can take one more example. For instance, let us take 49. I am taking a perfect square to make it to avoid and it is not necessary that I have to take S perfect square. So, its root is 7 here. I am interested in the divisors of 49, below 7. How will I find? This is one as a divisor, 2 is not, 3 is also not a divisor, 4 is not, 5 is not, 6 is not, 7 is a divisor. So, what then? Yes, so this is.

Similarly, if you above 7, you can only see 49 itself, right. So, here and here, we will have, correct, so the number here and number here. So, just one here the number of divisors above 7 is equal to the number of divisors below 7, right. It is what we say.

(Refer Slide Time: 03:43)

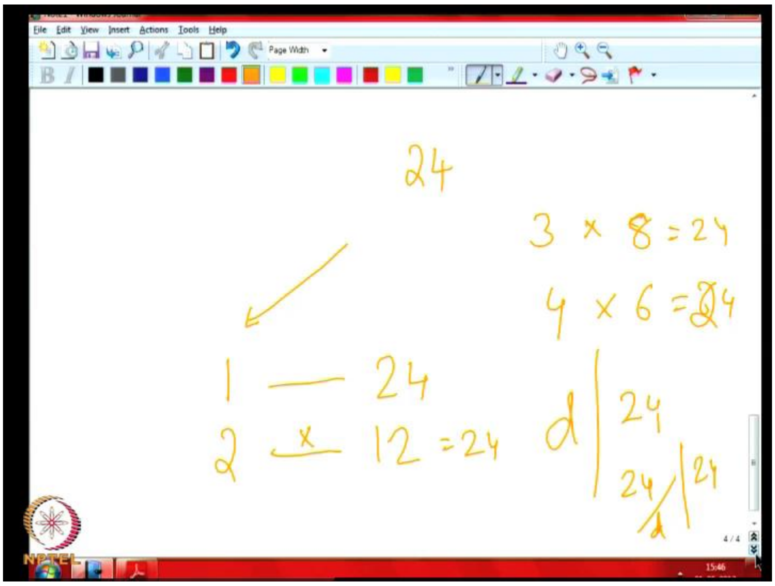


(Refer Slide Time: 04:25)



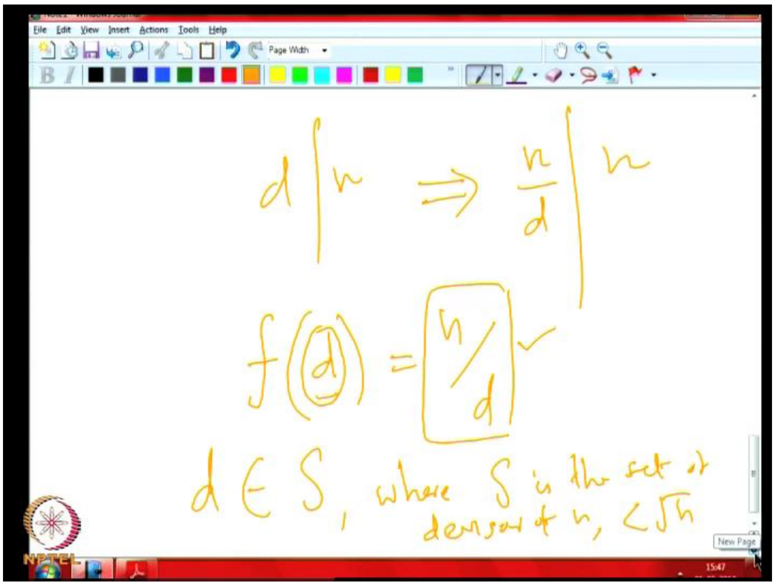
So, of case for small number S, this seems obvious, but is it true that for any number n, this is true. Given any number n, so the number of divisors of n, the number of divisors of n below root n, strictly below root n. So, this number is equal to the number of divisors of n, strictly about root n. This is what we want to say. What we do is, consider this as set S and we will consider this as set e and then we will establish a bijection f between S and T, right. So, we want a bijective function f from S to T. How do we do this thing? The point to note is that if 1 is the divisor of n, then n itself is also a divisor. So, if 2 is a divisor, then n by 2 is also a divisor of n because 2 into n is equal to n, right. Similarly, if d is a divisor, then n by d is also a divisor. This is what we are going to use in this bijection.

(Refer Slide Time: 05:03)



For example, if you take the divisors of 24 for instance, right. If you take 1, 24 is also a divisor, 2 is divisor. Then 12 is also a divisor, 2 into 12 is equal to 24. So, similarly, 3 is a divisor. Then 8 is also a divisor, right because 3 into 8 is equal to 24, right. If 4 is divisor, then 6 is also a divisor because 6 into 4 is 24. Now, the point is if any number d is taken and if it divides 24, then 24 by d also is an integer and that divides 24, right.

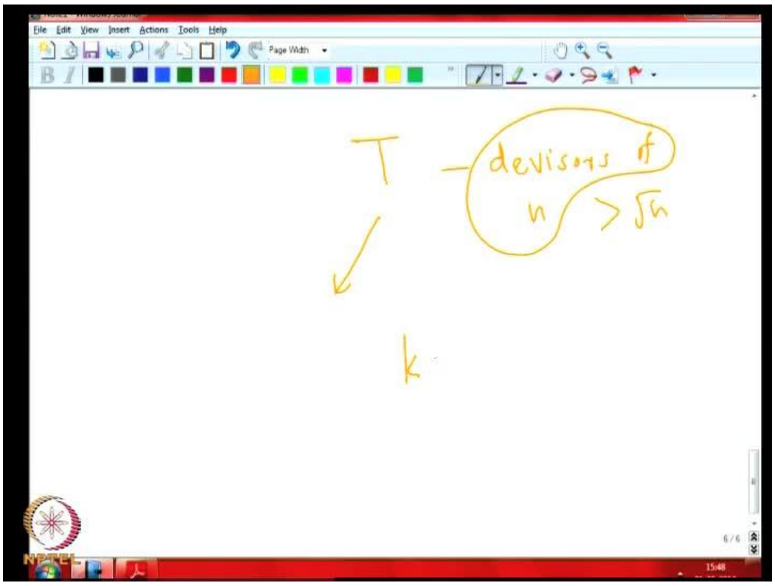
(Refer Slide Time: 05:48)



So, in general, if d is a divisor of n, d divides n, so that implies n by d is a positive integer and then that divides n, right. So, we can say that if d is a divisor, n by d is a divisor. So, our bijection is like this; f of d is equal n by d for d element of S, right where S is the set of divisors of n strictly below root n, right. So, we can map it to this. Now, we can check that this is a function first, what all things we have to check?

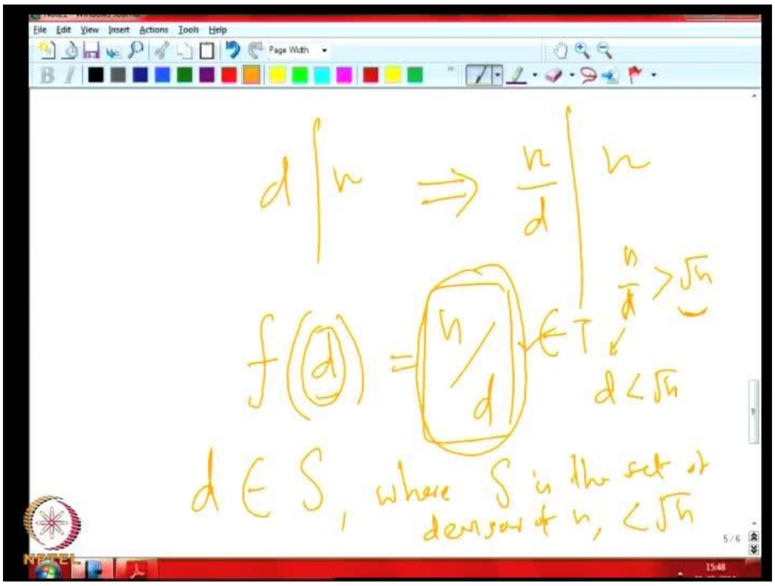
For every divisor, we do get this one. Yes, we do get because if d is a divisor, indeed n by d exist because you can divide n by this d n that an integer. That is why it is called a divisor, right. Now, for a given divisor of n below root n, this number is also uniquely defined and by d only, right. It is therefore, it is indeed a function, right. Now, what we have to check is this function is indeed a bijection. How do we check?

(Refer Slide Time: 07:20)



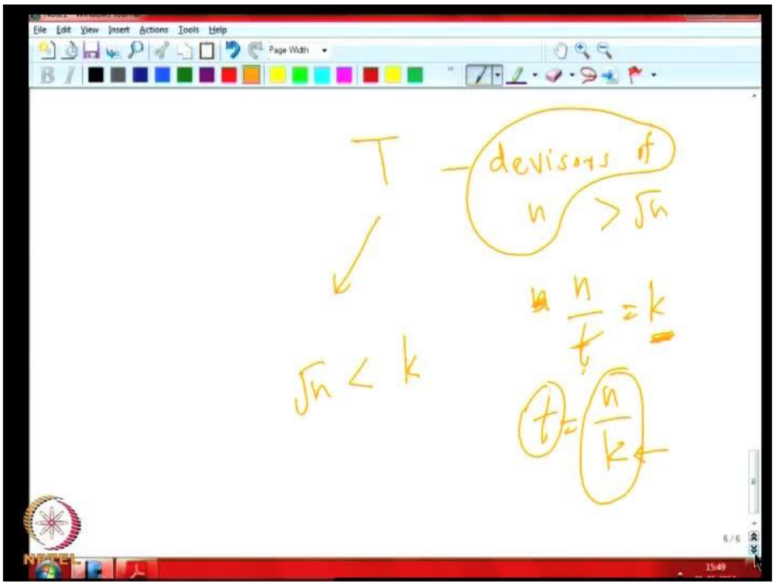
So, we go to this set T, namely the divisors of n strictly above root n, right.

(Refer Slide Time: 07:43)



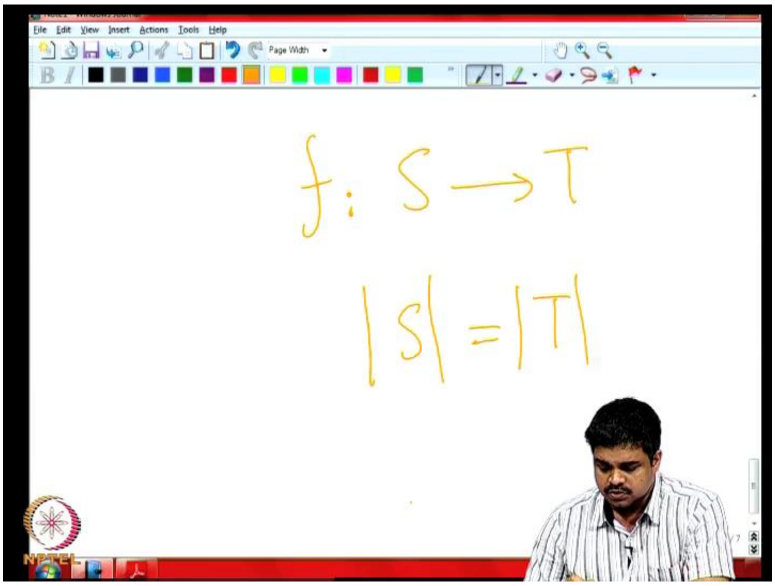
So, now you take any k, right. So, we also had to note that this n by d belong to T, right because our function is defined from S to T. So, you see that this d is strictly less than root n. Then n by d has to be strictly bigger than root n. Why? Because when d is root n itself, n by d will become equal to root n. If T was smaller than root n, then n by d has to be bigger than root n. That is why this is indeed a number which is a positive integer, bigger than root n, right.

(Refer Slide Time: 08:18)



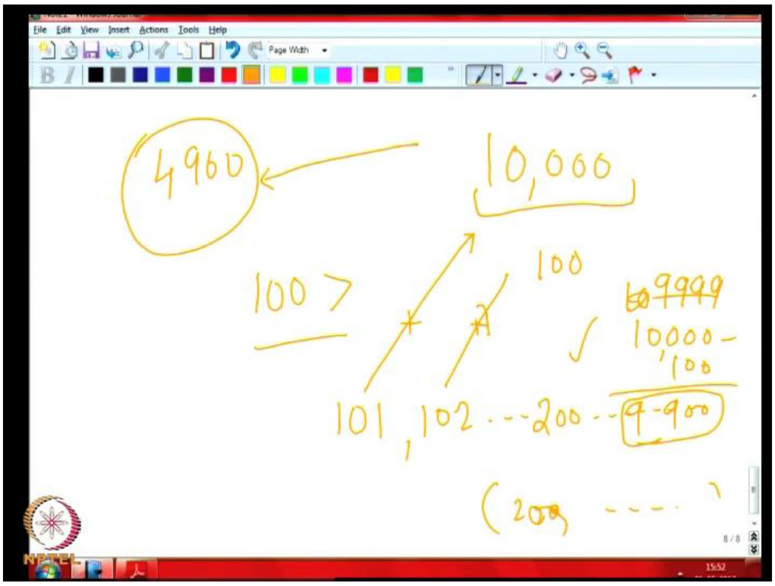
Similarly, here if we take any k, which is bigger than root n and a divisor of n if you consider n where it came from, right. So, this is from n by T for some, right for some T. That is what then. So, we want to find out which is that T. So, T is equal to definitely n by k. The same way, since k is greater than root n, n by k has to be less than root n, right. So, T will be a number which is less than root n and definitely, this T is a divisor of n because if I divide n by T, then we will get indeed this k the integer, right.

(Refer Slide Time: 09:11)



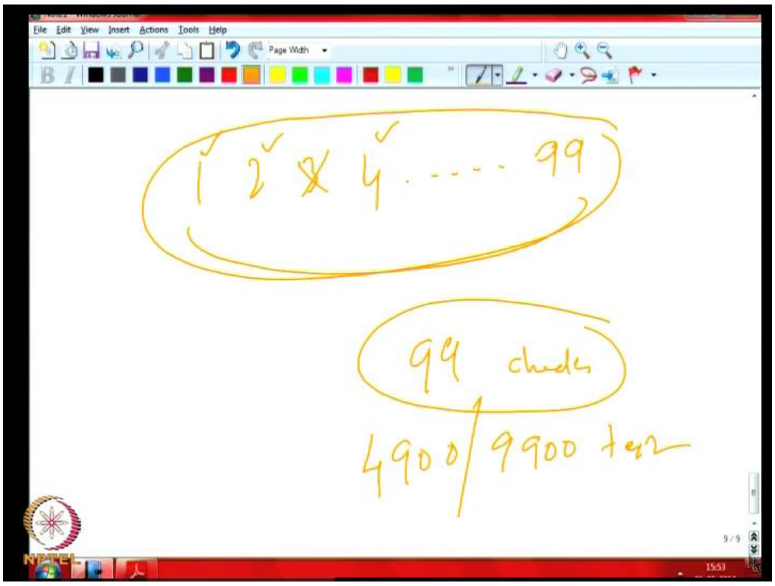
So, therefore, there is a pre-image and there is indeed one pre-image, right because this is a unique one we are getting, right. It is quite easy to see. Therefore, we have established a bijection f from S to T here, right. Therefore, the cardinality of S has to be equal to cardinality of the T, the way we wanted. The number of divisors of n, strictly below root n is equal to the number of divisors of n, strictly above root n. Now, you can see that is why it is useful. Of course, they are not just telling how many divisors are there for n, below root n or above root n, but then we are saying that both these numbers are indeed same, that is a number of divisors below root n and above root n. Sometimes, it can be useful.

(Refer Slide Time: 10:00)



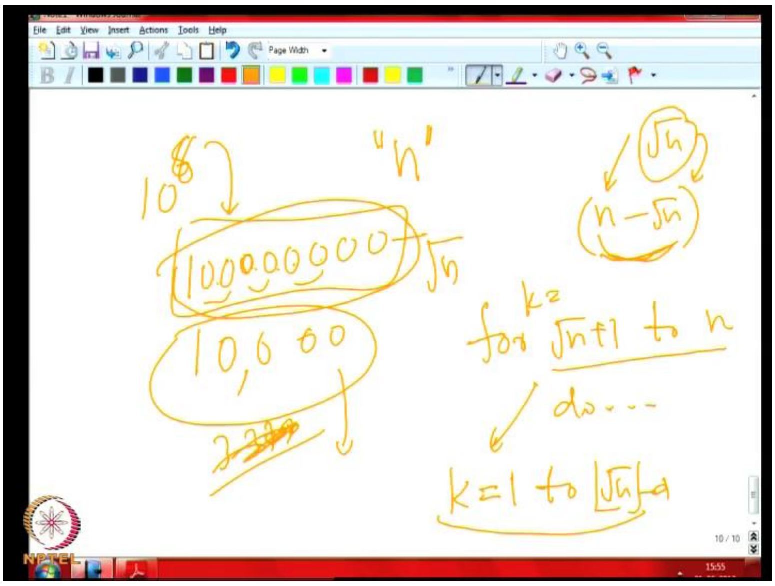
For instance, you can consider 10,000, right and then you ask somebody, may be a school kid to write down all the divisors of 10,000 which is greater than 100. Suppose, if you ask 100 is the square root of 10,000, right. So, then he may do like this. He may check whether 101 is a divisor of 10,000 or not, he says no. 102, he tries. So, like this he may try. So, when he reaches 200, he may write yes, and he writes 200, I got one. So, like that, he can keep listening things, but how many times he has to do these things. So, he has to do for all the numbers below 10,000 that means 9999 may be including 10,000 for that matter. 10,000 minus 100. These many numbers he have to do. That means, 9900, right. 9900 numbers he has to try.

(Refer Slide Time: 11:59)



So, you may say that why should I go above 5000 because I know that nothing above 5000 will divide 10000. So, may be even then I have to do 4900 times this thing, right. So, assuming that, you are clever enough to identify that he assume may end up doing 4900, right, but on the other hand, if we had used our theorem. That means the number of divisors, often 10,000 above 100 will be equal to the number of divisors of 10,000 strictly below 100 and then you could out done like this. You just check whether 1 is a divisor, yes. 2 is a divisor, yes. 3 is a divisor, no. 4 is a divisor, yes. So, like that you could have listed the first 99 numbers which are strictly below 100 and then we could have done with it. Just 99 checks, instead of that 4900 or 9900 tests, right.

(Refer Slide Time: 12:40)

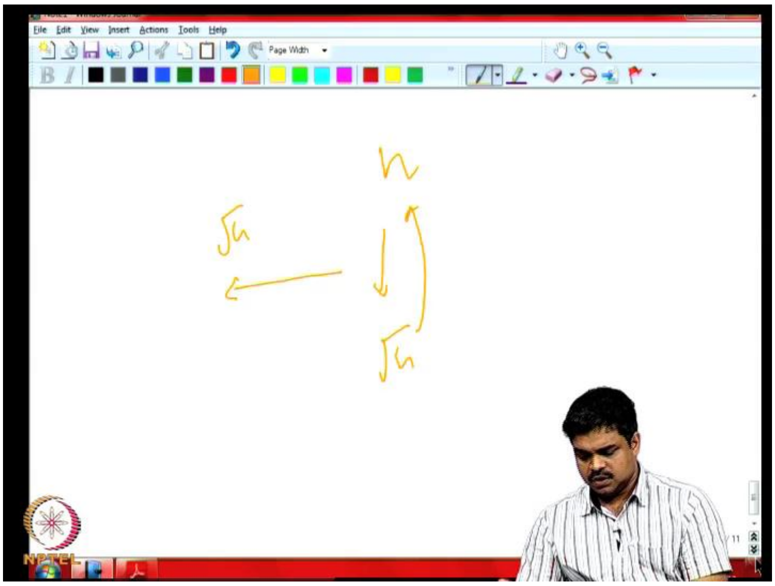


So, this is much better thing to do, right. So, of course, it need not be a school boy, it can be a computer. You may be writing a program for this to find out how many divisors are there for a number n which is above root n, right. Then you may write, say 4 root n k equal to root n plus 1, 2, n, do something like that saying that testing whether k is a divisor of n and then output k something like that, but then you have to do it for n minus root n times, right. If n is for instance, a very big number 10,00,000, so this can be very large, but on the other hand, the 10,00,000. How much is 10,00,000? So, say 10 raise to 3. So, 10 raise to 6 is a million, right. So, 10,000, right and 10,000, 1,00,000, 10,00,000, this is a million and then 10 million is one more 0. I do not know how much its root is.

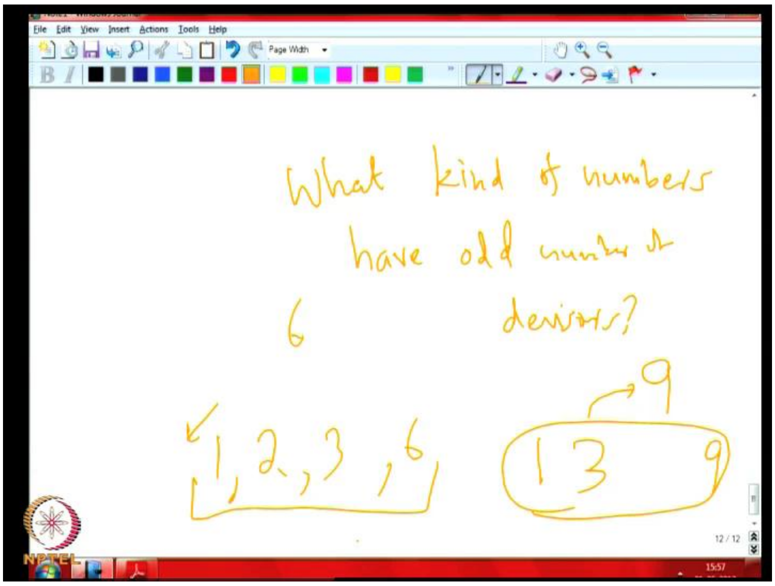
So, you can say, one 0 for every two 0's here. So, something above 10,000, 1000, may be 2, some 3000, 300 or something like that, right. 300 and 30, something like that. May be you can put one more zero here and make it a little bigger number. That means, 1, 2, 3, 4, 8, 10 raise to 8. Consider 10 raise to 8. So, then this is only 10,000, right. So, suppose the computer has to run, so this minus this times, this is almost this number and these many tests, it may take long time, but on the other hand, of course it depends on how fast your computer is, but still if you are clever enough to do it from k equal to 1 to root 10 minus k, we have to assume that root n is integer root n minus 1 or root n floor and then check the same thing.

So, you may do much faster because this root n here is much smaller than n minus root n because n minus root n is almost like n, right. So, for big enough n while root n is much smaller. So, this is one thing. Of course I always talk, suppose if you want to find how many divisors are there for n which is above root n. It need not be the question. Sometimes, you can simply get the questions, how many divisors are there for n, right.

(Refer Slide Time: 15:28)

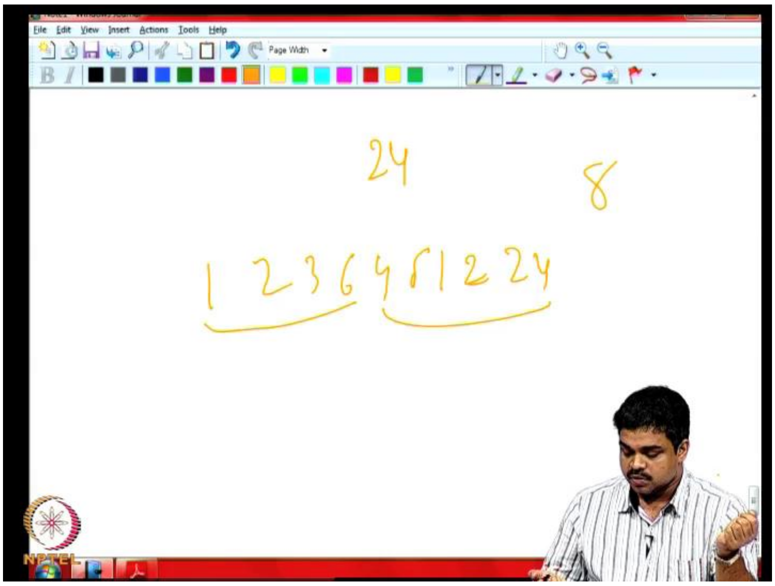


(Refer Slide Time: 15:50)

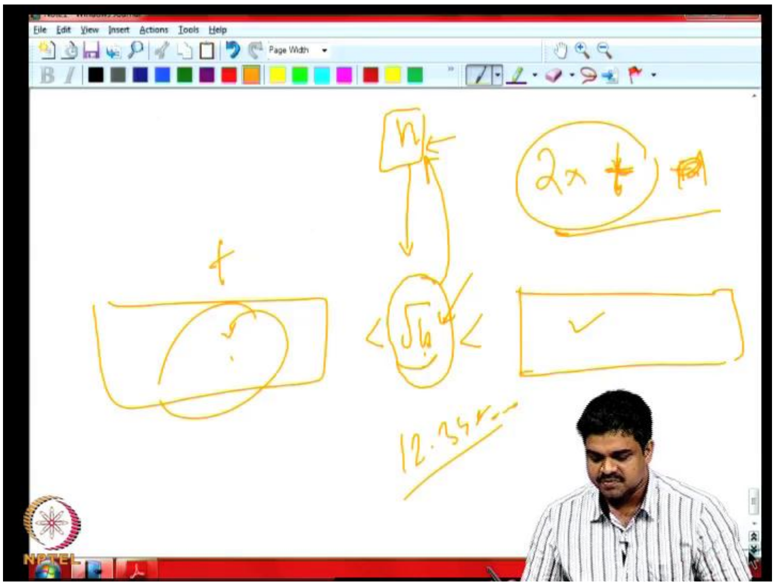


So, you can do the number of divisors below root n and multiply it by 2 because above root n equal number of divisors will come and then we can just test whether root n itself is a divisor of n or not. So, then that also indicates a very interesting thing. So, what kind of number has an odd number of divisors. For instance, 6. If we consider its number of divisors is 1, 6, 2, 3. It as an even number of divisors, right, four divisors, but on the other hand, if we consider 9, it has 1, 9, 3. Three divisors. These are odd number divisors, right.

(Refer Slide Time: 16:30)



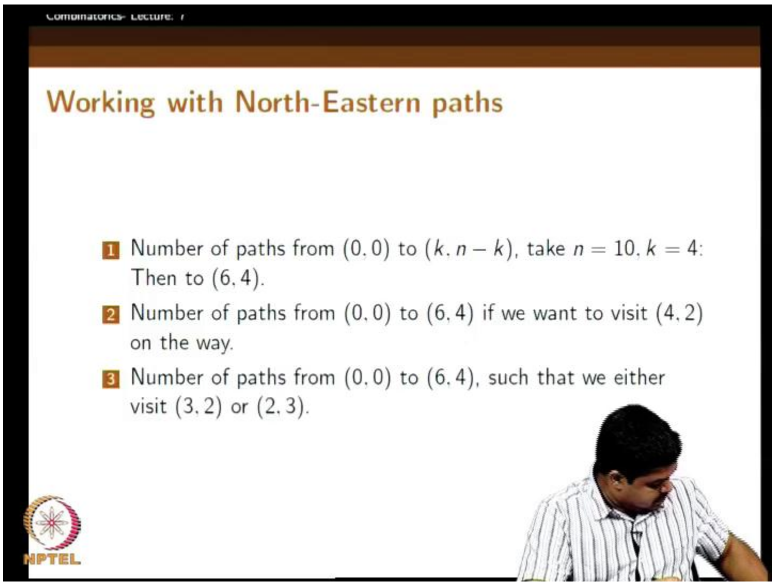
(Refer Slide Time: 16:54)



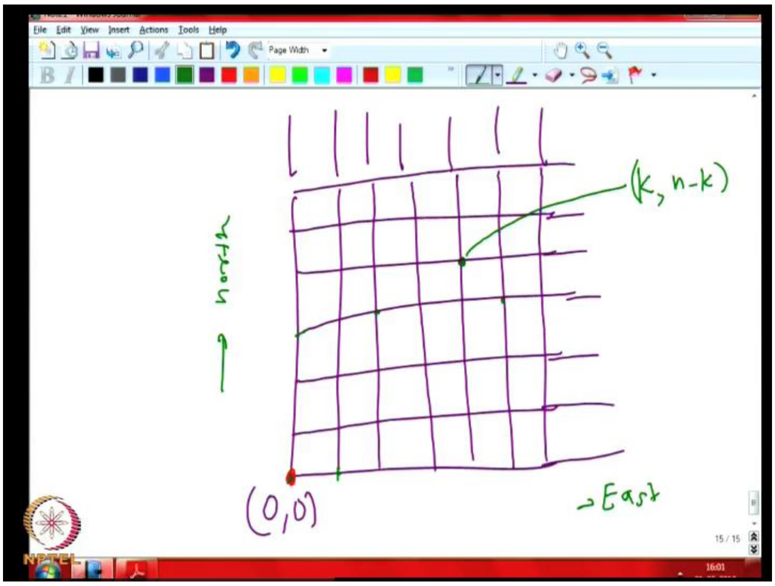
Similarly, if you take 24. If you have 24, 2, 12, 3, 8, 6, 4, these many divisors. 4 plus 4, 8 divisors, right. Now, some numbers have a even number of divisors, some numbers have odd number of divisors. What kind of numbers have odd number of divisors? Which is it is easy to see that depends on root n. If root n was an integer, then there will be odd number of divisors because if you consider the number of divisors below root n, that is exactly equal to the number of divisors above root n and when you count the number of total number of divisors, this plus, this will add, because these are numbers, that is two times this number here. So, that will be even number and then finally, we test whether this is a divisor of n or not.

If this is a divisor, then that 2, that even number here, 2 into this number, say t 2 t plus 1 will be total number of divisors which is solved. So, if root n is an integer that means n is perfect square, then it will have odd number of divisors. On the other hand, if root n was not an integer, it was something like 12.345, something like that, then definitely we do not have root n as a divisor of n. We just have 2 t. That means, even number of divisors, right. So, that way we can figure out easily what kind of numbers have even number of divisors and odd number of divisors. It is just a matter of looking whether n is an odd number or even number.

(Refer Slide Time: 18:25)



(Refer Slide Time: 18:46)

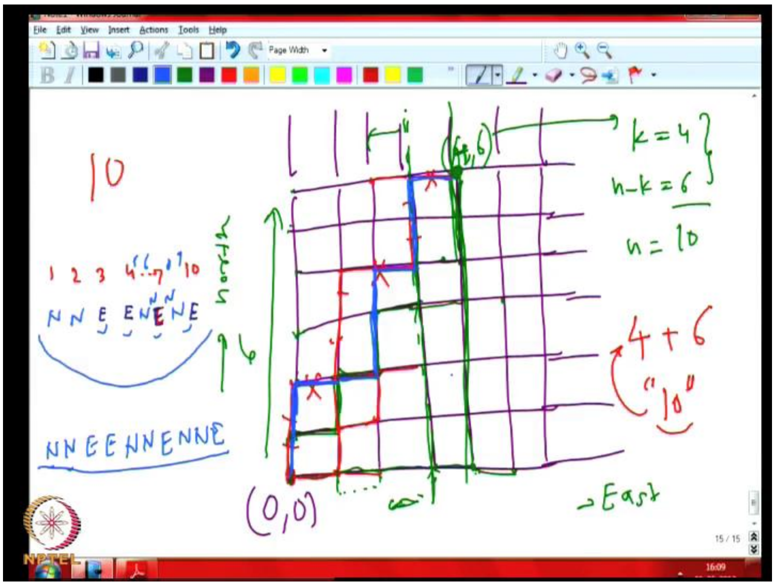


So, now next question we will consider is this. This is also a common kind of problem, a question which is considered in combinatorics. So, here we talk of something called north eastern paths. So, it is north eastern parts in a grid. So, consider a grid like this, something like this. It can be any length, but I will draw like this, say grid like this. You can draw further or here also you can draw further, right.

So, this is the point 0,0, this point and then suppose you can see, you can assume that this is some kind of roads, right. It is a collection of roads. These lines corresponds some roads and they are like this. They are some over in a, may be when somebody create a new city, modelled the city as a grid like all the roads in the city is like a grid. So, this is a 0,0 point. You always say that I want to reach this point or this point or this point, something like that, but suppose there is a restriction that you can either drive forward. So, initially you can say this is the east direction, right, forward east direction or you can drive north direction, right. So, either east or north, you can go. You can never go backward or you can never go downward, right. This is the restriction.

Now, the point is the question we ask is like this. How many ways you can go? There are how many routes to reach from 0,0 to certain point? So, the first question we ask is suppose, this is a point, say k, n minus k because I write like this. It is very clear. What I mean whether it will give us little bit of hint, but then I can take some numbers like in the grid.

(Refer Slide Time: 21:16)



Now, our question k n minus k. So, I put 10 equal to 10 and k equal to 4 and then right. So, this is 4, 6. So, I am interested in reaching 4, 6, 1, 2, 3, 4, 1, 2, 3, 4, 5, 6. So, 1, 2, 3, 4. So, 4, 6 here. 1, 2, 3, 4, 5, 6. So, this point is 4, 6. Suppose, I want to reach from here to here, right and I am driving the car. The rule is that I am driving the car eastward or northward. So, we are only interested in north-eastern paths. How many roots are there?

So, for instance, one example is I can try like this simply and then see straight here and then I will go straight here, but if I had taken one step further, I will never reach here because to reach here, I will have to comeback, right. So, that is not allowed or I could have taken a more zigzag path like this and then gone upward, right or if I could have just taken this path and here, I could have taken as this is the different path. So, the path is different. If we do not take entirely the same path, then we say that two paths are different, right.

So, the question is how many such paths I can take from each 0,0 to 4,6? In general, we write this 4, 6, k equal to 4 and we write n minus k equal to 6 and the sum of these two is 10, right, that is equal to n. We want to reach from k to n minus k. So, why do I write like that? So, the reason is that to reach 4, 6, you know whatever I do, however clever I become, I have to take four eastern directions. I mean four units I have to see one, see you can say that this is one kilometre. So, one unit here, one unit distance, one 4 units I will have to travel towards east because if I travel only 3 units, then I will reach maximum upto here, right, up to here. I will see vertical. I mean in the north direction, how much ever I travel, I will be before this line, right. I mean I will never cross this line, right.

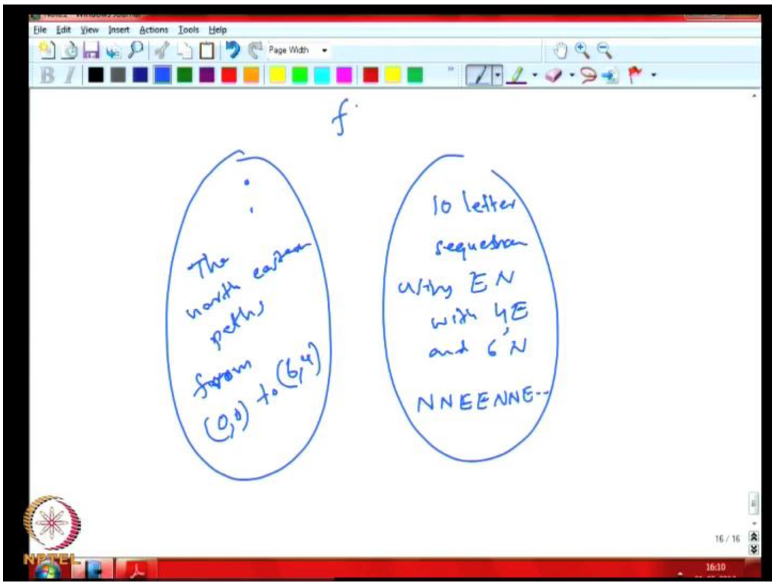
So, therefore, I will have to take 4. I cannot take five. If I take 5, I will cross this line and I will never be able to come back, right. So, to reach 4, 6, I have to take 4 units, say 4 kilometres east, I have to drive 4 kilometres east somehow. When I do is unimportant, before I reach there, I have to do that exactly 4 kilometres. Similarly, to each 4,6, I have to drive 6 kilometres north. I can do it like 1 kilometre east and then 1 kilometre north, 1 kilometre east, like this. 1 kilometre north, 1 kilometre east, 1 kilometre north, like that or I could have done 2 kilometre north, then 1 kilometre east, then another 2 kilometre north, then 2 kilometres east. So, I have finished 1 plus 2, 3. From it is one more kilometre, but here 2 plus 2 is 4 and another 2 here. Then I could have taken this. I will reach there, right, but whichever way I do not know. I should have taken these total 4 kilometres east towards somewhere.

Similarly, I should have taken this 6 northward units at some point of time, right. So, the question is total how many units I should take? I should take 4 eastward units and then 6 northward units. So, total 10 units I should take, I should move, right and then out of that 10, 4 is east. Now, how do I design my paths? So, one possibility is to try drawing all possibilities. It is not so easy because you know it is not easy to count because you see you can try. When I reach here, I have two possibilities and then when I reach here, I have another two possibilities. Like that I can count, but then some places it is the two possibilities, right.

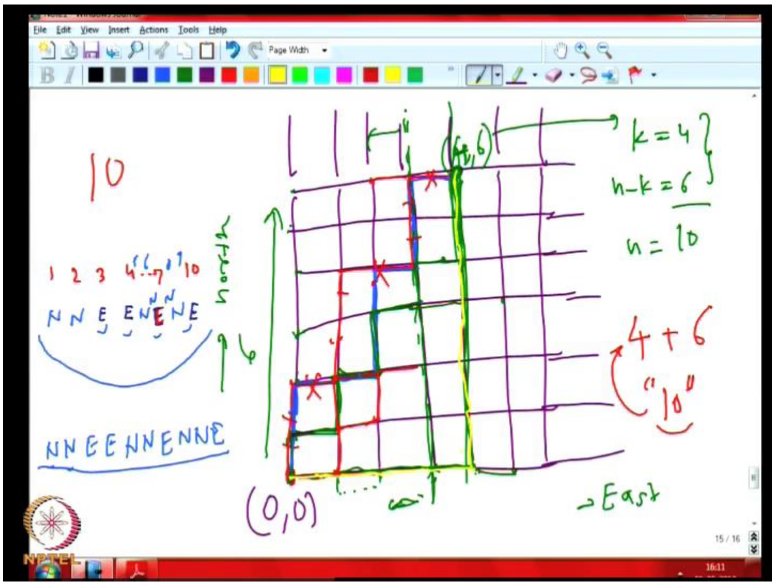
So, here we have two possibilities like that, right and then even if we count some, at some point of time what will happen here is when I reach here, there is only one possibility, right. So, we have to do some adjustment for that, right. So, one can try counting it in other words, but a clever way of looking at it is as I have mentioned is that I have to anyway take 4 kilometres eastward. The question is out of the 10 units, I take, I just arrange them like the first unit. This is the first unit, second unit, third unit, fourth unit and then finally, tenth unit. Out of this, I can say that I will take the third time, I will take the east, right and then say fourth time also, I will take the east. Tenth time I will take 10, in the tenth step I will take the east. May be again also during the seventh step, I will take the east. So, I will take the east, right.

So, if I have decided these are the three times, I take east, then other places I will have to put north, right. North, north, north, 6, at 6 north, north, north, like this, right. So, 6 north will come. That way 7, 8, 9 north and north. So, this is the way. So, once you fix my east, then the north will automatically be fixed. So, for instance, for this one. So, I see that the first step has to be north. So, this will be like this. The second step has to be north, I will go northward. Second step also as if the driver is looking at this chart and moving and the third step is east, so he will take the step and then fourth step is again east, he will take the step and then the fifth step is north. He will take this one and the sixth step is also north, he will take this one. Seventh step is east, he will take this. Eighth step is north and then tenth step is again north and sorry, ninth step again north and tenth step is east. So, he is reaching there, right.

(Refer Slide Time: 29:22)

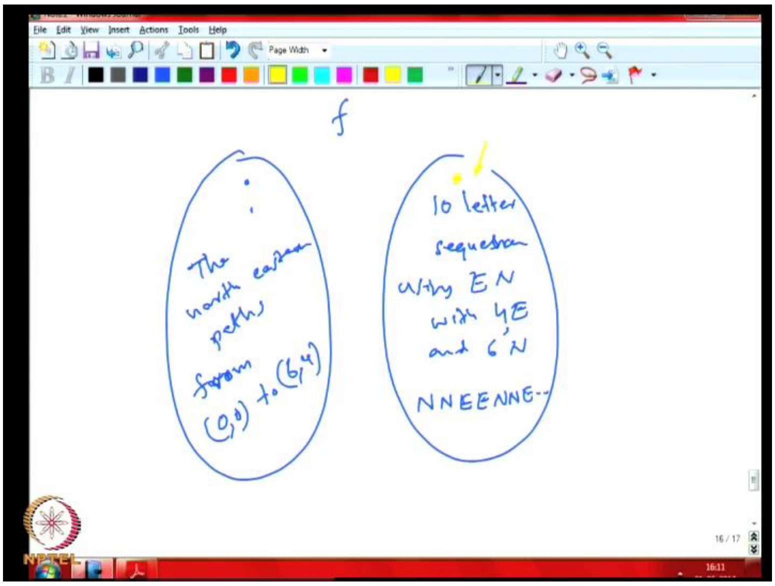


(Refer Slide Time: 30:16)

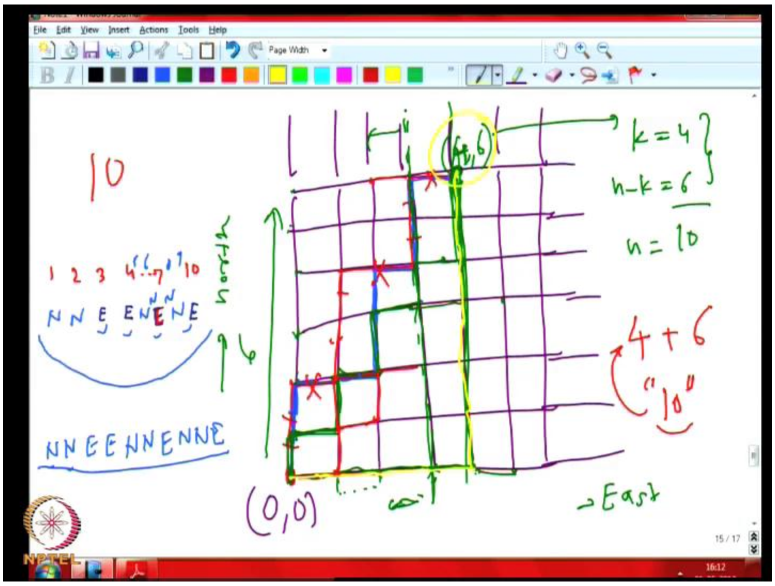


Now, what I am going to do is I am claiming that this kind of a chart, this kind of a pattern that you have written something like NNEENNENENNE. So, there is N and E chart. There are ten letters here. So, if we count the number of strings here which I can make like this, these strings corresponds to different paths and each path will correspond of its. In other words, I will setup a bijection between the north eastern, north eastern paths from 0, 0 to 6, 4 and the ten letter sequences, using sequences is ENN with 4 E and 6 n and something like this, NNENNE, this kind of sequences. Why is it so? Because as I have shown you, first say that this is a function we get. We map it. So, we look at path and we just write down, along the path we will go and write down the sequence. So, this is NNEENNENENNE like that for every path. For instance this path. This path, I can write N, sorry EEEENNNNNN, right.

(Refer Slide Time: 29:22)

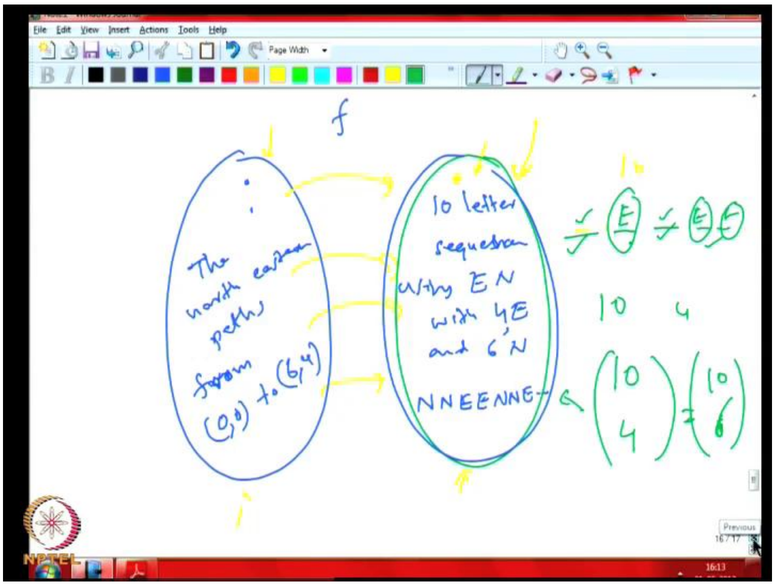


(Refer Slide Time: 30:16)



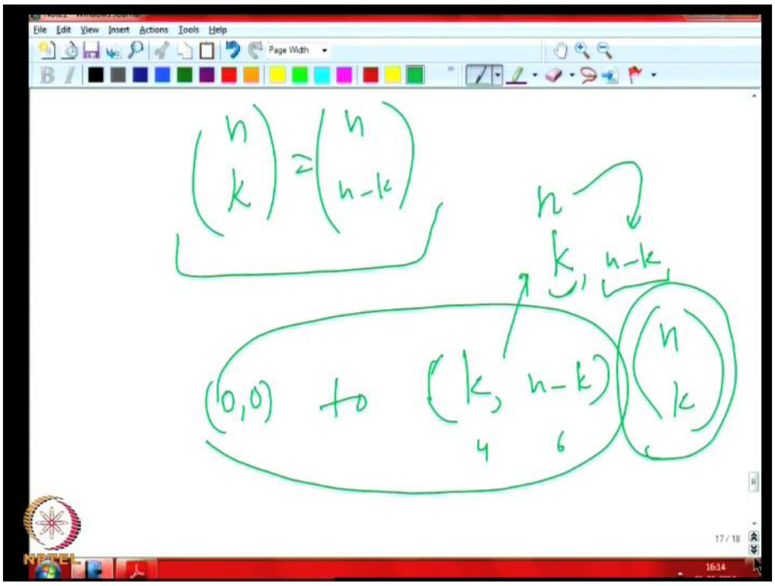
So, we will be able to write down sequence like that and you can clearly see that every path will get a sequence, right and for each path, just one sequence will come. So, it is indeed a function and then we want to verify that is a bijectional. Though, take any sequence. From that given sequence, we can look up the way I have shown the following sequence. So, the car driver can look at the sequence and decide the route and the route is unique, right and therefore, there is one path according to one sequence and there is indeed one sequence because you know four steps if he takes because in the sequence, there are exactly 4 E’s. If he takes whichever what step, it does not have to worry. If he takes 4 eastern steps and then 6 northern steps, he has to reach this four N because out of ten units, he has moved already 4 kilometres northward and 6 kilometres southward.

(Refer Slide Time: 31:48)



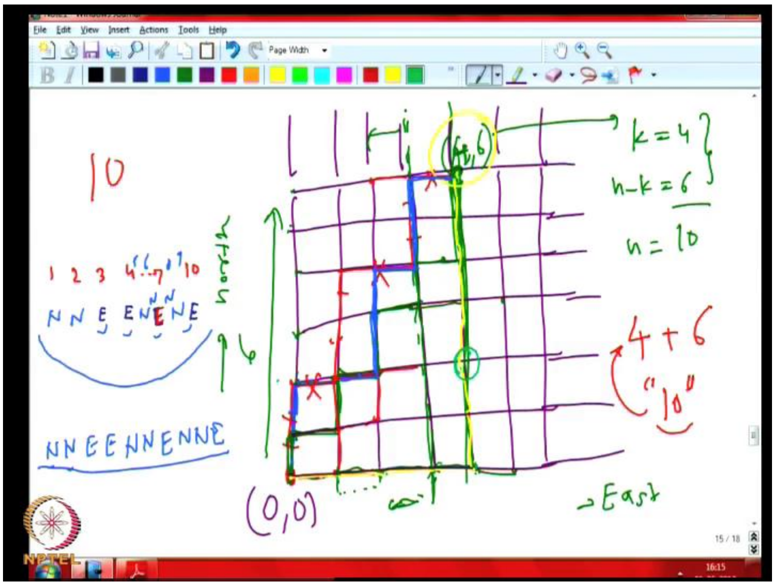
So, therefore, he will be reaching this if he follows that. So, that is indeed a bijection between this and this, right and then we see that this number and this number is same. Now, this number, what is this number? This number has actually there are 10 positions here. There are 10 positions here. So, we have to put east in four positions, right and the remaining position automatically will be filled by N. So, we just try to select the positions to set our E's, fit our E's in, right. So, out of 10 positions, how many? Four positions are to be selected, that is 10 choose 4, right. This number, cardinality of this number will be the set, will be 10 choose 4. So, that is natural 10 choose 4. So, you could also have selected the northern paths. So, that would be a 10 choose 6, right.

(Refer Slide Time: 32:56)



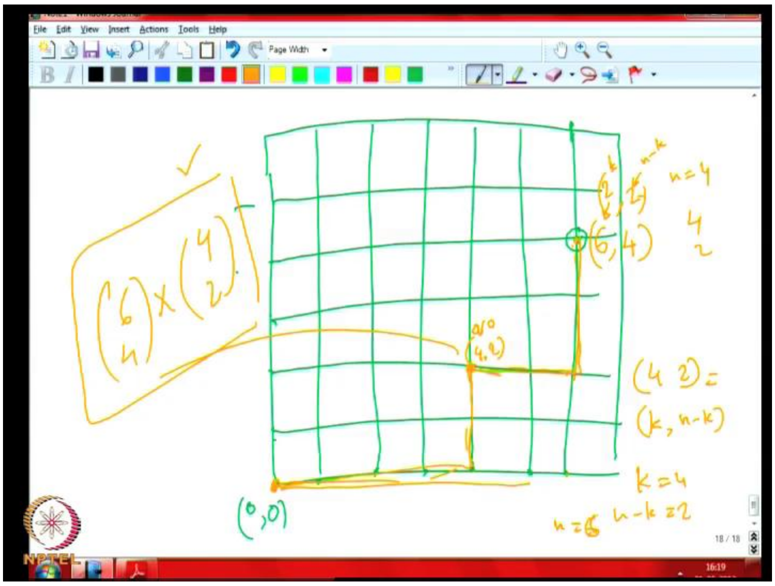
So, we will soon see that these are going to be the same. In another words, we can say suppose for a general problem, suppose if I want to go from 0, 0 to, sorry 0,0 to k n minus k, right. So, in our example, k was 4, this was 6. So, then that means, out of n units, we have to travel k units east and n minus k units north, right. So, it is just a question of fixing k in the sequence of n, that is n choose k. So, the number of paths from 0,0 to this is n is actually n choose k E, right. That is what we just proved. As we notice, though n choose k will be equal to n choose n minus k, then the same sequences can be obtained by fixing n's rather than E's. So, we have to fix n minus k, n. Then that would decide your k positions for fixing E's. Therefore, this is going to be equal.

(Refer Slide Time: 30:16)



Anyway, we will discuss it further. So, again we can ask this question. A little, slightly make it a little more difficult. So, what about the second question? Number of paths from 0,0 to 6,4. If you want to visit 4,2 on the way, right. I will go back to my picture. Here, I want to reach 4,6, but the constraint now is that I want to visit 4, 2 on the way. What is 4, 2? 1, 2, 3, 4, 1, 2, 3, 4, 4, 2. This is the point, ok.

(Refer Slide Time: 35:08)



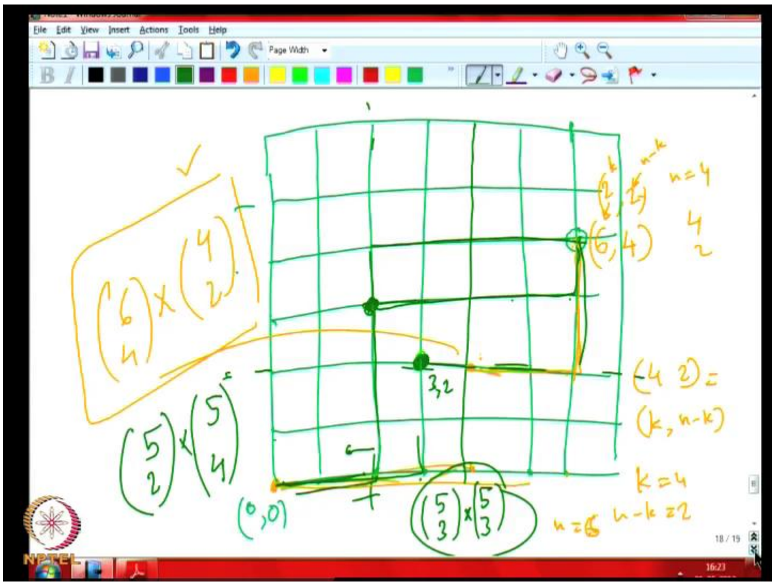
Let say, 2, 4 on the way because may be this looks simple. So, 2, 4 on the way. 2, 4 will be this. May be we will take a different question and then say this time I want to say, so this is 0,0 and see this is 1, 2, 3, 4, 5, 6, right. 1, 2, 3, 4, 5, 6, 7 to reach, sorry 1, 2, 3, 4. So, this is the point I want to reach, say this is actually 6, 4, right. 6, 4 because in the x axis, I see 6 and in the y axis, I see 4, right. Point 6, 4, but now the constraint is that I have to check, I have to hit 4, 2 on the way. So, 4 this and 2 is this. So, I have to drive from here to here, but I have to make sure that I stop here on the way. So, it is very clear that if I take this route, I cannot come back here, right. I already crossed. So, many routes that I will not be able to take.

So, how to handle this question? So, first because it is very clear that I have to hit that. So, first I reach there and then from there, I have to go. Now, I first count how many ways to reach here, right. So, this is 4, 2. So, I can take 4, 2. So, I will try to test k and minus k. So, that means k equal to 4 here, n minus k equal to 2 here. So, that means, what is n? N is equal to 4 plus 2, 6. N is equal to 6, right. Now, there is 6 choose 4 ways that is what we proved before. There are 6 choose 4 ways, routes to reach. From here to here, how many ways I can reach?

Now, this is the same problem because once if I look at I will translate this 4, 2 to 0, 0, so this will become the two units. Sorry, 4, yes 6 will become 2 and then this will become this because there is just two more units here and 4 will become 2, right. So, I just want to reach from 0, 0 to 2, 2, right. So, that is because k equal to 2 here, this is n minus k, so that is n equal to 4, right. Adding together, hence there are four units to cover and after that two units should be eastward, right, 4 chose to east. So, 4 choose to east that to reach from here to here.

We have 6 choose 4 ways and to reach from here to here, we have 4 choose 2 ways. Then by product principal, we can multiply because I can take a route from here. So, any root I can take from here to here which is valid and then combine it with any route which is valid from here to here, right. So, see of course here I argued that you can translate 4, 2, to 0, 0, but it is not necessary. You can notice that from here to here to total from where you have to take 6, sorry 4 more units to reach here. Out of which 2 units are eastwards and 2 units are northward. That is all we need to know, right. So, this is 6 choose 4 and to 4 choose 2, right.

(Refer Slide Time: 39:23)



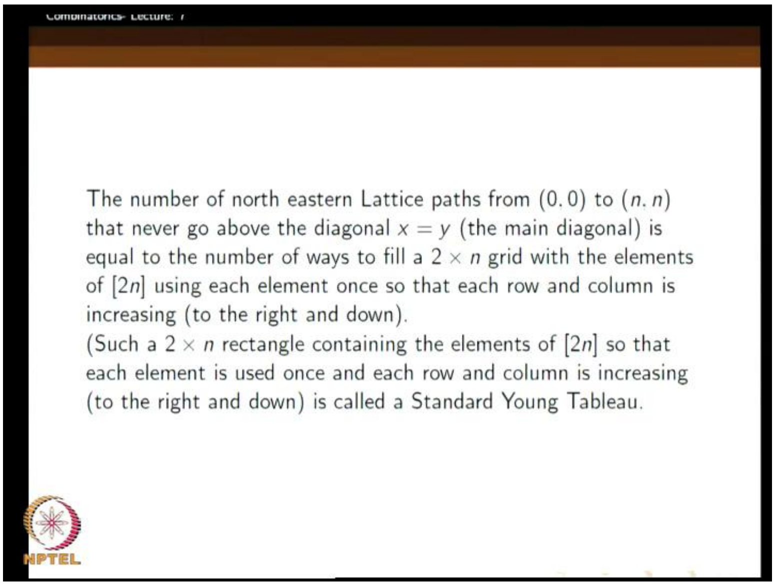
Now, another question here is this number of path from 0, 0 to 6, 4 is such that either it is 3, 2 or 2, 3, 6, 4, same destination. Now, this time the condition is that, so let us say I will remove all these things. So, either we fit 3, 2 or 2, 3. That means, it is not that 2, 3, where is 3, 2? 3, 2 is 1, 2, 3, 3, 2. Here, this is 3, 2. 3, 2 is this and 2, 3 we say this, right. 2, 3 is this. So, I should either touch this point or this point, not both. See, it is not possible to touch both because if you are already here, then there is no way of going back to here, right. Similarly, if you are already here, there is no way of going back between these two points. So, without lost of generality, you reached here first. That means you have already crossed this line, right and you are having hit. This means, you are below this line.

Now, you can try going forward, but you never reach here. Similarly, if you are already here, that means you have crossed this horizontal line here, above the horizontal line, but since we have not reached here, that means you are below this line. So, that means before this line and then you may try how much ever to move towards east, but you will not be able to hit this thing, right. So, therefore, if your car is stopping here, then you will, your car will never stop here, right. So, the paths which are going though this point are disjoined from the set of paths which are going through this point. So, the set of paths which are going through this point is disjoined from the set of paths that are going through this points. Therefore, we can add according to addition principal because they are disjoin sets.

Now, we just need to know how many paths will go through things. Same argument. This was 3, 2. 3,2 means total 3, 2 plus 5 units we have to travel here. Out of that, 2 should be, say northward or 3 should be eastward. So, that is 5 choose 3, right and then from here 3 to 6, 4 right. 3 more units in east and 2 more units. That means, 5 more units he has to travel, right. That is again 5 choose 2 because sorry 5 choose 3 because you know we are in 3. We need to take eastern directions.

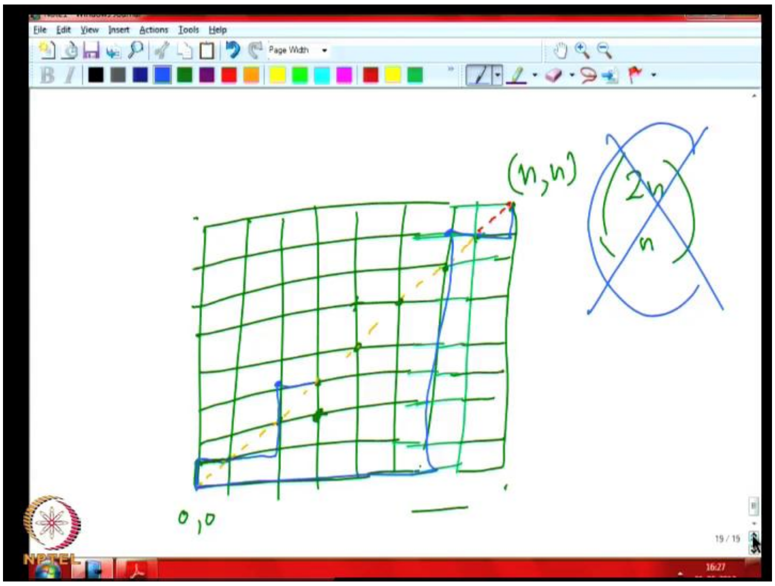
Similarly, here 2, 3 that is 5 unit you have to reach, but other two should be taken in the eastern direction, that is 5 choose 2 to reach here this many ways and from here to here, we have to take total again 1, 2, 3, 4, 5 units. Out of that 5 units, 4 units should be eastern, right. So, this is the number of routes going through this point, right. The number of routes going through this point is this, right. We reach here for somehow and then from here, you can reach here. So, by product principal, you can find the routes which are going through this thing. This corresponds to the last to remember and then through this also. Then by addition principal, you can add them together to get the total number, yes.

(Refer Slide Time: 43:17)



Now, we will look at another question. So, here is a slightly more sophisticated counting. So, this time we want to count the north eastern paths from 0, 0 to n, n, but then we should make sure that these routes our car never go, crosses the x equal to y diagonal, the main diagonal, right.

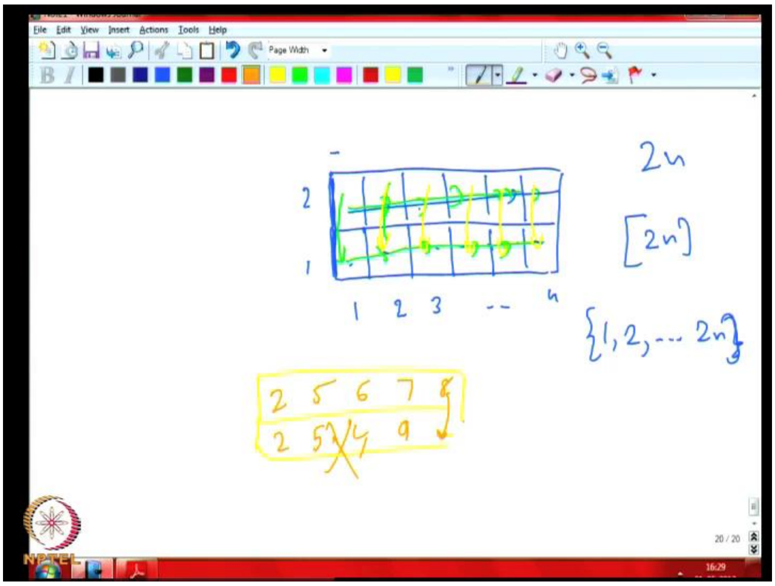
(Refer Slide Time: 43:50)



In other words, so we are suppose to reach from 0, 0 to n, n, right something 0, 0 to n, n. Our main diagonal will correspond to this. So, this is the main diagonal. May be the main diagonal will correspond to this. The n will be here. See, the point is we can tip this diagonal points. We can touch these diagonal points, but we are never allowed to cross, go above it. I mean, a point like this is above the diagonal point, like this is below the diagonal, right. So, we should always be below the diagonal, right.

Now, we want to reach this n, n. So, n, n means where n steps here and n steps here, right. Now, total two n steps we have to travel. If he just want to find out how many paths are there from 0, 0 to n, n, that is n to n choose n, right because we just have to take n eastern steps. Out of that total 2 n, but we are told that we are not allowed to take some routes. Those routes like this for instance, you cannot take a route like this, starting like this, starting like this. This kind of route is not allowed because it is growing above the diagonal or even this route is not allowed, right because here you are already above the diagonal, right. So, that means these many are not available.

(Refer Slide Time: 46:28)

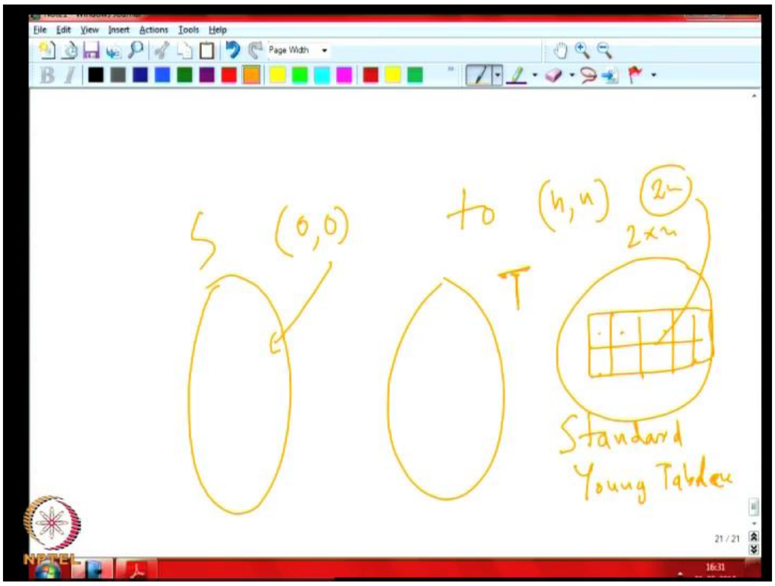


Now, we are going to show that we are not counting, going to count that exact number of routes here. That we will do later, but here we are going to show that this is to illustrate the bijective proof, show that this is equal to counting certain other things, namely something like this. You draw a table like this. So, this is a 2 by n matrix, say 2 by n matrix or 2 by n rectangular array, right. So, there are n of them, 1, 2, 3. Put n here. This is 1, 2. Now, total two n squares are there inside, 2 n squares.

Now, we are supposed to feel these squares using the numbers from this set. That means, the numbers from 1, 2 up to 2, 2 n. You have enough numbers, right. Exactly 2 n numbers, 2n squares. So, nothing should be repeated. Everything should be used exactly once, but when you feel we have to make sure that if you read out these numbers on a row from left to right, that should increase. Similarly, the second row, this row also should increase like this. Similarly, if you read column wise, it should increase downward like this. It should increase like this.

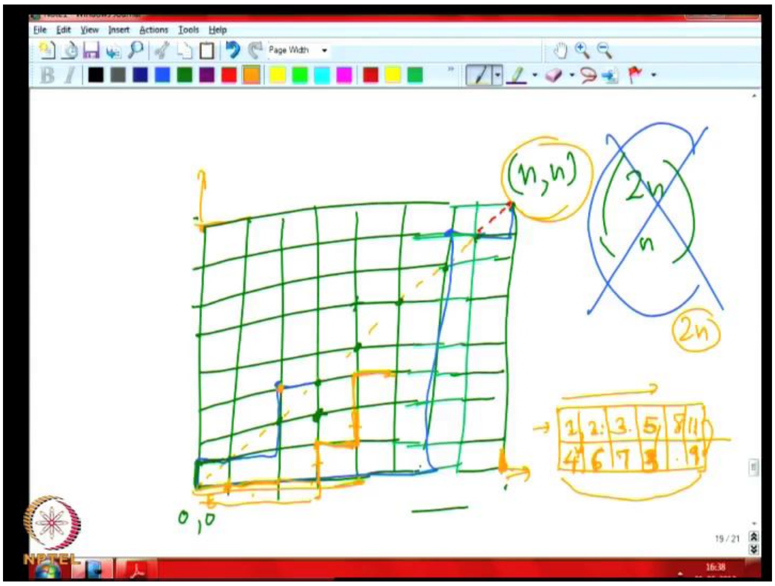
So, for instance, you can fill something like 2, 5, 6, 7, 8, but you cannot fill like 2, 5, 4. This is wrong or you cannot come down. Similarly, here if you have filled 7, then it should be 9 or 10 or something like that here, right. So, it should be a bigger number above, right. So, that is wrong. We ask how many ways you can fill such a rectangle using the numbers from 1 to 2 and using with number exactly once, and such that along the row, the numbers increase as we go from left to right and long column, the number increases from and we go from top to bottom, downward or that is only 2, 2 of them. Therefore, it should increase order. The upper number should be smaller and the lower number should be bigger.

(Refer Slide Time: 49:13)



So, we claim that the number or ways of filling this table is exactly equal to the number of ways of travelling from 0, 0 in that lattice, that grid structure to n, n, without ever crossing the diagram. How do we see this? So, again we construct a bijection between S and T. S being these paths. That means, the paths from 0, 0 to n, n which never crosses the diagonal and T being the number of ways, the number tableau. This is called a tableau. After you fill it, it is a tableau and the way we have asked to be filled. Finally, such a tableau will be called as standard young tableau, right. So, the number of standard young tableau 2 by n, n tableau. So, this will be equal to the number of paths, north eastern paths from 0, 0 to n, n such that it never crosses the main diagonal, never goes above the main diagonal. So, how will you setup this thing?

(Refer Slide Time: 50:43)



One thing is to look and say the north eastern path which never crosses the diagonal and show how to create young tableau from this thing, right. This is easy. What we do is, we as the driver move on, counts the number of strips he keep the track or the step he is doing. So, this is the first step. So, this is the second step, this is third, this is the fourth step, this is the fifth step, sixth, seventh, eighth like that.

The first step he has taken in the eastern direction, so that was an east, right. So, he writes that one here. Second step he has taken in the eastern direction. Whatever he does in the eastern direction, that step number he will write on the upper row. So, 1, 2 and then third step also he had made in the eastern direction. Fourth step, he made in the northern direction that you write here, right and then fifth step here because it was taken in the eastern direction and sixth and seventh are in the northern direction that he will fill here. Six and seven, the lower column, the lower row correspond to north and the step numbers he has taken in the northern direction in the order for in way whichever step he has taken earlier, will be written earlier.

So, this northern direction that means, seven, eighth, right and say like that he will fill. So, you can see that you can fill because the two n steps we will be taking and each step, he is filling either above or below, right and when we have filled, so it is possible that we will try to fill some after filling the entire n squares available in the upper row, he will try to fill one more number there. No, it is never possible because you know if he had entered n numbers in the upper row that means he has already taken n steps in the eastern direction. Now, we cannot, he will not take. He has already reached here, this last point and then he cannot cross that, right. He cannot because if he crosses that, he will never reach n by n, right.

Similarly, the rest of the things will come in. Similarly, if he had filled n, n, all the n squares in the lower direction, it is not possible for him to enter anything more because nothing will come in there because he has already reached this line. Then on other vertically he has gone this many northern direction, he has gone already this distance. Now, if he passes further, then he will never reach n, n, right. So, any valid path will give only numbers in this row and n numbers in this row and total 2 n numbers are there. It will be clearly giving tableau with all distinct numbers appearing distinct and correctly filling, but the only thing and definitely if you in a row, either in the upper row or row we have entered the numbers in the increasing order only. There is no problem there.

The only thing we have to check is, will we have a conflict in the column. That means when we read one column, is it possible that the upper number is smaller than the lower number. We claim that will also never happen because what is this upper number means? That means, a upper number being saying this fifth step, right. In the fifth step, when he has already taken the fifth step and I fright, so when he have already taken the fifth step, now if he is going to see a smaller number here, 3. So, suppose you see 3 only, that means when he wrote 3 here, the mains in third step. So, may be 3 means I have written 3 here that is let us say for a little later stage.

Suppose, I see 11 here and I see only a 9 here because I really want to show that whatever number I see here will be bigger number. Then that means 9 was entered because 9 is an earlier step. This was entered before 11, right, at that time that means at that time, we were definitely above the main diagonal is what we see here, right. So, that is why that increase cannot happen, right. So, what we see is here the young tableau comes because what the two properties of the young tableau indeed happen. That means, the row is increasing and the column wise also, it is increasing. So, for instance I can show this thing by example. So, the way we have written here, the first step, right. So, it is in the eastern direction, right.

So, now when I write the fourth step, it is coming here, right. So, it was taken later, right. Now, it is increasing here. So, it is fourth step is coming here. After several eastern steps, only we are already quite, so we are here, right by the time we first entered here, so 10. Therefore, we will be below the main diagram, right. Similarly, 2. Second, when I am seeing, so I am seeing a 6 here because again one more appeared. We put it here and then the sixth is coming here. That means, we are already here and that is why we are below the main diagonal. Only two steps are going to the northern direction. That is why this is increasing only, right. So, like that we can check. Now, it is time we will explain it further in the next class.