Combinatorics
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Lecture - 6
Elementary concepts; Binomial theorem;
Bijective proofs - Part (1)

Welcome to the sixth lecture of combinatorics. So, we continue with elementary concepts.

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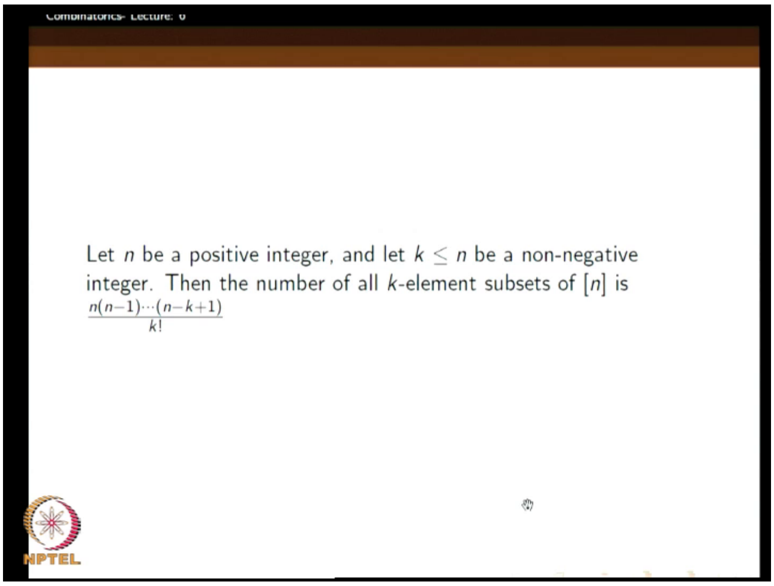

In the last class we were considering this problem how many k elements subsets are there for an n element set, right?

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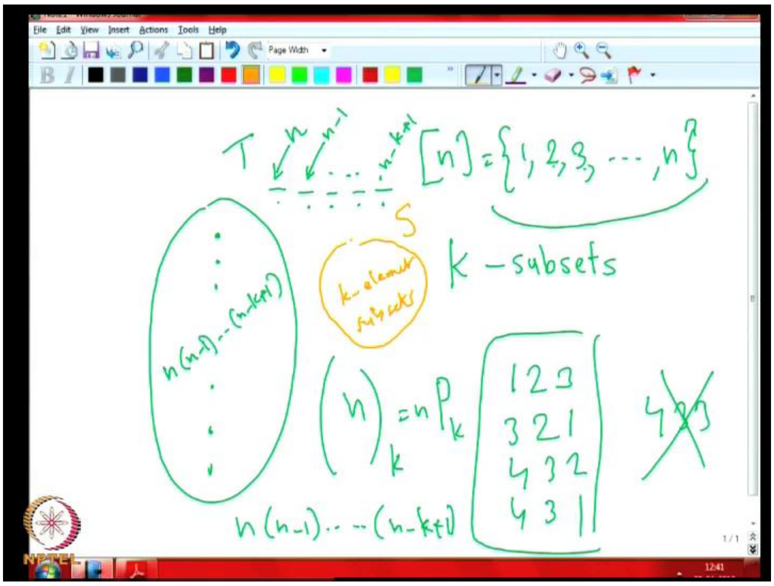

Here the n element set let us write as n equal to 1, 2, 3 up to n. And what we are interested in is the k subsets of it, k elements subsets of it I mean a subsets which has which contains k elements out of these n elements. How many we can form. So, our method was to consider two sets, one set we are already familiar namely the set of this is called T.

The set of all partial list from this 1, 2, 3 up to n containing k elements means, the partial lists means, we want members of the lists to come from this set at the same time the members should not repeat. For instance three element lists can be 1, 2, 3; 3, 2, 1 or 4, 3, 2 or 4, 3, 1, but never 4, 3, 3. So, there are we have already seen that we can get n k this many number of or it is also written as n P k this many number of such things, right?

So that is we so that if we want to make a list of k elements we can put the positions for the list k the first position is this, second position is this, third position is this, fourth position is this and this position can be filled by any of the n elements here in this 1, 2, 3 up to n. And this position once you have used a number here from this thing so we have only n minus 1 available things, and like that by the time you reach here we have only n minus k plus 1 available things.

So this number we have seen that this n P k is equal to n into n minus 1 into up to n minus k plus 1. And so this many number of there is n into n minus 1 into see n minus k plus 1 members are there in this thing, these are not points these are each of them is a list. Now we say that we consider the thing we want to count, the set S what is that the k element subsets of n subsets of n and we told we will construct a function from T to S such that it is a it is a, so called d to 1 function, where d is equal to k factorial this is what our plan is, so again we will draw a bigger figure.

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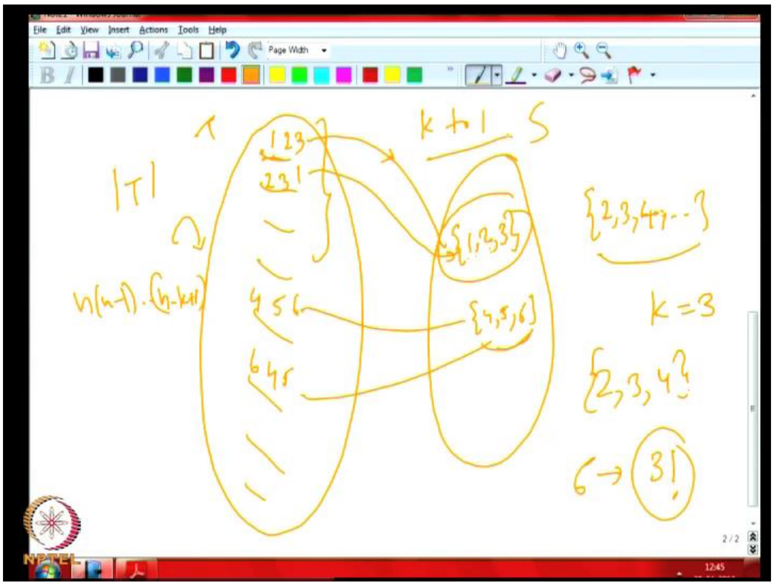

So this is our T here we have our partial lists. So this is each list is written here see, so partial so each list is k elements it is taken from the n 1 to 1, 2, 3 up to n, right? And the k lists formed from so the number is n into n minus 1 into n minus k plus 1. This many are there in this thing that is the cardinality of T. Now our set S this is the subsets, which have k elements in it, which are formed from 1, 2 up to n. Now give me one k element subsets any may be this 2, 3 something, some members here right. 2, 3 up to say 2, 3, 4 some k elements are here.

So, for instance, we can take a value for k, k equal to 3. So I will take 2, 3, 4 as a subset of 1, 2, 3 up to n. Now, from this thing the function is constructed like this if I see a list here say 1, 2, 3 I will map it to the set 1, 2, 3. So, this is different for this is a list, but this is a subset 1, 2, 3 here there is no order. So the mapping comes like this is mapped to this thing and also note that 2, 3, 1 this is a another member here, that also we get mapped to this because that is underlined set of numbers are 1, 2, 3 again.

So on the other hand this 4, 5, 6 is a list here this will map to another sets with 4, 5, 6. Say for instance 6, 4, 5 will also map to this. Now you see that this is indeed a function because each list gets mapped to a specific member here it does not get two different choices, it is clear that the member elements which are present there only that set only it gets mapped and it is a k element set.

Right because the k lists k element nothing is repeated here, there are indeed k things here. Now of case many things are getting mapped to this, the question is how many things will get mapped to the same thing for instance, this 1, 2, 3. How many things on the T side lists on the T side will get mapped to it, of case all the lists which can be formed using this 1, 2 and 3 will get mapped to this right there are six of them.

How do we get it? We just arrange all the three factorial consider all the three factorial lists we can so, permutations of 1, 2 and 3 those things will get mapped to this only and only to this. And we can say that every three elements set here will get mapped will have a pre-image. So we have six pre image, right? So just so therefore, it is a k to 1 function right d is equal to k here.

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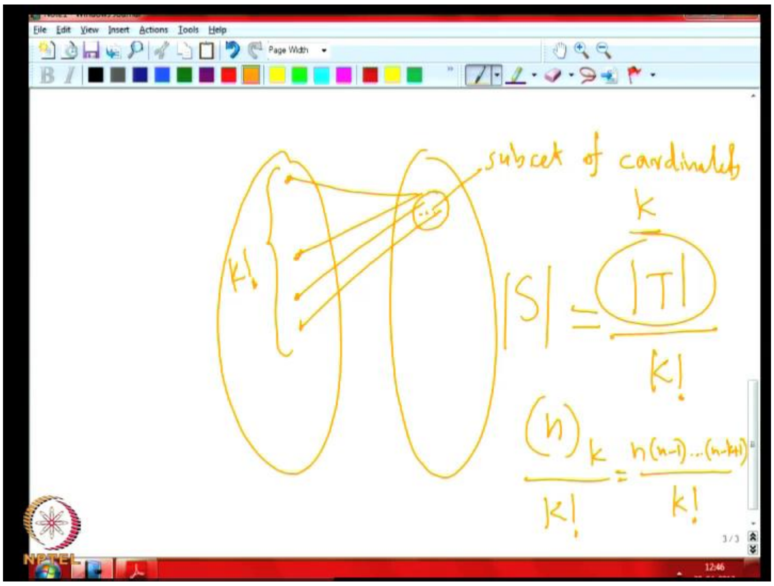

So therefore, here we can say that for each subset this is a subset of cardinality k. So this is right these things are getting mapped to this, the lists are the how k factorial of them will get mapped because this is a k element subset, k factorial permutations are there for the members here right each of those k factorial lists will get mapped here. Nothing else will get mapped here mapped it and each of this things will get mapped to this only. So this is such a function and by our division principle. Now, we can say that the cardinality of this set S right the cardinality of S is equal to cardinality of T divided by k factorial.

And we know the cardinality of T. What is that? This the all possible k lists partial lists that we can form using the members of 1, 2, 3 up to n without any of them repeating. That is n k divided by k factorial that is n into n minus 1 into n minus k plus 1 divided by k factorial, right? So this is exactly the number of subsets that we can form, out of the members of 1, 2, 3 up to n and which consists of only k elements the cardinality k subsets.

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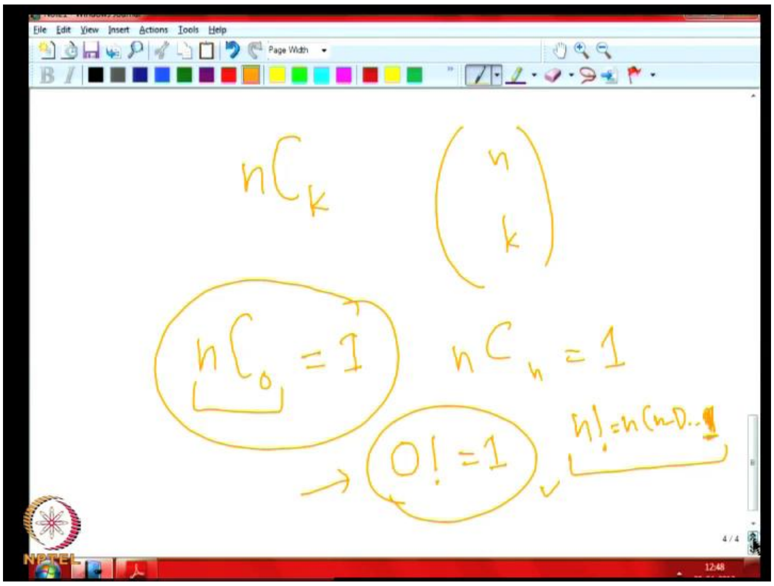

And this number is written like this n C k or another notation is n C k, this is the most important number in combinatorics of case n C k. So we can easily see that n C 0 is equal to 1. Why is it so? Because the zero elements subsets are only 1. So that so because we have to also write, so not that n C n equal to 1 right because only one subset can be formed out of n elements set, for not that I think when we define permutations we should have told that 0 factorial is by definition 1, though it does not make sense because what. So, usually we write n factorial is equal to n into n minus 1 into up to n 1. So, what do you mean by 0 factorial you just say that we define it like that to make things convenient. So, this is for all values of n greater than one greater than equal to 1.

This will make sense, this definition will make sense. So, then so then if you wanted to derive this n C 0 using the method earlier. So what is the number of zero element list from n element set, we should just say it is 1. So I do not know it is it is by definition or otherwise we should take it separately. So n C 0 is equal to 1 because zero element set we know that it is a empty set, the earlier method is telling us that.

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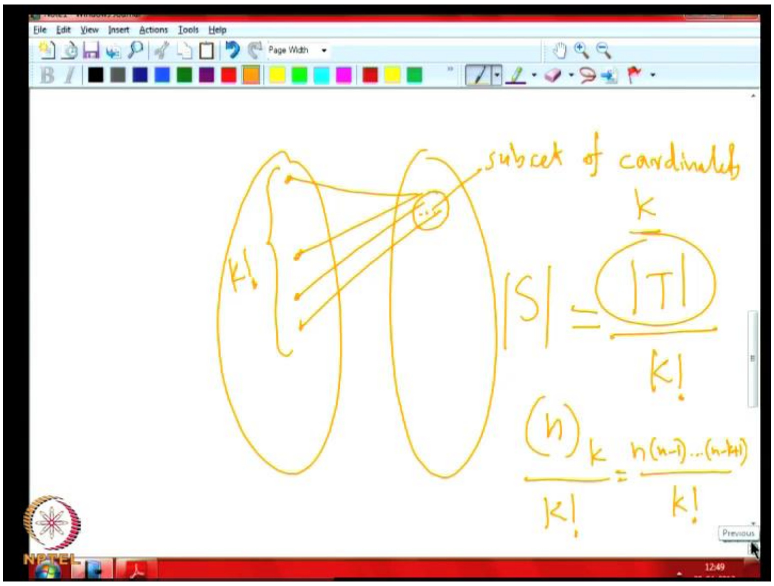

So, here we will list all the k partial lists so there are what zero partial lists right so that is an empty list. So it does not really make sense there, so what so you do not have to rely on that proof here. So we can just note that n C 0 is equal to 1 because we have an empty subset right. So that is a zero cardinality subset from this thing. So n C n is the full subsets, this definitely make sense because n factorial, y n factorial lists what that formula gives that gives us 1. So for all all all other things we do not have factorial.

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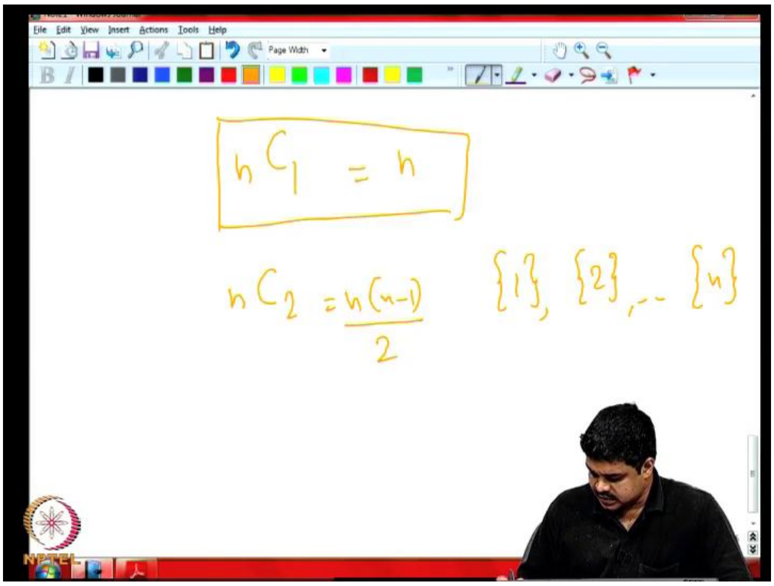

So we can easily note that n C 1 is equal to see that n into n minus 1 plus 1 is n right divided by 1 factorial. So this is n. So this is also easy to see how many one element subsets are there; that mean, how many single term subsets are there for n. So that is definitely subsets 1, 2 up to n. Now how many so then what is n choose 2, n choose 2 into n minus 1 that is the upper term 2 factorial is 2 n into n minus 1 by 2. So this is also easy to see it is just way of picking up two elements from n things. So for instance we can ask.

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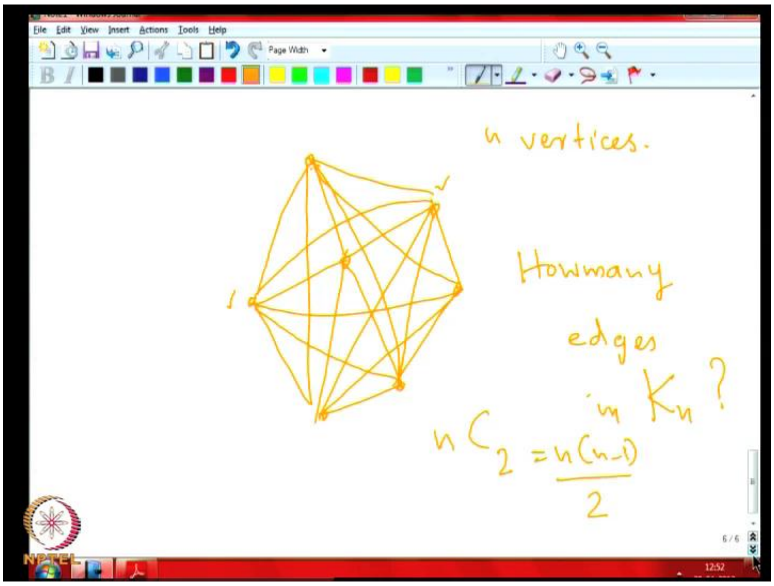

So, given a graph on n vertices n vertices and we are considering simple graph, no multiple multi-edges, no self loops. Now suppose, it is a complete graph, complete graph means all pairs are connected by some edge yes I mean if there are two different vertices there is an edge between them. This is the end of complete graph we drew right and then this I think I have drawn everything.

So, all possible edges are there; that means, if two vertices are there right which are distinct then we have we put an edge between them. So, how many edges can be there in this graph. This is the complete graph of n vertices. How many edges? This is also denoted by k n. So this is n choose 2 because we can just count it by asking how many pairs are there, how many two elements of subsets or vertices are there here, there are n vertices and number of two elements subsets are n choose 2 that is n into n minus 1 by 2. So, you can count it in different ways also, but just to illustrate one of this places where n C 2 comes and n choose 2 comes. So that is about n C 2.

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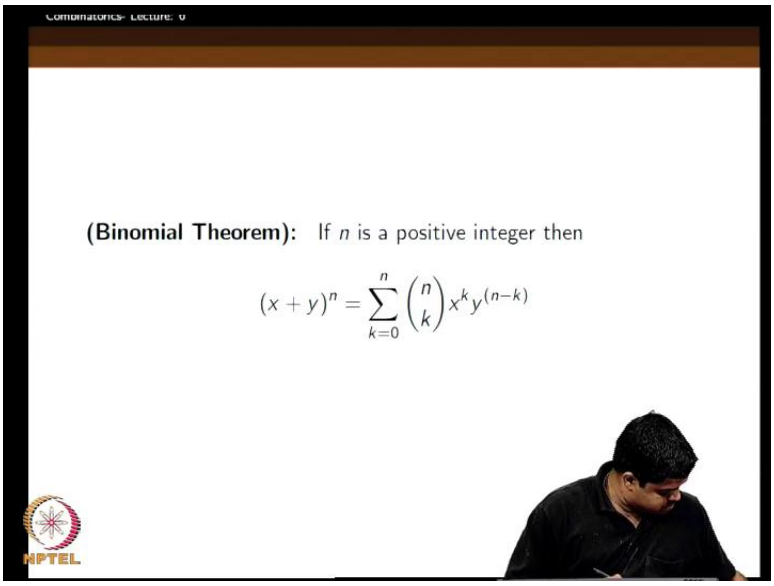

Now we mention one important theorem called binomial theorem. This is mentioned like this.

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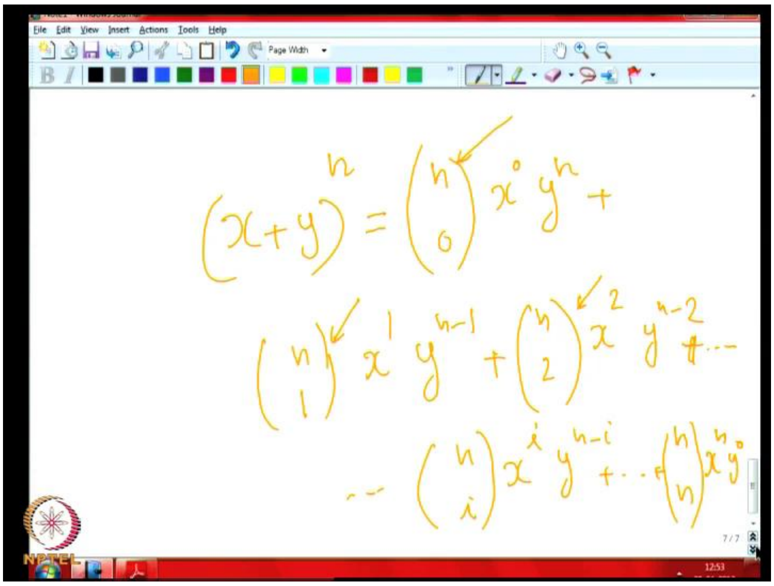

So x plus y raise to n, here n we are considering as a positive integer. So is equal to n choose 0 into x raise to 0 into y raise to n plus then n choose 1 into x raise to 1 into y raise to n minus 1 plus n choose 2 into x raise to 2 into y raise to n minus 2 and plus and so on, till n choose. A typical term will be n choose i into x raise to i into y raise to n minus i plus last term will be n choose n into this will be x raise to n and y raise to 0. This is the binomial theorem. So here this is a formula which involves this combinatorial coefficients and n C 0, n C 1, n C 2 etcetera. So how do we derive this thing?

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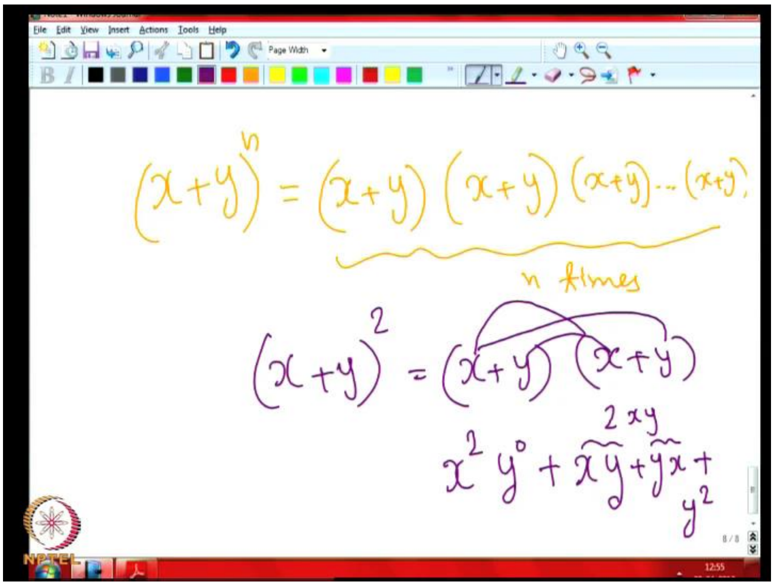

So the this is easy to see, see x plus y raise to n is actually what? That is x plus y into x plus y into x plus y like that n times right x plus y this is this is coming n times here. That is what right now of case, so we can we want to expand this thing, what all kinds of terms will come. For instance we can take a small example n equal to 2. So, x plus y square will be equal to x plus y into x plus y, so we multiply this thing x square. So this is x square into y raise to 0.

Say for instance if I just say what power of each term we will write in terms of both x and y. When y is not there it is y raise to 0 that is 1. So this is x square, so this corresponds to x y this corresponds to y x, y x and x y are same because of because we are thing it is commutative of cases x y and y x should be considered same so plus y square. So this will make 2 x y, this is the way we write.

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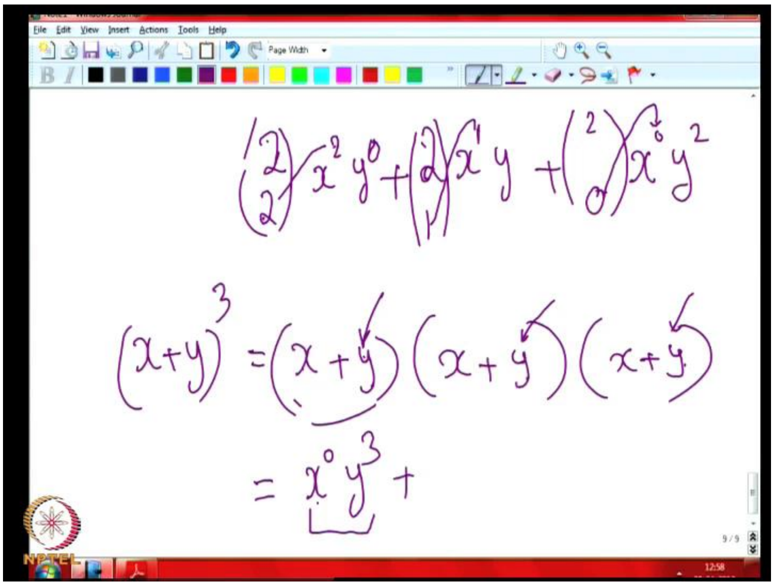

So this is we have x square y square y, y raise to 0 which is essentially only once that is we can see that that is 2 choose 0 plus. And then next is x y that is two times that is 2 times that is 2 choose 1 2 times is 2 choose 1 plus then next is x raise to 0 y square that is again 2 choose 2 of case. I should have written the other way because this is the way I put it 2 choose 2 here because this is this corresponds to this, and this corresponds to the power of x here and this corresponds to the sorry this should be 0 0 this corresponds to the power of x 0. So similarly, so let us say x plus y whole cube, this is equal to x plus y into x plus y into x plus y.

So, now this is see for instance you can consider the first term as x raise to 0 term. That means, all this we can take one y from here because for every term here, so this x plus y term will contribute to the product. So here it is y I can take and here also I can take y, here also I can take y. So, that will make x raise to 0 because I am not taking x anywhere, so that is x raise to 0 into y cube.

See now what is so now the question is somewhere for an if I start expanding like this will this term repeat again? We can say that it would not repeat again because this y has come from here, y has come from here, y has come from here for y cube to come there is no other way y has to be contributed from each of these case then only we can get a this y cube x raise to 0 y cube. So therefore, it will come only one way so therefore, that corresponds to yeah. So, for instance in other in another way what we can say is see x, x should be contributed by none here right. So that means, how many of these things.

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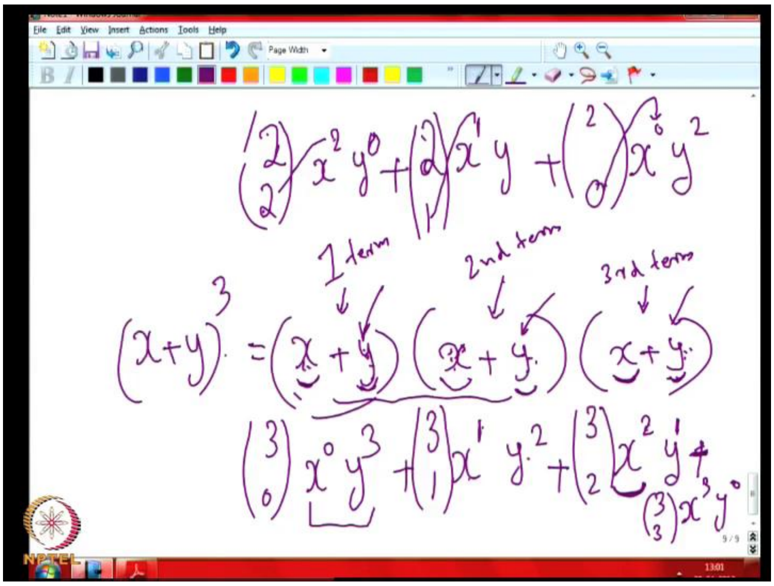

Say this is the first term here, first term, this is the second term here, this one, this is the third term here, out of this three terms how many terms contributed x? 0 terms contributed. How can you pick up 0 number of terms from three possible terms that is 3 choose 0 that is 3 choose 0 that is 1, only one possibility that is what and now the next thing we consider is when x is 1. So then y has to be 2 right, y is it because I can take x from here then here and here I can take a y x into y into y that will make a x y square, but then this term x raise to 1 into y square can come in different ways.

For instance, I could have decided to take x from here and y from here and here. So y x y that would also have given this term another way of taking getting this term is to pick x from here, and I could have taken a y from here and here that is y square into x same thing. Three different ways of doing it, why? Because x could be selected out of this three different terms, which term can contribute this x raise to 1.

So there I can pick up any one out of this three terms to contribute this x. So that is 3 choose 1, right? 3 choose 1 into x raise to 1 into x square. Similarly if I am looking for an x square then there is should be y why is it so? Because the total see x will come from two of this three terms and the remaining term. That means, the third term has to contribute y that is x square plus y because this some terms contribute x, some terms contribute y, each term has to contribute either x or y and never both. So, therefore, the total contribution has to be the number of terms here it is 3.

So, 3 2 plus 1 here right and the question is which all terms are contributing x? Two terms have to contributing x. How do we select those two terms from the possible three terms, 3 choose 2 is the number of possible ways you could take either this two to contribute x and this to contribute y. The first two to contribute x and the third to contribute y or you could select a the these two this and this to contribute x, and this to contribute y. Or you could have selected this and this to contribute x and this to contribute y right there are three choice to which equal to three possible ways.

And now similarly, they there is yeah so finally, we can finally, we can finally, we can see the last term that is the x cube term right x cube y raise to 0. That can only be selected in one way because each of these terms have to contribute x, three terms out of three right. That is 3 choose 2. So 3 choose 3 into x cube into y square. This is the way it is expanded. So this is I just took a small example to illustrate the point right now I can go to the general case, which we wrote earlier namely this one there are x plus y raise to n equal to this thing. Right now how will I find out the answer?

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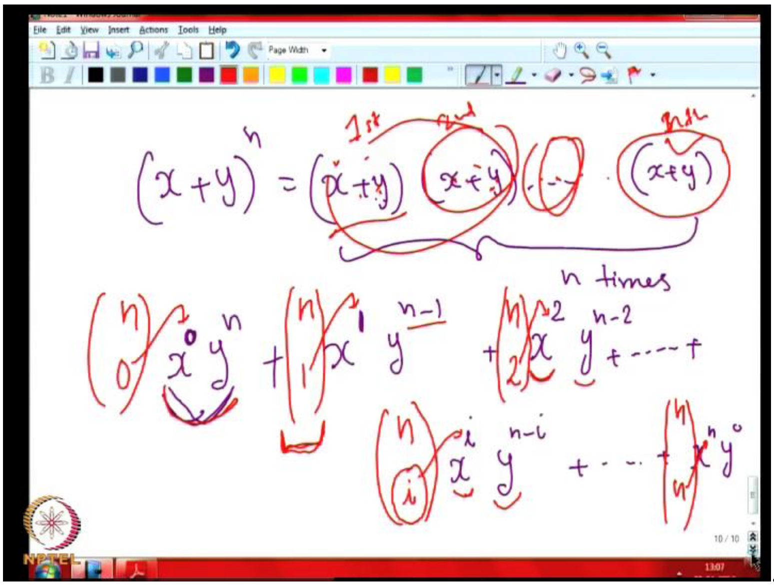

So of case so this is general term x plus y raise to n equal to x plus y into x plus y into. So, n times it will come x plus y. It is total n times n times. Now what all possible terms can be there is the first question, I we will arrange it like this there we can have we can have say x raise to 0, y raise to n because see each term has to contribute either n x or a y to the product when you expand it of case, and they if no term is contributing x then only x raise to 0 will come right is contributing x to this. So to I am thinking of forming a particular term x raise to 0. So every the every other term has to every term has to contribute to y. So therefore, y raise to n will come, right?

Now, suppose one of these things contribute x and the remaining things contribute y, this is this kind of terms. Now this is two of these things contribute x and the remaining things contribute y, this is this kind of things x square y raise to n minus 2 and so on. And in general see suppose i of the terms contribute x and n minus i of the terms contribute remaining n minus i terms contribute y.

So this is this kind of terms and finally, we have all the terms contribute x that is x raise to n y raise to 0 this kind of. The thing is several terms can be formed of certain forms but, look at this one, this term comes only once because x is contributed by no term that can happen only in one way because no terms. So everything is y so right so that is only one so that we capture by saying that it is n choose 0. So x is contributed by 0 terms this 0 corresponds to this 1.

Now, here x is contributed by exactly one term in this and all other terms are contributing y. So, this will out of the n term one term is contributing x, so this 1 correspond to this, you know when you take the product there it we will be like x into y into y into y into like that such a term. So, now y into x into y into y into y into such a another term will come. So then here y into y into x into y into y into such a term will come, so we have to add them all together. So just that we count and that count is put as the coefficient, how many times that x raise to 1 into y raise to n minus 1. So that is that is only we are doing. That is n choose 1.

Similarly, here because the question is x square and the remaining are so the which two terms will contribute x square is the question, out of the n possible terms here n possible terms here we take two and we say that those two are contributing x square this two correspond to this. And then the remaining terms are contributing can be this two terms or else can be this term and some term here right any two term those two terms are contributing here. In general i terms are contributing x. So that is n choose i right into x raise to y into y raise to n minus i. So here we choose n because all the n terms are contributing n. So this is what so this i correspondence right

So this is the binomial theorem formula. So you see here we have crucially used that is notion of n choose i, what is n choose i? We are saying that here instead of we always were saying that n element set, which are the n element set. So this is a first term, second term. So this is the way we consider third nth term that way we are saying that this is one, this is second, this is third, this is fourth though it is all written x plus y. But this order we see that this is these are different objects right this are first object, second object, this is the first object, this is the second object, this is the third object and this is the nth object like that.

And in that sense we can say that we are selecting which objects correspond to this i x raise to i; that means, those x's i x's are coming from which objects. That means, your selecting the i objects out of n objects right just to get this x raise to i because once you select the objects which give x, the remaining things will automatically give y. That is why x raise to i always comes in x raise to i into y raise to n minus i it is not possible that x raise to y into another power of y comes right because the remaining all terms has to contribute y.

So therefore, and then when we add up to see the coefficient of that we have to see how many times that has occurred, that correspond to how many ways we can take n i set out of an n elements set that is the point right. So therefore, it is a binomial theorem make use of this combinatorial coefficient it is written in terms of combinatorial coefficients. And now once we talk about binomial theorem so, it also gives us some...

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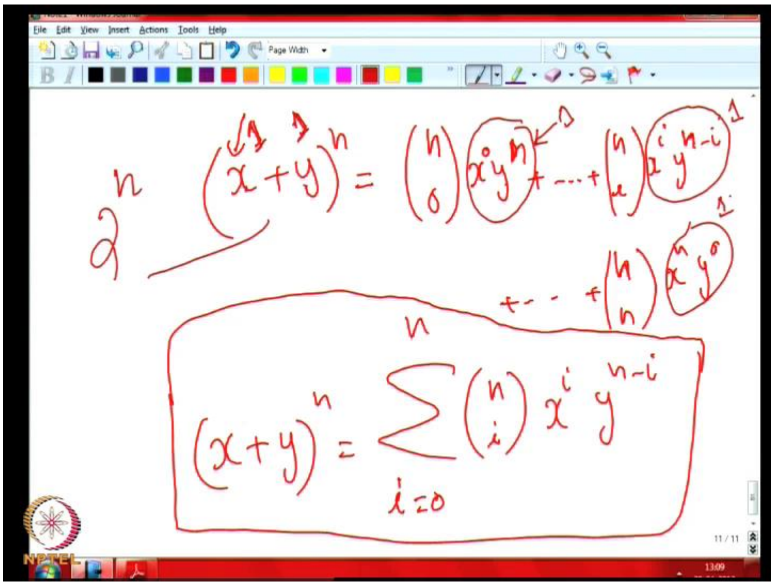

Because this in this equation x and y are variables, we can put some values for this thing, right? this is the e root n choose 0 into x raise to 0 plus y raise to 0 y raise to n plus right n choose i into x raise to i y raise to n minus i and finally, n choose n x raise to n y raise to 0.

If you want more compact way of representing this so writhingly like this you can write like this. So i equal to 0 to n n choose i x raise to i into y raise to n minus i this way. This is what we first wrote right. So now we see we can try substituting some values for x and y. So what if i put x equal to 1 and y equal to 1. So what if I put x equal to 1 and y equal to 1 what will happen? So, then this will become this, this will become 2 to the power n because x plus y 1 plus 1 is equal to. And what will this become here you put x equal to 1 y equal to 1 this will become 1 this will become 1 and here, each term like this will become 1 because x raise to i is now 1 raise to i that is 1 only y raise to means, 1 raise to n minus i is again 1. So all of them are ones right now this will become just this, right?

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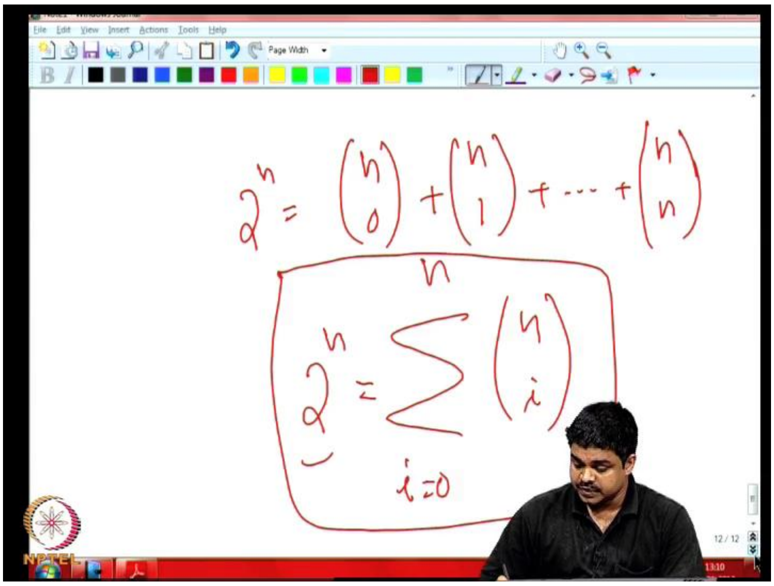

So, we get 2 raise to n is equal to n choose 0 because that x and y's are disappeared now, n choose 1 plus n choose n so or in short form we can write 2 raise to n equal to sigma i equal to 0 to n, n choose i, right? This is we got a relation between the the the summing up the combinatorial coefficients, we are fixing i and fixing n and varying i from 0 to n this will give us 2 raise to n as what it says. So the way we can interpret is it is that the number of subsets number of subsets of n number of subsets of n is 2 raise to n. How do we see that?

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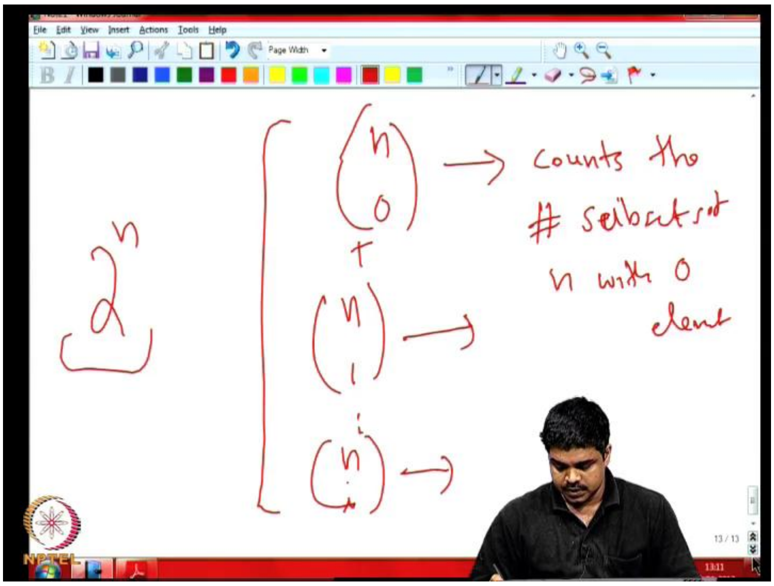

Because what is this n choose 0, this is the empty the counts the empty subsets, counts the number of subsets of n with 0 elements. So, n choose 1 as we already mentioned it counts the single ton sets. That means, the number of subsets of n with just one element. Similarly, n choose i counts the number of subsets with exactly i elements in it

So, now if you sum up all these things up to n we are actually counting all the possible sets because any subset has to be has to have either zero element, or one element or two element or n element right. So therefore, this number is the total number of subsets of an n element set, and we are seeing that that is 2 raise to n by binomial theorem right binomial theorem says that, we can again discuss these things later because different proofs, different comparisons, different strategies we can see. So I think we already discussed it once this number of subsets. So very basic thing before it will keep coming again and again.

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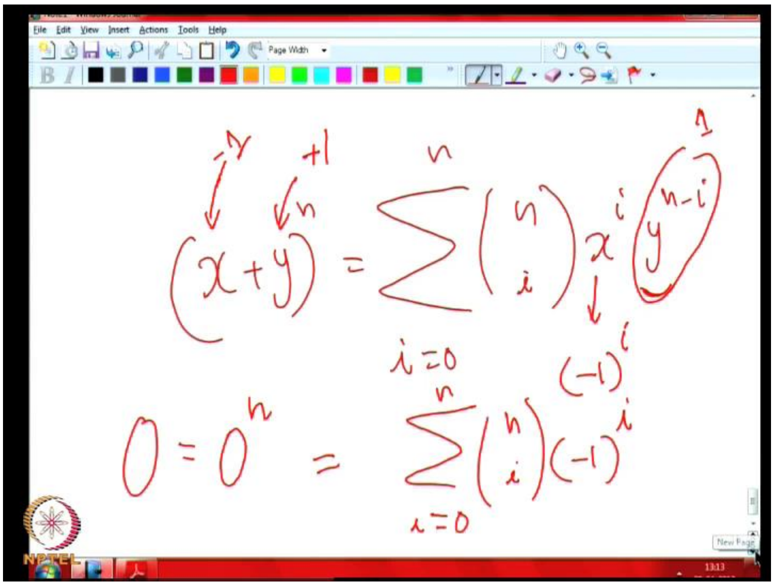

Now so coming back to the binomial theorem namely x plus y raise to n is equal to sigma i equal to 0 to n n choose i this is the binomial theorem. Sorry x raise to i into y raise to n minus i. Now we can think of putting x equal to 1 say assigning x equal to 1 and y equal to minus 1 what will happen? So then this will become 0 raise to n which is 0 right here because 1 minus 1 is 0 that will become i equal to 0 to n n choose i. But, here what will happen because whenever we have a initially this i equal to 0 case, here we are getting yeah because we start with this thing.

For conveniences let us take this as minus 1 and this as plus 1. So therefore, here we can because this will not contribute any because plus y is plus 1 plus 1 raise to this thing will remain as 1 right always this will be 1, but this case so x raise to i that will become minus 1 raise to i right. This will become so therefore, this will be written as i equal to 0 to n n choose i into minus 1 raise to i this is what we will get. So if you write it in detail.

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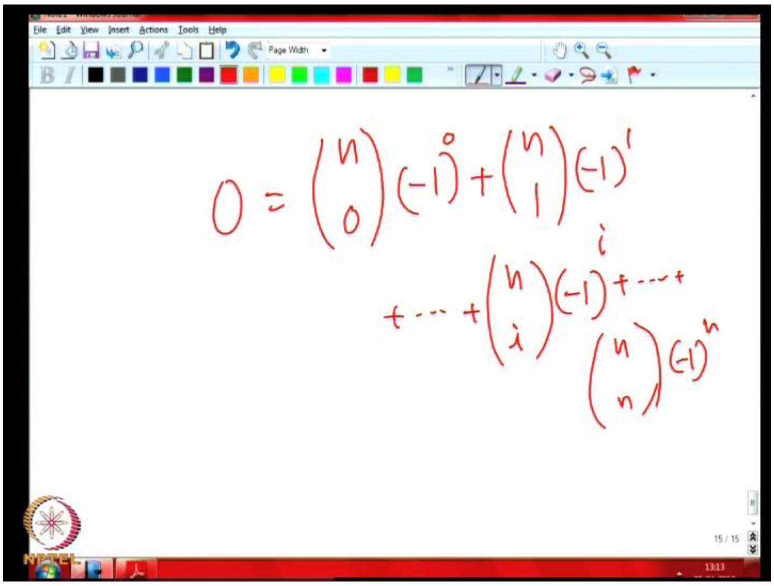

We are getting 0 equal to a n choose 0 into minus 1 raise to 0 plus n choose 1 into minus 1 raise to 1 plus. So, finally, n choose i into minus 1 raise to i plus n choose n into minus 1 raise to n, right? So this is a what does it mean? But, this if you look at this term this is just 1 right we can just delete it. So that is just 1 on the other hand this thing is minus 1 raise 1 we cannot delete it, but we can rather remove it from here and make this a minus here minus 1 raise to n is minus 1.

And similarly, here this depending on whether this even number of pose number or this thing for in a number or odd number we can have plus or minus sign. So this will be again choose 2 here and next will be n choose 3 and so on until we reach here then I we do not know whether it is odd or even. If it was even this number we can remove this thing and put it plus only here this can this will remain plus only, but if it is odd but if i is odd then we will remove this thing and we can put minus here minus here like that right.

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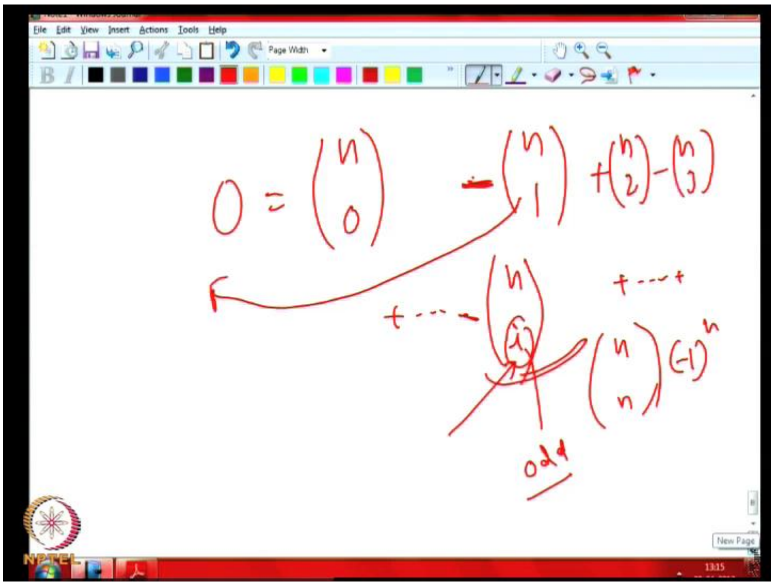

So therefore, we will get something like this so we can say that because this is 0 all the negative terms can be taken to this side. So negative terms are those n choose i where i is odd right negative terms corresponds to those n choose i's where i's are odd therefore, what we get is something like this.

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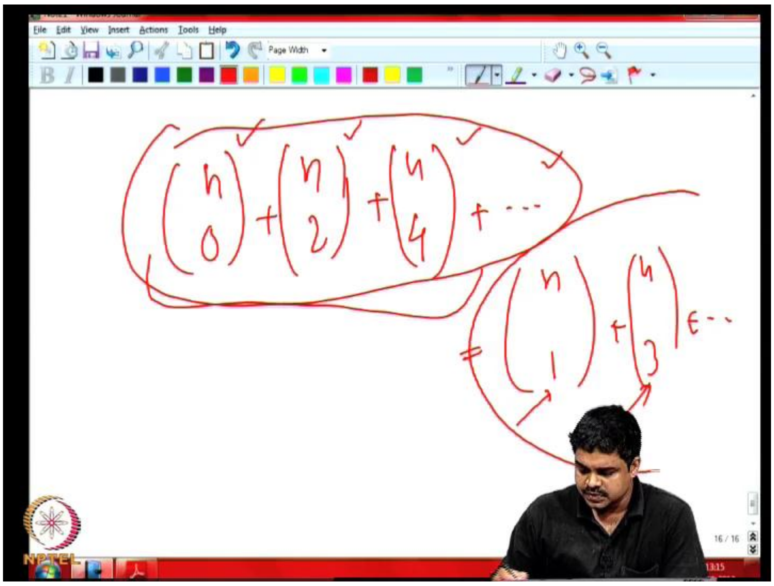

n choose 0 plus n choose 2 that is even numbers I am collecting these are all positive signs n choose 4 plus like this. So, and then this will be equal to n choose 1 plus n choose 3 plus. So the odd that the where this i's are odd things we can take it to the one side when our i's are even things we can there going to be equal or in the terms of subsets what we are saying is this counts the number of subsets whose cardinality is even, even cardinality subsets.

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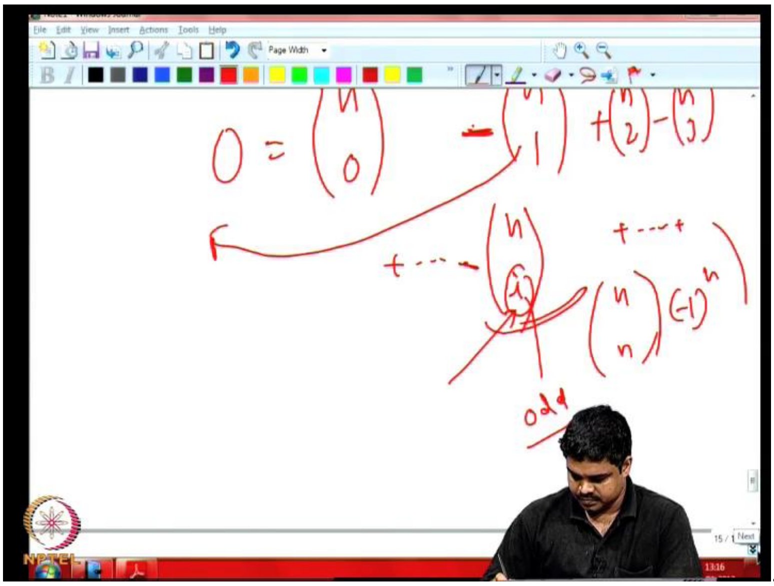

This one and this counts the number of subsets of odd cardinality right. So we are saying that they are same right, the number is same.

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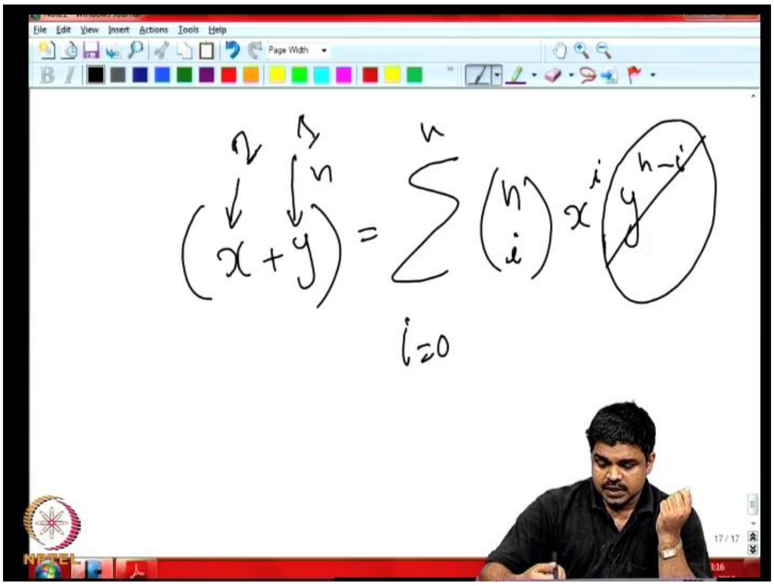

Again we can see other fancy relations also coming out of this binomial theorem. For instance x plus y raise to n equal to yeah, this was the formula right n choose i x raise to i into y raise to n minus i where i is going from 0 to n. So what can happen if I put x equal to 2 and y equal to 1 so what can happen? So these terms will go away then because y raise to n minus 1 is 1 raise to n minus sorry y raise to n minus i so it is 1 raise to n minus i always this will be 1 so this would not be there. So then it will be totally like this so this will go away and here this x raise to x will be 2 right so it will become 2 raise to i.

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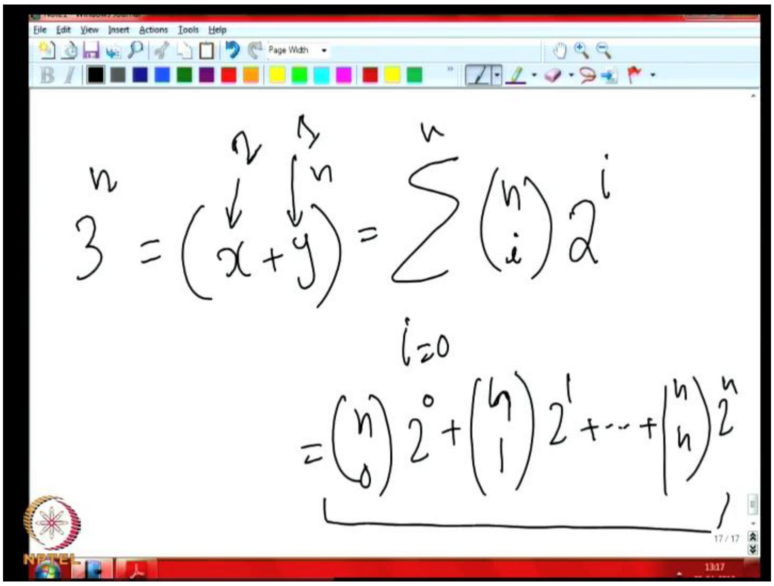

So this will become something like this if I put it here this will be 3 because 2 plus 1 is 3, 3 raise to n here 3 raise to n is equal to i equal to 0 to n, n choose i into 2 raise to i that is something like this n choose 0 into 2 raise to 0 plus n choose 1 into 2 raise to 1 and so on up to n choose n into 2 raise to n interesting. If you add up these things so you should get 3 raise to n as what exits. So we can produce such interesting equalities here using this binomial may come very useful sometimes.

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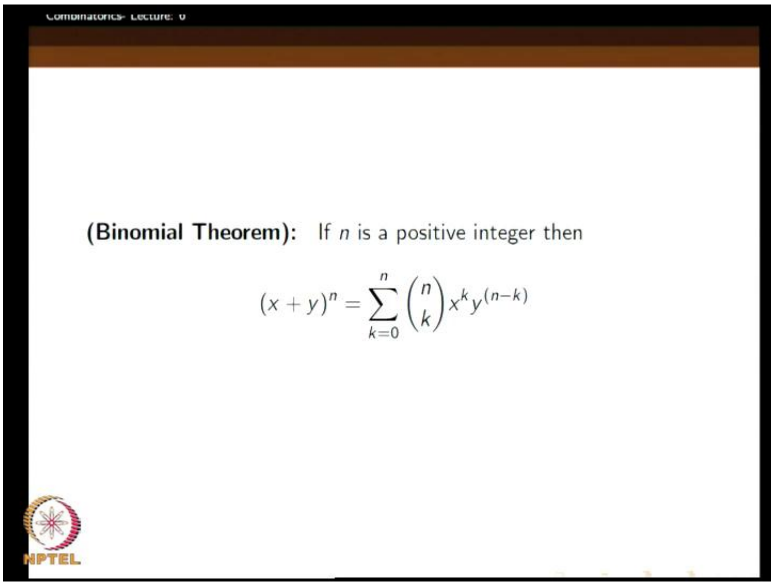

So, we will move on to the next topic next idea. So which is so we have to these are very important thing we have to remember that.

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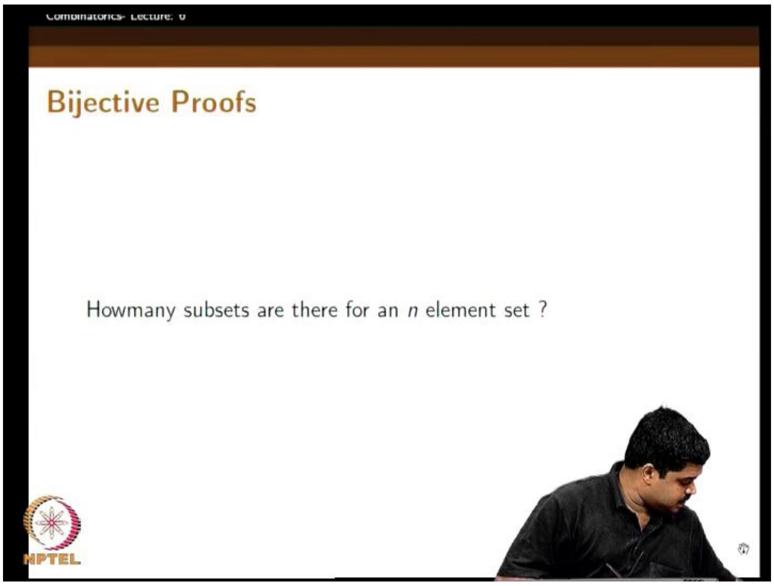

The next notion for instance coming to the combinatorics. So, one of the main things is to count how many are there, how many objects of second type are there. So, there are several techniques we use to do these things. So what we have learned up to now is some elementary ideas so initially, we were discussing about several principles like addition principle, then we know that two disjoint sets are involved in the whole and then each separate set we can count and then to get the total we just have to add.

Similarly, when we know the whole the total and we know the size of the cardinality of a subset of it, then we can see the difference by see the number of elements which belongs to that big set. And number of elements does not belong to the small set which is included in it is given by subtracting the cardinality of the big sets. Subtracting from the cardinality of the big set the cardinality of the small set, like that this is the subtraction principle. The multiplication principle was actually counting the number of members in the cartesian products of two sets, right?

When certain objects have several components and one component comes from one set and whose cardinality we know and the second components comes from another set and so on. Then we can multiply the cardinality though say of the individual sets to see how many are there. Then we also discussed the addition principle and we were using the division principle to show the circular arrangements, and then even to derive the n choose the number of subsets from the so from the number of partial lists consisting of k things taken from n things. So, now we are talking about a certain a general strategy here so this finally, when you when we discussed the division sorry the division principle.

We told what we are thinking is like we have a set and T and we have a set S and there is a d to one function from t to s. In the sense that every member of in S every element in S has d pre images in T and they are all definitely, they have to be distinct pre images because this because is a function. So, every member of T will get only one image in S, but in S can get several pre images, but then they get exactly d pre images, but then we so that if you want to count S one way is to count d and divide by d.

So, like for instance initially we consider the example of families several families coming to a party and each family has two children, if you count the number of children then we can get the number of families also by just dividing by then by 2 right because each family has two children. So, then here is a very special case of this d to one function case.

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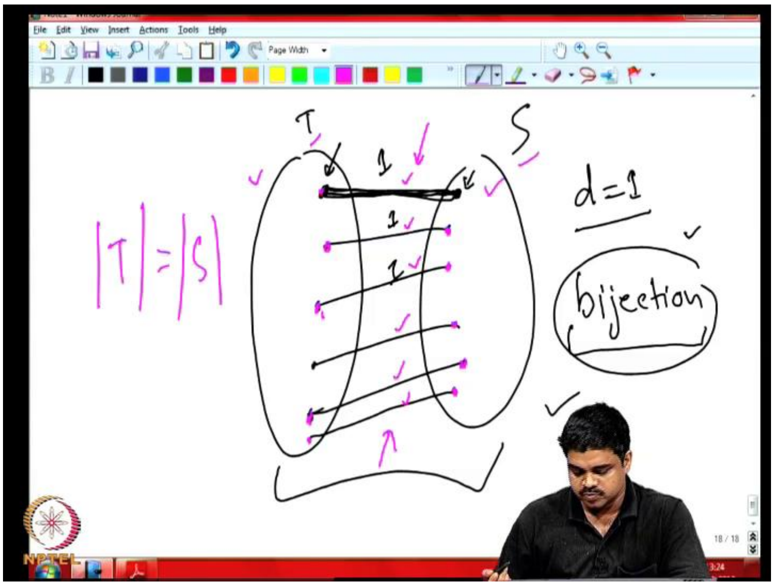

Namely when d equal to 1 right T and S so that means, here so this is a function, but we know that every member of S has exactly one pre-image exactly one pre image. That means, d equal to 1 case so this is called a 1 to 1 function d equal to d to one function is 1 to 1 function now, and this another name is bijection. So, you should revise what the chapter on functions and relations and then record what was bijection, surjection, injection and things like that. I would not get into the details of those bijection is that the name for one to one function, but this is probably one the time to just recall the definitions of like on to functions, into functions, surjection as different means so surjection, injection and things like that.

So, this bijection is one to one map. So every member of T is mapped to this thing, but of case something here exactly one thing here and every member of S is mapped back. That means, the pre image of it is unique one, there is only one, there is one and only one that is that then we say that between T and S there is a bijection there is a bijection. So this is the picture, so you remember there is would not be anything straight thing here sitting here without getting a pre image.

And also there would not be anything which will have something like this; that means, there would not be any guy who has two pre images this is also not there right, this exactly like this in the graph theory like. So, it will for instance if you imagine this as a bipartite graph and this T is one side of the bipartition. And S is another side of bipartition and this is a collection of these are considered as edges right and this is usually called a matching perfect matching. Because everything here is matched and everything here is matched and then that is a exact one to one mapping here all right. So if you are familiar with graph theory otherwise you can forget about it.

So now this is a bijection, now you see by the division principle we have already observed that if we to count this thing you need to count this right and then say that T equal to S because there is no division now, or whichever is easier to count once we establish a bijection. Whichever is easier to count we count and then we say that the other also as the same number of elements. So sometimes it can be easy to do certain things can be easy to do, while certain other things need not be. Once we see a bijection we see that it is the same thing the same, so the number of things should be the same that is the plan.

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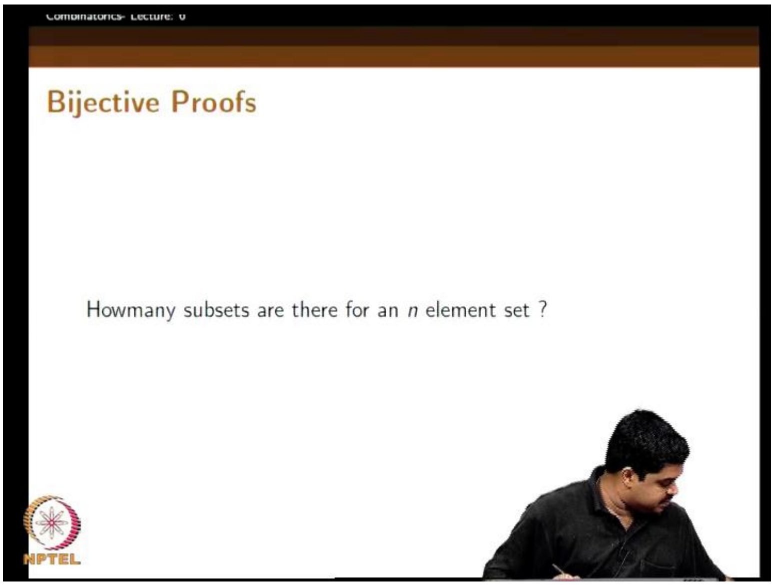

So the simplest example again we come back to counting the how many subsets are there for an n elements, this is what we are always seeing. How many subsets are there for an n n elements set? So, we had seen one strategy now like.

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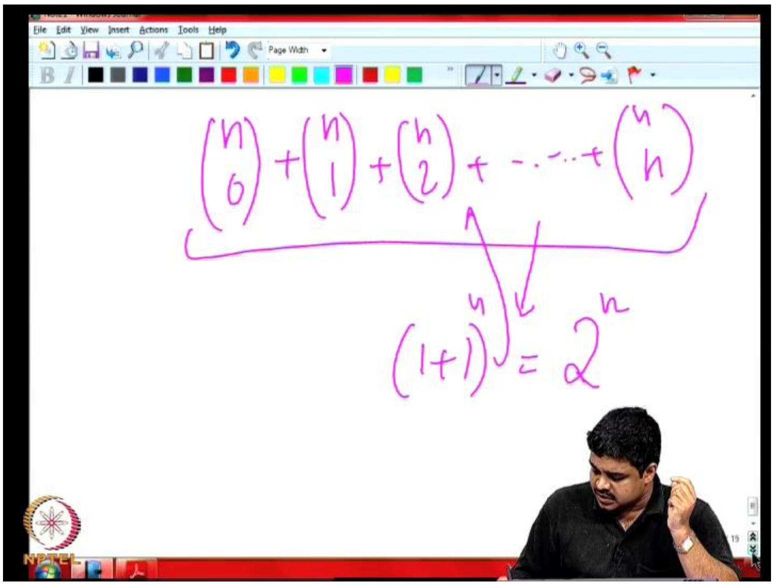

We show that it is n choose 0 plus n choose 1 plus n choose 2 plus so n choose n. This is because this n choose k means, we are counting k elements subsets that is the meaning of that n choose k this is the way it is divined. And then we are summing up all these things we are actually getting the number of subsets and we has seen by binomial theorem that this is actually 2 raise to n, because this is this will be the expansion of this will be the expansion of 1 plus 1 raise to n is 2 raise to n. Now we try another method right we will do is to establish a bijection between a subset of the n element set.

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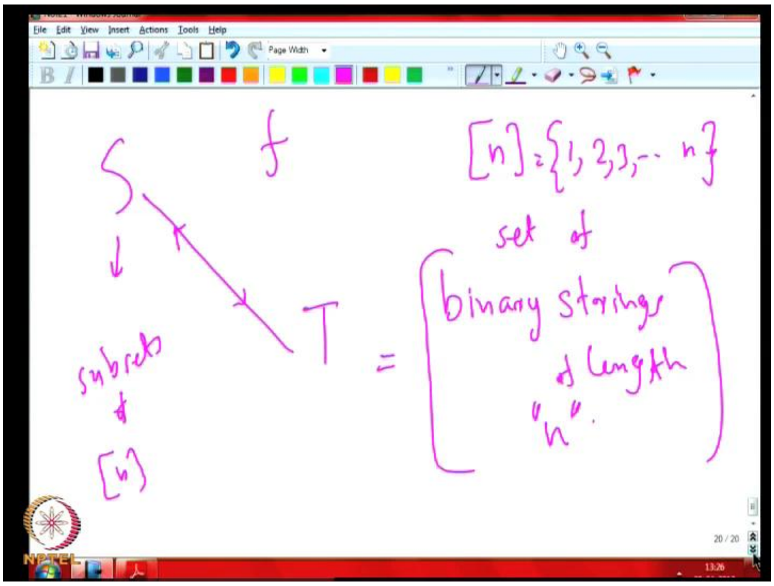

So subsets of these and n equal to 1, 2, 3 up to n take any subset of this we will put a bijection to the binary strings of length binary strings of length n. So this will be our T right the set of binary strings of length n and S will be the subsets of this one n right we will now establish some bijection f between S and T. This is what this is our plan is. How do we do this? So first we note that you.

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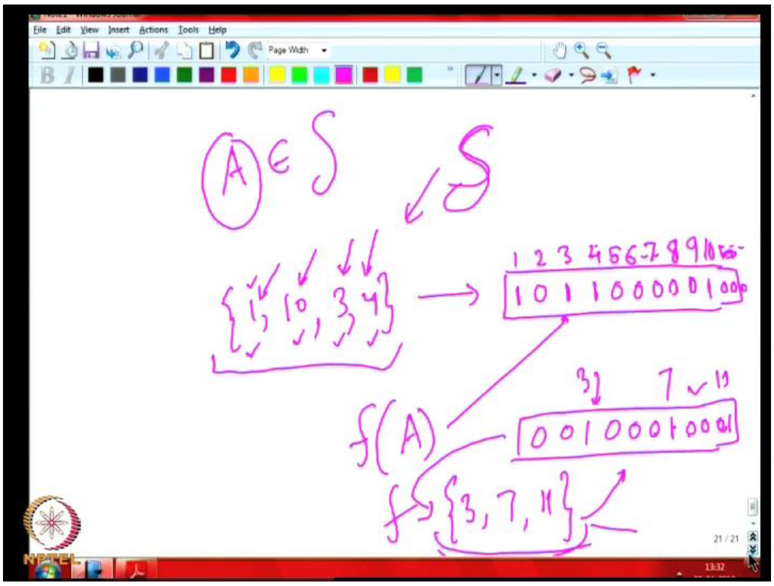

So consider some subset A from S right; that means, something like 1, 10, 3, 4. Now, we say that looking at the subset I can devise a binary string, the in the following way binary string I will write like this is binary string I am going to write this is first position, second position, third position, fourth position and this is the nth position. Now will see one here, so I will put 1 here in the first position now I will look for 2, 2 is not here.

So, I will put here 0; that means, 2 is not here I will let 3 is 3 present here yes 3 is present. So I will put 1 here is 4, 4 4 present here yes I will put 1 here then 5, 5 is not present here I will put 6 0 6 is not present here then I will put 0 then 7 is not present here I will put 0 10 sorry 8 0 9 0 and when I come to 10 I will see this here I will put 1. And later I will put only zeroes right because like this because they are not going to be here.

So I see that I am defining the bijection like this given this A my f of A that function value is this string. So like that you can see that given any subset, I can define a string like this by putting once it is exactly those positions, where those numbers are the members of this subset and putting zeroes 4 those positions in those positions where the position numbers are not in this subset. Now you can see that given subset we can only define one such string because it is exactly defining, what the string is right because the positions where one is one is to be put that the positions to be set are defined by the members of the set.

So and everything will get a string. So this is indeed of function, now the reverse way we can look at any string and from this string we can read out, the see where in which all positions one has appeared. So here 3 third position there is a 1. So I will make a subset like this I will write 3 and then I will write 4, 5, 6, 7, 7 th position there is 1 so I will write 7. And then say 8, 9, 10, 11, 11 th position there is again a 1 right the position numbers where I is a 1 I will collect and that becomes my subset that is the reverse. You can see that only one way of getting it back and this is the function, if I apply the function on this thing exactly the string should come. So, this indeed of bijection.

So, when we say a bijection we have to verify that what we are defining is indeed a function. And then from the other side we have to make sure that every member there is indeed getting a pre image, indeed has a pre image. Here for instance, you take any binary string you can create a pre image for it. You can see a pre image for that and also then we have to say that no two different objects will get the same pre image. That means, for instance if you take two different binary strings it cannot get the same subset back because the position numbers where, it is set one is unique right, it you can two different strings cannot have the same position numbers, where the ones are set and right.

So, if you collect all the positions where we have put one that will give exactly one set here right. So two different strings cannot unless they are the same string cannot get the same set back. So therefore, that uniqueness has to be checked this is what we have to verify when we prove that something is a bijection. This two that pre there is only one there is a pre image there is only one pre image sorry sorry there is a pre image and then sorry I think I made a mistake here, I am saying that two subsets we just have to verify that two subsets cannot map to the same element.

That means, given one member you do not get two different pre images. So the other thing comes on the function side to see to check that it is a function here to say that everything every subset here is indeed getting a binary string, I means it has to get a value and second. Thing is two different sorry given one subset you should not be able to map it to two different things should be unique mapping that is it, right? So this will establish the bijection right. So, this is one way of contact so here what have we achieved so then we have achieved that.

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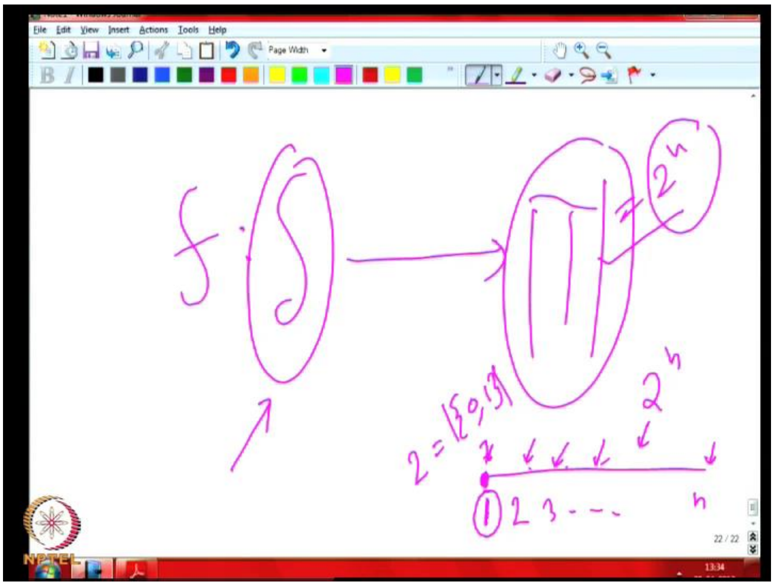

So, S is the number of so S is the subsets and from that I have defined this bijection as I have described to T, where T is the set of all binary string of length n. Now to count S we can just count T because if we so that we is counting S. We require this binomial theorem and all. Now we do not require binomial theorem because we rather count T right we rather count the number of binary strings of length n. And how many number of binary strings can be there?

Here we can use the product role because see this n position so first position, second position, third position, n position. First position we have how many choices there are only two choices it can be either 0 or 1, right? the cardinality of this choice is the set of choices is 2. Here also the same situation two choices, two choices, two choices.

So, now we can apply the product rule and multiply it out 2 into 2 into 2 into that will become 2 to the power n, right? So therefore, there are 2 to the power n the cardinality of T equal to the 2 to the power n right and therefore, the number of subsets is equal to 2 to the power n. So essentially this choice is here 0 or 1 here understand that what, when we put 0 we means we mean that we are including it in the subset, this one this position number whatever element corresponding to the position number is not being included in the subset.

Otherwise, if you are putting 1 here then; that means, this is this element is being included in the subset right. So therefore, it is a question of whether we want to include an element in the subset or not to include these are the choices. So that is why when we want to count the number of subsets, so this choice is for each element there is the choice whether we have we should include it in the subset or not include. So that makes 2 raise to n possibilities right. So what a from this bijection it should be clear, how the counting is done.

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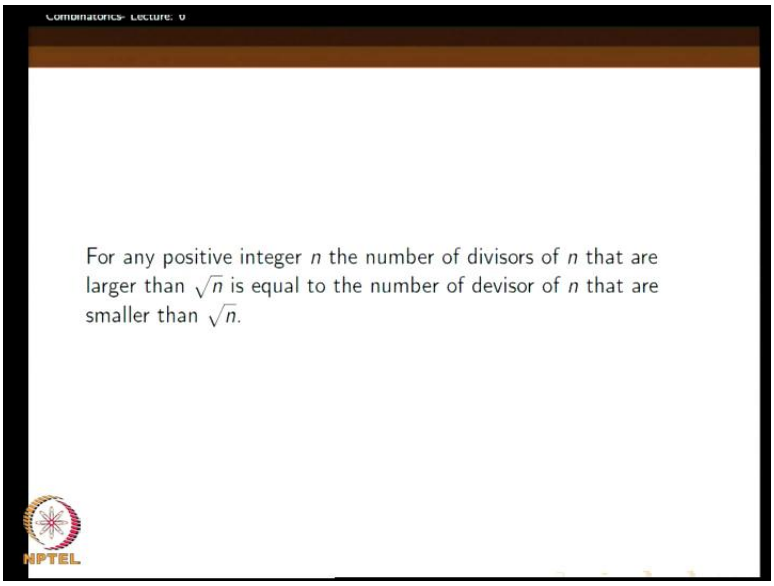

The second the next one is so another example for the bijection based proof is this one. So for any positive integer n the number of devisors of n that are larger than root n is equal to the number of devisors sorry number of devisors of n that are smaller than root n. So what do we mean by this thing?

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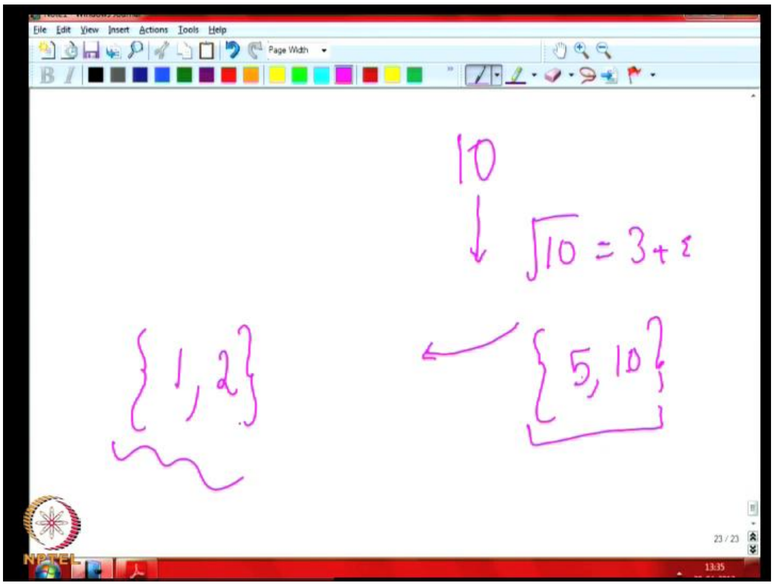

See you take positive integer 10, 10 does not have a is not a perfect square, but what is root 10? Root 10 is something like 3 point something some 3 plus epsilon I do not know what exactly it is 3 points something right. So I am interested in the devisors of 10 which are below root 10. So; that means, up to 3 so, 1 is a devisor, 2 is a devisor and then I am interested in the devisors of 10 which are above root 10 that is 5 above 3 above 3 plus it is above 4 or more it is 5 and 10. So it so happens that this number of devisors here is above root 10 it is 2 here and this is also 2 so, this are equal we are claiming that for every number and it is like that.

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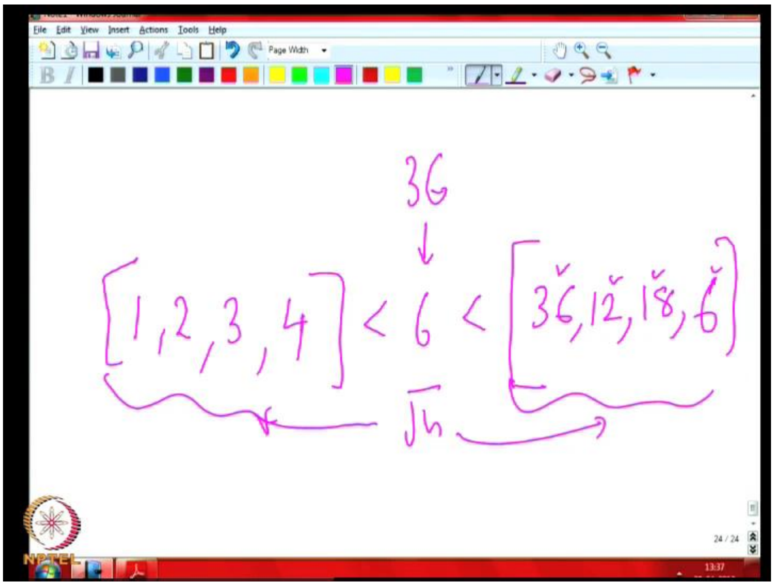

So for another example we can take say 36 root 6 the number of devisors of a 36 which are below 6 are 1, 2, 3 and 4, four devisors. This one devise, two devise, three devise, four devise, five does not divide and the number of devises of 36 which are above 6 is 36, 12, 18 and 9 sorry 18 and a 3 sorry not 18, 36 this is 12, 18 yeah sorry 36, 12, 18 and 6, right? There are four here, there are four here. How come it is happening? Below root n the number of devises is same as the number of devisors above root 10. So, we want to show that, this numbers are same we will again use our bijective proof method to show this things in next class.

Thank you.