Combinatorics

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Lecture - 40
Sterling Numbers

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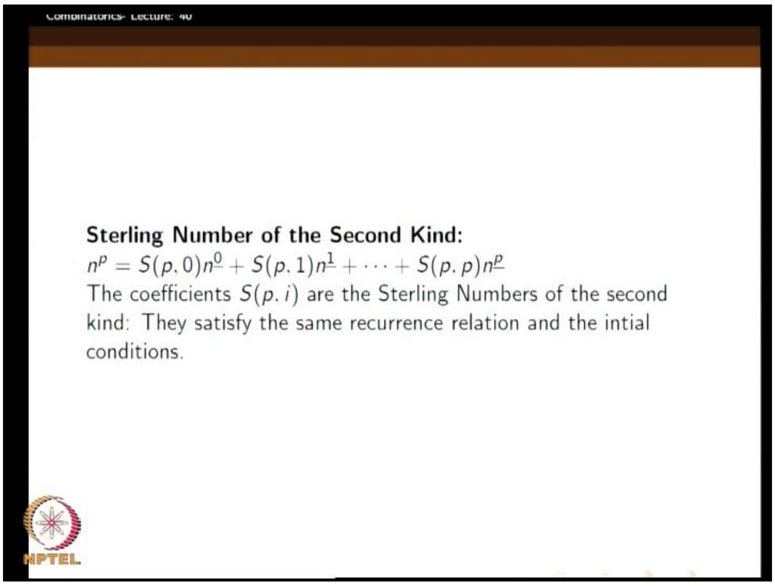

Welcome to the fortieth and last lecture of combinatorics. So, in the last class, we were discussing sterling numbers of the second kind. We started by defining the sterling numbers of the second kind in a combinatorial way.

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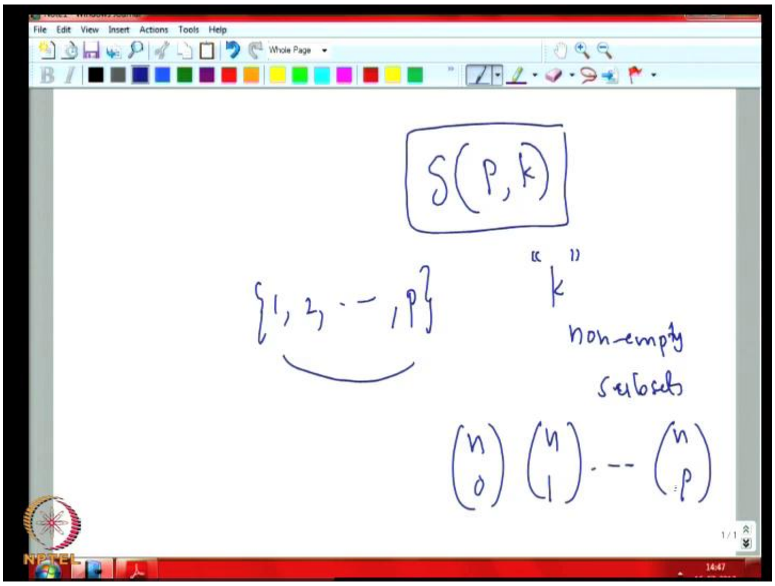

So for instance, the numbers S (p, k) was defined as the number of ways that we can partition the set 1, 2, 3 up to p into k subsets of this, right. We have to get a partition of 1 2 3 up to p into k non-empty subsets. In how many ways, we can do that? That was S (p, k) by definition. Now, we introduce the concept of difference sequences. We studied what is difference table and we understood that there is a special significance; there is some special significance for the numbers, which appear in the difference table along the 0^{th} diagonal, because these numbers can be used, right.

For instance, if the general term of the sequence is representable using a polynomial in n of degree p, then these numbers will be possible in non-zero till p and pth row, right. After pth row, that means, first elements only we are taking from each row. They will become 0, p plus 1th row onwards. Then, this p numbers, we can use coefficients into presenting that polynomial in using the bases, where the members of the bases are n choose, so 0 n choose 1 up to n choose p, right.

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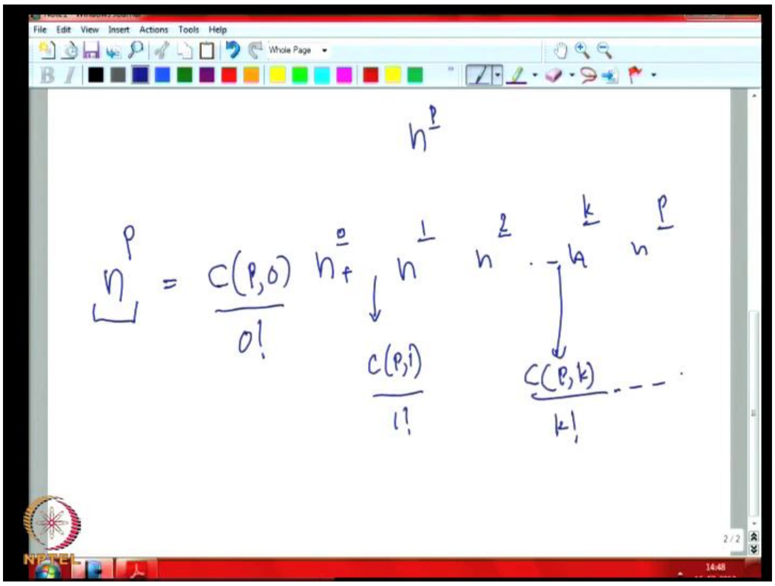

Then, we also saw that, we can use instead of n choose p, so, this n falling factorial p, right, n falling factorial 0, which is 1, n falling factorial 1, n falling factorial 2, n falling factorial p and then, the coefficients change. For instance, here the coefficients will be that c p, sorry, I was trying to represent the specific polynomial n to the power p, right.

Then, we wrote the co-efficient c 0 c p 0 to indicate that we are actually talking about, representing, as well; n to the power p, right. So, then this will become 0 factorial here plus here, we will have c p 1 by 1 factorial. In general, for n k falling factorial, we have the co-efficient of c p k by k falling factorial and so on, right and we told these numbers S p k.

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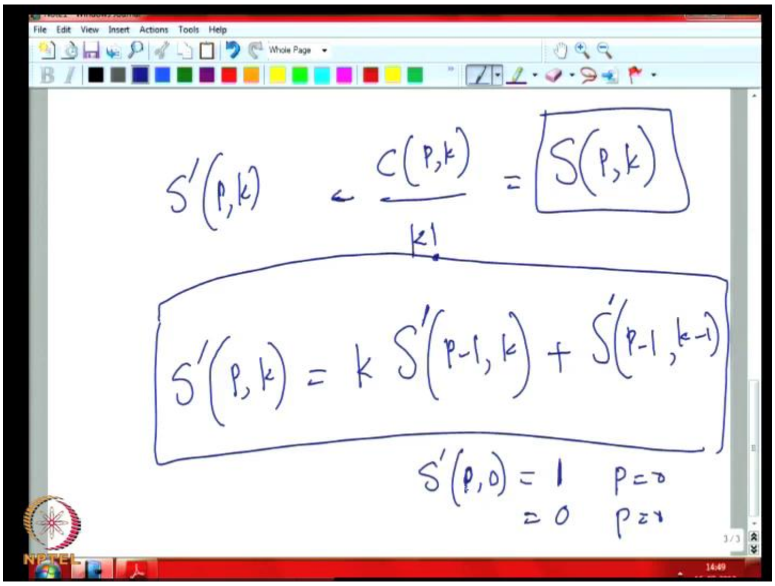

So, these numbers c (p, k) by k factorial, actually are equal to S of p comma k. We have not proved it, but, we claimed that this is right. So, we are going to verify that claim now. First, we will show that this number c p k, for the time being, let us called it as dash of p comma k. This numbers S dash of p comma k satisfies the following recurrence relation for k 1 2 p minus 1. This is equal to k into S of p minus 1 k plus S of p minus 1. So, you will write as dash here for the time being. S dash of p minus 1 k minus 1, which is the same recurrence relation the sterling numbers of the second kind was holding.

So, when we define it using the combinatorial in the deportation, right. We just have to verify that the initial conditions are also correct. Namely, S dash of 0 comma, so, S dash of p comma 0, right, is equal to 1. If p equal to 0; that means, a 0 0 is 1 or otherwise, it is 0; p equal to 0. Is this correct?

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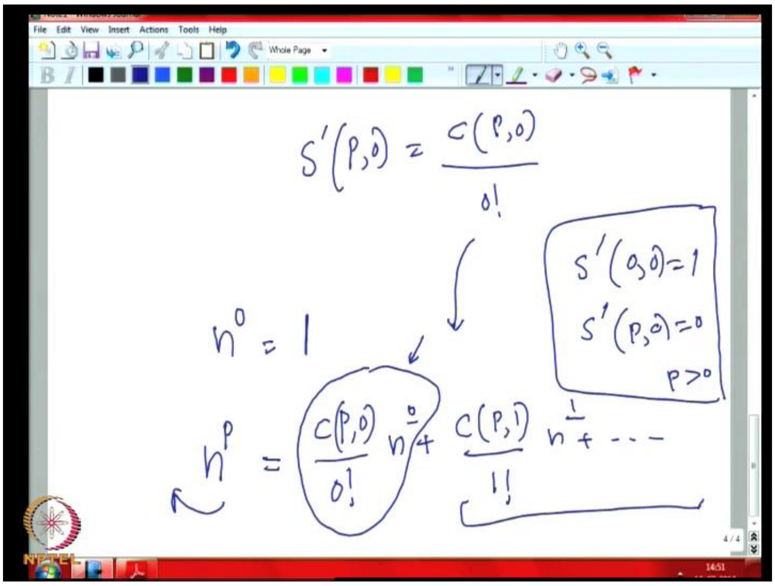

For instance, this is what, S dash of p comma 0 is c p 0 by 0 factorial, which is, you know if p was equal to 0, we are trying to represent n raise to 0 and we just have one term. This is 1, right. So, this has to be 1 as usual, because e p 0 has to be c 0 0 has to be 1. So, we can say that S dash of 0 0 is 1. So, let us assume that p is greater than 0, right. If p is greater than 0, what will happen.

If p is greater than 0, then we know that this term has to be 0 because, n raise to p does not have any constant term, right; p is greater than 1 greater than 0. So, there is no constant term. So, if we write this as c p 0 c p of 0 by 0 factorial into n raise to 0 plus c p of 1 by 1 factorial into, so, n raise to 0, n raise to 1 falling factorial and so on. Then, we know that all this later terms have n in it. So, the only constant term is coming from here. So therefore, because there is no constant term on the LHS, so, this has to be 0, c p 0 by 0 factorial. So, that means, S dash of p comma 0 will be 0, if p is greater than 0. This is what we get, right. So, that was the same initial condition we had for the sterling numbers of the second kind, right.

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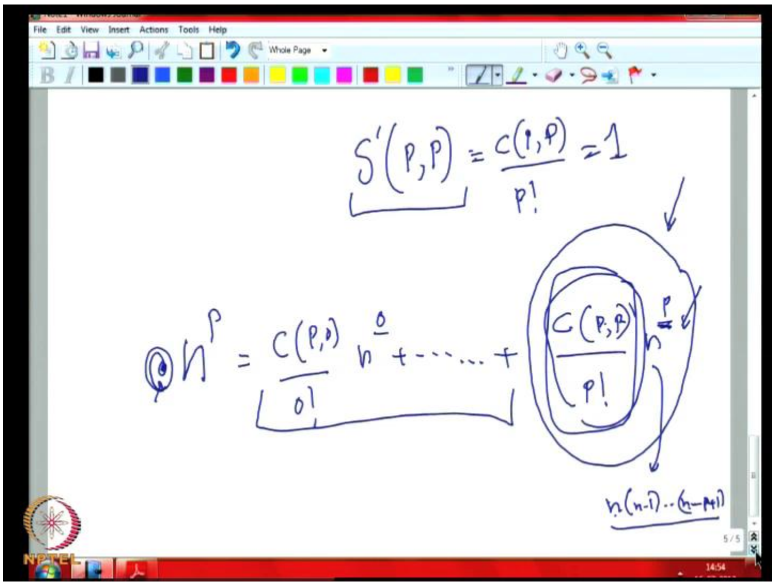

Now, also we have to worry about S dash of p comma p, right. We know that in the sterling numbers of the second kind, when we considered p comma S dash S of p comma p, that was equal to 1 because, we were considering the number of ways to partition the set 1 to p into p non-anti-subsets, because, it has to be non-anti. Everything has to go into different subsets, because, so actually that has to be singletons of subset. That is only one way, right. So therefore, so, which essentially meant that this is was equal to 1. Then, how do I know that S dash of p comma p is equal to 1. So, when you write n to the power p equal to c of p comma 0 by 0 factorial into n 0 falling factorial and final term being c comma p by p factorial into n comma p falling factorial. See, all the previous terms up to here has only powers of n to the power p minus 1, up to p n raise to p. Up to the last term will have n p minus 1 falling factorial, which has maximum n to the power p minus 1 as power.

So, if at all we have n to the power p, that should come from here, because you can see this is n into n minus 1 into up to n minus p plus 1. So, n to the power p comes from this. So, that is, the co-efficient has to be this, right. But, the co-efficient for n to the power p is actually 1 from the LHS.

So, S dash of p comma p is equal to c p p by p factorial is equal to 1 by comparing the coefficients on both sides here, right. This co-efficient is the same as this co-efficient, right. Because, that is the only place we get coefficients n to the power p, right. So, because n to the power p minus 1 falling factorial or n to the power p minus 2 falling factorial, none of them will contribute to the co-efficient of n to the power p. Only n to the power p falling factorial can contribute and the contribution is exactly this. Because, if you look at this thing, there is no other n to the power p comes in this thing with 1 as co-efficient. So, this will multiply by it, right.

So, both the initial conditions are same for the S dash and S, right. So therefore, we just have to verify the recurrence relation, namely whether this S dash satisfies this recurrence relation, which is satisfied by the sterling numbers of the second kind. If it satisfies, we are done. So, how do we verify that, right? Now, let us start from this thing.

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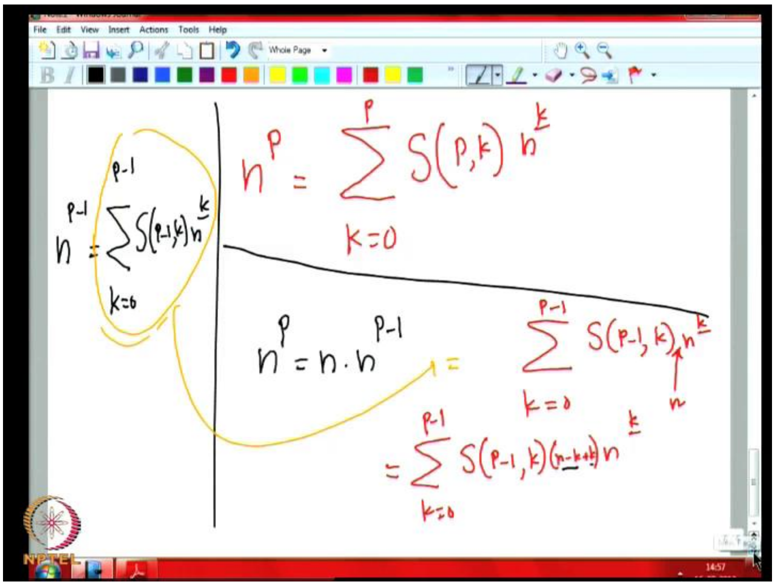

So, we know that, this is the way we define as p k. So, this is what, sigma k equal to 0 to p S p k n to the power k falling factorial. This is the way we define our n to the power p. For reference, let us remember that, I will write it here, n to the power p minus 1 can be written as sigma k equal to 0 to p minus 1 S of p minus 1 comma k into n k falling factorial, right. Now, if I want to construct n to the power p, which is n into n to the power p minus 1. So, I can multiply this, right, with n.

So, this means n into sigma k equal to 0 to p minus 1 S of p minus 1 comma k into n k falling factorial. This is what we can get. Now, see this n can go inside. So, n can go and join here, right. n can join here. So, it will look like this now. sigma k equal to 0 to p minus 1 into S of p minus 1 comma k into n into n k falling factorial. So now, I can also rewrite this n as n minus k plus k, for the purpose of splitting. So, we will write like this. n minus k plus k, right. So, this will look like n minus k plus k into this thing. Now, we can split it in two. So, we can multiply with this into this plus multiply by this into this, right.

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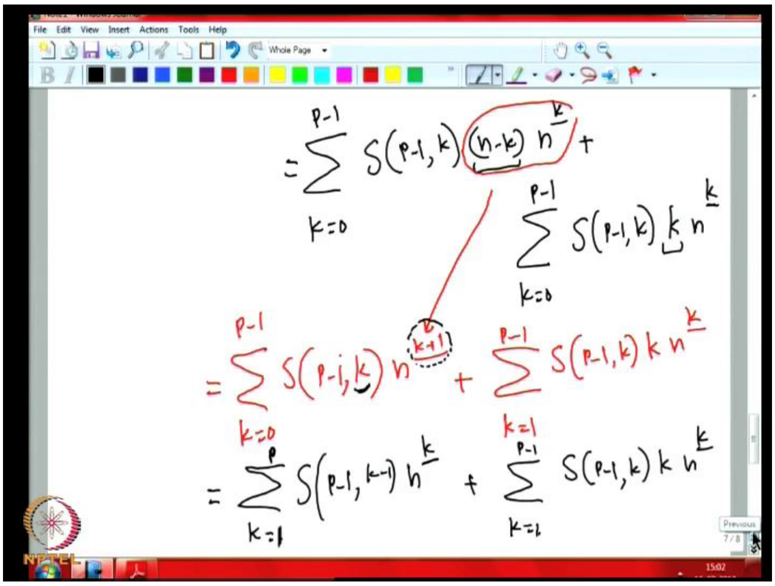

So, this will look like sigma k equal to 0 to p minus 1 S (p minus 1, k) n minus k into n k falling factorial plus sigma k equal to 0 to p minus 1 S p minus 1 k right k into n k falling factorial. So, this n minus k and k was actually n there, n minus k plus k. So, we just separate into two different terms. Now, what we can notice is, these two things can be combined because, n, what is n k falling factorial? This is n into n minus 1 into up to n minus k plus 1, k terms downward, right. Now, if I multiply by n minus k here, so, here also I can put n minus k. Notice that, this n minus k is the next number below n minus k plus 1.

So, it is as good as saying that this thing, is this thing, is equal to n k plus 1 falling factorial. We are starting from n, we are going down by 1 each n minus 1, n minus 2, like that k plus 1 terms now. Earlier, it was only k terms and we considered n k falling factorial. So, by multiplying by n minus k, we are going down k plus 1 step downwards. So, it is n minus k plus 1 falling factorial, right. So now, we will use that will use that.

So, we will rewrite this as sigma k equal to 0 to p minus 1 S p minus 1 comma k into n k plus 1 falling factorial plus k equal to, see we can just start from k equal to 1 to p minus 1 here, because when k equal to 0, this entire thing will go away because, 0 into something is 0. So, we can start with k equal to 1 to p minus 1. That is, S of p minus 1 k into k falling factorial. Now, what is this? You can do a change of variable here. This k plus 1 can be now called k, right or maybe you can use a different variable if you want. You do not have to. We just rename k plus 1 as k; that means, now, k will go from 1 to p because, when k is equal to 0, k plus 1 will be 1, right. k plus 1 will be 1. So, when k is p minus 1, k plus 1 will be p. So, that is why it is going from k equal to 1 to p and this is S p minus 1. When, because now our k is k plus 1, so now, this should be called k minus 1, right, because, we have changed the meaning of the variable. So, here it will be n k, right, plus this is ok, k equal to 1 to p minus 1 S of p minus 1 comma k k into n k falling factorial. This is k falling factorial.

Now, what do we see here? We see that, so here, this is going from k to p minus 1. This is k going from k k equal to 1 to p minus 1. Here also k equal to 1 to p. So, we can separate one term here.

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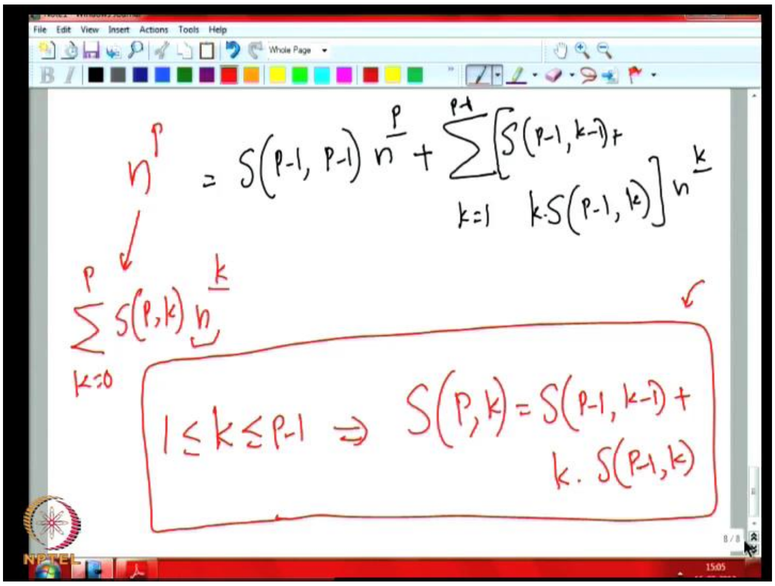

So, namely the pth k equal to p case, S equal to, that will look like S p minus 1 comma p minus 1 and k equal to p this will p minus 1 n p falling factorial plus sigma k equal to 1 to p minus 1. We can combine both the things together, right, this and this together. So, I just removed the last term from this first, this part.

So, now that is S p minus 1 k minus 1 plus k into S plus S p minus 1 k into S p minus 1 comma k into n k falling factorial. This is what we get. This is the term, right. We collected together. Now, this is other n to the power p. But, we know n to the power p is actually sigma k equal to 0 to p S p k n k falling factorial. Now, we can compare the coefficients of n to the power k falling factorial from both sides, right, from LHS and RHS.

So, for instance, when k equal to 0, this p k is going to 0. It is not going to have any, so, we have already seen that S p 0 is equal to 0. So, it is k equal to 0 is not coming here at all. So, that is fine, When k equal to p, here it will be S p p, right, will be the co-efficient of n p falling factorial. Here, it will be S p minus 1 S p minus 1. You know both are one. So, that is they are equal.

So, we are only interested in the k, when k is in between 1 and p minus 1. In both these cases, comparing the coefficients we get on the LHS, we get S p k; on the RHS, we get S p minus 1 comma k minus 1 plus k into S p minus 1 comma k. This is what we get and this is exactly the relation that we had for the sterling numbers of the second kind. Since, the initial values are also same, we infer that these numbers, this see a of this I should of put a S dash here, right. I forgot to put S dash, right. Here, I should of use of S dash, but, fine, it is not a problem. So, you have to assume that there was S dash over here. So, it follows that, because this S dash satisfies this condition. The earlier S, we have this other sterling numbers of the second kind, right. Fine.

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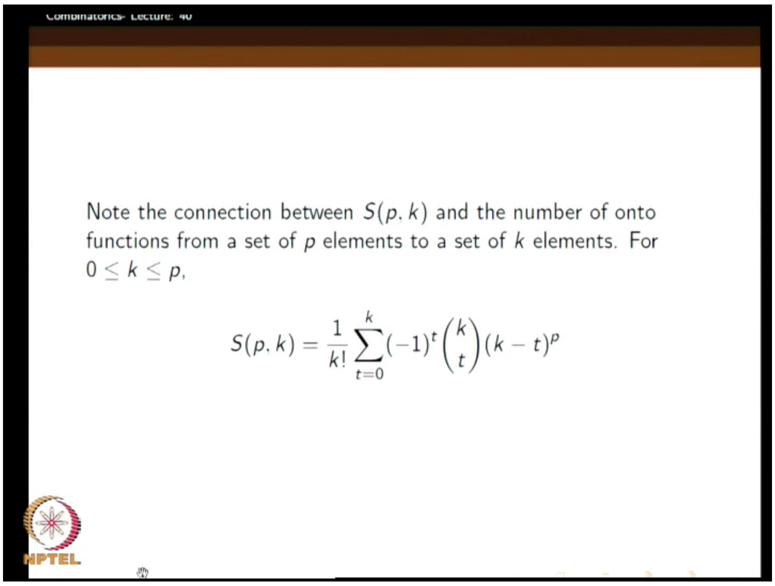

Now, before moving further, we will just mention that there is something called a, sorry, so, we will notice that, since this S p k is a; remember, that it is a combinatorial interpretation for this S p k, namely the number of ways we can split partition the set one to p into k different, k subsets, partition is to be k subsets, where each subset is non-empty, right or how we can put p different things into k boxes, where the boxes are indistinguishable, right, such that no boxes empty. That is what we interpreted.

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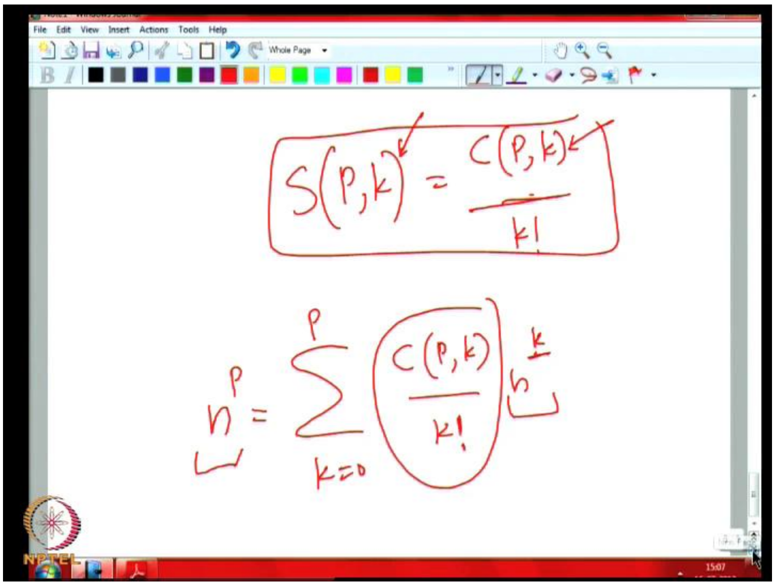

Now, we also know that from the previous discussion, we also know that this S p k is equal to c p k by k factorial, where the c p k is the co-efficient appearing in this, when we try to express in p k like k equal to 0 to p, right, k factorial into n k falling factorial. When we are trying to, then, we want to represent the n raise to p using n raise to k falling factorials. This is the co-efficient, which is appearing for n k falling factorial, right, c p k by k factorial. Now, that is S p k. Then, because this has a combinatorial interpretation, the c p k also should have a combinatorial interpretation. Can we explain that?

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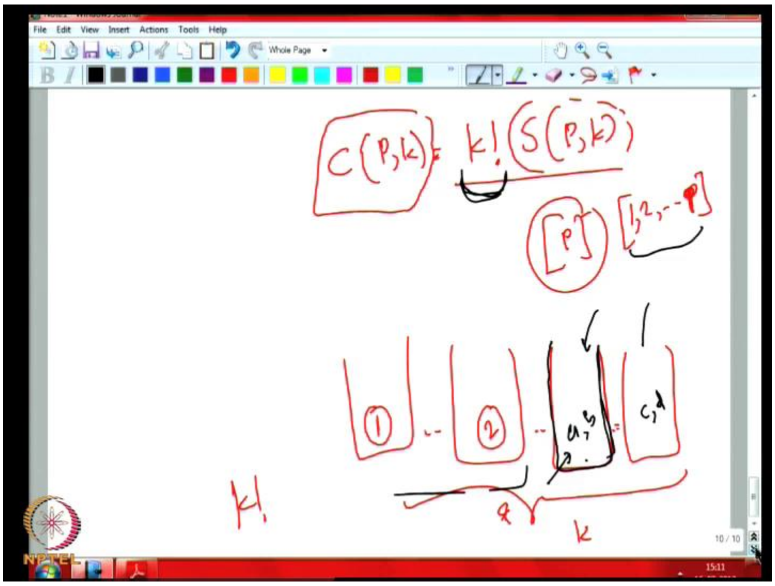

So, what is the combinatorial interpretation for c p k? Whether a c p k is actually k factorial into S p k? This thing is therefore, so, you know, we have the set 1 to p and we are distributing it into k indistinguishable boxes, such that, no boxes, none of the k boxes are empty.

So, k boxes, we are taking, there are k of them here, We are putting this p things into this. How many ways we can put? The condition is that, no box is empty here. Now, think of giving a label to each boxes; that means, maybe we can apply a sticker on each box. This is the first box, this is the second box, this is the third box and this is the kth box. Now, you know, so, it is now depending on this S o four one arrangement corresponding to, see the indistinguishable boxes, now this stickers can be given in k factorial ways. Each will be considered a different arrangement, because, now the boxes are distinguishable.

So therefore, the c p k is the number of ways to assign this 1 2 up to p to k distinguishable boxes, k distinct boxes, right. k distinct boxes. k label box. That is what we see, because that is k factorial and S p k. Note that this assumption, that is p k counts the cases where p things are distributed k boxes, where none other boxes are empty is very important. Otherwise, some labeling, for instance, suppose this and this was empty. So, when I label it 3 and this as 4 or when I label it this is 4 and this is 3, so, this different kind of labeling, so, one labeling, I suggested 3 4 using black here. So, or 4 3. Both would have looked same, because the other things being same. Here both are empty. How can, I mean, whether fourth box, this box and these both boxes empty, then it would not, whether I call it 3 4 of 4 3 will not would not made any difference because, anyway in both cases, third box is empty and forth box is also empty.

But, even if there is there is one thing here, the things are different because, the objects, this 1 to p, which thing here, already makes this box and gives this box an identity. For instance, if I put a and b here and c and d here and if I place 3 on this and 4 on this as numbers, stickers, if I paste, then that is very, that is really, that is definitely different from calling it 4 and 3 because, now the forth box contains a b. Earlier, the third box contained a b, right. So, that is why, it is, so, that is why the non-empty condition is very important in, say, because in this argument, that now when you multiply by k factorial, we actually get the number of ways to assign p distinguishable. So, p things, p different things to k boxes, such that, none of the other boxes are empty.

So, k distinct boxes, such that, none other boxes are empty. That is very important. None other boxes are empty is very important. If some boxes are empty, we cannot tell this thing, because then we have to say, we can only say that, so many things are empty, then by, say for instance, h things are empty, we have to divide by h factorial. So, that would not make it equal to c p k. So, this c p k is that, but, we can also remember that, this number we are very familiar with.

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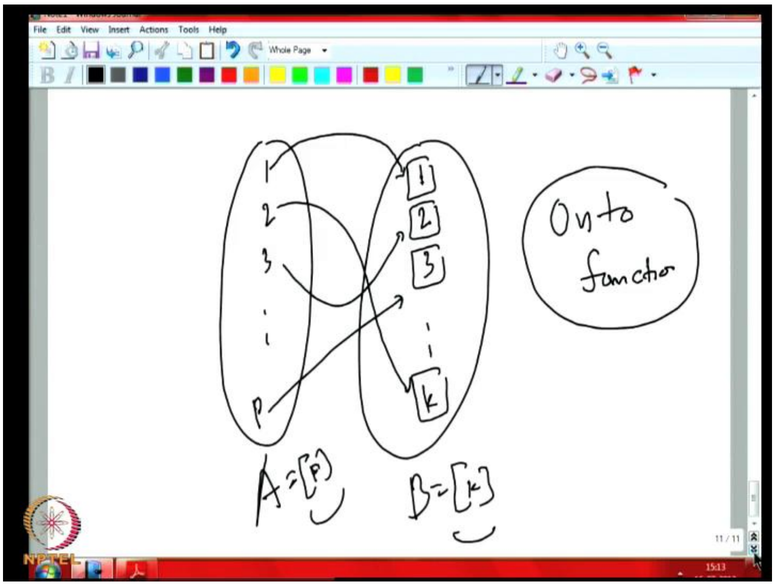

For instance, this putting, so, if I write 1 2 3 up to p here and this boxes, this label boxes, see the 1 2 3 k, these are the label boxes. Box 1 2 3 k, if I put here and this assignment is a function, right. 1 goes somewhere, 2 goes somewhere else, 3 goes somewhere else, right. So, this is perfectly a function. It is a function. Not only a function, but onto function. So, this a function from A to B, say A B in this and so, A B p and B B k, right. So, this is function, onto function because, no boxes now empty, because every number on the B side, everything on the B side, should get pre image, right. That is what it says, right. Other non-empty boxes are not allowed. So, we are actually, this same as counting the number of onto functions, from the set p 2.

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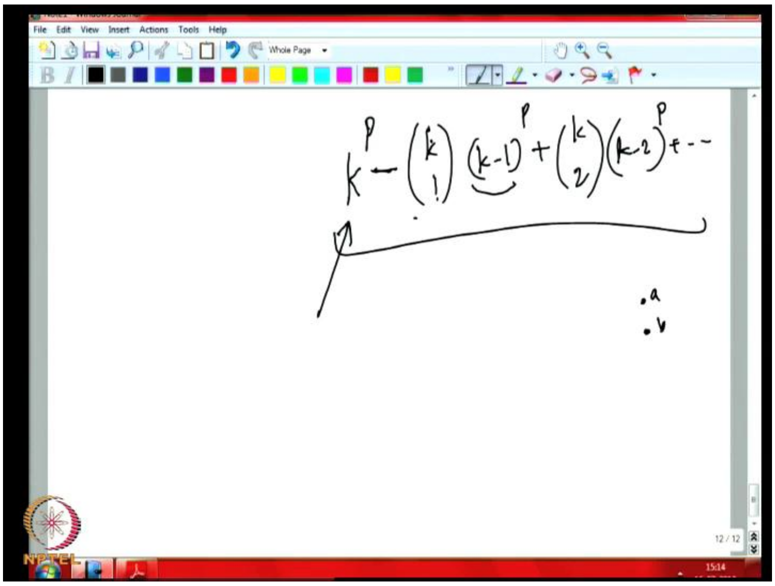

This to this, right, and we know this number. How much is that? This is k to the power p minus k choose 1 to the power k minus 1 to the power p plus k choose 2 into p minus 2 to the power p n. So, k minus 2 to the power p and so on. This we have done in two different ways. One is by considering the x inclusion exclusion principle because, there are total of k to the power p functions, onto or not, counting everything, right. Then, we minus those functions, which are skipping just one element, which are not giving a pre image to exactly one element.

So, given one element, right. So, that is not exactly; some fixed element. So, that can be, so, element can be selected in k choose one ways. So, and the number of functions, which are mapping p things to the remaining k minus 1 things is k minus 1 raise to p. So, that is what this thing. But, then we have over counted because, there can be two things, say a and b, which are not getting pre image, the same terms. So, we just re add it, k choose 2 into k minus 2 raise to p comes and so on, right.

This formula, you can just go back and see from where we have studied. We had also done it using exponential generating functions, if you remember. So, we have done it two times. So therefore, S p k will be 1 by k factorial of that number, namely t equal to 0 to k minus 1 raise to t k choose k minus t raise to p. This is the count of, without this 1 by k factorial, the rest is the count of number of onto functions from p things to k things. So p, 1 to p to 1 to k, right. So, right. So, we do not re write this formula. So, and therefore, this number of onto functions from 1 to p to 1 to k is actually equal to c p k.

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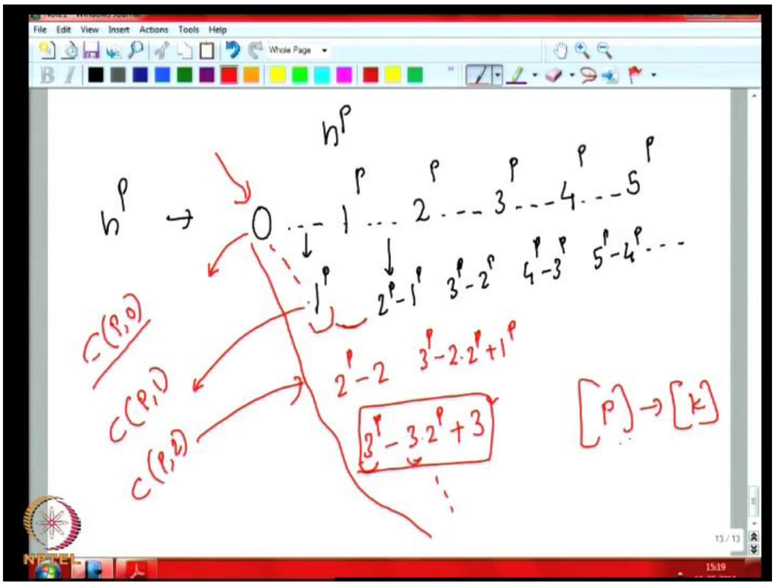

So, if you are thinking that it is surprising, so, it is advisable to look and write down the sequence, say for n to the power p. So, what is the sequence corresponding to that? When I put n equal to 0, it will look 0. This is 1 to the power p. May be, I can put little gap, 1 to the power p, then 2 to the power p, then 3 to the power p and then, 4 to the power p, then 5 to the power p.

So, this is the sequence corresponding to n to the power p, right. So now, if I consider the first order difference sequence, this will be 1 raise to p minus 0, right. This is 1 to the power p. So, this is the gap here, right, the corresponding to. So, here 2 to the power p minus 1 to the power p. Here, it is 3 to the power p minus 2 to the power p and here it is 4 to the power p minus 3 to the power p and here it is 5 to the power p minus 4 to the power p, right, like that.

So now, if I look for the second order difference sequence, so, you can see that this number, this number already, c p 0 is already the number of functions from p to the empty set, right, onto functions; number of onto functions. There is no onto function from (()) we can it is, right. That is not even a function. So therefore, that is 0. Similarly, this number is already, this is the second number, this is c p 1, this is the number of onto functions, from p to say 1, right. There is only one function, and that is 1 to the power p because, everything has to be mapped to this one, right.

Now, let us look at c p 2. c p 2 will be obtained here in this gap. That is, 2 raise to p minus 1; minus 1, one more, one more minus 1, right. So, 2 raise to p minus 2, which is exactly the number of onto functions from p to 2, right. So, because there are total 2 to the power p functions and then, you can select one of the two things and we can decide to map everything to the other.

So, in two ways, we can select that and then, there is just one function, which is this is what. If you remember, we counted the number of onto functions. This exactly is that, right. So, 2 to the power p and this number, for instance, is 3 to the power p minus 2 into 2 to the power p plus 1 to the power p, right. Now, if you minus this from this, what you get? 3 to the power p minus 3 into 2 to the power p plus 3, right, plus 3. So, is it same as, so, 3 to the power p minus 3 into 2 to the power p and this plus 3 because, total number of functions now, so, from p 2 3 we want to consider onto functions, right.

So, this is 3 to the power p total. Now, one can be avoided. Just fixed one, fixed one of the elements and avoid that. So, that is 3 choose 1 ways we can fix it and 2 to the power p functions. Now, avoid two things together, that is 3 choose 2 is equal to 3 ways you can do that, right. So, and then there is just one function to map everything to the remaining. So, this is the number. So, like this. So, it is not very surprising. So, if we carefully look at, this is indeed the number of onto functions. Along this 0^{th} diagonal, we will see the number of onto functions from p to k, right.

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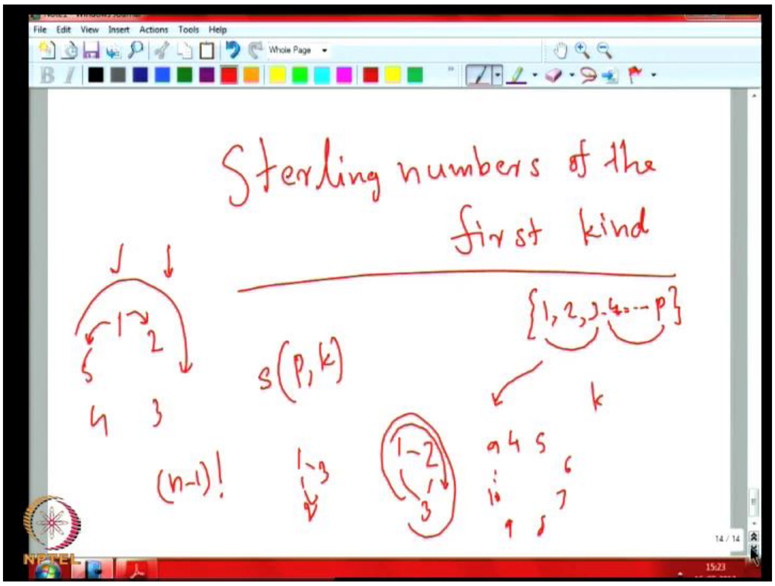

So now, we will look at the sterling numbers of the first kind; sterling numbers of the first kind.

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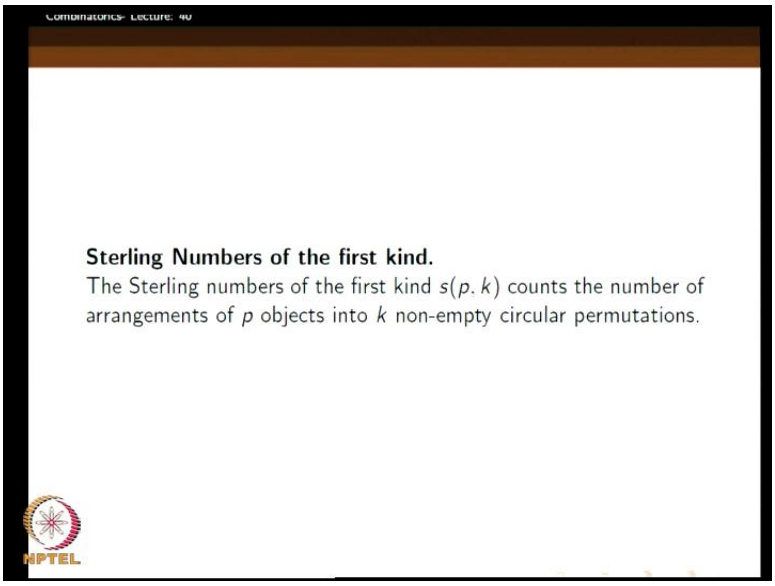

The sterling numbers of the first kind we will use small s, small letter s to denote it, s of p comma k. For the sterling numbers of second kind, we had use the capital S. That, by definition is count the number f arrangements of p objects into k non-empty circular permutation. What does it mean? So, we have s p k. We will say that. So, we have this 1 2 3 up to p and this things, this p things, we have to arrange into k nonempty circle of permutations. For instance, we can take say 1 2 3 1 2 3 say 4 5 up to p, right. This 1 2 3, we can arrange in a circular way like this, right and the remaining 4 5 6 7 8 9 10 up to p we may arrange this way. But then, there are several ways of arranging.

So, 1 2 3. So, we can also have 1 3 2. This is a different arrangement. Remember, it is a circular way means, see if I rotate in counter clock-wise direction, it is not going to make any change. For instance, we will say, you remember the way we defined the equivalence of the circular permutations. In the sense that, if I always, for instance, if I look at that clock-wise direction, if I always see the same person, same number, then from every number, then it is the same thing. The way we write is important. For instance, if I write 1 2 3 4 5 and if I just write 5 1 2 3 4, this is the same, because, we have just rotated it like this. Because, any number if you look, its clock wise neighbor is the same. Similarly, if will look at the counter clock wise neighbor, that is going to be the same, right, Then, we say there is a same circular permutation. But, on the other hand, if this number, if you had written the reverse, for instance, 1 2 3 4 5, this is different. This is different because, you know the counter clock wise neighbor and clock wise neighbors are different in this arrangement and in this arrangement. Say, if you consider 1, here 2 is a clock wise neighbor, 5 is a counter clock wise neighbor, anti clock wise neighbor. Here, it is 5 is the clock wise neighbor and 2 is the, so, they have reversed the order. So therefore, that is a different arrangement. So, how many circular permutations are there. We have discussed it in one of the earlier, at the beginning classes, when we studied the circular permutations, right; the division principle. So, when we studied that, we had discussed these things. So, we know that, if there are n things, there n minus 1 factorial ways to circularly permute that, right.

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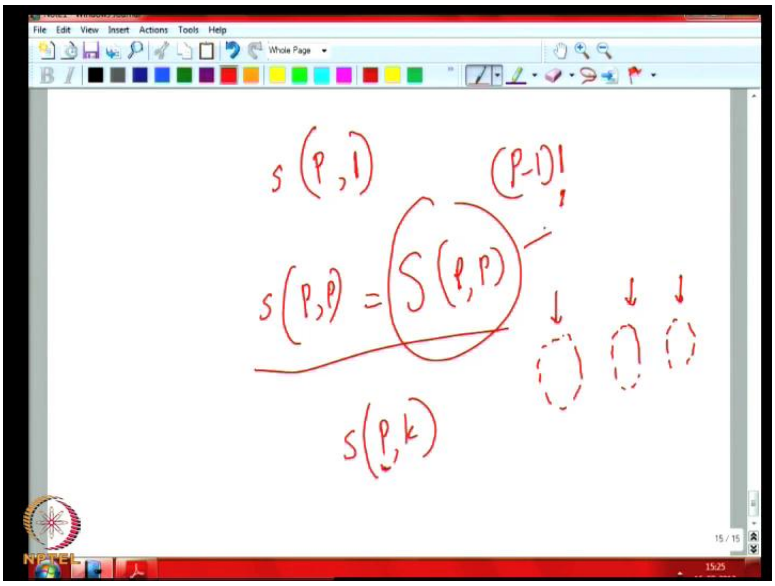

So, the first question we can ask is, what is S n 1 or S p 1. S p 1 say; that means, if I want to arrange p things, 1 2 3 up to p into just one circular permutation, then there are p minus 1 factorial ways to do it, because we are only allowed to use one circle. All of these p elements has to appear in that circle and then, we have to circularly permute it. We know that it is p minus 1 factorial. But, on the other hand, what will be S p comma p. Note, this is, I have to use small s, this p comma p. What is S p comma p? S p comma p is the same as S p comma p. Why? Because, in order to make, one observation we can easily make suppose of. Suppose, if I want to, say if I write, if I am looking for S p comma k, so, the p things are there, 1 into p. They have to form k different circles, which are non-empty, right. So, but, for that, first we have to decide which goes to, we have to first divide the p things into k subsets. Then, each subset has to be arranged in circular way. The first step corresponds to the sterling number of the second kind, right. So, for each partitioning of 1 to p into k subsets, such that, each subset is nonempty.

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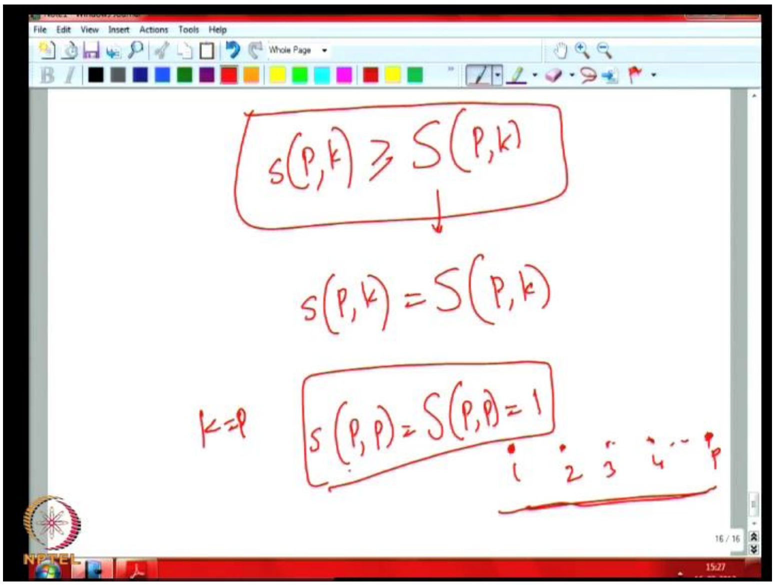

That is counted as p S p k, right, the sterling number of the second kind, right. For each subset, we can arrange the numbers of the subset in circular permutations in various ways. For instance, in a particular circle, if there are i things, we can arrange it in i minus 1 factorial ways; each of them, right. So, typically we will expect that this small s of p comma k will be much bigger than capital S of p comma k. But, this is in anyway correct for all p k greater than 0 from the combinatorial reputation because, first of all, we have to partition p into k subsets, which are all nonempty. Then, for each such partition, we can generate lots of circular permutation, for each such part in the partition, right. But, there are some cases, which, this small s of p comma k, that is the sterling number of first kind is equal to the corresponding sterling number of the second kind.

When is that? For instance, when k equal to p; that means, when we are talking of s of p comma p. s of p comma p means, see, first we have to divide into p subsets. So, the p things only are there. But, each subset has to be single term, right. There is no other way. So, each one becomes a single term-set, right. If each one become is a single term set, then the single term set does not have many circular permutations along themselves, right. Because, it is a one element thing and what can happen to that. So, only one, right. So therefore, that partitioning into subsets itself will correspond to the ways we can circularly permute into k p different circles. So, this is one such case, where we do not get more. So, this p p is equal to S of p p, which is equal to 1. Anyway, there is only one way of doing it.

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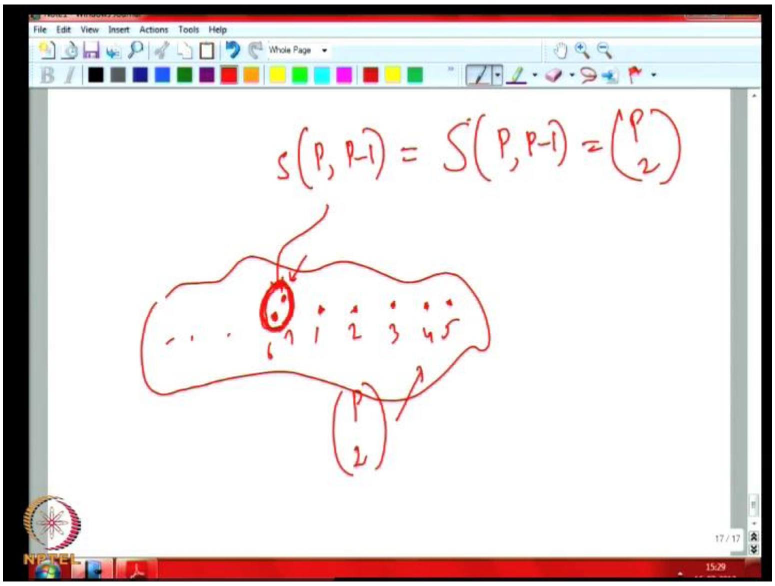

Another case, which this, where this happens is s p, so, when I consider small s of p comma p minus 1, right. Then, what happens is, we have to first partition p things into p minus 1 groups. So, except one among them, all others has to be single term, this time, right, because we have to get p minus 1 things. Only one of them has a chance to become, one of them has to become that two element subset now. All others has to become single term subsets. Like, may be 1 2 3 4 5 6 7 something like that. So, but, which two stick together that decides the entire partition and that can be decided in p choose 2 ways. Any 2 can be selected first and then, make them two elements subset and all other are single term sets. This is the way we can partition.

Now, we say that, each of them can, we have to allow circular permutations of them. But here, singletons cannot circularly permute in more than one way because, there is only one element there and circular permutation is possible, that is other than just that. Even for two elements it is true because, we do not have more opportunities. Because, we just have this, if this is 1 and 2. We have this circular permutation. If you write 2 and 1, it is the same thing, right. So therefore, here also we do not have any difference between sterling number of the first and second kind, when they correspond to p comma p minus 1 as parameters, right. So, this is also S of p comma p minus 1, which is p minus p choose 2, right.

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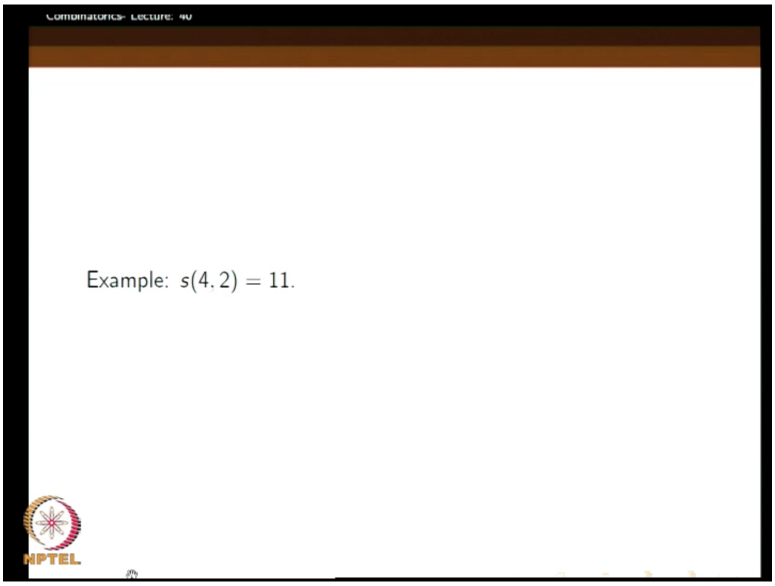

These are two cases, where, but then, there are situations, where it can be different. For instance, s 4 2. So, you can look at s 4 2.

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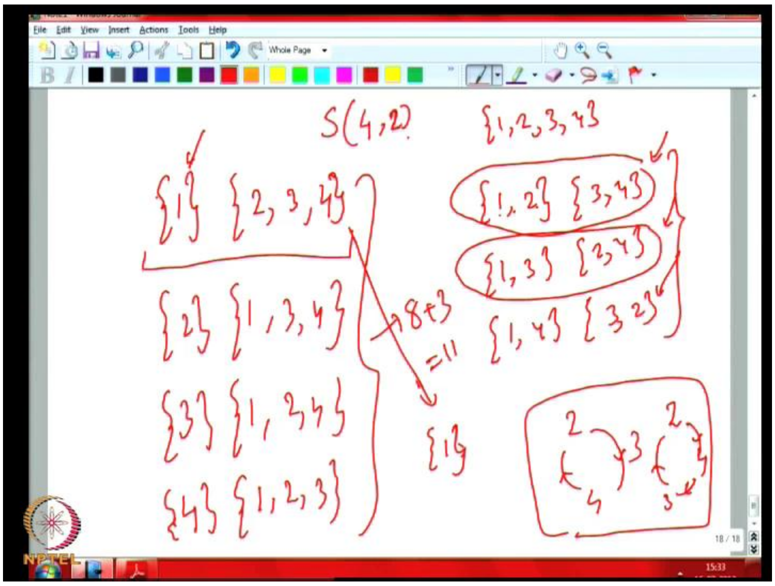

In S 4 2, first let us say, we will, 2 parts would come, right. So, we can put a singleton set and the 3 element set. So, what is the way? So, we will take 1 first. So, we will first partition it. 1 and then, so, this is the basics, so, 1 2 3 4. So, I will take 1 and 2 3 4, right. Now here, see, if it is just for participation, this can be done or it can be, say 2 1 3 4 and 3, this can be written as 1 2 4 and this can be 4 1 2 3. There are 4 ways of putting into two subsets. So, this all counts for sterling number of the second kind and also, we have 2 into partitions, right. So, these 2 things can go on one side and there is only way we can do. Either 3 three things, one thing, one singleton set and 3 element set. Otherwise, just two elements set and two elements. So, that can be done in 3 ways, right. 1 2 3 4 one or it can be 1 3 2 4 or it can be 1 4 3 2, right, because you know the other way.

For instance, 3 4 1 2, we need not count because, it is the same thing as 1 2 3 4. This will correspond to the sterling number. But then, here, these things do not contribute more. So, when I am counting sterling number of second kind, when we look for the sterling number of second kind, we will only be looking for how many, this partitions only I will be counting. But then, in the case of sterling number of the first kind, we have to consider each of them and then, see how many ways I can circular permute them. That is 1 2 3, this, we do not get anything more because, 1 2 cannot be circularly permuted and permuted in more than one ways because, there are two elements here.

Similarly, so, all these things, we only just one h. But here, we can give more. For instance, if I take this, right, this 1 does not give me any scope, right. It will just be like that. But, 2 3 4 can be done in two different ways. Either, we can put it as 2 3 4, like this, this is one circular permutation or it can be written as 2 4 3. This is different. So, we know that it is 3 4 3 2 factorial ways of permuting. So, starting from 2, next can be 3 or next can be 4, right. No other way. So, these are the 2 ways. So, we get, from each of them, we get two each. So, we get 4 into 2 8 circular permutations corresponding each of this partitions plus here 3.

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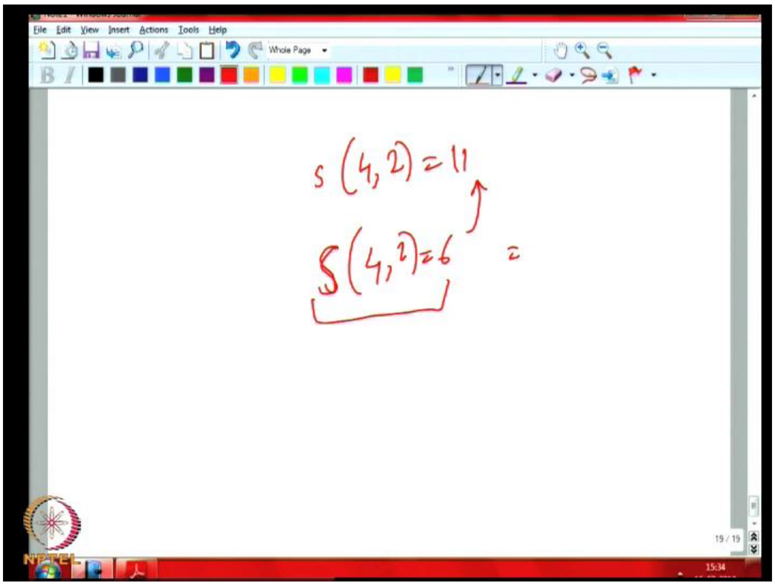

So, 11 will be the answer for this s 4 2. s 4 2 equal to 11, while capital S 4 2, will only be 6, right. The capital S 4 2 will be only 6. So, for instance, we had a formula for finding capital S 4 2, namely, S p 2 was equal to 2 to the power p minus 1 minus 2, right. So here, if you put 4, so, this is 4, this will be 8 minus 2 equal to 6. We had discussed this formula last time, if you want to cross verify, whether this indeed is 6 or not.

So, for capital S, it is 6, but, you see that small s, it is 11 because, some of those partitioning of the 4 elements, especially when we have split them into singleton and three element sets. So, each set partition gives rise to more circular permutation, this thing, more arrangements because, 3 elements can be circularly permuted in two different ways. That is why 4 of them give 8 and then, plus 3, right. That is why this gave.

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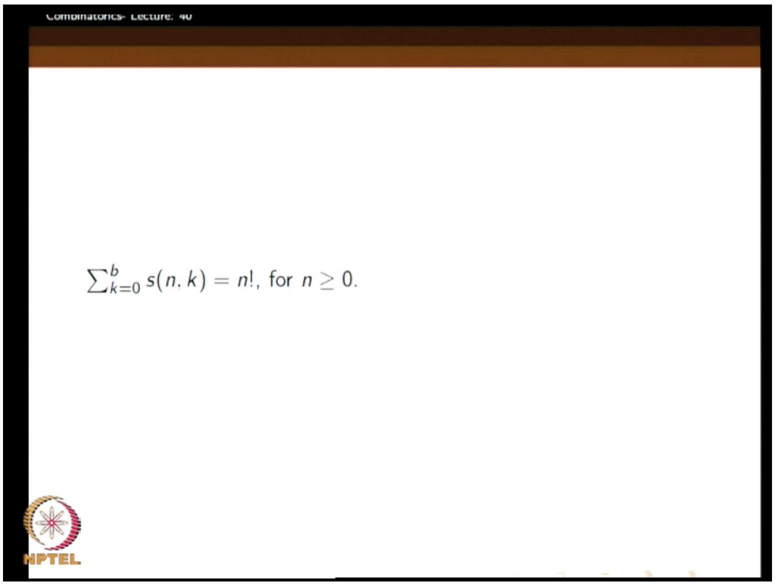

So, now the next thing we want to discuss is, some, this identity may be interesting. So, k equal to 0 to n, not b, k 0 to n s, so, we get k equal to 0 to n.

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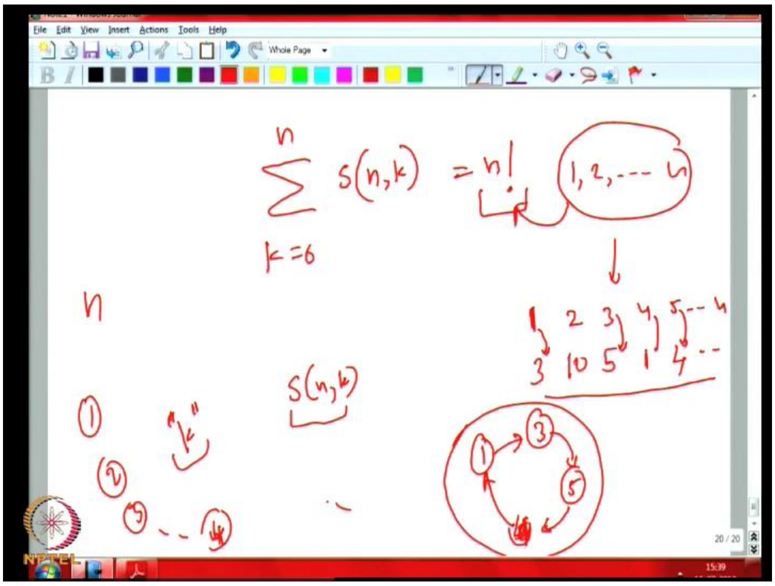

This one, 0 to n, small s of n comma k, if you add up all these things, we will get n minus, sorry, we will get n factorial. Why is it so? Because, if you take, you know, this is the number of permutations of 1 to n, the number of permutation of 1 to n. If you take any permutation, you have a cycle structure in it. For instance, you can consider 1 2 3 4 5 n, so, for instance, this is 3 5, right, 3 3 5 4 and 1, suppose something like this. So, suppose this kind of a permutation, say this, so, I see that 1 goes to 3 3 goes to 5 5 goes to 4 5 goes to 4. So, 1 goes to 3 3 goes to 5 5 goes to 4 and 4 goes to 1, we close this cycle.

So, then we start from somewhere else and we will create another cycle. So, this permutation can be represented by a collection of cycles like that. If you write down the 1 to n of there, the values, so, how I write is, i will always goes to pi of i. For, pi is the permutation and then, pi of i will go to pi of pi of i and so on, right. So, definitely it has to come back and if you think for some time, we will see a cycle structure there. We have not discussed it in detail before, but, if you give some thought, you will realize that, every permutation will give rise to a circular arrangement of 1 to n into some k, some number of cycles.

So, suppose it is right. So, of case, now this can be, depending on each permutation, can be grouped into, so for instance, we can group the set of permutations into n groups. Namely, if the cycle corresponding to that permutation has 1 cycle or 2 cycle or 3 cycle or n cycle, now if you count 1 of the groups, then what will you see? You will see that we will be producing lot of circular arrangements into exactly say k cycles, if you consider the group, where the number of cycles are k and this exactly corresponds to S of n comma k because, n e, conversely n e circular arrangement of n into exactly k circles will correspond to one of the permutations also. I will leave to you to figure out how this is happening. So therefore, if you sum up this small s of n to k, from k equal to 0 to n, we have to get total number of permutations of namely n factorial, right.

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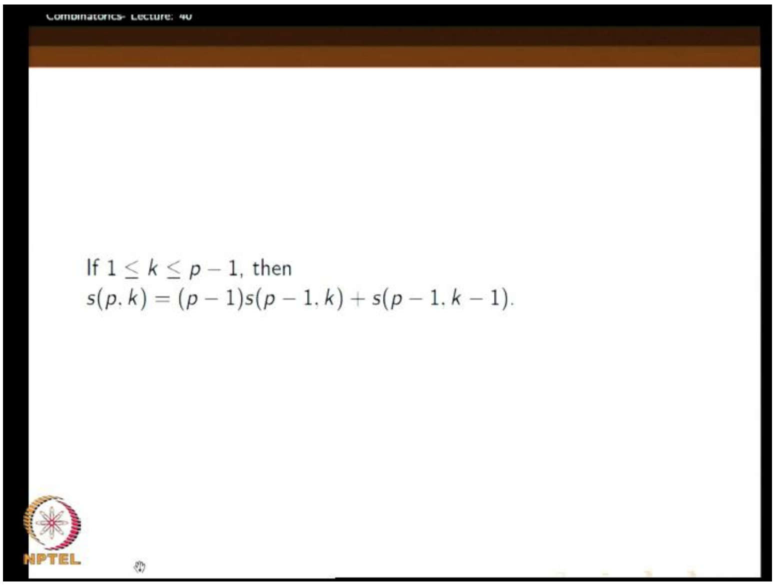

Now, the next thing I want to discuss is, so, this is b should be made to n. So, the next thing I want to discuss is, so there is a similar recurrence relation for s p k, namely from 1, if k is can be 1 and p minus 1, we can write s p k is equal to p minus 1 times s into p minus 1 comma k plus s in s of p minus 1 k minus. This is same, almost the same as the recurrence relation for the sterling number. Just that here, we have p minus 1 and earlier it was k. In other words, this if we consider, this particular term in the sterling number of second kind, we were multiplying with the lower index.

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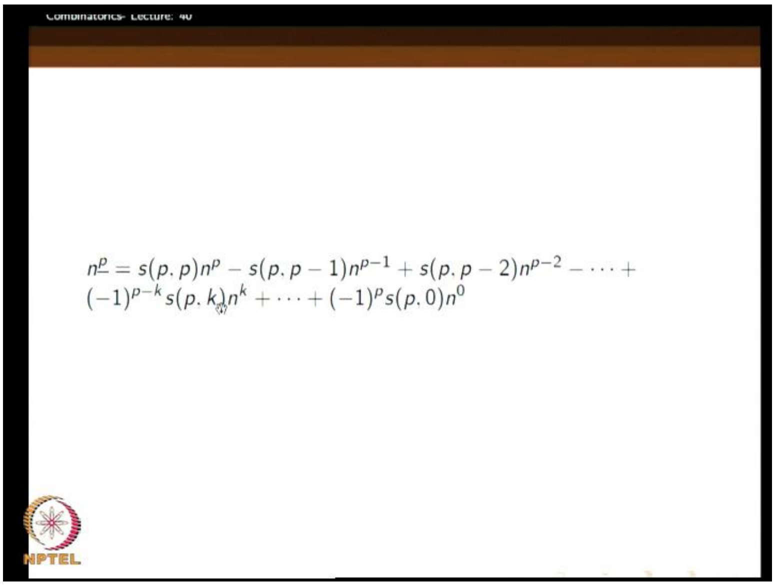

But here, we are multiplying by the upper index, right. To remember this recurrence relation, we just have to remember, if you just discard this multiplication, it is the same as the recurrence relation. So, n c r. So now, in sterling number of the second kind, we have to multiply by the lower index, in the first term s p minus 1 k, that term and the lower, k should be the multiplier and the other one, the higher index namely p minus 1 should be the multiplier. In the proof of this thing is more or less this same as the other one. We will consider s special element p, we will ask whether that appears as a singleton cycle or not. In that case, it can happen in s p minus 1 k minus 1. On the other hand, if it is not coming as a singleton, then we remove that and the remaining things can be arranged into k circular permutations, circles and each of circular permutations s p minus 1 into k ways. Now, this last element can be inserted into any of them in p minus 1 ways because, we can put it behind any, for instance, if you assign an order, so, clock wise order of something, we can think of putting this thing behind any of the element. So, that will result in a circular arrangement of p things. So, it is little fast because, we do not have time. We are just introducing this thing, but, you can figure it out yourself.

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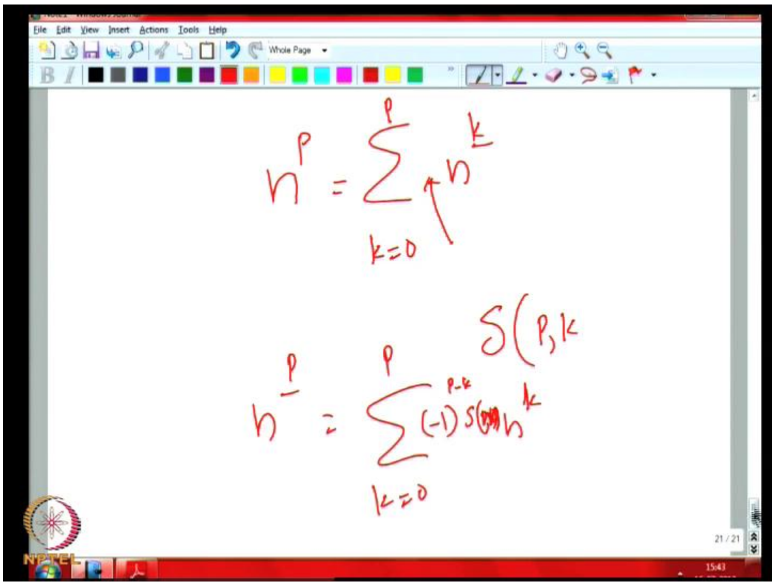

Finally, we will say that corresponding to the formula, so, we told that, when I wrote n to the power p as, so, using n k falling factorials right, k equal to 0 to p, the coefficients are actually sterling numbers or second kind. Now, on the other hand, if you write nth p falling factorial, using n to the power k, right, so, k equal to 0 to p. Except for a negative sign, we get minus 1 p minus k here. So, the other thing is the sterling number of the first kind. So, I will just show you there in the slide. This is the way it will come. So, this also you can read from, say the text book by Richard Waldie or may be any book. So, this is the last thing. We end the course by this thing.

So, thank you very much for listening to the course.