Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 4
Pigeon hole principle - (Part 4)

Today we have the lecture number four of combinatorics course, and in the last class we were discussing one final problem in pigeonhole principle namely the problem from Erdos-Szekeres, so that is what pigeonhole principle part four.

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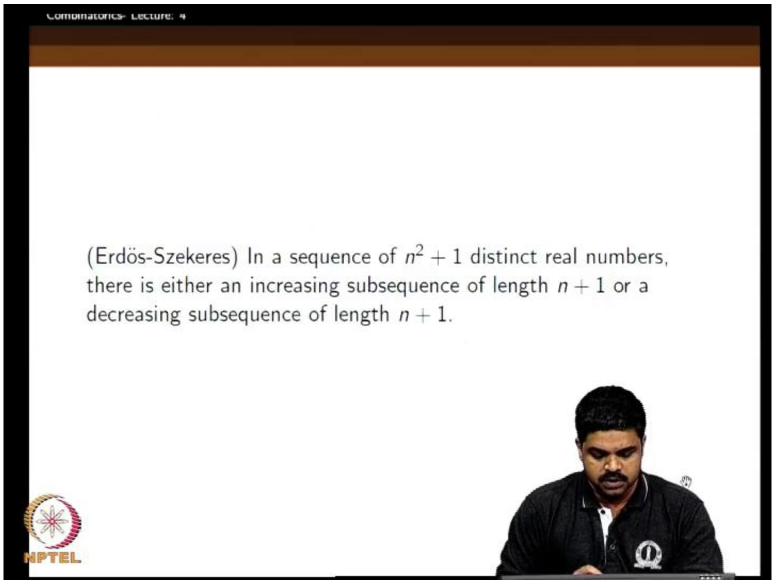

This statement says that if we have n square plus 1 distinct real numbers, there is either an increasing subsequence of length n plus 1 or a decreasing subsequence of length n plus 1, we can try with small.

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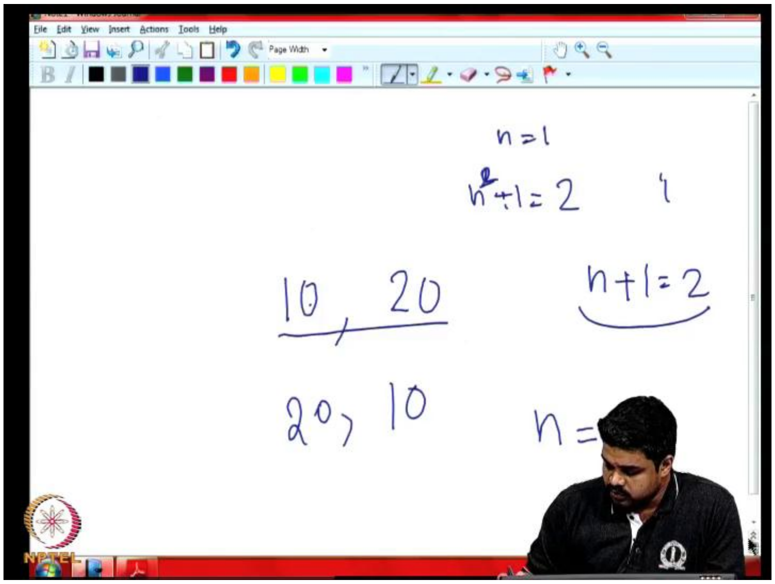

Suppose we can take, say n equal to 1, so that is n square plus 1 is equal to 2 here. So, if you take any two numbers, say 10, 20. So, in whichever way we order them, say either there are only two ways of ordering it, right, 10, 20 or 20, 10. So, in this case we have an increasing subsequences; this itself is a sequence 10, 20 it increases, so of length here n plus 1 is equal to 2. Similarly, so in this case we have a decreasing subsequence 20, 10 here, so of length n plus 1 namely two, so but n equal to 2 is not a good case because it does not illustrate the point much, so let us say n equal to 2.

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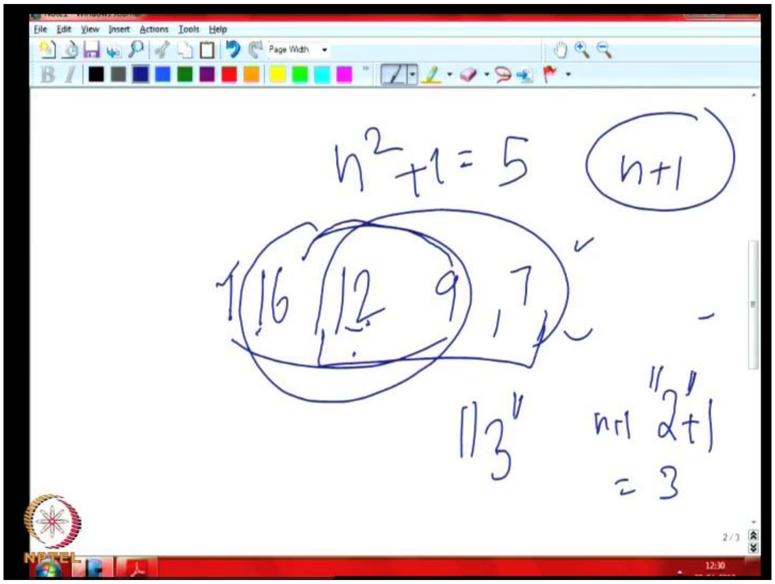

So, that means n square plus 1 equal to 5. You can take any five numbers, say, for instance we will take 1, 7, 9, 16, 12, say, five numbers. You can order in any ways, so this I have put in increasing order. So, we try to what we do is so the claims says there is either a subsequence of length n plus 1 which is increasing; subsequence means you can for instance this is a subsequence, say, just cut it off 7, 16, 12 is a subsequence or you can write 1, 7, 9 this is also a subsequence or 1, 16, 12 is a subsequence or 1 itself is a subsequence whatsoever. So, suppose if we delete some numbers from this list, the way we have ordered it, we just keep the order but we delete certain numbers from it. The remaining thing is a subsequence; that is what but what we claim here is if we have n square plus 1 numbers in this case five numbers or arrange in any order we will get two things in an increasing order or in a decreasing order, means sorry two plus 1, n plus 1 here is 2 plus 1, three things increasing or decreasing.

So, we can try it in various things; for instance, say, so this one, say, 1, 7, 9, so I can try to what do i? See 1, 7, 9, 12 here this itself is increasing, so there is nothing big about it. Anyway in fact the entire thing is increasing. Let us try to avoid increasing sequences of length three, right, so what will happen let us see. So, we can try to bring it here 9, right, 1, 9, 7, right, but then what happens so this will go away. So, 1, 9, 7, 16 but so we still have, say, 7, 16, 12 here or anything

1, 16, 12. So, we can try to bring this 12 and write here, say, but we still have 7, 12, 16 here. So, we can try to what will you do? We can try to take this and put it here, alright, but then we still have 9, 12, 16, this increasing subsequence of. So, now what can I do? I can take 16 may be this is the problem. So I will take 16 and put say here.

So, now but even 1, 9, 16 is increasing, so I will put it, say, here s16 here and 9 here. So, then what happens? 1 but 1, 9, 12 is something like that. So 1, 9, 12 is increasing so I will put 9 here, 12 is here and 9 here. So, 1, 16 of course nothing can increase from 1, 16, 1, 12, nothing can increase now. So 1, 9 but it is only 7 now. So, there is no increasing sequence of length 3 starting from 1 now. Now let us look at, say, can anything start from 16 and increase. Now 16 is the biggest number, nothing can increase beyond that. 12, no because the only biggest number is below that, so now starting from 9 we have only one number here.

So, that is also decreasing in fact. So, we have managed to kill all the increasing subsequences of length three in this thing but then the theorem says okay, if you have done that you have truly made a decreasing subsequence of length 3 which is obviously true because you can see that 12, 9, 7 is a decreasing sub 12, 9, 7. Over here we have 16, 12, 9 something like that, right. So, these extremes are you cannot avoid at it; that is what it is, as long as there are sufficiently large numbers in this case n square plus 1 numbers, right.

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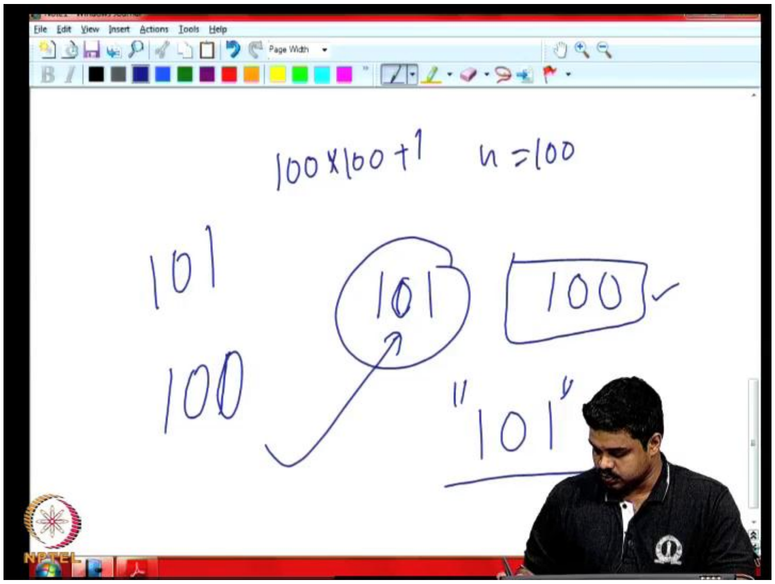

So, n can be anything n can be 100, right, n square plus 1 would mean 100 into 100 plus 1, right, ten thousand and one. If you have ten thousand and one numbers and if you arrange them in any order you have distinct numbers, right, so nothing is equal to each other and if you try to avoid an increasing subsequence of length 100 plus 1 in it 101 in it, right; that means you make sure that the biggest increasing subsequence you can get from the ordering right is only a 100 but then you are sure to have decreasing subsequence of length 101; that is what it says.

So, one of these things from the other way suppose if we avoid all decreasing subsequences of length 101; that means you make sure that every subsequence if you count I mean you have in this thing is only maximum 100, then we will definitely have an increasing subsequence of 101 is what it says, right. So, therefore this is unavoidable. One of the configuration one of the things should happen; we cannot avoid ah getting one of these two things, why does it happen and what is the proof for this? So, here we have a very simple pigeonhole principle argument for this thing; this is the way we do it, so what do we do?

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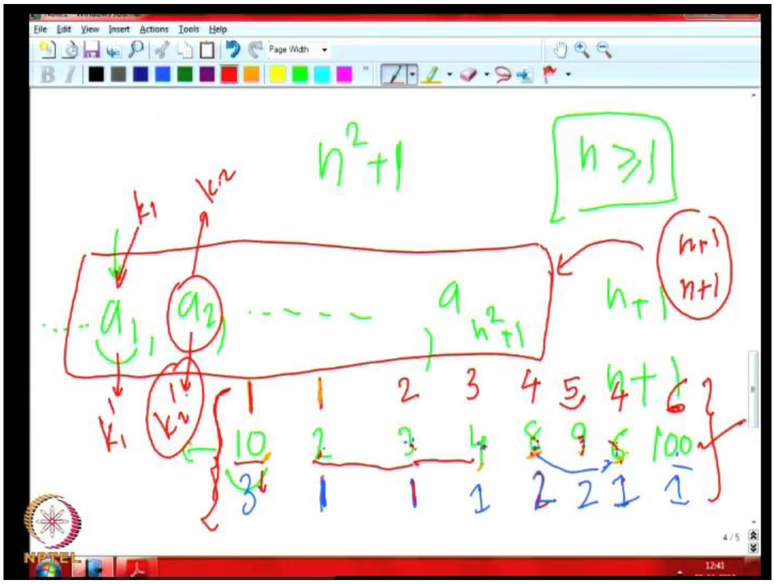

We have n square plus 1 numbers here, right, and theses things are so where n is greater than equal to 1 positive integer. Now we consider any ordering; suppose somebody claims that it does not have an increasing subsequence of length n plus 1, neither it has a decreasing subsequence of length n plus 1. Suppose somebody manages to create it, we will produce a contradiction by pigeonhole principle. So, suppose this is that ordering, say, a 1, a 2, a 3, this is a n square plus 1 are the numbers the way we see it in this ordering, right, so-called good ordering, right. Now what we can see is so we go here a 1 and see what is the largest increasing sequence ending at it. So, that will be definitely 1 because there is nothing coming from here. So, therefore this should be just a single increasing sequence if you want some number certain numbers.

For instance if I write 10, 2, 3, 4, 8, 9, say, 6, 4, sorry 4 is repetition, right, 100, right. Now if I asked what is the biggest longest increasing sequence ending at 10 here. So, definitely this is just one because there is nothing before it. So, the longest increasing sequence will be I will mark it with this write here just one. Now on the other hand I can see which is the largest decreasing sequence starting with 10. Now that will be for instance 10, 2; so nothing is decreasing from here to no, you can go further now. 10, 3 no, that is also length 2, so I can start with 10 then say what about next thing being 4, 10, 4; no, nothing is decreasing here also but then if I had taken 10 as the first, 8 as the second one, then see 9 is not possible, 6 you can take, right. 10, 8, 6 this is a three length decreasing sequence. You have a longer decreasing sequence but if you had gone 10, 9, 6 that is also equally good. So, three length decreasing sequence you can get; nothing better you can get.

So, decreasing sequence you can get here 3, right, a three length decreasing sequence you can get for this thing. Now with 2 if you ask what is the biggest increasing sequence? So, it is again 1 because you know nothing; if you start with ten it is a decreasing sequence for this thing it is not increasing till 2. So, the biggest increasing sequence ending with 2 is just one length and the biggest decreasing sequence ending with 3 is also one, right, because if you start with 2 nothing is decreasing here; with 3 here you will get two because you know 2, 3 this a increasing subsequence, right, 2 to 3 it is increasing. So, on the another hand if you look for the decreasing sequences here, so see starting from three nothing is decreasing here, right. It is just 1 because 3, 4 it is only increasing, 3, 8, it is only increasing, 3, 9 it is increasing, 3, 6, it is increasing.

So, now with starting with 4 we see that the biggest increasing is 2, 3, 4, you have this much 2, 3, 4; that means you can write 3 here that is an increasing sequence of length 3 starting here but here we have 4, no nothing is decreasing. So it is just one starting with 4 there is only one length decreasing sequence but with 8 the thing is like with 8 we will get 2, 3, 4, 8. Here it is four but starting with 8 we have 8, if you go to 9 it is increasing. 8, 6 it is a two length decreasing sequence here, so 8, 6 like this. Now with 9 we have again 2, 3, 4, 8, 9, there is a five length increasing sequence ending at 9 but for decreasing sequence it is again 2 because 9, 6, right; with 6 we have 2, 3, 4 and then 6; that is a four length increasing sequence, decreasing sequence is just one, starting with six only one decreasing sequence is there. With 100 we can take the biggest here, right, 9; that is five, 2, 3, 4, 8, 9 and then 100; that is six here, right. Then here we have only one, right, because starting with 100 there is just that 100, there is nothing decreases beyond that. So, we get a certain collection of numbers here above it above the numbers which are written in red color here.

So, red color here, right, indicates the biggest the longest increasing sequence which ends at that number starting from left of it and then ending at that number, right the numbers which we have written below this using blue color; it indicates the longest decreasing sequence that you can get starting with that number towards the right. So, this is the numbers; for instance here this is to illustrate this thing look at 9, 9 the longest increasing sequence which ends at it is of length 5. For instance you can start with 2, 3, 4, 8, 9, it is a long five length sequence which is ending at 9 but on the other hand the longest decreasing sequence starting from 9 is only 9, 6, nothing more, right. So, that is what.

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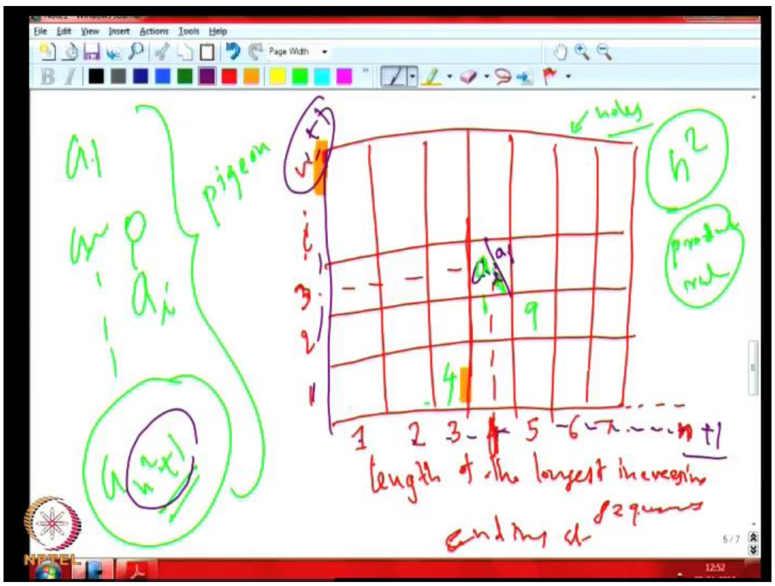

Now what we do is we make a table like this, table is like this, right. So, see here of course I just illustrated what this numbers are I mean what we are looking for using some sequence of numbers, right, but the person who claimed that their excess that Erdos-Szekeres theorem which we are trying to prove now is wrong; that means he can come up with a sequence four n square plus 1 distinct real numbers such that there is no increasing sequence of length n or no decreasing sequence of length n plus 1. So, suppose we have written this numbers like this, this is the way sequence; see this is the actual sequence we have written here 10, 2, 3, 4, 8, 9, 6, 10, 100 is not a good example because one, two, three, four, five, six, seven, eight numbers here. So, to illustrate n square plus 1 we should have taken a correct number, so this n is only 2 here, right, 2 square plus 1. It is already bigger than 5, so we do not look at this stuff. This was just to illustrate a point that we are trying to consider certain kind of numbers but here whatever we did here we can do here also.

So some number k 1 we will write here, some number k 1 dash we will write here, some k 2 we will write here, some k 2 dash we will write here like that. This k 2 will indicate the longest increasing sequence which is ending at a 2 and this k 2 dash will indicate the longest decreasing sequence which starts from a 2, right. Now what we are going to do is to create a table like this. What does table signify? This table signifies see here the length of the longest increasing sequence ending at a number, this is what we are going to mark here, right. So, here these numbers we will write because you know these numbers on the horizontal axis this I will write it as 1, 2, 3, 4, 5, 6, 7, this corresponds to, up to here it can go n, up to n because you know we are saying that there are no increasing subsequence of length n plus 1 here.

So, here we will for instance this is like this 1, 2, 3 up to n. So, we will for instance in the i j th this is i and this is, say, j so we can put maybe 3, 4. In this square here if we start to identify the square, so we will write a number a j here if the longest increasing sequence ending at a j coming from the left and ending at a j is 4 and a the longest decreasing sequence starting at a j and going towards right okay, so that is 3, right. So, that means what we are going to do is in this squares we are going to place the numbers say a 1, a 2 up to we have a n square plus 1, n square plus 1 numbers are there. These numbers we are going to place in these squares. So, a particular number a i will be taken and then we will look at what was written above it; remember in the previous thing we were writing the numbers above it and number below it, right.

This above number we will find and corresponding column we will find out; that means this is the first column, second column, third column, corresponding column we will find out in that column and then we will look at the number which is written below. So, this for instance is 4, this 3, 1; 4 will be written in 3, 1 here we will write 4 here, right. For instance this 9 was 5, 2; 5above and 2 here, right. 9 will be written in 5, 2 here, here 9 will be written. Like that each of this a 1, a 2, a n square plus 1 will be written on this squares, right. Its column number will correspond to the longest increasing sequence ended the length of the longest increasing sequence ended at that number and the row number will correspond to the length of the longest decreasing sequence starting from there, right, that way.

Because we have two numbers corresponding to each a i in the sequence each number in the sequence, so we can find out a square for that, right; each one will go into one square here but the point is that because we know that there is no subsequence increasing subsequence of length more than n, n is the biggest, so these numbers are only 1, 2, 3, the number of columns are only n. Similarly here we have only n, yeah, we have only n rows also, right. So, all the numbers should occupy one of these n square n into n n square columns, right.

So, there are n rows here, n columns here, how many squares are formed like that because any square should be in one of the column, there are n possible columns and each column can contain n squares. So, n into n this is called product rule. So, by this applying this product rule so we can see that there are n squares only, so these numbers should occupy n square but then there are n square plus 1 numbers. So, considering these as pigeons, right, this a i's as pigeons, and the squares where it has to go are the holes, these squares are the holes here, right. These are the holes. Now there are n square holes and n square plus 1 pigeons.

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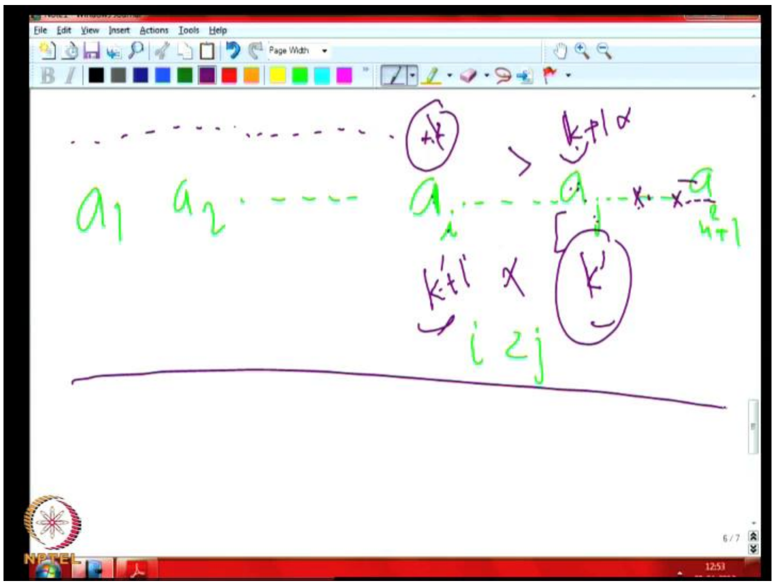

So, two pigeons should occupy the same column; that means there should be some number a i and some number a j. Let us say without loss of generality j is greater than in that ordering, so they are starting from a 1, a 2 like this. So, a i and then a j, so until a n square plus one, right, which occupied the same square; which occupied the same square means, say, may be this a i, yeah both this a i and a j came here in this column; that means the number which was written above it in that sequence was also same maybe here 4 and the below it was also the same for both a i and a j, right.

So, here for instance we had marked a number here, some k i was marked here, some k i dash was marked here. This corresponded to the longest increasing sequence starting from here and ending at a i and this corresponded to the k dash corresponded to the longest decreasing subsequence which is starting from a i and going towards this thing, right, longest subsequence. Same number will come for a j also, this k j and k j dash, right, corresponding to the longest subsequence which is a going towards this and sorry longest subsequence increasing subsequence which are ending at a j and this will correspond to the length of the longest decreasing subsequence going towards this and starting with a j, right. So, we are saying that this a i and a j occupied the same column; that means k i should be equal to k j, let me just write k for both of them, right, and then here this should be same because there the lower numbers are also same; that is why they went to the same square, right.

Let me write just k dash for this both this thing, right. So, there is problem here but then a i, a j they are different numbers. Suppose this a j was bigger than a i, this a j was bigger than this thing, then there is a contradiction in this instance if you count, say, the number of l m n’s in the biggest increasing sequence up to here a i, so that was k. Now with that increasing sequence you can put a j also; after a i in that increasing sequence we can put a j also, we will get one increasing sequence ending at a j with at least k plus 1 elements in it. So, how can it be k; so this was wrong. So, that means this was definitely like this, this was like this; that means a i should be correct.

Then here there is a problem because the meaning of this number is that the longest decreasing sequence starting from a j. So, something like a j here I identified a number like this, like this if it goes, there was k dash numbers starting with a j and decreasing towards this, right, the longest one but here a i also says it is the same for it also; k dash numbers starting from a i but then a i could have been attached just before the longest decreasing sequence some starting from a j, right. This sequence a j this one we could have started from here, a i, a j and these things that would have given this k dash, here you should have got k dash plus 1 because this itself is k dash so we should get one more here, right, but we say that these are equal this two are equal. So, that is a contradiction, right.

So, it is not possible for two pigeons to occupy the same hole is what we have seen because either this has to be greater in which case one of these two things have to be different. If they are same then that means it was the opposite this was bigger, right. Now one of these two things have to the same it would be different. It is not possible for this pair and this pair to be same. If this both the pairs were same then only we can put both this a i and a j in the same hole namely same square there in the previous picture like this, right. But then that is bound to happen if there are only n square if there are n square pigeons here, n square numbers distinct numbers here and the hole are only n square, right, n square plus 1. It is a very nice application of pigeonhole principle here.

So, what can we infer from that there is something wrong in assuming that there are only this many holes. So, that means either here we should have in this horizontal this thing; that means the number of columns should be n plus 1 or more or the number of rows should be n plus 1 or more, right, n plus one or more. So, that is what we can. So, number of columns being more than n; that means n plus 1 or more means that we should have, so we should allow the number of increasing sequences to take values from 1 to n plus 1, right, 1 to n plus possible values it has to allow, right. It not just n, right, if the biggest increasing subsequence was only n, then here you can only have n possible values right but that is wrong.

So, we should have n plus 1 possible values; that mean they should be one increasing subsequence with the n plus 1 elements in it. Similarly if that is not true the number of decreasing subsequences which are marked in the rows, right, that should have n plus 1 possibilities; that means there should be a decreasing subsequence of length n plus 1, this is what it says. Now if you carefully look at a these statements, so we can see that here we can slightly change the statement namely instead of n and n we can use r and s here. See earlier we told n square plus 1 distinct real numbers; now let us say r into s plus 1 distinct real numbers. Then there is either an increasing subsequence of length r plus 1 or a decreasing subsequence of length s plus 1, in generalization we do not have to say n and n so that symmetry can be broken, right; that is what it says. So, that proof is same because we will do all the same things.

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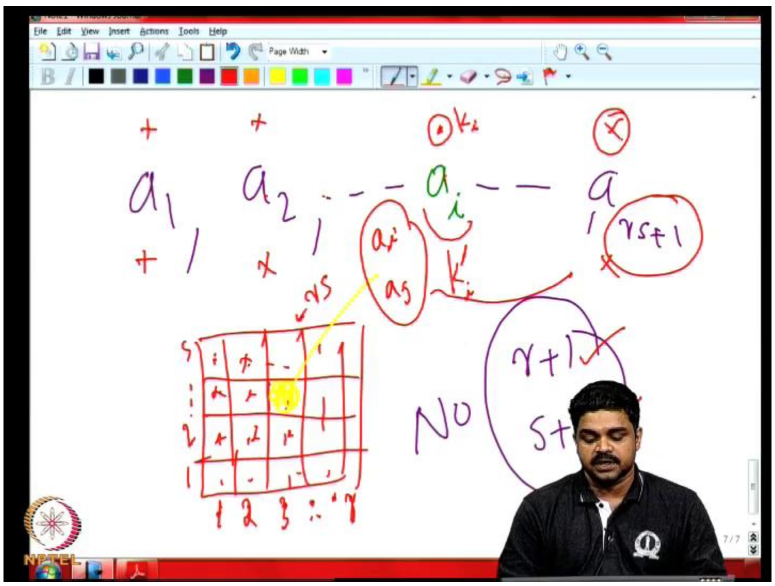

So, these are the real numbers. Suppose you got a sequence which does not satisfy this property; that means a r s plus 1 numbers are written such that there is no increasing subsequence of length r plus 1 and there is no decreasing subsequence length s plus 1, right. In that case we can show a contradiction by again writing say some number here, some number here, some number here, some number here like this. So, what does it mean? The upper numbers the numbers what we write here indicates for instance for a typical a i the number we write here this k i indicates the longest increasing subsequence that ends here up to here. So, the biggest one we take, right.

Similarly, the number we write here k i dash this will indicate the longest decreasing subsequence which ends here, right. Now we again draw that; now this time a rectangular grid we draw and here these columns will correspond to the numbers which we are writing here. So, if we do not have any r plus 1 length increasing subsequence, it is very clear that this can be 1, 2 up to r only. If we do not have s plus 1 length subsequence it is clear that it is 1 to up to s only, right. The total number of squares possible will be r into s this number, right, number of squares will be only r into s, right, but we know that there are r into s plus 1 numbers distinct numbers here. So, these being the pigeons and these being the holes, right, the number of holes are one less than the number of the pigeons.

So, two pigeons should be occupying one of the holes wherever it is, maybe it is this, right; maybe it is this here, right. Some a i and some a j has occupied the same hole here, right; that means both of them got the same pair of numbers above and below it, right. So, here above numbers are same below numbers are same, right. So, then we saw that because those numbers are distinct either a j is greater than a i or a i is greater than a j. In both cases there is a contradiction either from above or below, right.

So, that we have seen because we can if we assume that if it is above if you assume that up to here the longest increasing sequence was this number, then because if a j is bigger than that we can add a j also to that increasing sequence you can get one more, so that the assumption that the increasing sequence length of the longest increase subsequence ending at a i and a j are same is wrong because a j being bigger than a i can be added; that means a j is smaller. So a j is smaller than and in which case we will get a contradiction from the lower part, right. The longest decreasing subsequence starting from a i will be at least one more than the longest subsequence starting from a j that is what, so this way we can generalize it.

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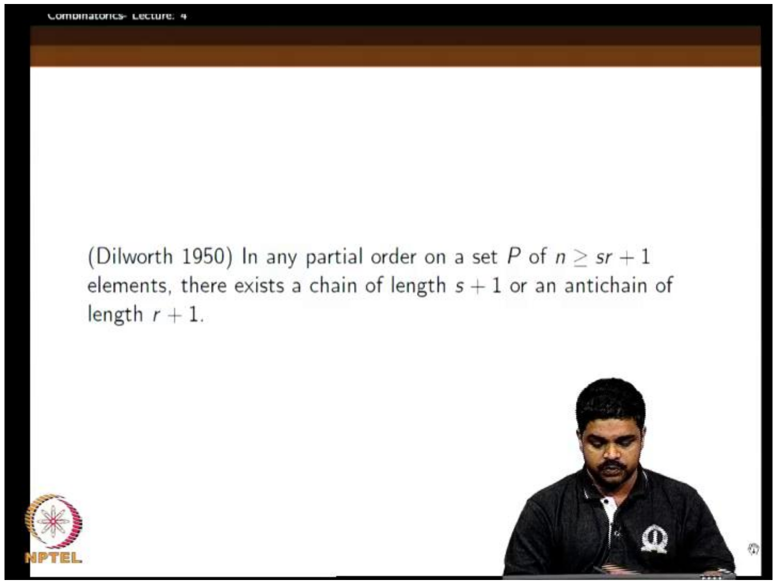

Now we will see one more generalization of this theorem for partial orders. So, the partial order is also a notation I assume you should know before coming to this class so I advice you to go through the chapter in Grimaldi and Ramana. So, we have the introduce relations, partial orders and things. So, there will be some lot of examples there, you can just have a look at it or any other discrete Math books; the first chapters will introduce relations and so there they will talk about partial orders, right. So, I am assuming that you are to some extent familiar with it not like just the notion is this thing but still I will just remind you what it was partial order. The partial order means, so there are certain objects.

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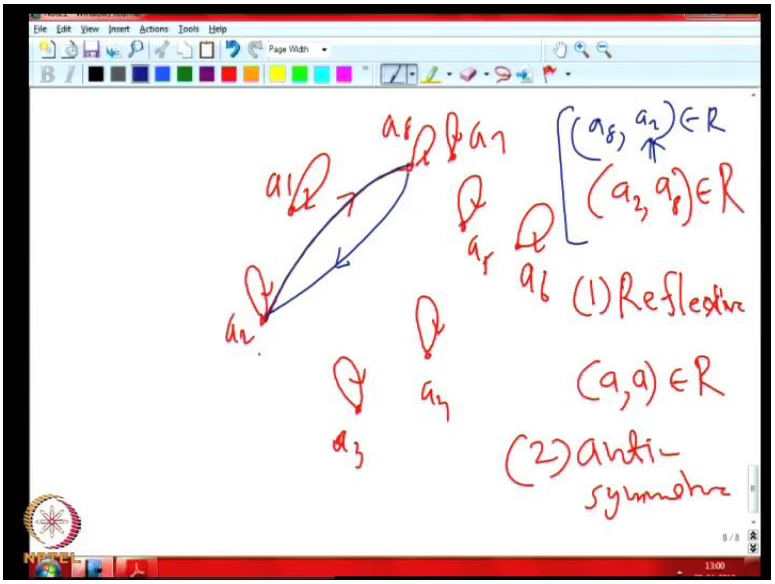

Now we have a relation on this objects means maybe let us draw this objects here these are just by using points let me this thing. When i say that there is a relation I will put an arrow between them. So, typically this is a 1, a 2, a 3, a 4, a 5, a 6, a 7, a8. So, this arrow means this a 2, a 8 belong to our relation now, right, and the kind of relations we are interested in is when they satisfy three properties. One is they should be reflexive; when i say reflexive that means for each of them I have this kind of arrows; that means so for everything we have this kind of arrows, okay; that means we have all the pairs of the form a, a in r and now the relation is antisymmetric.

So, when I say the relation is symmetric we mean that suppose there is a relation here; that means this arrow is there; that means a 2, a 8 belongs to the relation then a 8, a 2 this also should be there, a 8 and this would imply that a 8, a 2 also belongs to r; this is what is symmetric. This is what we mean by stating that a relation is symmetric; antisymmetric means this is never true, means for the relation to be symmetric this should be true, whenever there is an arrow from to this thing that reverse arrow should be there; that means whenever there is a pair a, b b, a also should be there.

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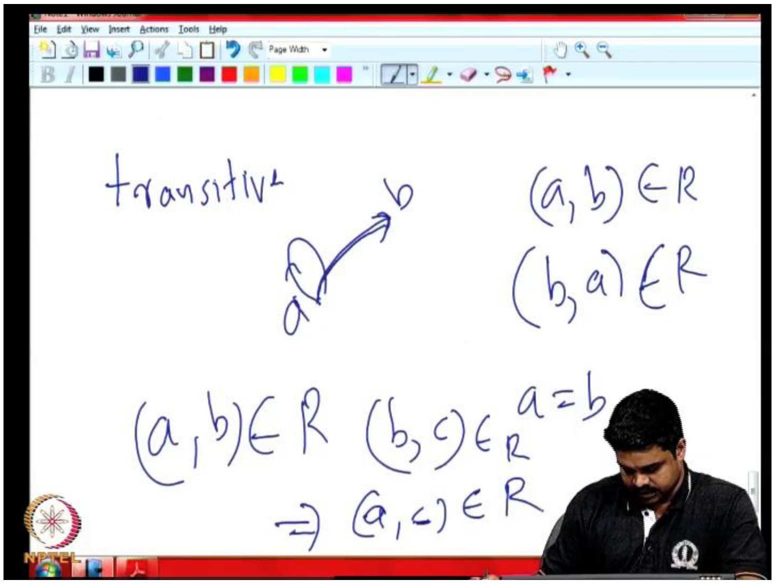

Antisymmetry says if like if a is related to b then b will never be related to a. In case we get a, b is r and b, a both are in the relation, then that would mean a equal to b; only in the case of this that means a and b being same, then only we have that pair where a, b and b, a together, right. So, in other words when we draw this graph diagram, so here for instance this will never be there; for instance we are clear that it is only we will put this edges always one direction will be there maybe this thing not both, right.

Whenever between two points we have drawn here then we draw an arrow here, the arrow may not be there but if at all arrow is there then there will be only one direction; it is not that both ways we can put the arrows, right. So, that is antisymmetry and the third thing is something called transitivity, the relation is transitive. What do you mean by relation is transitive? It means if a, b element of the relation a, b is in the relation and b, c is also in the relation and then a, c also, then that should imply that a, c also is in the relation.

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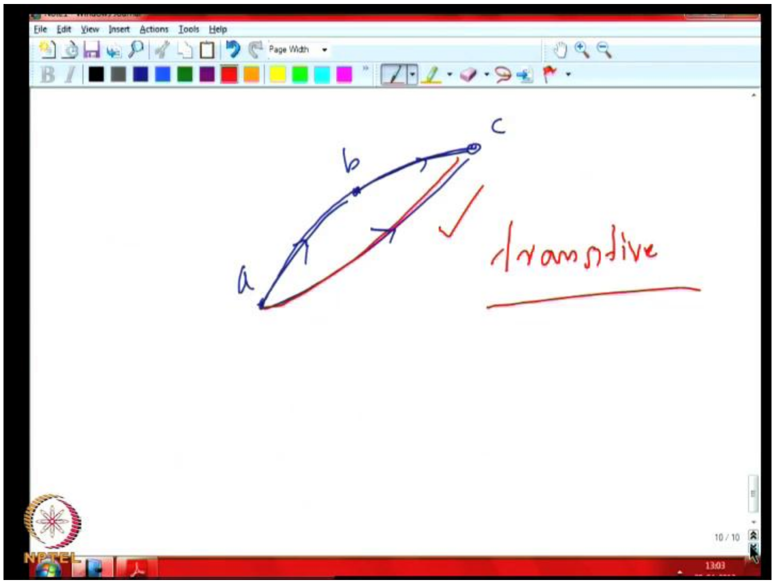

So, in the diagram form it should be much easier to see. Suppose we have this belongs to the relation, right, and this also belongs to the relation. So a b c, then this should be there in the relation that is what it claims, claims and asks for it; that is when if for every such a situation that a, b belong to this and b, c belong we also have this one, then we will say it is a transitive relation, right. Partial order has these three properties; a relation is called a partial order and we have when the relation is reflexive, antisymmetric and transitive, right. So, remember we will call it an equivalence relation when instead of antisymmetry we have symmetry, okay; instead of this thing we have symmetry, right. So that is the slight difference, right.

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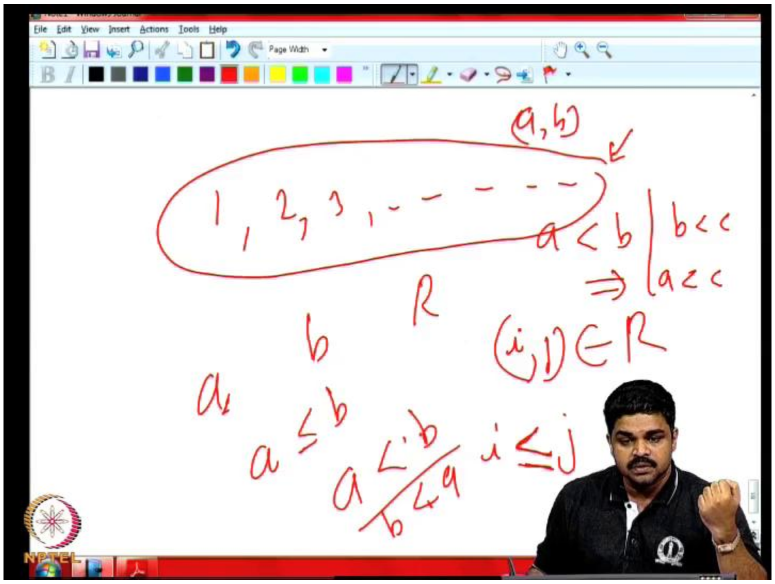

For example we can see that if we take natural numbers, say, 1, 2, 3 so the set of natural numbers we can just define a relation in this thing, so I say I, j belongs to r if i strictly less than j, the natural order which we have on the natural that is the relation, right. So, you know by that relation every pair is related, we will say that every pair is comparable, right. So, if you take any a and b, right, so we can say like this, sorry less than equal to any a and b even when a equal to b, so we have a relation a less than equal to b or b less than equal to a, never both. It is reflexive because a is less than equal to a; it is antisymmetric because if a not equal to b and if a less than b, then we can never have b less than a also, only one of these two things true, right, antisymmetric and transitivity is very clear because if a less than equal to b and b less than equal to c, then that would mean a less than c, right.

So, therefore this natural order that we have in the natural numbers or real numbers that is a partial order; actually it is a total order, because their partial order is called a total order when every pair is comparable, between every pair there is that relation is valid either a, b belongs to the relation or b, a belongs to the relation. It is never true that both neither a, b nor b, a belongs to the relation, right. Such a relation is called partial order. Now so for instance this was a total order what we consider now the example but non-trivial example can be obtained.

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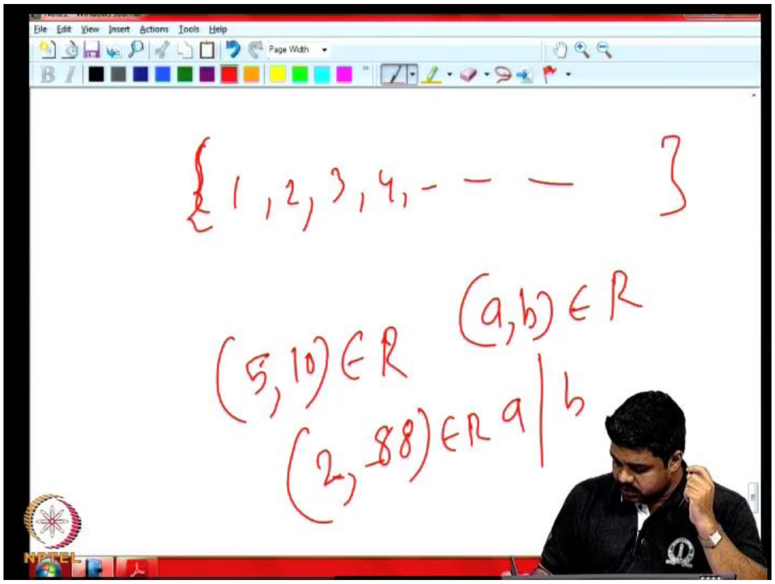

Suppose if we take the natural numbers say 1, 2, 3, this is a set of natural numbers 1, 2, 3, 4; here we can define a relation by, say, a, b belongs to the relation if a divides b; that means a is a factor of b. For instance 5, 10 would belong to our relation because 5 divides 10 or 2, 88 will belong to our relation because 2 divides 88, right. So, now here we can see that this is a partial order.

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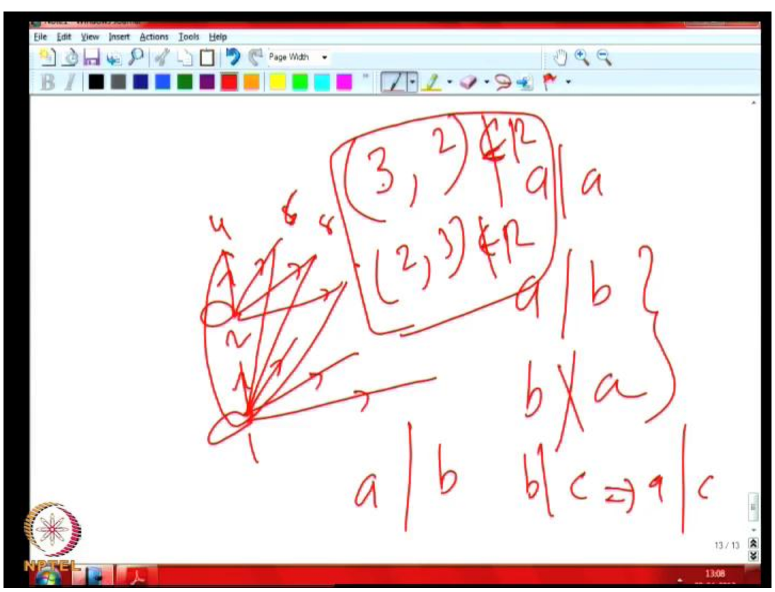

So, let me for instance for some numbers I can write it 1; 1 will be a divisor of everything, right. So, we have this arrows for everything and 2 will be a divisor of all even numbers, so all 4, 6, 8 like this we can if you draw it; of course we cannot draw everything, so this also of course here it will be to this also, right. So, now this is the way the partial order you will see; of course we can check whether it is a reflexive antisymmetric transitive relation. It is very easy to do because a divides a, so there the reflectivity is coming. See if a divides b that means b is bigger number than a, and assuming that a naught equal to b.

Then b does not divide a, right, and therefore that there is an antisymmetry here; either a divides b or b divides a, never both, right. It is possible that either a divides b or b divides a but if a divides b then b cannot divide a, right, that is antisymmetry and the transitivity is also apparent because if a divides b and b divides c, a has to divide c also, right. So, here definitely this is not a total order because there are many pairs which are not comparable; for instance if we take 3, 2 neither 3 divides 2 nor 2 divides 3; that means this is not in r, this also not in r, right. Such pairs are here. Here in this case we cannot say that is a total order, right. I just gave another example where partial order need not be total order, right.

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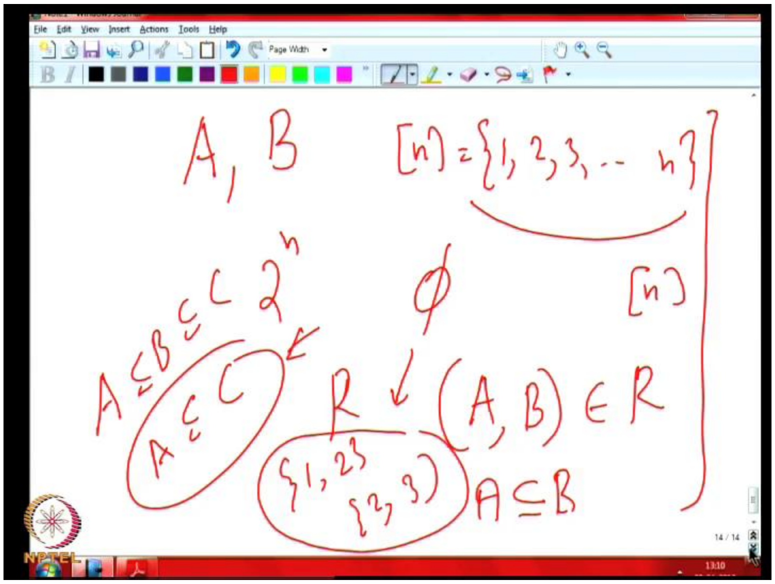

So, we can other several other most important example is set system. So, for instance we can take n that means 1, 2, 3 up to n and then you can consider all subsets, say, 2 to the power n subsets; we have already mentioned that there are 2 to the power n subsets available for the set this n element set, right. Now because this phi there is the full set itself and then there is so many of them to remind you and now we can have a relation defined like this. So A comma B, A is one subset of this thing, B is another subset of this thing. This belongs to r if A is a subset equal to B, right. So, here also we can say that it is a partial order because reflectivity is easily checked because A is subset of A for any subset A and if A is a subset of B and B not equal to A, then B cannot be a subset of A because B is bigger now, right, cardinality wise bigger.

So, antisymmetry is true and similarly it is very easy to see that if A is a subset of B and B is a subset of C, then A has to be a subset of C, right. So, therefore transitivity is also true here, this is another thing. Here also we can see that this is not a total order because we can easily find pairs of subsets of the form A, B such that neither A is a subset of B nor B is a subset of A; for instance we can consider these 1, 2 and say 2, 3. So, 1, 2 is not a subset of 2, 3 nor is 2, 3 a subset of 1, 3. So, they are incomparable elements. So, therefore since all pairs are not comparable so we say that this is not a total order but it is a partial order, right; all partial order need not be total orders; that is what I told.

Now coming back to our question we gave a theorem for Dilworth. The theorem is this; in any partial order on a set P of n greater than equal to s into r plus 1 elements, there exists a chain of length s plus 1 or an anti chain of length r plus 1. So, now whatever partial order I talked about initially that natural numbers, they have infinite number of elements and similarly the second one the division partial order are there. So, one is a divisor of other. There also I consider the entire natural numbers the elements for the partial order.

The third one was an n element set the subsets for the member of the partial order; that is only 2 to the power n that was finite the finite number of elements only were there in that partial order, right. So, here we consider a partial order on a set where the number of elements is greater than equal to s into r plus 1; s and r are two positive natural numbers, say, at least s r plus 1 elements. Then it says there exists a chain of length s plus 1 or an anti chain of length r plus 1, then what is a chain? In a partial order, a chain means it correspond to an increasing subsequence for instance the natural order corresponding to this thing, any increasing subsequence is a chain, right.

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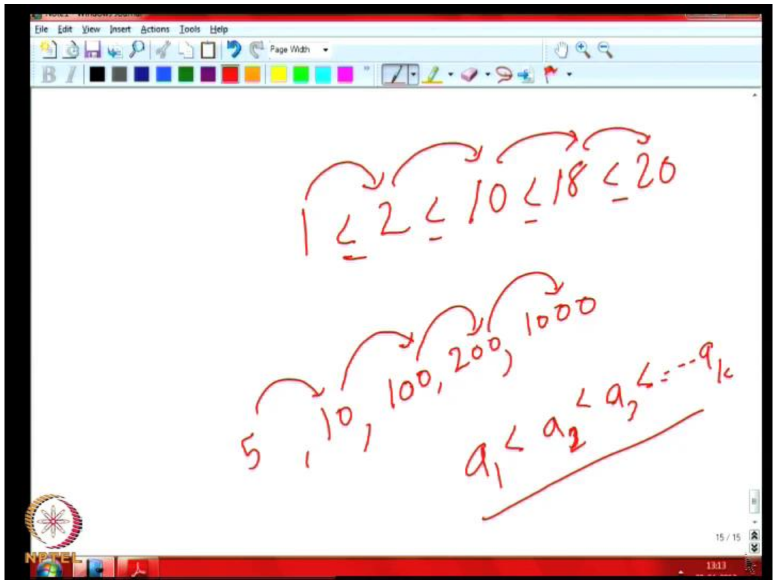

So, for instance you can say here 1, 2, 10, 18, 20, this is a chain because it is like this. You can put the relation like this, right. So, always the relation is forward, right. In the division partial order for instance 5, 10, 100, 200, then 1000, right, this is a chain because 5 divides 10, 10 divides 100, 100 divides 200, 200 divides 1000 and so on, right. A chain means some members of the partial order, so we can let us say arrange it in this way, right, a k in set such a way that this is related to this, this is related to this, this is related to this and so on. This is called a chain and we can talk of longest chains here, right, which is starting from some element ending at s 1 element something like that, right. So, it makes sense; if the partial order is finite definitely it makes sense, right.

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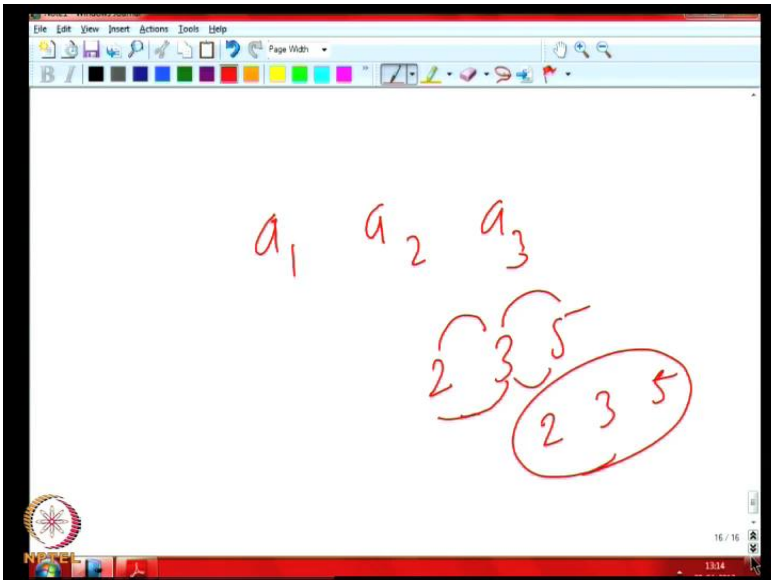

Now the antichain; antichain means a collection of elements which are pair-wise incomparable. For instance we say that these three elements form an antichain and a 1 is not related to a 2, a 2 is not related to a 3 and neither a 3 is related a 1. For instance in the division relations and in divisor relation we considered in the partial order defined by a divides b. So, you can take these 2, 3, 5 for instance; 2 does not divide 3, 3 does not divide 2, these are incomparable, similarly 3 and 5 are incomparable, similarly 2 and 5 are incomparable. So this 2, 3, 5, they do not have any connection between each other, right, any relation between each other, right. So, for instance if we take the set of prime numbers they will form an anti chain, because no prime number will divide another prime number, so that is what, right.

So, if you want to look at the graph form; for instance initially we drew this kind of a graph to represent the partial order saying that for every pair which is in the relation, we can put a directed edge. Then an anti chain will correspond to an independence set, right, something like this, right; between any pair this yellow things there will not be any connections. So, forget about direction edge itself will not be there, right. They are not comparable, right; that is anti chain. Chain will correspond to what here that is interesting think to think about. Of course it will correspond to direct path; more than that it will correspond to a complete graph here which a student can think about, right.

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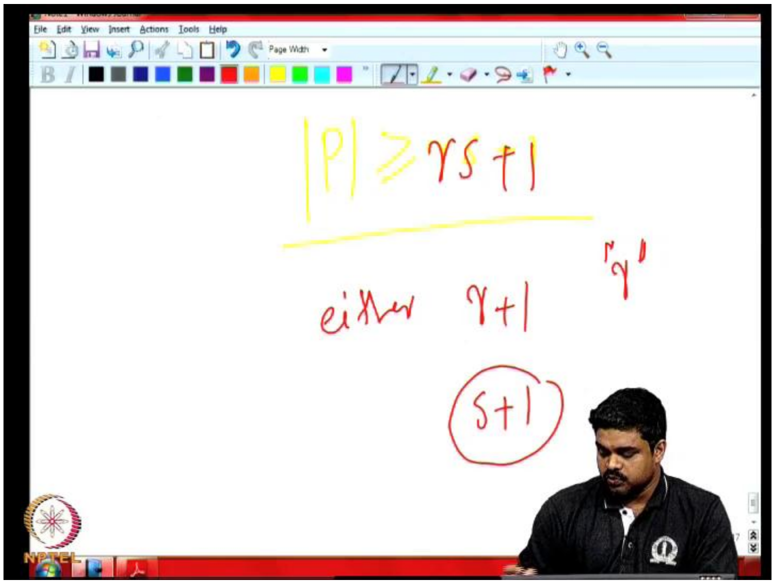

Now coming back to our problem, so the statement says suppose the number of the elements in the partial order is at least r into s plus 1 r s plus 1, right, then either we have a chain of length r plus 1 and actually if we do not have any chain of length r plus 1, all possible chains are of length r only, then we are sure to have an anti chain of length; length means cardinality s plus 1, how is it true? So this is the way to look at it. So, we can consider each element like almost the same way we considered the proof of Erdos-Szekeres slightly because now we are not talking about a total order but we are a talking about partial order, right.

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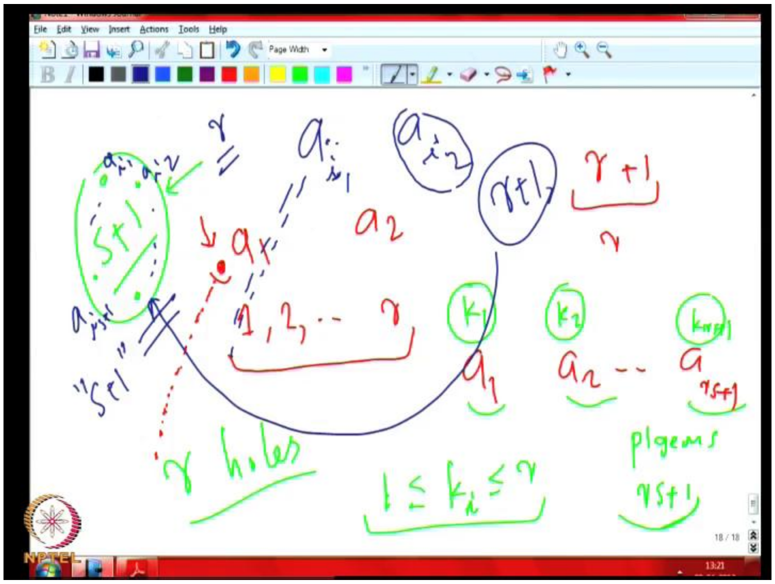

So, for instance we can take element a 1 and then ask which is the longest chain ending at a 1, you now because we are assuming for contradiction that there is no chain of length r plus 1; that means the biggest has to be r. So, whatever it is the biggest chain ending at this a 1 has to be of length utmost r; it should be some number that length should be a number 1 or 2 up to r, right. It cannot be zero because at least a 1 itself is there. So, it can be 2, it can be 3, up to r it can be. So, now for each element we can calculate and for instance so these are the members a 1, a 2, a r s plus 1 are the members of the partial order. For each of these members I can write that number above it, say, namely this is the biggest longest chain which ends at a 1, right, up to a 1 this is the biggest chain biggest possible chain.

So, some number k 1 will be written here, some k 2 will be here, some k r s plus 1 will be written here; not that each k i is in between 1 and r. So, r possible values are there. So now this a 1, a 2, a r plus 1 we will consider these are the pigeons, we have r s plus 1 pigeons and the holes are this possible values that k i can take; that means 1 to r. There are r holes here, r holes and r s plus 1 pigeons. So, we can apply our generalized pigeonhole principle here. See r s plus 1 pigeons and r holes, then that means they should be at least one hole in which s plus 1 or more pigeons re have occupied. So, one hole which is occupied by s plus 1 or more pigeons because all the holes were occupied by utmost s pigeons only; since there are only r holes r into s pigeons only will be there but we have r s plus 1 pigeons, right. So, now this s plus 1 pigeons we claim that this s plus one pigeons they correspond to some numbers, right, a i 1, a i 2, say, a i s plus 1.

So, these pigeons should have a property; that means the property is that they form an antichain that is what we have trying to tell. There will be s plus 1 sized antichain is what we want to prove but why is this antichain? This is an anti chain suppose this some a i 1 and a i 2 we take from this thing; they all claim that the longest chain which ends at them is of length r. Now suppose there was a relation between this and this; that means this was related to this, a i 1, a i 2 was in the relation; that means in our graph diagram if we could put an arrow here suppose for contradiction. Then the chain which ends at a 1, we could have extended to a longer chain which such that it ends at a 2, right. This we will read out this chain, reach a 1 and then read out a 2; that will be a bigger chain in the a 2, then how can this a i 1 and a i 2 have the same length for the chains the longest chains ending at them. This should be one more right one more at least can be even more; so that is a contradiction.

So, this edge cannot be there; that is what we are saying. This edge cannot be there, this edge cannot be there, right. So, therefore any pair if you take they cannot have an edge between them because the reason is one of them is related to the other mean there is an arrow from one to the other in that graph diagram for instance, then we can extend the chain ending at the former with a the latter getting that bigger chain for that. That will contradict the assumption here; for instance from here what we are getting is all of them are in the same hole, all correspond to one number, that number correspond to the length of the biggest chain longest chain which ends at them, right, and they have to be different but then here they are same. They are in same hole, right, same length it has to be.

So, therefore if we put an edge between them they have to be different. So, that means we cannot put an edge between them. So, we infer that ah there is an antichain of length s plus 1. So, the contradiction came because of assumption that there is no chain of length r plus 1 or more, the biggest chain was of length r, right; that means either we have a chain of length r plus 1 in the thing or we an antichain of length s plus 1 in this. So with this, this was the last problem we wanted to consider in the pigeonhole thing but though we took several lectures for pigeonhole principle but along with this thing we introduced lot of concepts like we introduced a little bit of graph theory and we talked about partial order, we talked about some properties of numbers, right,

So, that is an initial thing we spend on several things but you are supposed to read these topics in more detail from either beginning first chapters of Grimaldi and Ramana or one of those other discreet math’s books you are following in your university, right. So, which ever book is okay, so just be familiar and comfortable with this terminology, what is a relation, what is an equivalents relation, what is a partial order, what are the main things about graphs like degrees, edges, so like independent sets, some preliminary notions about graphs and then even this what is a permutation such notions you should be aware. So, we will meet in the next class.