Combinatorics
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Lecture - 39
Difference Sequences

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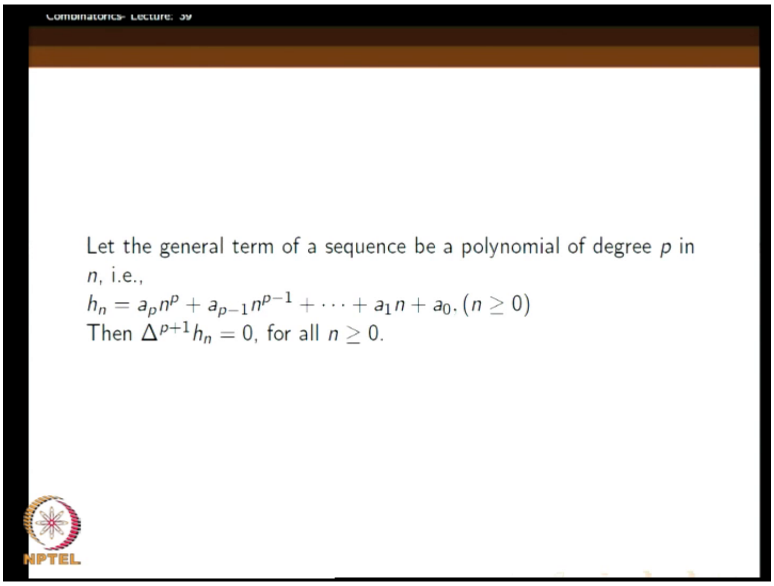

Welcome to the 39th lecture of combinatorics. In the last class, we had seen different sequences. So, this was the thing.

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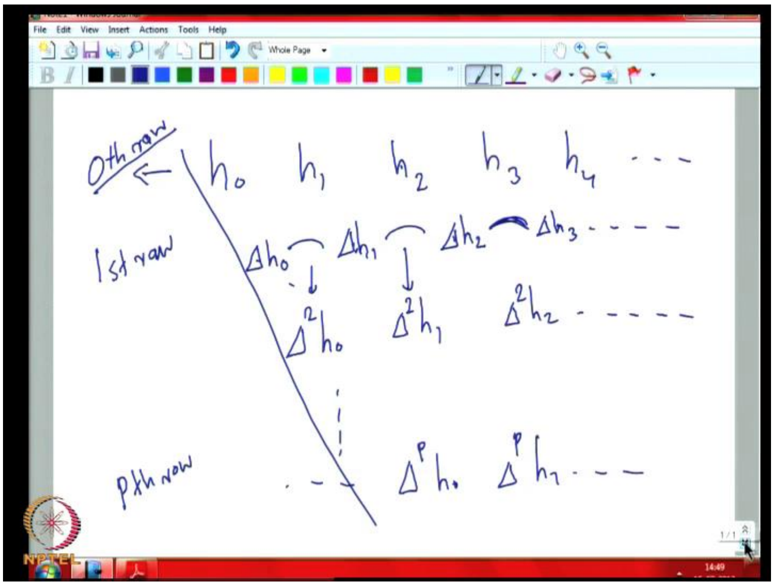

So, we had some sequence, say something like h 0 h 1 h 2 h 3 h 4 and so on. The first order difference sequence, say written in the, this we say that is the 0^{th} row and it is the 0^{th} order difference sequence. So, the sequence itself is called the 0^{th} order difference sequence. Here, it is h 1 minus h 0, which we write as delta of h 0. We will write delta of h 1 for h 2 minus h 1 and delta of h 2 for h 3 minus h 2 and so on, right. So, this will be delta h 3. So, this sequence in the first line, first row, is the first order difference sequence.

Now, if I take the difference between delta of h 1 and delta of h 0, delta h 1 minus delta h 0 is delta square h 0 and this is delta square h 1. This is delta square h 2. This means, delta, so, the gap here, right. Similarly, this corresponds to the gap here. This corresponds to the gap here. This minus this; this minus this; this minus this, like that. This sequence is a second order. Like that, in the pth row, we will have the pth order difference equations. So, this will be like here, somewhere here. pth order delta raise to p of h 0 delta raise to p of h 1 and so on, right. So, this table is called the difference table. It is a difference table. This is what we saw in the last class. These concepts were introduced.

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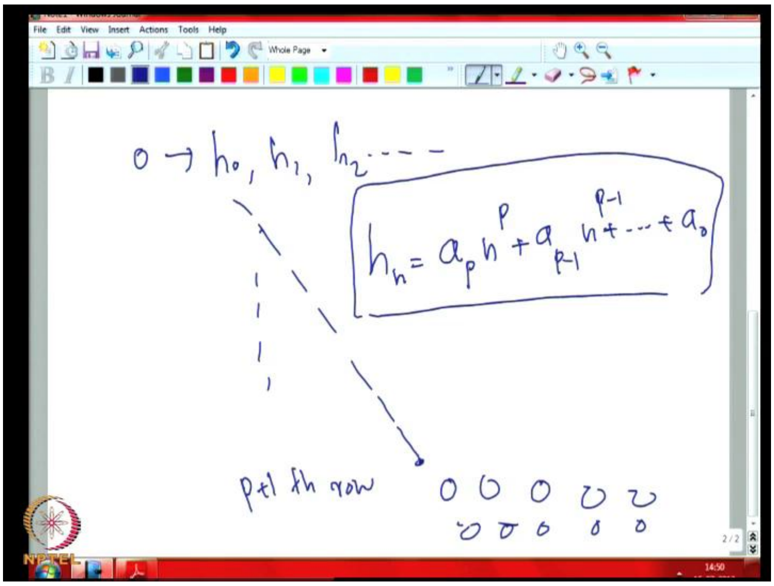

Now, we will look at one property of the difference table, when the underlying sequence h 0 h 1 h 2 etcetera are given by the general term h n, say it is a polynomial in pth degree, say n p, say it is like something a 1. So, suppose it is something like a p into n raise to p plus a p minus 1 into n raise to p minus 1 p minus 1 plus, like that until a 0. So, if this general term can be represented using a polynomial, the pth degree polynomial, in n, so, your a p not equal 0. So, that is why it is a pth degree polynomial. Then, if you make the difference table, we will see that this being the 0^{th} row the p plus 1th row onwards is fully 0. p plus 1th whole will be 0 0 0 0 0, something like this. Until the pth row, we will have some entries probably. So, but, after that some non-zero entries. After that, we will see only 0's 0 0 0 0.

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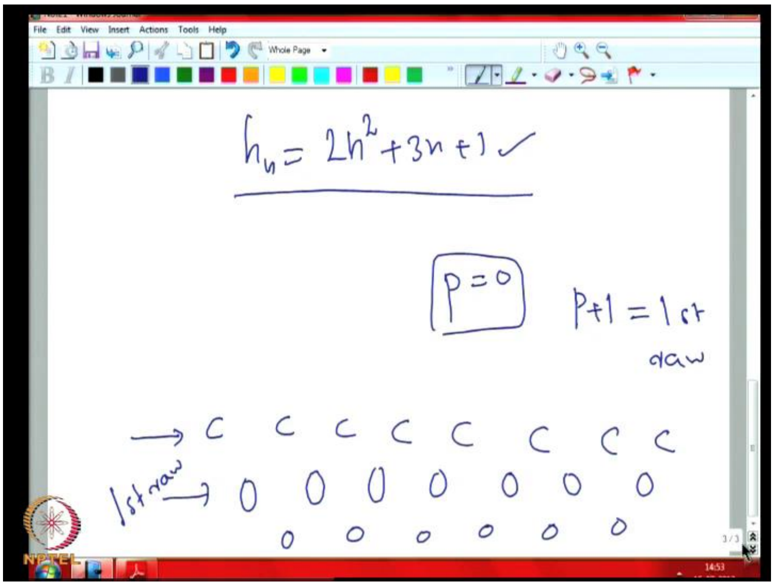

This we saw in the last class by taking an example, where it was a n square 2 n square plus 3 n plus 1. So, when h n was equal to 2 n square plus, we saw that from the 3rd row onwards, third row, when first row is called the 0th row, so, if you really count the rows, it is the 4th row. Because, if the first row is the sequence itself, but, we call it 0th row. So, first row, second row, third row onwards, we were seeing all 0's for this thing, right. This is a general fact. That is what we were trying to prove. So, we will do it by induction. Suppose, the degree of the polynomial is 0. So, the degree of the polynomial, the p, it is a pth degree polynomial. So, degree p polynomial, suppose p equal to 0, then it is a constant. Then, the sequence itself looks like c c c c c c c.

Now, if you take the first order difference sequence, then we will get 0 0 0 0 0. That means, these gaps, these gaps are always 0. So therefore, we see that this being the, so, this here being a 0 degree polynomial, so, this is the 0th line. This first line is the 0th line. The second line will be called the first line, first row, right. So, the p plus 1 equal to 1 here, the first row is 0 and then onwards, we only get 0's, right. Because, the second degree, second order difference equation is going to be also 0's because, anyway it is all 0's from starting from here. So therefore, we infer that, so, when p equal to 0, the statement is true.

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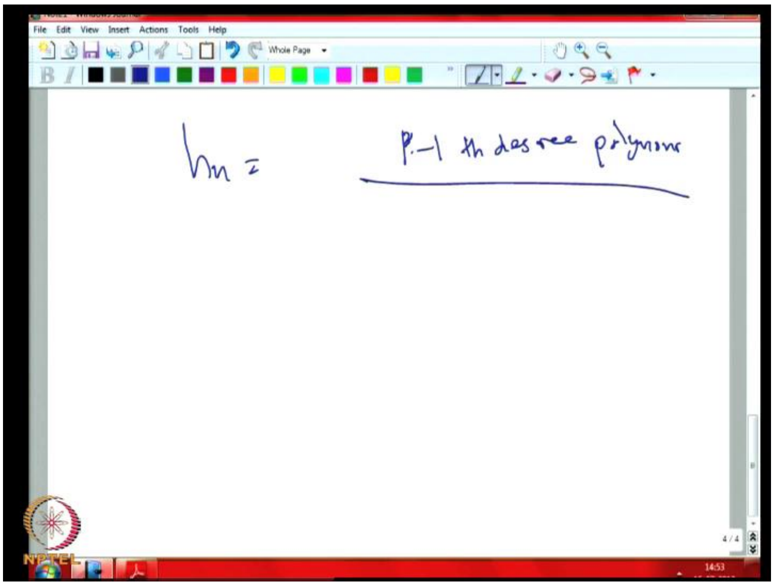

Now, we will prove, we will assume that up to p minus 1th degree polynomial, so, 1th degree polynomials, it is true. That means, if the general h n can be written as p minus 1 degree polynomial, then we can, we assume that the statement is true. That means, if you go to the pth row of the difference table, that will be 0 and from there onwards, it will be 0 only.

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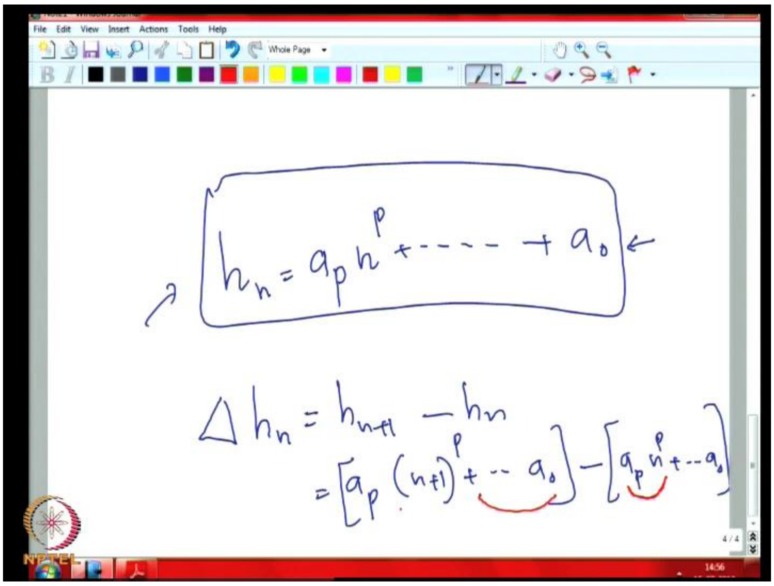

Now, let us consider the general term, where the degree of the polynomial representing the general term is p, right. That means h n equal to a p into n raise to p plus a 0. Now, let us look at the general term. Sorry, we will, by induction hypothesis, we know that for any p minus 1 degree polynomial or less, when the general term is a polynomial of degree p minus 1 or less in n, then the statement is true. That means pth row onwards is 0. But now, here for this case, we have to show that the p plus 1th row onwards is, p plus 1th row onwards is 0. This 0 is what we want to show, right.

Now, if I find delta of h n, what do we get? Means, the first order difference sequence for this thing, if I evaluate for any n, right, so, it will look like this. So, a p into n plus 1 raise to, because this is what this is, h n plus 1 h n plus 1 minus h n. So, this is p into n plus 1 raise to p up to a 0 minus a p into n raise to p a 0. Sorry, this is what it is. Now, if you are looking for the co-efficient n raise to p in the expansion, right, so, after taking the difference, so, what will you do? So, because from here onwards, none of terms of n raise to p because, there all p minus 1 degree onwards. Here, only one term has n raise to p. So, they only contribute to that.

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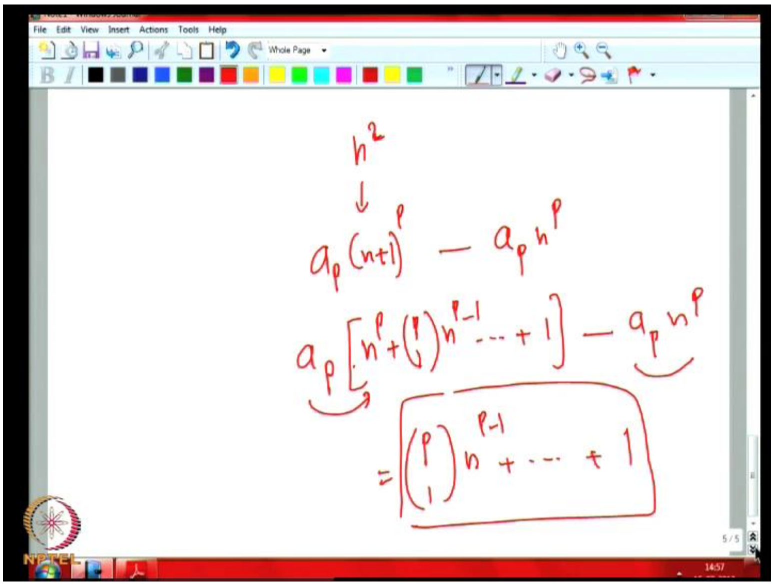

So, the co-efficient of n raise to p in the resulting polynomial will be coming from these two terms, a p into n plus 1 raise to p and minus a p into n raise to p. But here, if you take the binomial theorem and expand it, this will be n raise to p into p choose 1 into n raise to p minus 1 and so on. So, finally 1, right. So, minus a p into n raise to P. So, here this n raise to p and this a p into n raise to p and the a p into n raise to p will cancel. We will end up getting p minus p choose n raise to p minus 1 plus 1. So, we only have terms involving n raise to p minus 1 or less in this difference. So, in other words, the co-efficient of n raise to p will be 0, when I take this thing.

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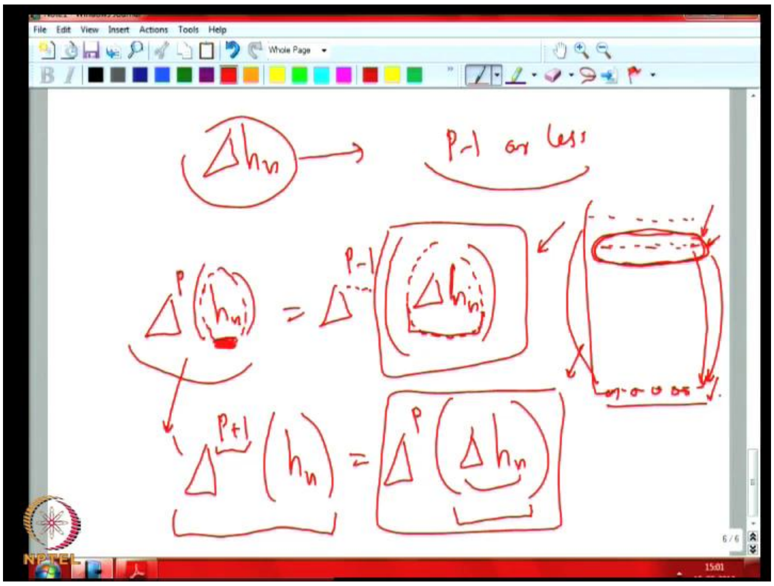

So, this delta h n is the first order difference sequence. Its general term is polynomial in n of degree p minus 1 or less. So therefore, we can, but now, for such polynomials, this for we know the delta p of h n is actually delta p minus 1 of delta h n. So, if you want to get the pth order difference sequence for h n, so, we only have to consider this sequence delta of h n and take p minus 1 order, p minus first order difference. The difference table is clear. So, for this sequence, where this corresponds the 0^{th} row, the difference table, we are looking for the pth row. So, in the differentiable corresponding, if this is considered as 0^{th} row, then we are actually looking for the b minus row. These rows are same. So, these both are going to be equal.

So therefore, we can, but then, this we have already seen. That is represented by a polynomial in n, whose degree is p minus 1 or less. Now, we can apply induction hypothesis. So, we are only thinking about p minus 1th degree p minus 1th order difference. That means, if corresponds to the pth row, pth row of the difference table for this one. So here, so, this is essentially the p plus 1th row. It is, we want, actually this is correct.

So, what we want show is that, so the p plus 1th row, namely delta p plus 1 of h n is equal to 0 is what we want to show, right. We want to show that this is equal to 0. So, this will be equal to delta p of delta of h n, right. Now, this being a p minus 1 degree polynomial, this row, this delta p of this thing, this pth row here is going to be 0. That we already know.

So therefore, delta p plus 1 of h n is also going to be 0. So, from the difference table, what we have done is, we considered the first sequence, right. We want to show that the p plus 1th row is going to be now 0. So, first row being considered 0^{th} row. But, to prove that we produce the first row, right, first row happens to be a; can be represented as by the general term, which is the polynomial in n of degree p minus 1 or less. Now, this p plus 1th row for the first step, the other table, this table, is going to be the pth row for this, right.

So, we know by induction hypothesis, that because it is a polynomial of degree p minus 1 only, so this row is going to be 0. So, the original, this thing is also 0. So, this notation you have to write accordingly. The p plus 1th row corresponds to the delta raise to p plus 1. So here, starting from here, when I tell delta of h n, this correspond to this thing, delta p of delta h n correspond to this row, right; this same row. So therefore, it is an easy proof. Just apply the induction. One has to carefully write the symbols. That is all. Then, this is one property about the polynomials, when the sequence is actually, the general term of the sequence is actually a polynomial in n of degree p. So then, the p plus 1th row, the difference table is going to be 0 and there on it is 0. Another property of the difference is table is its linearity property.

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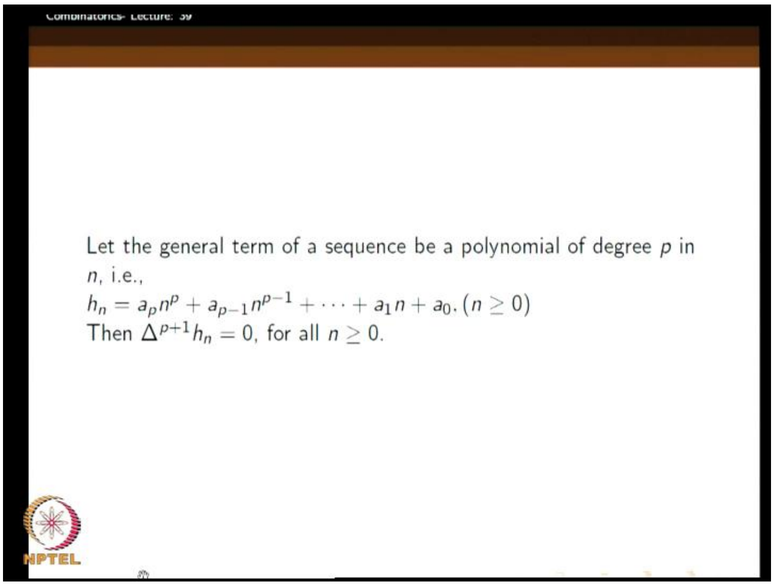

What is the linearity property of the difference table? This is very familiar, if you are seeing the same kind of properties before.

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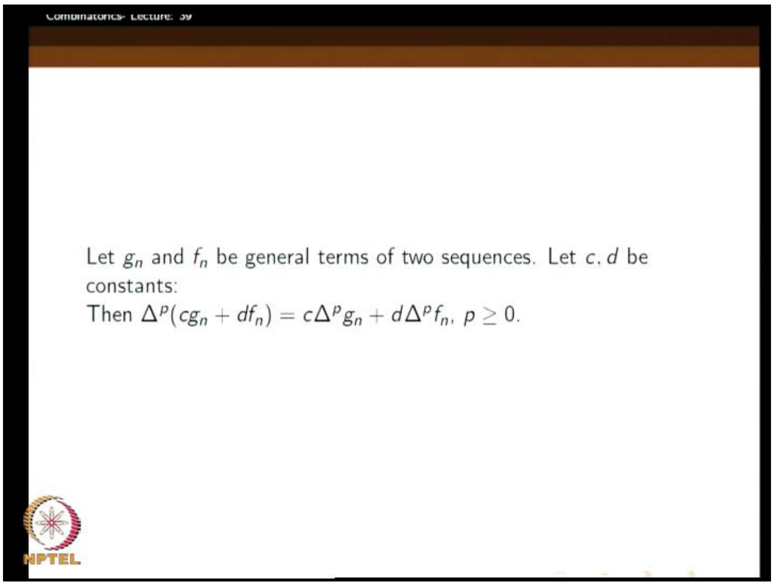

It can be represented like this. Suppose, g n is the sequence and f n is the sequence. That means, g n represents the general term of a sequence and f n represents the general term of another sequence. Let c and d be constants. Then, delta p of c times g n plus d times f n is equal to c times delta p of g n plus d times delta p of f n and p greater than equal to 0.

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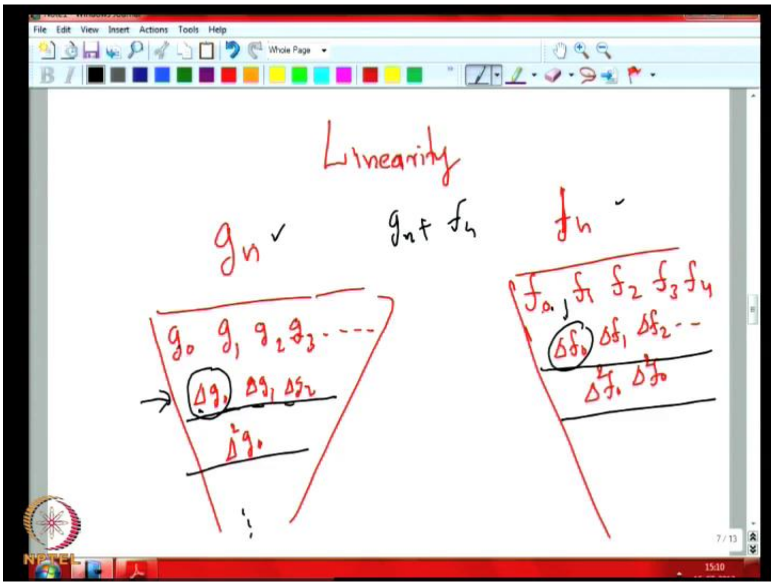

Now, if you want to carefully, suppose we have this g n sequence. So, when you write the difference table, it will look like this. We will write this g 0 g 1 g 2 like this and g 3 something like this. So, this will be delta of g 0 delta of g 1 and delta of g 2 and so on. This will be delta of square of g 0. So, this will be the difference table corresponds to g n. So, similarly we will have one difference table for, we can write the difference table f 0 f 1 f 2 f 3 f 4 and so on. This will be delta f 0 delta of f 1 delta of f 2 and so on. This will be delta square of f 0, and delta square of f 1. This is the difference table corresponding to f n. Now, what we are saying is what about adding this g n and f n together. That means, every term here g 0 will become f 0 plus g 0. The sum g 0 plus f.

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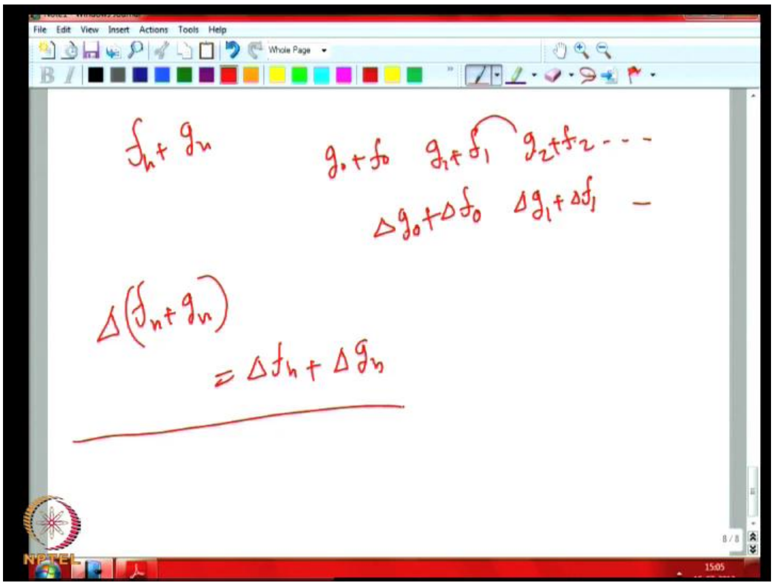

So, if you add g n plus f n, what we get is, see g 0 plus f 0, right, g 1 plus f 1 g 2 plus f 2 and so on. This will be the new sequence. Now, if you take, if you want to consider this first row of this thing, difference table, the new difference table, difference table for f n plus g n, so, we have to minus this thing. This is g 1 minus g 0, right. g 1 minus g 0 plus f 1 minus f 0, which is nothing but, g 1 minus g 0 is actually delta g 0 and this is actually delta f 1. So, f 0. So, this is delta 0 plus delta f 0. Similarly, this will be delta. So, if I take the minus here, we can easily, this g 2 minus g 1, which is delta g 1. This is f 2 minus f 1. That is delta f 1. So, this is delta g 1 plus delta f 1, right. So, delta of f n plus g n is clearly delta of f n plus delta of g n, right. This is one thing, which you can easily observe.

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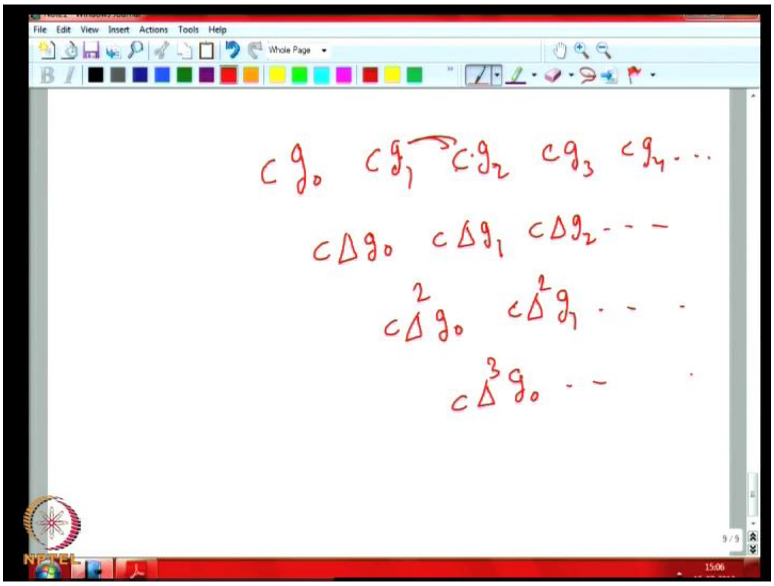

Similarly, we can easily see that, if you have sequence g 0 g 1 g 2 etcetera and I multiply with a constant; that means, c, this is the new sequence, suppose, right. So, this is the new sequence c times g 3 c times g 4 and so on. Now, if you take the difference thing, so, if I consider c g 1 minus a, this is delta of g 0, that is c into g 1 minus g 0 only I will get, which is actually c times delta of g 0. Similarly, when I take c g, so here, the gap, c into delta of g 1. This will be c into delta of g 2 and so on, right. When we take the difference here, this is c into delta square of g 0 and this is c into delta square of g 1 and so on. This is c into delta cube of g 0 and so on, right.

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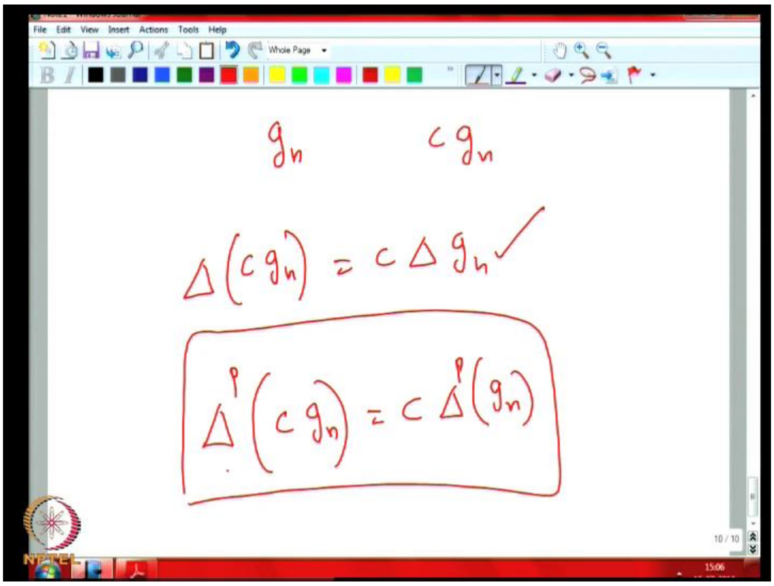

We can easily see that from, if g n is a sequence and c g n is the new sequence. So, then delta of c g n is going to be c times delta of g n and actually we can extend it to delta of p of c times g n is equal to c times delta p of g n, right. This second statement can easily be proved by induction, because we proved it for the p equal to 1 case.

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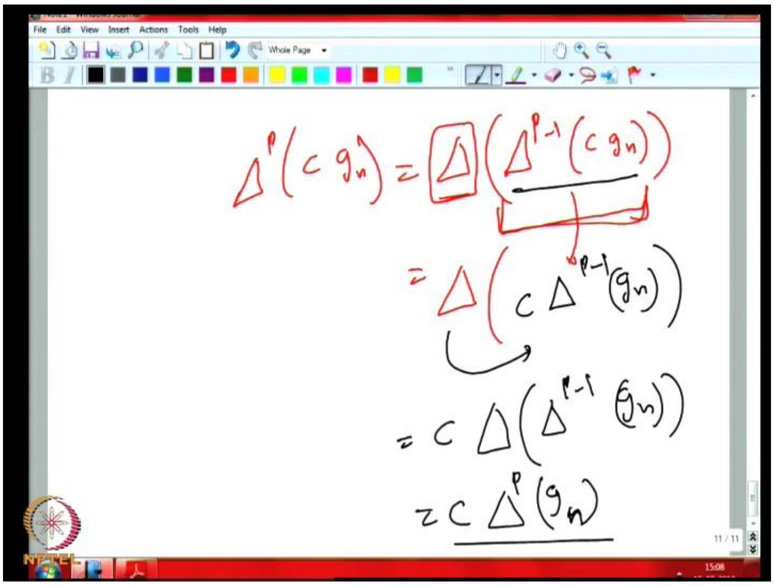

Now, if you want to prove it for general p, what we see is that delta raise to p of g n c times g n is actually, we apply this as, this has delta into delta delta of delta p minus 1 of c times g n, right. This is what it is. So now, if you apply this thing on this, right, we know already, we have proved already for the p equal to 1 case. This is essentially, sorry, first we apply here because, by induction, we know that this is, this we can write, this portion we can write as c times delta p minus 1 of g n. So now, we can apply this thing. We have already, because for p equal to 1 we have already proved. So, that is c into delta of delta p minus 1 of g n. So, which is essentially c times delta p of g n. This is the situation.

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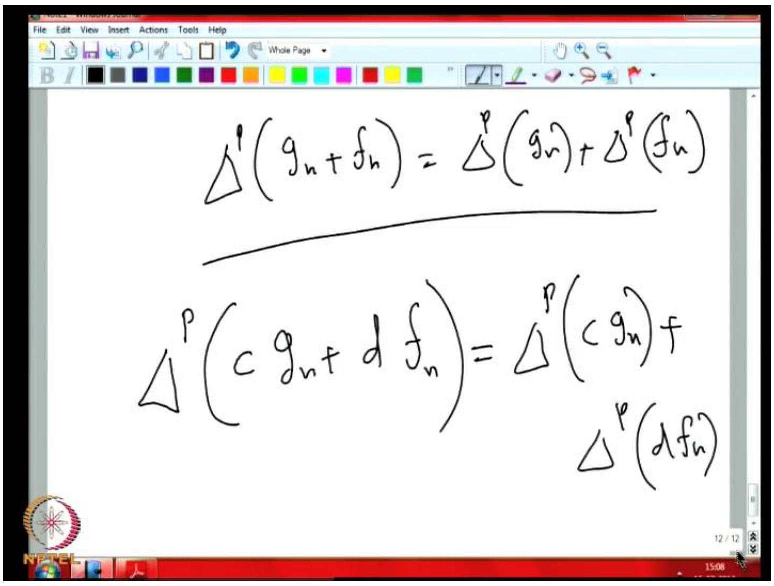

So now, similarly, we could have proved delta p of g n plus f n is equal to delta p of g n plus delta p of f n. So, this is, this also follows by induction. The same kind of argument, right. Now, from both of these things, what we can infer is delta p of the c times g n plus d times of n, where d and c are some constants, this is what; first, we apply the first rule. So, this will be delta p of c times g n plus delta p of d times of f n. This way we can write.

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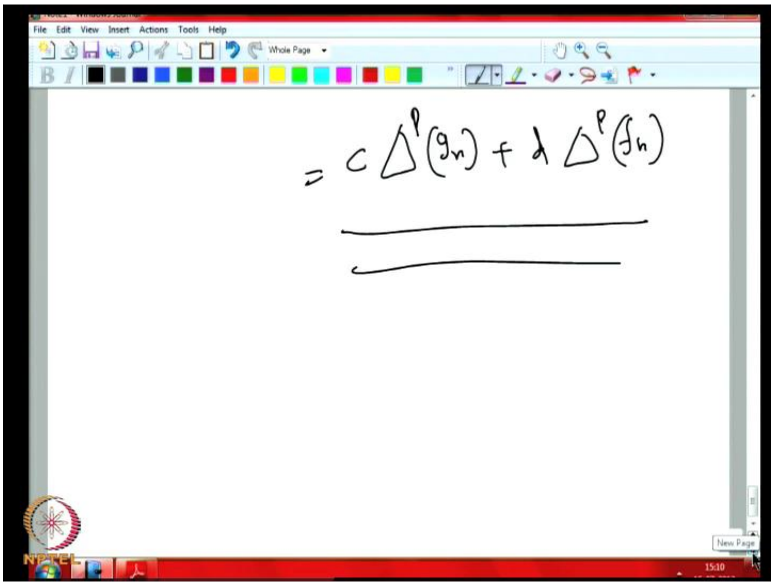

Then, we can apply the second rule, namely, this is equal to c times delta p of g n plus d times delta p of f n, right. This is what it is. This is linearity property, right. So, which essentially means that, if you have this difference tables, right and this table and this table, when you create, if you want to create the difference table for g n plus f n, we can simply add, from term by term can add; g 0 plus f 0 is added here.

So, the first row of obvious because, that is by definition, but, the second row also we can add. So, this and this will add and will stay just below this gap, right; the first gap. So similarly, or in other words, the first order difference sequence also is a sum of this first order difference sequence and this difference. Similarly, the second order difference sequence also can be added. So, that is what we are seeing.

Actually, if you, similarly, if you are multiplying the difference table, the first row of the difference table by a constant, every term is multiplied. Then, all the coming rows also will get that multiplier and then, when you want to add them together, naturally we can add the tables. That is what. If you want to visualize using the tables, this is what it sees, right. That is a very simple property. If you just think for some time, you get it very clearly. Now, this is the linearity property of that.

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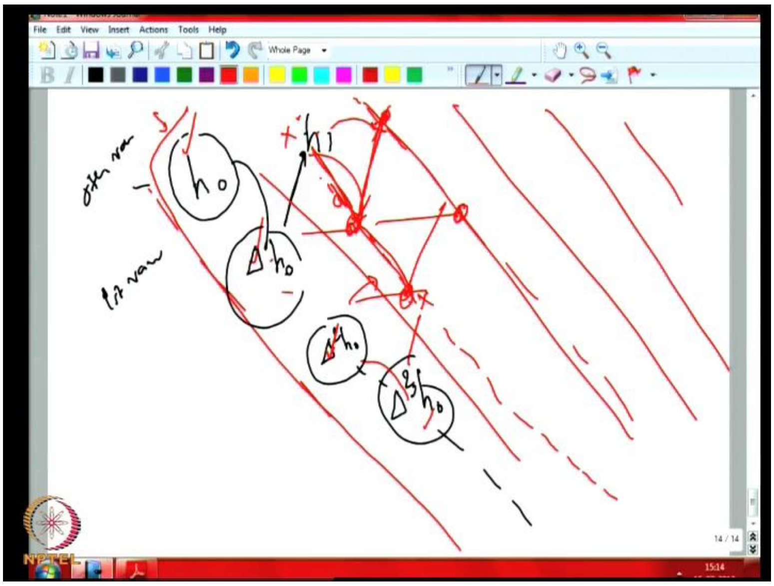

Now, another interesting property of this difference table is that, this difference table can be completely written down, if you know the starting terms in each row. We know very well that, if the sequence itself is given, the difference table can be, for instance, if the first row of the 0^{th} row of the difference table is known, then the difference table can be completely written down, right; completely written down as long as, so, we are ready to write it. There are infinite rows, then of case, we can, so, it is completely determined anyway, right.

So, but, that is because that is the way we are defining because, once the first sequence is given, then the first order difference sequence written, the second order difference sequence is written and so on. Now, we are telling we can also do, create this difference table, if you had known these things, delta of this one and delta of h 0, which is essentially the first term in the second, the first row term in the first row. This is the 0^{th} row. This is the 0^{th} row. This be in the first row, this first term and here, delta square of h 0. This is delta cube of h 0. If these terms are known, then also we can complete this table. Why? I am assuming that I do not know anything here, right, I know only this, this, this, this and so on. How can I write?

So, first I write this. This is what I want, because I know this term is, this h 1 minus h 0, right. So, this h 0, I know, right. Clearly I can write h 1, right. delta h 0 and h, sorry, delta h 0 plus h 0 will create h 1. So, that means, this can be created by adding. So, this is h 0. Adding these two things, right, I can, if I add these two things, I will get this. So, this will be h 1 because, h 1 minus h 0 is this. So, this plus this is this. Similarly this, here, this number here, this number here can be created by adding this and this, because this minus this number minus this is this. So, this plus will be this.

So, this will be what? delta square of h 0 plus delta h 0 will be this because, we know both of these things. I can write now. Similarly, I can write this number by adding this and this, right. So, like that I can create this entire second diagonal. This being the 0^{th} diagonal, the first diagonal we can create, this left most diagonal. So, this left edge of the table you can say. This is first and after that, the next diagonal can be created and then, next diagonal can be created. How? We will add this and this and we get this thing.

So, we will add. Why is it so? Because this number is produced by this minus this, right. This number minus this was this. So therefore, to get this number, we just have to add this and this. Similarly, this number is produced by this number minus this number is this. So, in this row, right. So therefore, to get this thing you just have add this. So, once you know this diagonal, you can produce the next diagonal and so on, right. So, it is very clear that, if you know this left edge completely, then we can create the rest of the difference table.

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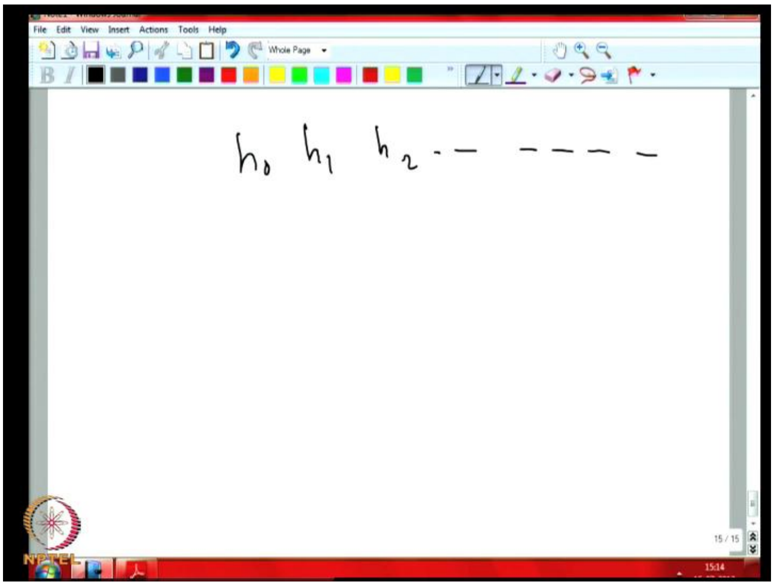

To summarize, what I am telling is, you can, your difference table is completely determined, either by your first 0^{th} row, which is essentially h 0 h 1 h 2 see we can tell, which is not at all surprising. That is where we define the difference table.

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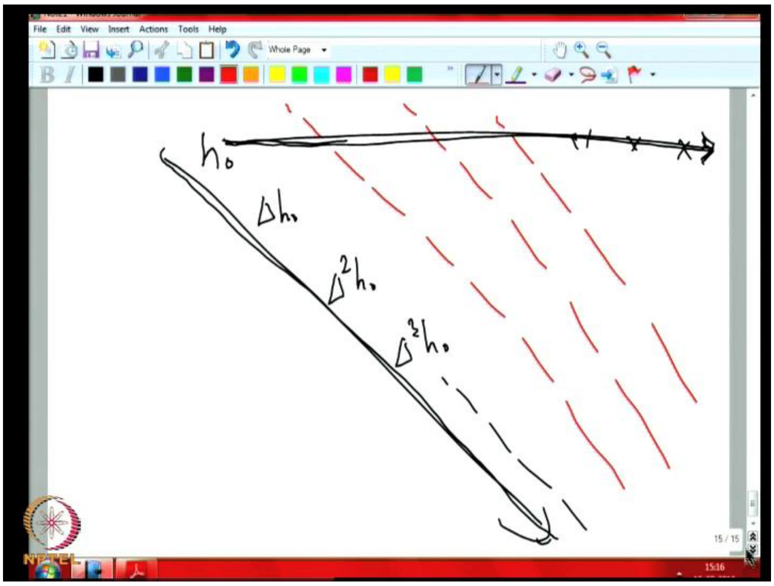

But, what is interesting is, it can also be completely generated, if somebody gives you only this left edge or 0^{th} diagonal of the difference table, namely, these numbers h 0 delta h 0 delta square h 0 delta cube h 0 and so on. If these numbers alone are given, then we can write down these numbers. Then, we can write down these numbers and then, we can write down these numbers and so on. Completely we can generate the entire table from this thing. So, the difference table is determined either by this line or this line, right. The rest automatically follows.

So then, what is the advantage of having this thing? The advantage is that, this line can be, say, in for some cases, if this first 0^{th} row is actually any infinite sequence, in the sense that, there are infinite number of non-zero entries there, right. It may so happen that, in this diagonal, we have only a finite number of non-zero entries. That may make our life easier. We may wonder what situation, in what situation it happens. We have already mentioned one situation, which is important enough; I mean to motivate the study.

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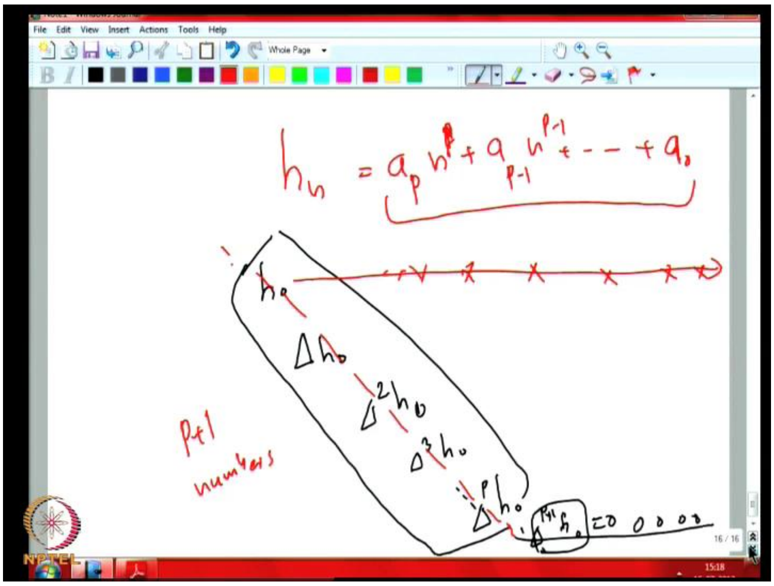

Namely, when h n is of the form a p n raise to p plus a p minus 1 a p minus 1 n raise to p minus 1 plus a 0; that means, h n has the form of a polynomial in n, right; of degree p c. Then, we have seen that, if you look at this sequence of numbers namely, h 0 delta h 0 delta square h 0 delta cube h 0 and this will go up till delta p h 0. But, delta p plus 1 h 0 onwards, we are going to have 0's. This is going to be 0. Why? Because, I know the p plus 1th row is going to be 0. That is what we proved in the previous theorem, right. We have shown that, if the general term is given by a polynomial in n of degree p, then the p plus 1th row onwards is going to be 0; p plus 1th row p plus 2th row and so on

In particular, the first term of that rows, namely delta p plus 1 of h 0 delta p plus 2 of h 0 delta p plus 3 of h 0, all of them will be 0. So, we will have non-zero values possibly up to here and not beyond that, right. So, this is well, if you write down, if you try the 0^{th} row it may go on and on because, there may be several non-zero entries, infinite number of non 0 entries, so that, that way we cannot concisely represent this table. On the other hand, if you look at the left edge on the other hand, 0^{th} diagonal, we know that starting from h 0, we will have to consider delta p of h 0, namely p plus 1 numbers, p plus 1 numbers along the 0^{th} diagonal, right. This will completely determine the difference table. This is the advantage of understanding that. So, this full difference table can be represented either by the 0^{th} row or by the 0^{th} diagonal. So, now that we have this understanding, let us look how we can make use of all these things, right.

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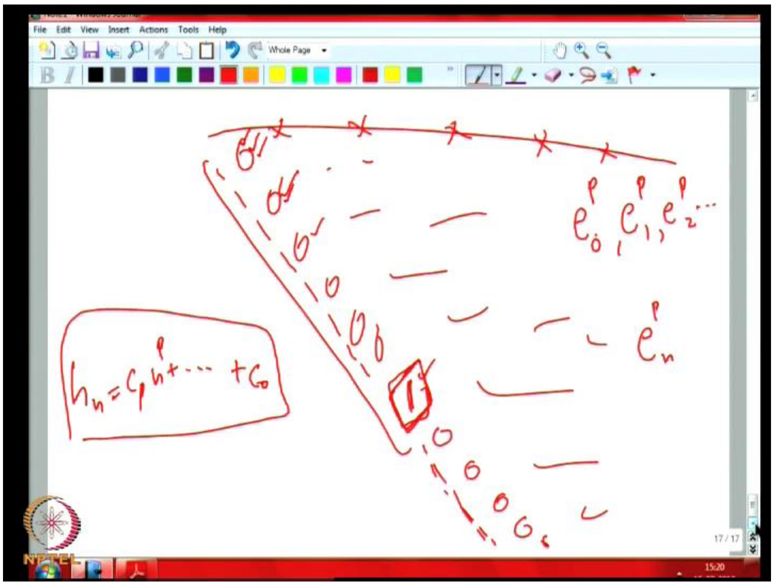

So, the good thing is, the 0^{th} diagonal may look like this 0, right, c 0 c 1 c 2, say some c p and beyond that it may be all 0's, right. Suppose it is a case of polynomial. In the case of polynomials, right, say for instance, if my general term h n was some polynomial of this form; that means, nth degree polynomial, then these numbers 0 c 0 c 1 c 2 c p, these numbers may be non-zero. But, beyond that all of them will be 0, right. But now, suppose if this one was 1 and all the others were 0's, right, if all others were 0's, let us say we can get, so, let us call it the sequences e p, right. e p means the sequences e p, say e p, right, e p 1 e p 2. So, what I mean is, in the pth row, this being the 0^{th} row, this being the first row, this being the second row, this is the pth row. Pth row we have 1 and all other places we have 0's. All other places in the 0^{th} diagonal here, here, here and we are not bothered about what is going to be in these places as of now. So, then this sequence, which we may write it as e p 0 e p 1 e p 2, right, this sequence, suppose if I can figure out, right, the general term be e p n, right.

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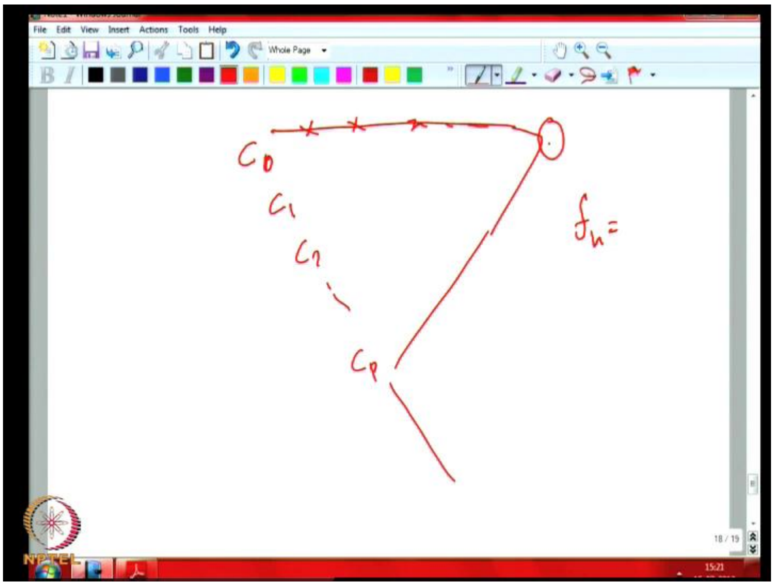

Then, we can easily see that the table, which is generated by c 0 c 1 c 2 c p in the main diagonal, this kind of a table, right, this table can actually be represented. So, this table will be giving, say for instance, some f n, right. Some general term will be coming here for this thing, if I work out this thing.

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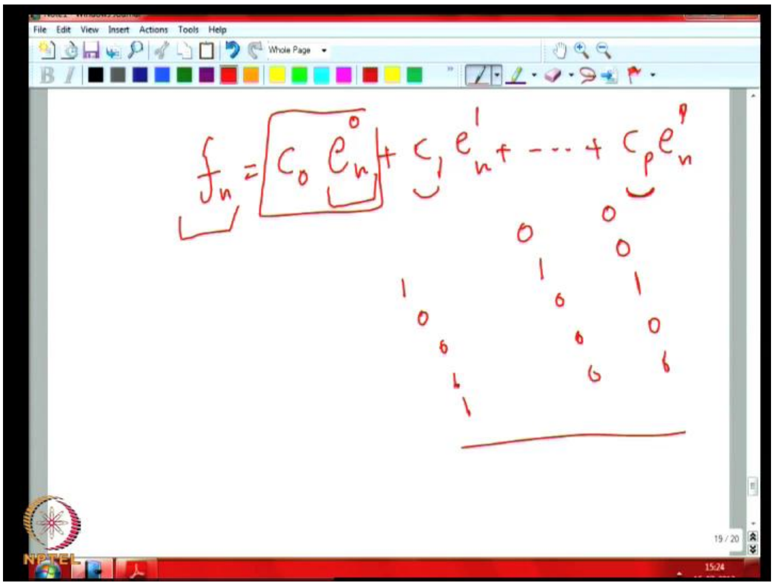

But, the general term can be written as f n equal to c 0 into e 0 n plus c 1 into e 1 n plus c n c p into e p n. Above that, we do not have the c p's all 0's. So, this can be written like this. Why is it so? Because, of the linearity property of this thing, right.

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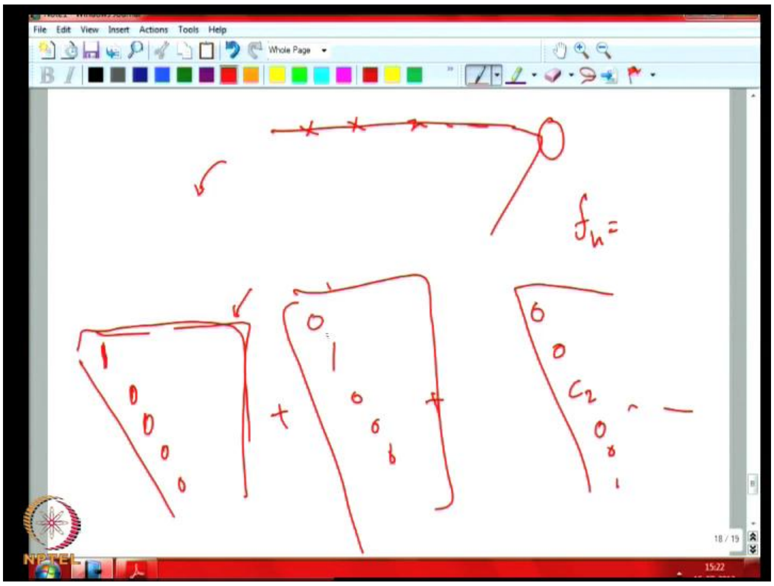

Because, this term is actually c 0 times 1, right and you could have decomposed this tables like c 0 1 1, sorry, this 0 0 0 0 table corresponding to this 0[th] diagonal and c 1, so, another table corresponding to 0 c 1 0 0 0 and so on. Another table corresponding to 0 0 c 2 0 0 and so on, right. If you add these things together, this row will be c 0 c 1 c 2 c 3 up to c p, right and this particular table, c 0 itself could have been produced by multiplying by c 0, the table which is produced by 1 0 0 0 0 1 0 0 and so on, right. This table can be produced by multiplying the table produced by this left diagonal 0 1 0 0 0. So therefore, overall, see if you want to get the general term corresponding to the difference table, where the 0^{th} diagonal of the difference table is c 0 c 1 c 2, etcetera, what we do is, we create the table corresponding to the 0^{th} diagonal of this form 1 0 0 and then, multiply the table by c 0.

Then, we produce the table, which corresponds to the 0^{th} diagonal of this form and then, multiply by c 1 and then, I add it to the previous one. Then, we produce the table corresponding to 0 0 1 0 0 and then, multiply by c 2 and add it to previous and so on, right. Then, we have do it p plus 1 times, where once for c 0 1, once for this, once for this, once for this and that will create the general term for this, right.

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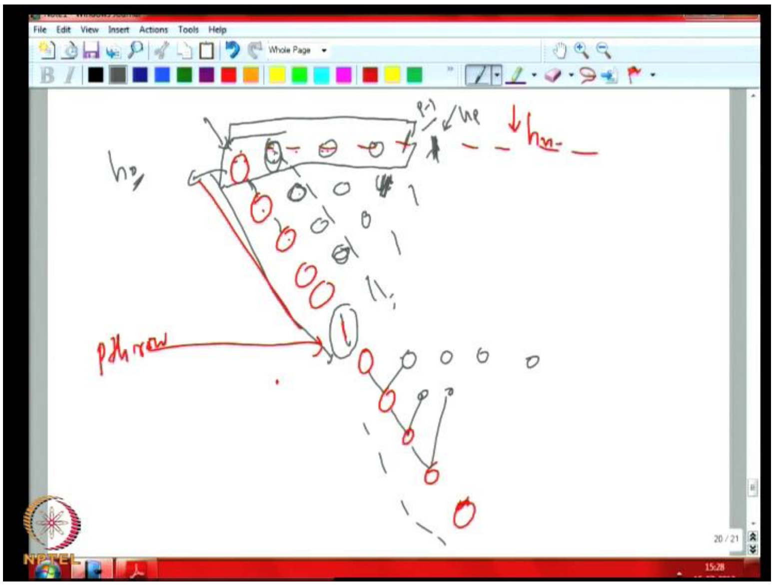

So, but for this to be effective, we should have a very concise and neat formula for the general term, corresponding to the for the sequence corresponding to the difference table produced by this kind of a 0^{th} diagonal, right; this kind of a 0^{th} diagonal. So, here it is the pth position. So, here pth row, we have 1, right. This is 0^{th} row. This is first row, second row, this is pth row having 1, right. Now, if you want to reproduce the general term here; that means, the first row, what will be h n here, right?

Now, so you know that how will this second, sorry, the first diagonal look like. This is the 0^{th} diagonal and first diagonal will look like, definitely this and this should be added to get this and this should be added to get this and it will look like this, right. Then, here we will get 0 and 1. This will be 1, right. Now, here also it will be 1. So, let us discard for the time being. From here onwards, it will be all 0. We know that. So, this will be all 0 plus 1. But here, this row we will have something. But, for the time being, let us not worry about this portion.

Similarly, if I create this thing, this will be 0, this will be again 0, this will be again 0 and this will be 1, right and here, it will be 0 and then 0 and there will be a 1 so on, right. So, we will be getting something like this, upward, right. So, 0 1 and then, then we have a 1 and then have a 1, like this, we will be will be going upward. So, if we have p here, then same thing we will get, right, here. So, like this, here we will have a sequence of 1's going upward like this and 0 this same way, right. So, here we have a, this is the pth one, right. So, here we have up to p minus 1. This is $0^{th,}$ this corresponds to h 0, right. This corresponds to h 1 and this will be h p minus 1, right.

So, and this will be h p. h p will be always, h p will be 1. Now, looking at the structure of this thing, so, let us, because you know from, till p th, on the pth row we have 1, then beyond that we are getting 0's. So, we are not sure whether, so, we can also, examining this thing, we can also get this. All these 0's are coming here, right, because that is the way, this and this will add to this and this will add to this. So, it is very reasonable just to think that the corresponding general term will be a polynomial in p. So, we will assume so. So, it is a polynomial in p, and then, can we find that polynomial? It may look like, can we find that polynomial?

But, one good thing is that, we know that when we substitute n equal to 0, that polynomial evaluates to 0 and when you substitute n equal to 1, that polynomial evaluates to 1. When I substitute n equal to 2, that polynomial evaluates to 0 and so on. So here, because first row is something like 0 0 0 0 0 0 and suddenly the pth term is becoming 1.

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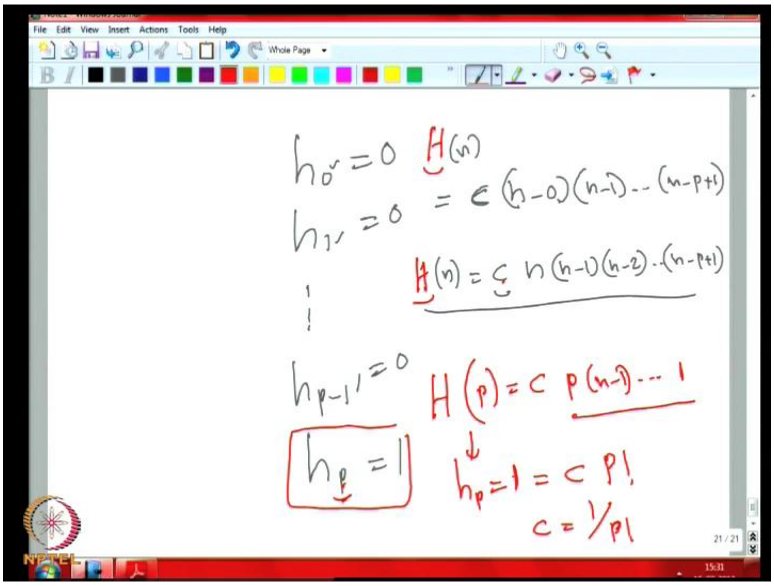

So therefore, we can see that, we have h 0 is equal to 0, h 1 equal to 0, and like that, h p minus 1 is also equal to 0 and we have h p equal to 1. This is what we see. So, what does it mean? So, we know that, if it is a polynomial in n, then the roots are 0 1 p minus 1. We got all the p roots of it. So, it looks like 1 n minus 0 and that polynomial look like n minus 0 into n minus 1 into n minus p plus 1, right. n minus p plus 1 and some constant. This will be that polynomial. So, p of n will look like this, right. Some constant times n minus 0 into n minus 1 into n minus. So, this looks like, c into n into n minus 1 into n minus 2 into n minus p plus 1. This is the way it will look like.

Now, but how will this is p of n? But, how will I determine the value of c then? So, to do that, we have one more value available is h p equal to 1. That means, when you substitute n equal to p, so, this I can use a different symbol for this. Let say h of n. So, h of n, sorry, f of n. For that polynomial, let me use this kind of a p, right, p of x, right. Because, I am confusing with this p. Now, I will just substitute n equal to p, then this will look like c into p into p minus 1 up to p minus p plus 1 up to 1.

That means, our constants, but, value of this thing is already known, right. So, right or I can always use the h itself for the polynomial. Only thing is that, let me use the capital h to denote that it is a polynomial and h of n, right. So, n is to be considered as a variable here in this polynomial. So, this h 0 h 1 etcetera, are the terms of the sequence. This polynomial, I just use capital letters to, in order that we are talking about the polynomial.

So, let me see. This is h of p. So, h of p actually will evaluate h p, right, which is actually 1, right. This is equal to c into; this is what? p factorial. So, this c is equal to 1 by p factorial, right. So, what is our answer? So, we can substitute by 1 by p factorial here.

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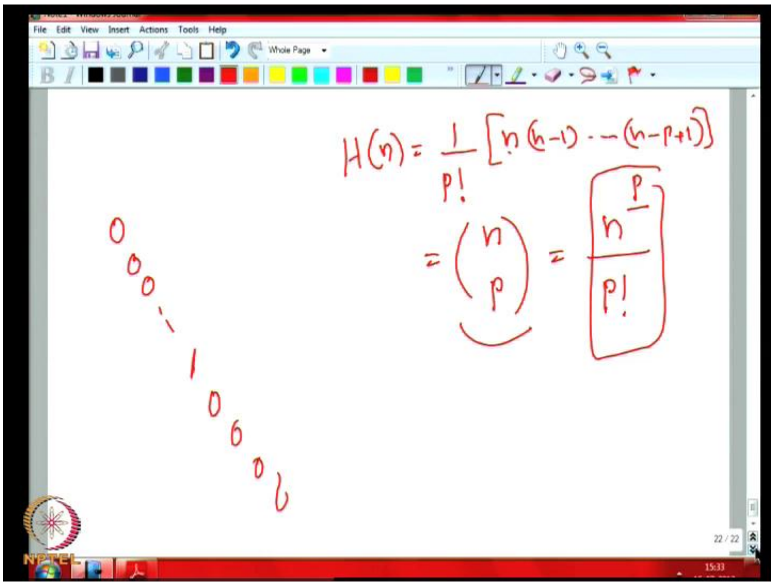

So, we will get H of n is equal to 1 by p factorial into n into n minus 1 into, so, n minus p plus 1. This is very familiar to us. This is actually n choose p, right. So, n choose p, which also can be written as n p, falling factorial by p factorial. So, this is the n falling factorial and starting from n, n into n minus 1 into n minus p plus 1, right. So, this way, we can right or just write n choose p, right. So, the polynomial corresponding to the sequence, which corresponds to the difference table, whose 0^{th} diagonal 0 0 0 0 up to p. So, up to the pth position we have 1 and then, 0 0 0 and pth position, we have 1 and then, 0 0 0 0 will be n choose p, right, n choose p; n falling factorial p. n is taken as variable here and falling factorial p divided by p factorial. This is what.

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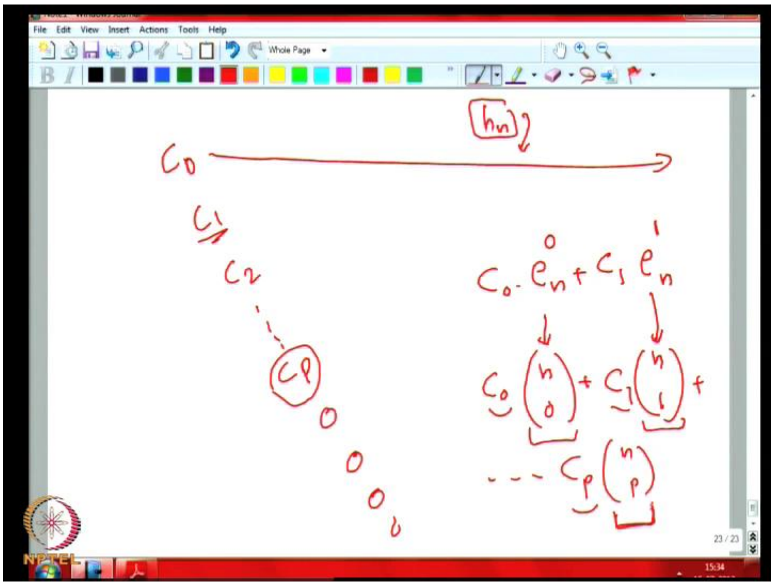

Now, we know that, if we are considering the difference table for c 0 c 1 c 2, this suppose 0^{th} diagonal look like this c p and then, we know that 0 0 0 and then, if you create this first 0^{th} row and then, what will be h n. What will be the general term here, right? We know that this will correspond to by the linearity property of this difference tables.

So, we can multiply the c 0 into e 0 n, which is we know already the n choose c 0, right. c 0 into n choose 0 plus c 1 into, this c 1, into e 1 n. We know this one is c 1 into what? n choose 1 and up to p, we can do. c p into n choose p, right. Because, if we just add 1 in this position and everywhere else it was 0 and that will correspond to n choose p. p being the row number, where the 1 has occurred. So, if 1 had occurred in the 0^{th} row, we will get this one, first place of the first row. So, it will be this and the first place of the and the multipliers are definitely c 0 c 1 to c p and when you add up, you will get like this, right.

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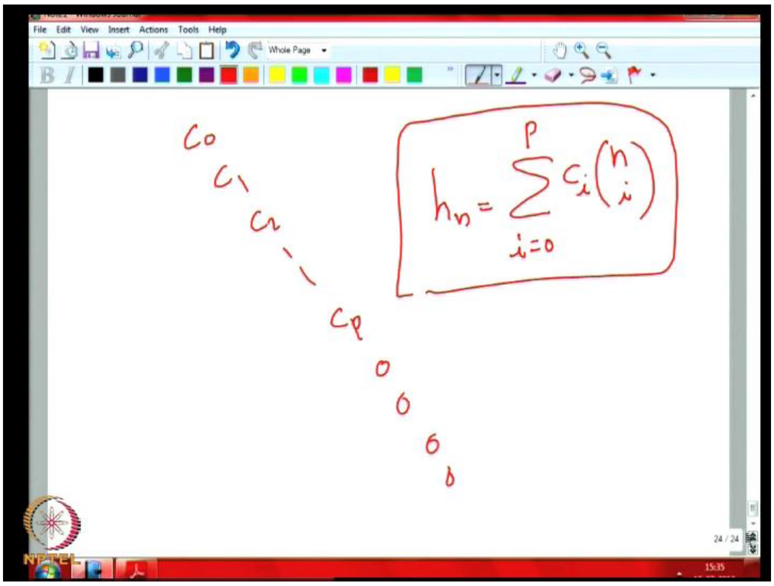

So, if the difference table look like c 0 c 1 c 2 up to c p and then, 0 0 0 etcetera, so, we see that our h n, the general term corresponding to the sequence coming from this difference table is actually sigma i equal to 0 to p c I, n choose i, right. So, this will be the general term for such a sequence. So, that is what we are getting.

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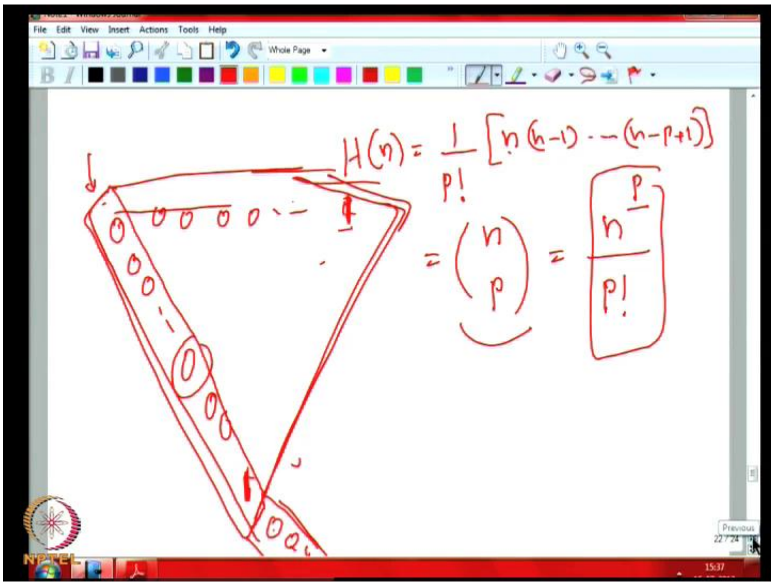

So now, so, the only one thing we have forgotten to discuss is that, right, we have figured out a polynomial, which will correspond to this general, this kind of a 0^{th} diagonal and then, right, so, right. It is easy to verify that this polynomial n choose p actually corresponds to this 0 table. We guessed it first. But then, we saw that up to here it is satisfying, up to the p plus and, so, 0^{th} term up to pth term will satisfy. Now, we can see that this thing, this portion of this difference table is following from that. It is satisfying this much and up to here it is satisfied, right. So, and then, we know is a, this n choose p is a polynomial of degree up to here.

So, this is, what I mean is 0 0 0 1. This is the peak portion. So, this portion of the table, where this is 0 to p terms here. So, h 0 h 1 h 2 h p and this is up to 0 1 2 up to pth row, right. This much anyway we will follow, right. If you can definitely see that this is 0 0 0 1, then we will get 0 0 0 0 0 1 and 0 0 1 and this portion of the table will come and from here onwards, therefore, up to here, it is actually giving the 0^{th} diagonal as we wanted. Since, n choose p is a polynomial in degree p, p plus 1th row onwards, we should have 0 0 onwards.

So, this will also satisfy the remaining things. That is why we just have to verify that after getting this answer, n choose p, that actually corresponds to the table, corresponds to a table, where the 0^{th} diagonal is looking like 0 0 0 and 1 and pth row and then, 0 0 0, right. You know, once this is matching, the 0^{th} diagonal is matching, then the entire table will match, right. So, we do not have to worry more than that. Then, once that is true, then it was only the linearity property we used and then, we just can multiply the corresponding terms with c 0 c 1 c 2 etcetera and then, we end up with this formula for the general term. Now, we can simplify a little bit.

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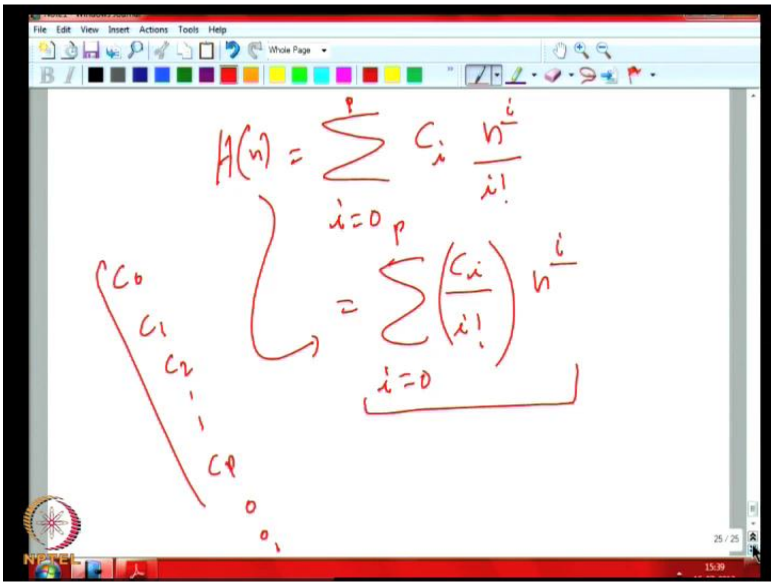

Suppose, so, this can be written as H n equal to, right, sigma i equal to 0 to p c i, assuming that the diagonal, so, the main diagonal look like c 0 c 1 c 2 c p and then, 0 0 0, right. The first 0 diagonal look like this. c p into, this is n choose i, right. For this n choose i, as we have mentioned is n i falling factorial divided by i factorial. So now, this can be written as sigma i equal to 0 to p c i by i factorial into n i falling factorial. So, this polynomial can be represented by this, c h of n. This h of n can be represented by, is actually is nth general term. So, general term for the n. So, if you want to write it as polynomial, we can write h of n. So, it can be represented like this.

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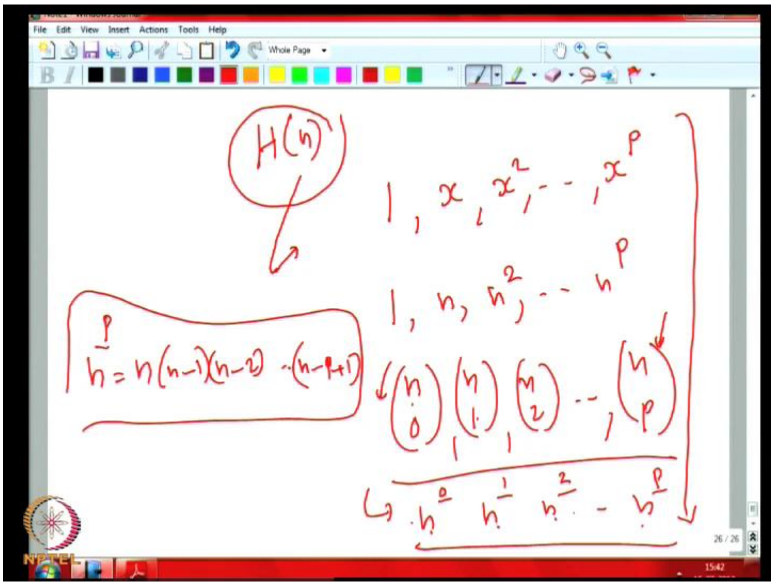

Now, we will just notice that, if you are familiar with the linear algebra, it is not very surprising because, if you are talking of polynomial, say H of n, which is of degree p, right. So, definitely if you consider the vector space of polynomials of degree p, then you know, 1 x x square x raise to p form a bases for this thing, this polynomials. If you want use n, 1 n n square n raise to p form a bases for this thing.

Similarly, it is not difficult to show that n choose 0, n choose 1, n choose 2, n choose p, which are also polynomials. This is a constant and this being degree 1 polynomial, polynomial of degree 2 here, this is degree p polynomial. This also form a bases for H of n. Naturally, you can simplify this thing and say that n 0 factorial, which is a constant, n 1 falling factorial, n 2 falling factorial, n p falling factorial. I hope you recall the definition of falling factorials. When I saying falling factorial, I mean n minus 1, n minus 2 up to n minus p plus 1.

This is what. This is also a bases for the vectors space and therefore, it is not surprising that any polynomial of degree p can be represented as some constant into this, plus some constant of this, plus some constant this, plus some constant with this, right. It is not at all surprising. So, but again, we would not go to linear algebra much. So, the presentation gives, of case, we could have told like this as any coefficients are there.

But, the previous presentation in difference table gives a little insight into what kind of numbers these things are and also, it is not only about existence we are talking about. We are actually seeing the numbers, which are appearing here. So, fine. So, then what we have told now is that any polynomial can be represented like this. So yes, the coefficients of the falling factorials. So, in the bases of using the falling factorials, we can represent any polynomial. Now, what we are especially interested in is this simple polynomials like n raise to k n raise to p, let us say.

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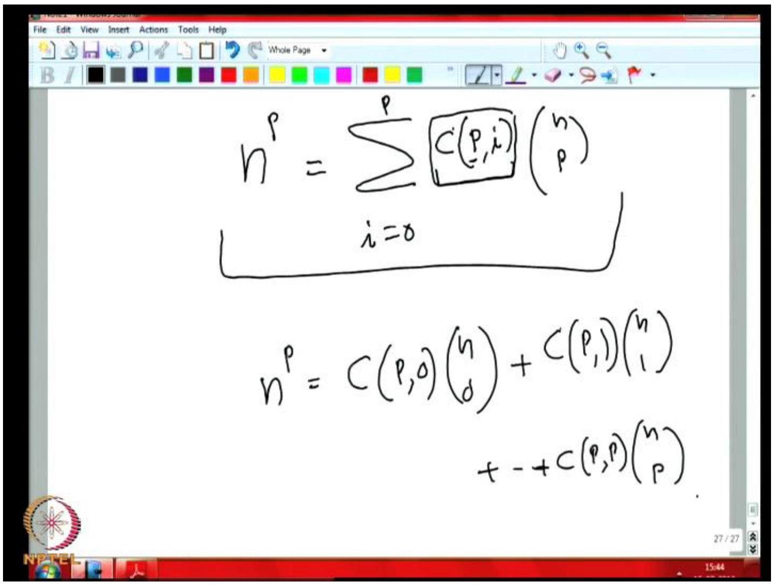

So, you want represent n raise to p as i equal to 0 to p, right, we need only up to p for this thing. So, here this is, so, we will use c p 0 because, we are talking about n raise to p, c p 0, rather than c p 0 c 1 etcetera, we will call c of p comma 0, p of p comma 1, because we are talking about the coefficients, when we want to represent n raise to p. So, into n choose p, right. So, these numbers are special.

So, because we are studying this n raise to p, so, we can represent n raise to 0, n raise to 1, n raise to 2 and so on. So, in general, this is true because, this is true from the previous discussion because, n raise to p is just a polynomial in degree p. So, whose degree is p. Therefore, this is also possible. These coefficients we have changed just to note that we are actually representing n raise to p. That is why c of p is coming and then, this will go from c p i, i equal to 0 to p. It will go from i equal to 0 to p. So, it will look like n raise to p equal to c p 0 into n choose 0 plus c p 1 into n choose 1 and so on, till c p into n choose p.

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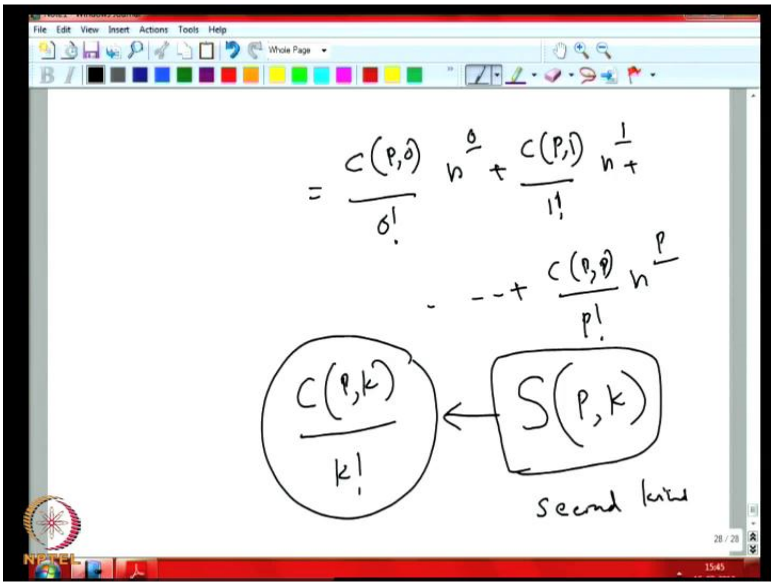

Now, so this also can be, as we have already mentioned, this also can be represented c (p, 0) by 0 factorial into n 0 falling factorial; so it is just 1 constant and then, c (p, 1) by 1 factorial into n 1 falling factorial and so on. c (p, p) by p factorial into n p falling factorial. These numbers, c (p, k) by k falling factorial appearing here will be the sterling numbers s (p, k).

So, we will show that this is the same sterling numbers we have defined. Sterling numbers of the second kind that we have defined before using combinatorial definition. What our strategy is to show that, this s (p, k), this numbers actually satisfy the same recurrence relation and initial conditions as the previous sterling numbers of the second kind, which we defined using the combinatorial interpretation, right. So, that is what we will do in the next class.