Combinatorics
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Lecture - 38
Catalan Numbers - Part (3)
Sterling Numbers of the 2nd Kind

Welcome to the thirty eighth lecture of combinatorics. In the last class we were discussing the Catalan numbers and about balanced parenthesizes.

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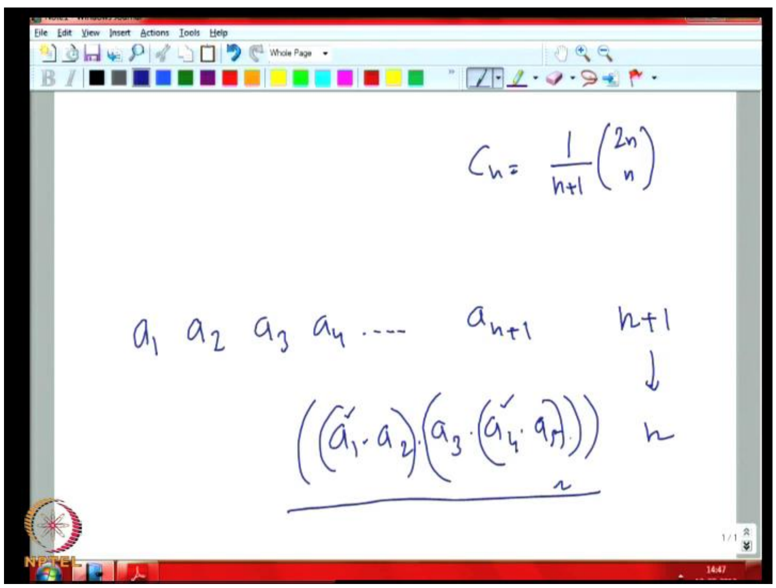

So, if there are n pairs of brackets, so we show that the number of ways to arrange them, in a balanced way, is the Catalan number namely 1 by n plus 1 C n 2 n choose n, right, C n Catalan number. Now consider some n plus 1 numbers a 1 a 2 a 3 a 4, say, a n plus 1, right. Now this parenthesis may be to evaluate the product, say, take some example a 1 a 2 a 3 a 4 a 5. So, you may multiply this together first, then you may multiply this and this together, then you may multiply this and this, then finally this and this may be multiplied. So, how many pairs for five numbers? So, 1, 2, 3, 4, so n plus 1 numbers we will need n pairs of parenthesis to complete the multiplication, not that we keep this order; we do not multiply a 1 with a 4 in the beginning and things like that.

So, the order in which the numbers a 1 a 2 a 3 a 4 a 5, etcetera, are given is important, we do not change them, right. Just that we can start our multiplication process from anywhere in that or in the way it is written, right. Now you see this question also how many ways you can multiply this multiplication schemes, how many ways you can do this thing; also will be the nth Catalan number before when we consider n plus 1 numbers, right, why is it so?

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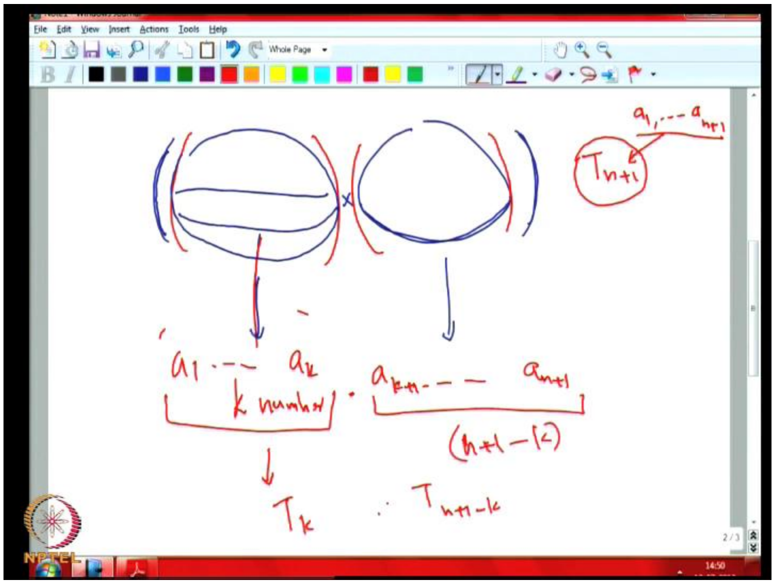

Because you know if you consider the last parenthesis it is always like this. So, which will correspond to the multiplication of the last two numbers, right? So, this first number in this thing may be, right, for instance we created one number here, created one number here, and then we multiply them together, this is the last one right. So, this is the nth multiplication, right. So, now this so we will have a pair of parenthesis here, a pair of parenthesis here, right. Now within this thing we are multiplying k numbers, say, a 1 to a k and here we are multiplying a k plus 1 all the way to a n plus 1 that is n plus 1 minus k numbers, right. So, now if I represent by T n to make it faster I will say that so T n represents the number of the multiplication schemes when we consider n plus 1 numbers a 1 to a n plus 1, right.

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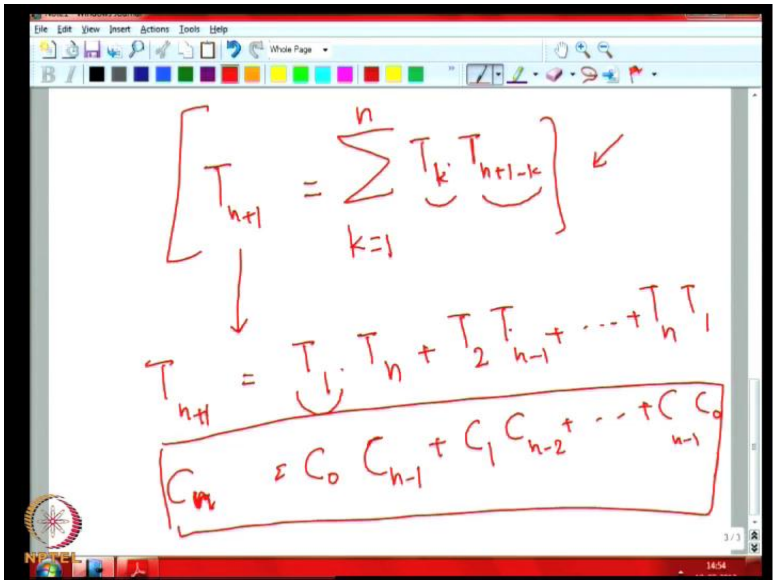

Then we say this T n can be written as, right, so the product of k equal to 0 to n minus 1. So, it would be easier to understand it is an n quality n plus 1 when we have n plus 1 numbers and then later changes. So, T n plus 1 is with corresponds to this thing when we are multiplying n plus 1 numbers, right. As we have seen here it is the final multiplication when we consider there are k numbers being multiplied first k numbers being multiplied to produce one number, and the remaining n plus 1 minus k numbers being multiplied to produce the second number, and finally we are multiplying them together, but this can be done in T k ways, and this can be done in T n plus 1 minus k ways, right, so but not that k can be 1, 2 up to n, right.

So, T n plus 1 will be equal to I will write like this T n plus 1 is equal to k can be 1 to n, right, so T k into T n plus 1 minus k. So, for instance if there is only one in the first side then it is T 1 into T n. So, if there are two of them it is T 2 into T n minus 1 and so on, right, or we multiply a 1 up to a n first and then multiply it with a n plus 1, then this is T n into T 1 like that. Now we will note this recurrence seems to be very similar to the one which we wrote for Catalan numbers, so just that we are starting from 1 to n. So far we just note that this is identical to C n; for instance if I write T n plus 1 equal to C n, right, then we can write the recurrence relation as k equal to 0 to n minus 1, right, because this will become T 0, and this will become T n plus 1 minus k minus 1 that is n t n minus k, right. Here there are n plus 1 minus k of them we have n minus k things here, but only thing is here see here so we have to start with, so this is the equation.

So, this is like when we multiply one element that is T 1 into T, this is T n plus 1 is equal to T 1 into T n plus then T 2 into T n minus 1 plus up to T n into T 1, but here actually there is normal depilation we can call this as C 0, and this is actually T n minus 1, right, and this can be called as C 0, we are just reducing one, C 0 into C n minus 1 and this is T 2, T 2 can be called as C 1 and this can be called C n minus 2 and so on, this is C n minus 1 into C 0, right. This will correspond to just C n, right. So, this is exactly the recurrence relation we had for the Catalan numbers; we just identified that T n plus 1 is same as C n, right.

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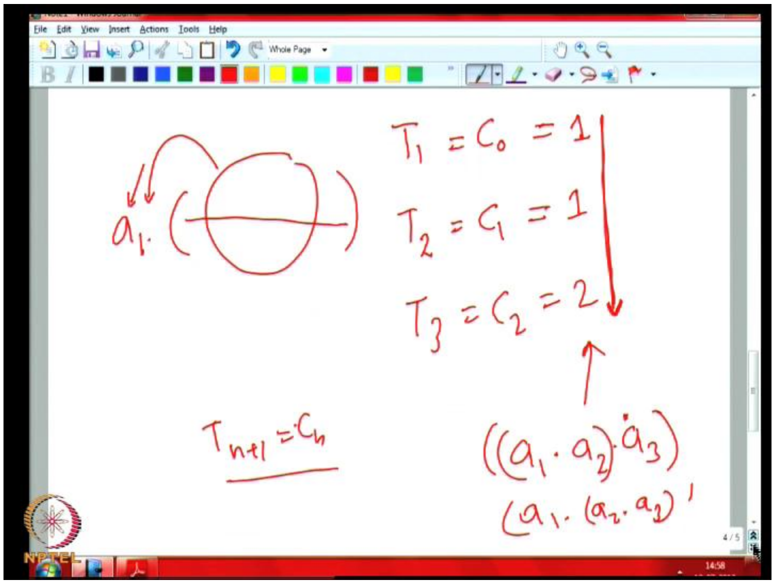

So, therefore and also the initial conditions; we will see that T 1 is actually equal to C 0. So, T 1 is this is just no multiplication, but we can define it as 1, C 0 is 1, and then T 2 is equal to then there are two numbers, then is again C 1 is 1, right, and T 3 if you look T 3 is equal to when there are three numbers a 1 a 2 a 3, how many ways you can do it? So, we can do it in C 2 ways two ways, why? Because either you can multiply this first, and then multiply with this thing or you can multiply a 2 and a 3 together, and then multiply by this thing. There are two different ways of doing this. So, the initial conditions are same. So, therefore we see that this will give you the Catalan numbers once again, right. So, it is essentially it is equal to the number of balanced parenthesis, right, when n plus 1 numbers are there we have n pairs of balanced parenthesizes.

So, while we are defining T 1 equal to 0, because this recurrence relation also wants this, right, if you define T 1 is equal to 0 then this is not valid, right, because we have one multiplication in corresponding to this term, right. T 1 has to be equal to 1, though it is just a matter of definition to product to get the product, right, because we are just taking a 1 and then the remaining things are all multiplied together. Now if this was zero then that term, right, how many possible ways are there? That will become zero into something that will become zero but we need one as the answer, right.

We want T n as the answer, right. We have these many ways and then produce it with this one multiplication, right, the final multiplication, right. So, for each corresponding multiplications to produce this product we should get one as the answer. So, we have to have T 1 is equal to one, right. So, therefore this follows exactly the same initial conditions as the Catalan numbers just that we have to define T n plus 1 equal to C n and the recurrence relation is same; therefore we get the Catalan numbers nth Catalan numbers as the answer to the number of multiplication schemes possible for multiplying a 1 to a n plus 1. Now there is another question where yeah so this is the question.

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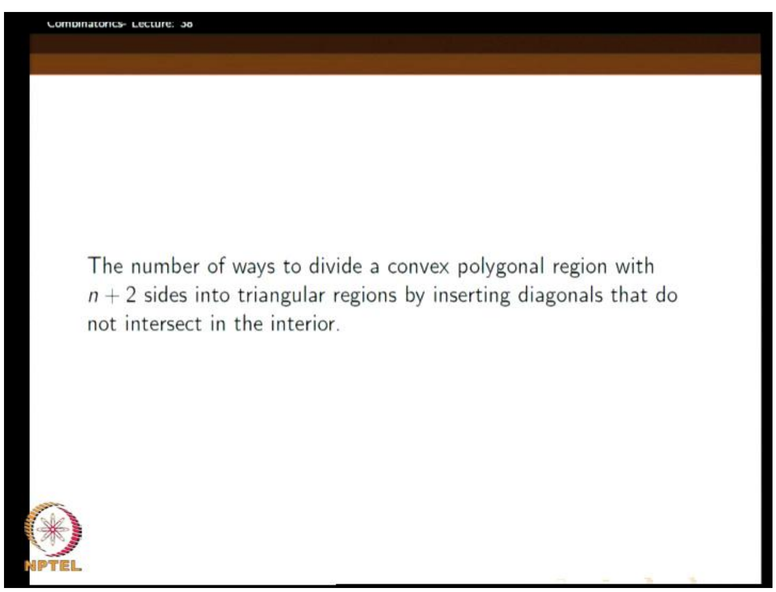

So, find the number of ways to divide a convex polygonal region with n plus 2 sides into triangular regions by inserting diagonals that do not intersects the interior.

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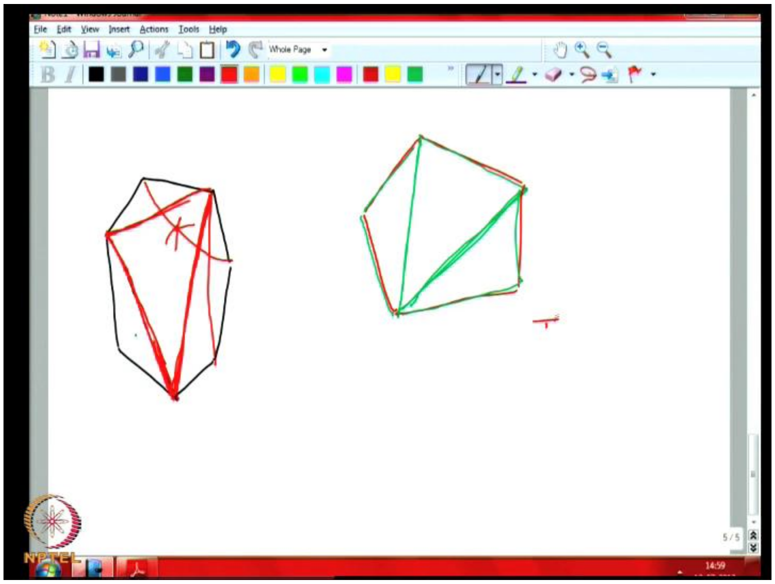

So, what we mean is so you consider, say, this is a polygon, right. So, we can triangulate it by putting some diagonals by inserting by some diagonals, say, something like this. So, we can put a diagonal like this, so all are now triangular regions. This is a triangle; this is a triangle; now this is also a triangle, right. So, similarly so now let us say so this is a polygonal region, so we can triangulate in, say, may be one diagonal can be added here, another diagonal can be added here, say, now another diagonal can be added here and, say, another diagonal can be added here.

So, this is a one way of triangulating this thing. So, the question is given a polygon with n sides how many ways we can triangulate this but not that we are putting always diagonals, right, internal diagonals, right, we do not this kind of things, but they do not intersect with each other, we never draw. So, once we draw like this we cannot draw something like this, this is not allowed, right. So, how many ways we can do this thing? This is the question.

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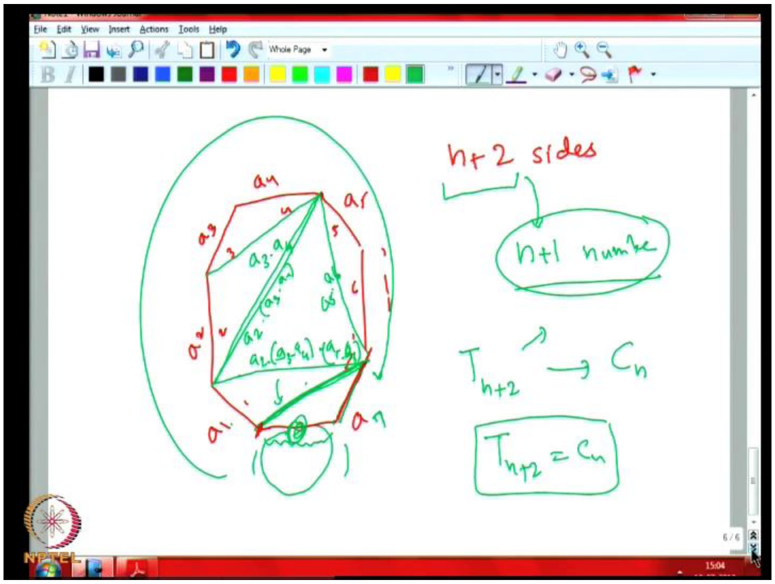

So, now of course so we just for the ease of presentation let us take a polygon with n plus 2 sides, right, so two sides and then we will note that this is somehow similar to the problem of evaluating the expression. Suppose so we start this as the first side, second side, third side, fourth side, fifth side, sixth side and this is the n plus 2 side. Now I will put a 1 here, a 2 here, a 3 here, a 4 here, a 5 here, a 6 here and till this will be a n plus 1. So, we take this n plus 1 numbers and write like this. Now if we decide to multiply two numbers, say, a 3 and a 4 then we can draw this diagonal here which means that this is a multiplication of a 3 into a 4, right, and also if you are deciding to multiply a 5 and a 6, then we draw a diagonal like this a 5 and a 6. Now if I want to multiply a 2 with a 3 and a 4 I will draw a diagonal here which corresponds to a 2 into a 3 into a 4, right, so diagonal corresponds to that.

So, now you see if I draw a diagonal here this would correspond to or maybe I can take so this n plus 2 lets without loss of generality this is 6 and 7, this is 8, right. Now if I draw this thing this is a 7 into a 8 and now if I draw this thing this will become a 1, sorry this a 8 is not there. So, we have till here; this is not given, up to a n plus 1 only we are using, this is kept blank, right. So, now we can put it like this or may be for a change we can take this terminal. This will correspond to a 2 into a 3 into a 4, right. This is this diagonal into this, so into a 5 into a 6. So, this way we can represent this multiplication schemes for corresponding to each multiplication scheme, we will get one triangulation, right, corresponding to each triangulation we can get a multiplication scheme also, right.

So, I would let you work out this details, right, so it is rather than wasting time drawing this thing, but I just wanted to give you the picture that corresponding to each way of multiplying we can keep drawing the diagonals and get a triangulation in the end. Finally here we will get the entire answer, right, because here this will come and so this into this will be copied here. So, this will represent the product of a 1 into this one and this will represent the product of this into this like that. So, finally here we will get the final expression with full parenthesizes, right, I mean indicating how we can complete the multiplication scheme. So, you have to work out the details of this bijection between the multiplication schemes discussed in the previous problem and this one.

So, note that here there are n plus 2 sides; in the previous problem there are n plus 1 numbers, right. So, now this n plus 1 numbers are written on the first n plus 1 up to here and this last one is for the final answer, right, final parenthesized expression, right, and so the number of ways to triangulate a polygon of n plus two sides will correspond to the multiplication schemes for n plus 1 numbers and which will essentially be C n. So, we will want write T n plus 2 is equal to C n, right. This is what we want, but we can also directly see this thing this is C n.

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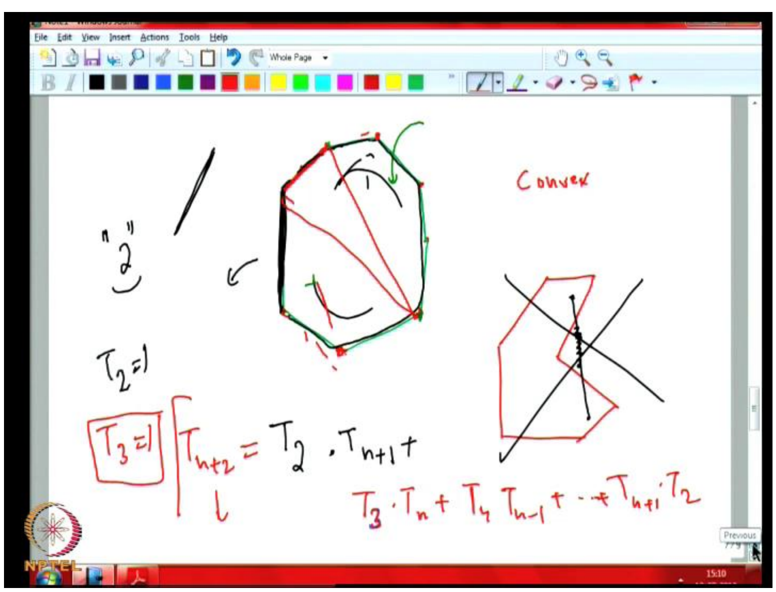

For instance if you look at a polygon an one way to write the recurrence relation is, see once you triangulate and just look at this side. This has to be part of one triangulate, right; it is not possible that this side is not part of any triangle after we fully triangulate this convex polygonal region, right. So, you note that I have not mentioned what is convex this convex see the shape should be similar which formally means that if you take any two points inside the region and draw then that full line will be inside. For instance if I consider this kind of a polygon this is not convex because if I take this point and this point and then draw a line this line will go outside the polygon, right. So therefore, this is not convex, but this kind of polygons are convex here, right, see whichever two points I take inside and draw a line between them that will be fully inside this thing, so that is what is convex.

So, I thought you will be familiar with this concept from geometry. So, now let us say coming back so here we have this side and this after fully triangulating this convex region, so this should be part of one triangle, right. Now let us say so what is the other end point of this triangle the triangle which contains this one, right. So, the other end convex point can be this, it can be this, or it can be this, or it can be this, or it can be this, or it can be this, right. There are so many possibilities. For each possibility we will decide for it typically. So, for instance if this was the other end point, now we have got one triangular region. Now it is clear that we can triangulate this polygon, say, that this polygon and multiply the number of ways of triangulating this polygon with the ways of triangulating this polygon, right. That will give you the number of triangulations of this polygon such that this triangle finally comes, right.

So, like that depending on where this vertex comes; that means this other end point of this triangle which contains a side comes we can get all the schemes. So, we start from here. So, when we start from here so what happens is it will look like this, so it will look like this the first triangle. This is the triangle, but on this side we have because its total number of sides is n plus 2. So, once you remove this thing we have a polygon with n plus 1 sides, right, because two are removed but one is added. So, T n plus 1 will be the number of ways to triangulate this portion while after removing this thing this side we do not have anything but we can say that this is just a single edge polygon; that is a polygon with two sides, such a polygon does not exist.

We will define the T 2 the number of ways to triangulate a polygon like this. We will just say that T 2 is equal to 1 to make this recurrence work; because we need the number of ways to triangle is actually the number of ways to triangulate this portion, right. So, for corresponding to each triangulation of this thing we can add this triangle and we get one valid triangulation of this entire thing. So, this we will define T 2 is equal to one, right. Now the next case is when you take the vertex to be here, so then it will look like this. So, now this will be T because here we have this triangle, right, this triangle. So, this triangle is T 3, right, T because it is a triangle it is a polygon with three sides; T 3 is the number of ways to triangulate it.

We know the T 3 is again one because given a triangle we can only triangulated it in one way, so there is same namely that keeping it just like that, and on the other side we have you know T n plus 1 minus, so how much how many will be there? So, because we have one, two, three are gone but one is put back. So, we have ah T n, right, because total in this side we have three lost here but one added; that means total two lost T n plus 2 minus T n plus 2 minus 2 that means T n. So, next will be when we allow the vertex to be here, right. So, now here we have a four sided triangle that will be T 4 and we have T n minus 1 and so on till finally we will reach to T n plus 1 into T 2 in the end, right. So, now this is going to be T n plus 2, right.

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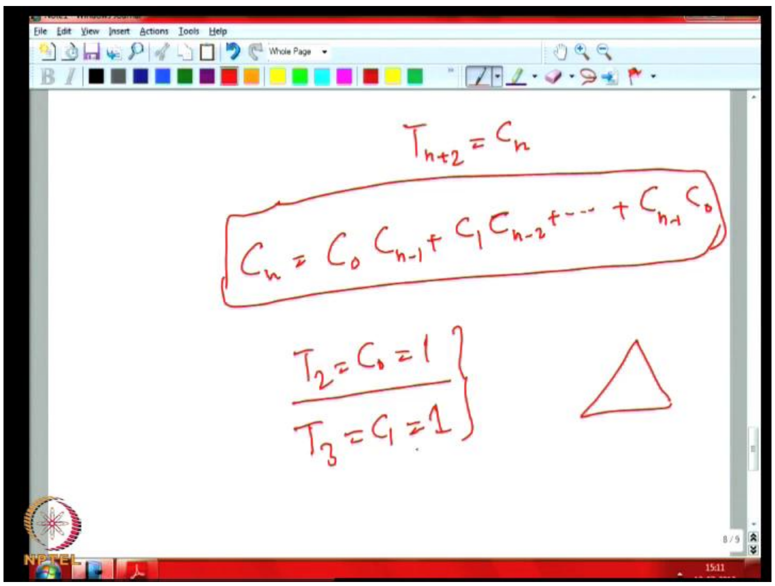

But now suppose you identify that T n plus 2 is equal to C n then we can rewrite this as C n is equal to this will become C n, here C n is equal to the T 2 will become C and T n plus 1 will become C n minus 1 and then C 1. So, T 3 will become C 1 and T n will become C n minus 2 and so on. Finally this will become C n minus 1 into C 0, right. This is the same recurrence relation as we had for the Catalan numbers and initial conditions are same as we have seen, because this T 2 is going to be C 0 is equal to one as we defined it and T 3 is equal to C 1 is equal to one as we have seen because once you have given a polygon of three sides it can be triangulated in only one way and so the initial conditions being same and the recurrence relation being same.

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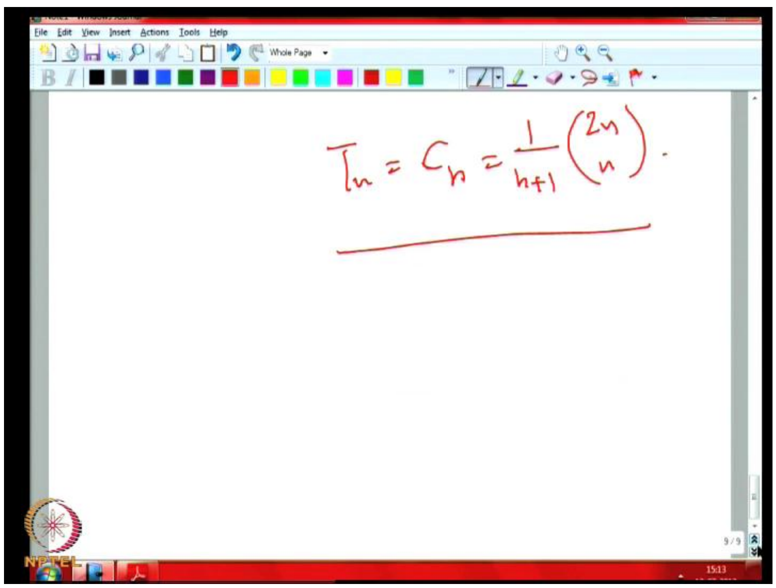

This T n is going to be equal to our nth Catalan number namely 1 by n plus 1 into 2 n choose n. So, this is what. So, we stop the discussion of a Catalan number with this last example that is it. So, in this final problem we try to sketch out two portions; one is to show that the basic recurrence relation underlying recurrence relation is the same, the other thing was to get a mapping to the previous problem namely the number of ways to put parenthesis so that we can multiply it out, right. So, it is a 1 to a n plus 1, right, the number of multiplication schemes keeping that order intact, right, but that is easy. You can just easily work out how it can be done the bijection can be set up, right. So, these are the two methods we used to say that the number of ways to triangulate the convex polygon of n plus 2 sides is actually C n namely the nth Catalan number, right.

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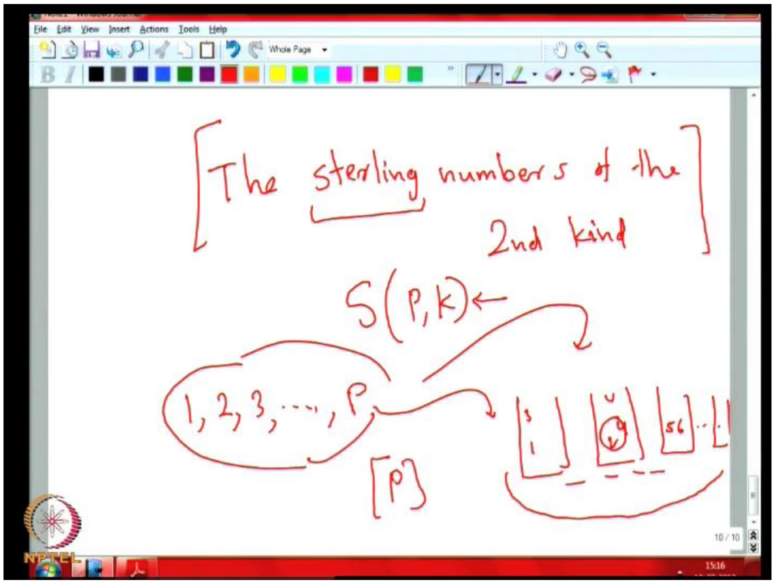

So, we will go to another topic now namely the Sterling numbers of the second kind second type, right. So, we will discuss about the first kind later. So, it is for James Sterling and yeah this is the way we define it.

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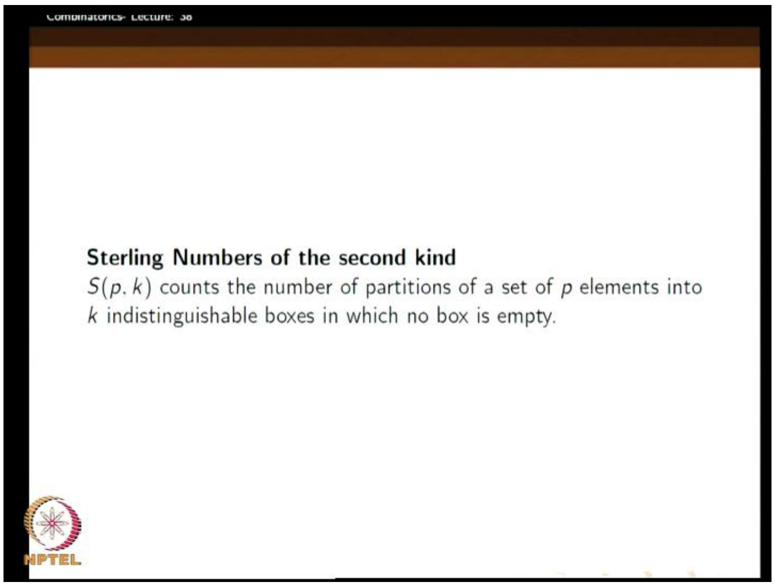

So, this counts the number of partitions of a set of p elements into k indistinguishable boxes in which no box is empty. So, we have number called S of p, k, right. So, we will have p, say, number, say, 1, 2, 3, so different objects p different objects. We want to put them into k indistinguishable boxes, so the boxes cannot be distinguished from each other by looking from outside, right, k of them 1, 2, 3. So, they are not labeled of course there are k of them, right. Now we want to distribute these p things in this k boxes. There are two condition; one is each box has to get at least to one, each of this k boxes has to get at least one.

Second condition is the boxes are indistinguishable; that means we cannot tell suppose if you do 1 2 here and 3 4 here 5 6 here and so on, and then if I take 1 2 from here and put it here and then 3 4 here, that is not going to make any difference because for us an outsider sees it will be the same, because he sees 1 and 2 together in one box, 3 and 4 together for in one box, right. So, you have to change say for instance if you take 3 from here and put with 1, so you make it 1 3 and put 2 4 here then that is different, because you will see that okay this was different from them, because earlier it was 1 and 2 was always together, right, but now he does not see 1 with 2, 3 is coming together, right. So, it is essentially you want to decompose p into k subsets non-anti-subsets, right, in how many ways you can do this thing? This is what this count is given by S P, K.

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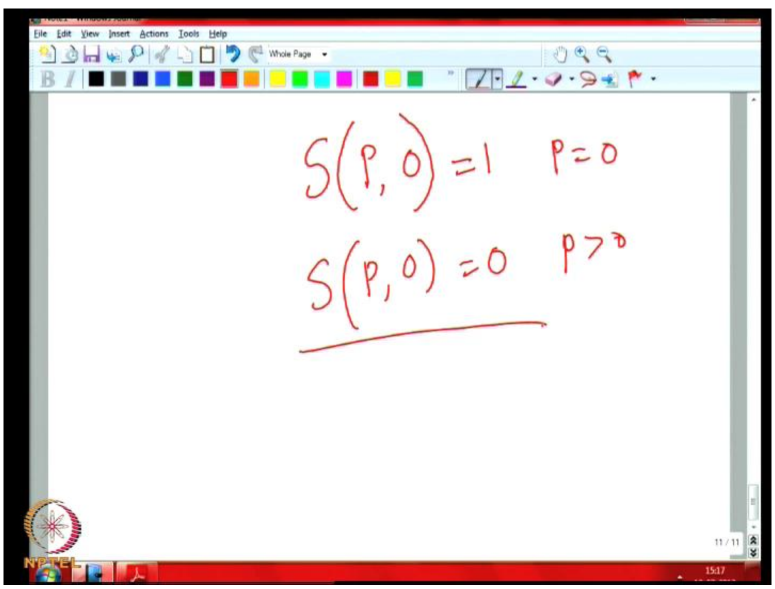

So, some of the simple observations we can make about given a p; so let us say p is at least 1, so there is no point about distributing zero things into something, right, or negative things into something. We will just say if p can be 0 but let us say p is something and then if you consider S p, 0, right, this will be if p was equal to 0, then it is okay we can put 1, right. If p equal to 0 we can put 1 because zero things can be put in zero boxes in one way, but if p is greater than 0 then this has to be 0 because if there is at least one object and the zero boxes we cannot distribute them into zero boxes, right. So therefore, this is zero.

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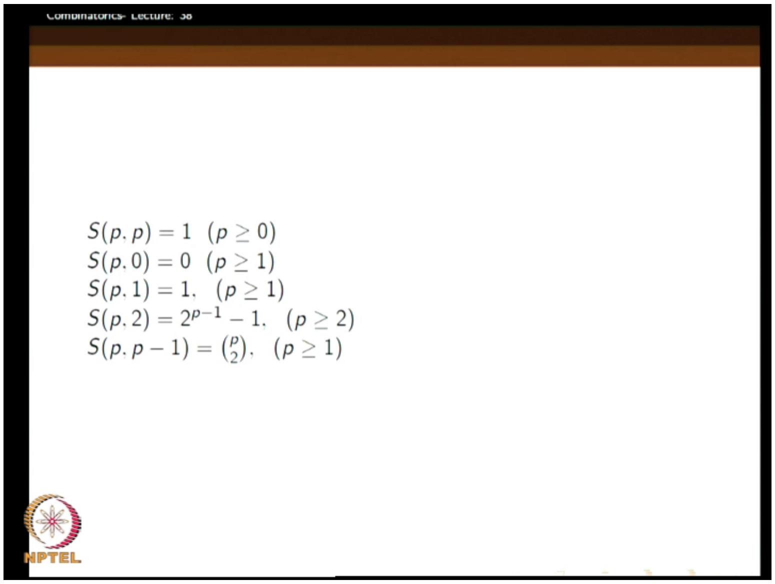

S p, 0 is equal to 0 for p greater than equal to 1, S p 0, 0 we will take it as 0 and another thing is S p, 1, what will be S p, 1?

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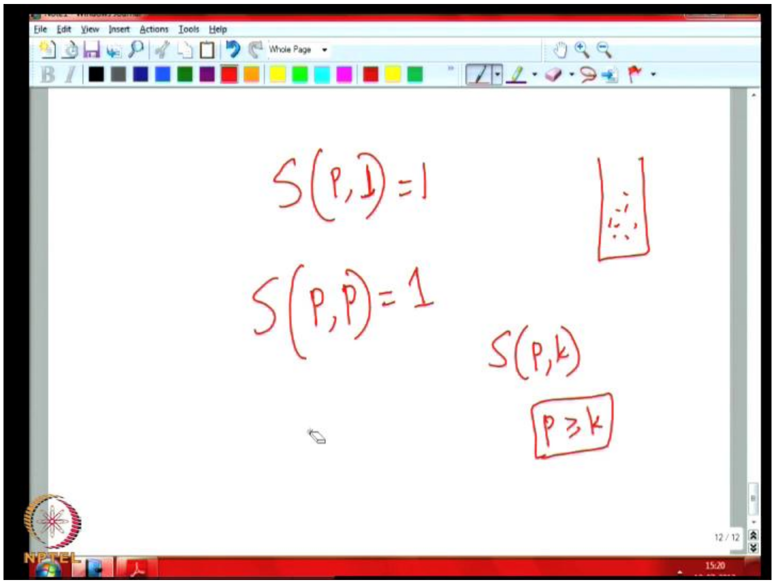

What will be S p, 1? There are p objects and there is just one box, right. See the point is if p 0 or something then we cannot put it in one box, so that this has to be 0 because you know this box has to be non-empty finally, right. So, therefore we will assume that p is greater than equal to 1. We will assume that yeah p is greater than equal to this number, because if I consider S p, k where k is greater than p, then if every box has to be non-empty we cannot do that, then this will become zero, right.

So therefore, we will assume that now for s whenever we consider S p, k this p is greater than equal to k, so that we can keep every box non-empty box; that is one of my major concerns, right. So, with that condition S p, 1 so p, 1 we want to put p objects in one box, right. So, that can be done only in one way; we have to put everything in that same box, right, there is no other way to do it. Now another thing, what about S p, p? Sum on special cases when p equal to k, right, S p, p. So, the only way is because every box has to be non-empty; we have to give one object each to each other boxes whether there are exactly p boxes, there are only p objects, then this has to be one. There is only one way to do that, one object each to each box.

So, we cannot see for instance these are this p boxes; we can put 1 here, 2 here, 3 here, 4 here. So, you may wonder why I cannot put 4 here and 2 here? Will it not give or produce a different arrangement? No, it will not, because the boxes are indistinguishable; so, whether you keep the box here or here it does not matter. What matters is the content inside the box. So, as for as we are concerned this 1, 2, 3, 4 is partition into this subsets 1, 2, 3, and 4, right, whether you write this subset first or this subset first is unimportant because as far as they are concerned. So, this is the way we have partitioned this thing into subset. That is only one way of doing it, right, so that is S p, p, therefore, has to be one, right.

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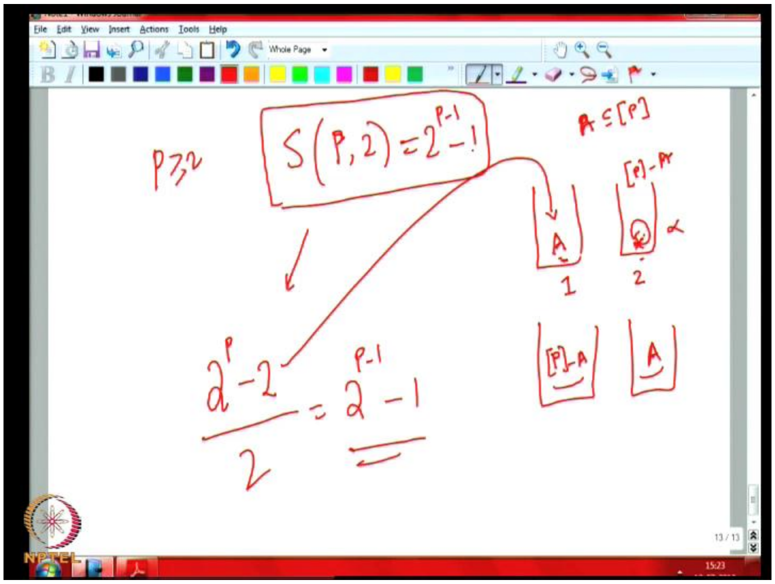

Now what about S p, 2? Now S p, 2 there are p objects and there are two boxes. So, the question is in how many ways this p objects can be split into two subsets. So, it is easy to see that; for instance you try to take phi as a subset that is not allowed because the boxes have to be. So, what I do is for the time being I will assume that this is the first, this is second though the boxes are indistinguishable. So, if I take phi and try to put here phi out of p and an empty set out of p, it is not allowed. So, it has to be non-empty set, not only that we cannot take the entire p and the entire set of objects and put it in this thing because if I do that this will remain empty; that is also not allowed, right.

So, apart from this two any other subset of this thing can be taken which are actually 2 raise to p minus 2 possible subsets like that excluding the empty set and universe full set, right. So, this many are there. If you give any of those subsets here the remaining can be given here, then both will be non-empty. The only thing is that so if I give an A here and say p minus A here where A is some subset of A is some non-empty subset of p where A is not equal to p, right. Then we are also counting this case p minus A is here and A is here, right, but you know because this boxes are indistinguishable this A being here and p minus A being here is the same as the p minus A being here and A being here.

So, we are over counting, so we have to divide by two because for every pair we are counting twice. So, therefore the final answer will be 2 raise to p minus 1 minus 1 where this is going to be this one, right. This many ways of doing it, right, p greater than equal to 2; this is valid for p greater than equal to 2, right. So, p was one then we know we cannot put it at all, because we cannot keep so that will be zero; if you put one also this will work 2 raise to 0 minus 1.

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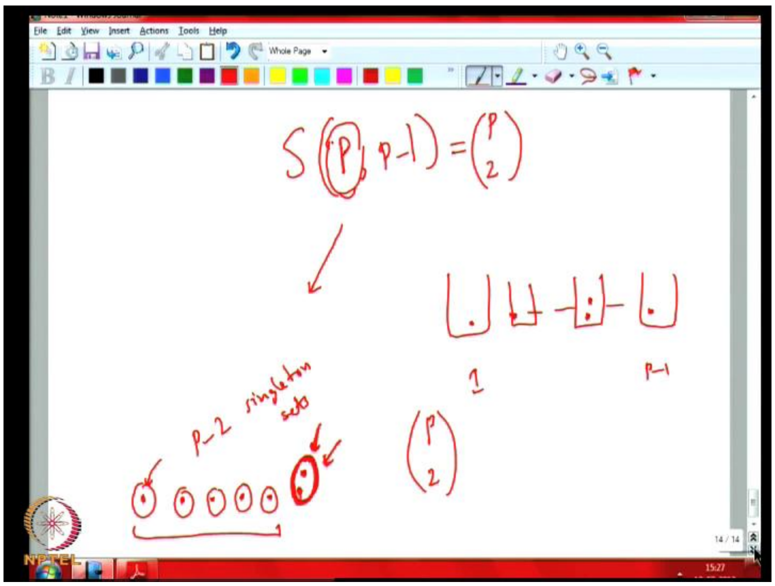

Now the next one S p, p minus 1, what about S p, p minus 1? So, we have p minus 1 boxes. Now p things are to be distributed here; yeah p minus 1 boxes are there. We have p things, and we have to distribute these p things into p minus 1 boxes, how many ways we can? So, we can see that because there are p minus 1 boxes; it is very clear that each and each box has to be empty. So, we have exactly one box in which two objects appear and all other boxes have just one object, right. So, the kind of partitioning into subsets, this p is to be partitioned to subsets. The kind of partitions we are looking for subsets we are looking for is it is a p minus 2 singleton sets and just one two element set, this kind of option is.

So, to count these things it is very easy to see that we just have to decide which two elements go into the bigger set, right, and the remaining things can be put single ton, right, that can be done only in one way. So, for each selection of this two then we have one valid partition, right, partitioning of this p objects, so assignment of this p objects into boxes. So, how will you decide these two objects? So, out of p we can select in p choose 2 ways this two elements, then once you select those two elements we put them together, and all the remaining objects we put one each into the remaining p minus 2 boxes.

So, therefore S p of p minus 1 is equal to p choose 2 right. So, this has some of the, yeah this will work for, right. So, for instance if you are taking p equal to 1 box there is just one object, it should be put into zero box then it is zero. So, here we will see that 1 choose 2 are 0. So, that is correct from p greater than equal to 1 onwards. If p is equal to 2 what happens is we have to put two objects into one box that can be done only in one way; that is 2 choose 2 is going to be 1, right. So, we can verify some initial cases so that is all correct.

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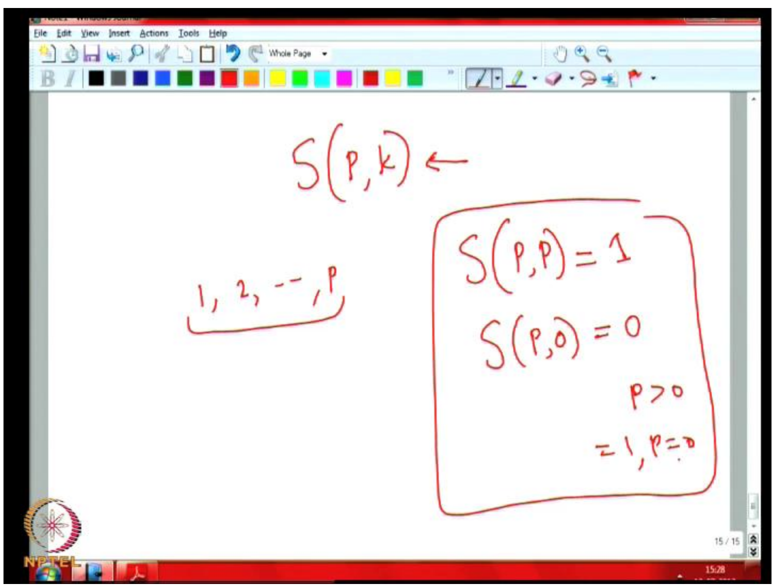

So, yeah this was some example to make you familiar with this concept this new concept namely S p, k this is S p, k, right. So, we are putting the numbers 1, 2 up to p into k boxes such that none of the k boxes are empty and this boxes are indistinguishable, right, how many ways you can do, this is what it is. Now this S p, k satisfies one recurrence relation; let us look at what this is. We will take two things as initial conditions namely S p, p equal to 1 as we have seen before. If there are p boxes and only p things we can only assign them to boxes in one way, right, because we are not distinguishing between the boxes. It is just that each one goes to a different box that is the only arrangement, right. So, another condition is for S p, 0 is equal to 0 when p is greater than 0 and is equal to 1 if p equal to 0, right. These are the two initial conditions we have already discussed this thing.

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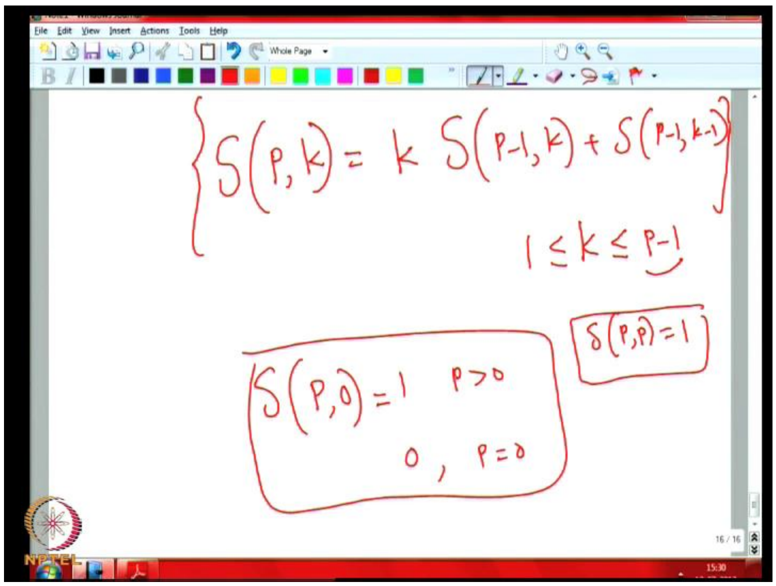

And the recurrence relation we claim is this S p, k is equal to k times S of p minus 1 times k plus S of p minus 1 times k minus 1. So, this is true for k for 1 and p minus 1. So, we are not giving this recurrence relation for k equal to 0, because we have already defined the initial condition S p, 0, right. This is 1 or 0, 0 if p equal to 0; otherwise p greater than equal to 0 this is 1, right. Similarly, we are not giving for k equal to p k's because S p, p is equal to 1. This is given as an initial condition, right. So, now for the remaining numbers before when k is in between 1 and p minus 1 we are defining this, I mean we are claiming that this is the recurrence relation valid for S p, k. Why is this valid for S p, k, right.

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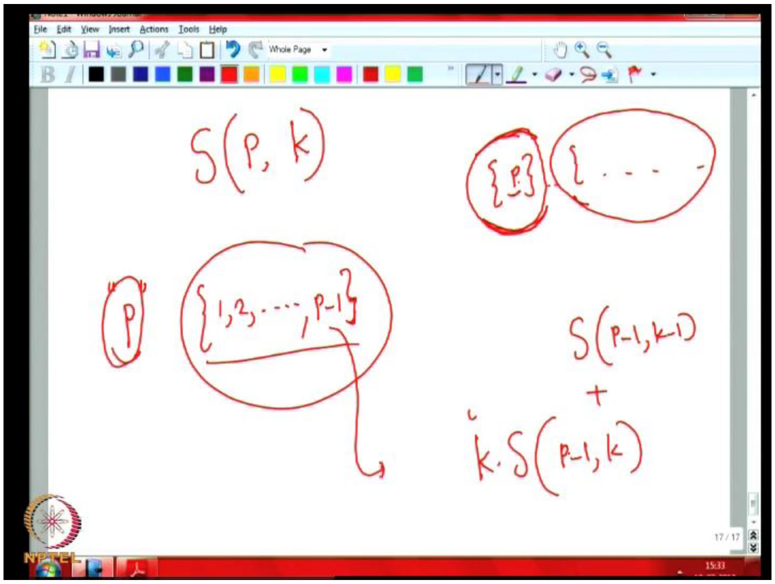

So, this is because what does S p, k mean? S p, k means we have p numbers, say, distinct objects and we want to assign them to k boxes these boxes being indistinguishable. So, we just take the last number namely p; we just separate it out, right, and the remaining numbers say 1, 2 p minus 1, right. So, now we will take remaining numbers is this. So, we will take two cases. So, some of the arrangements may have this number last number p as the single ton set, right. So, it may have p like this and then the remaining one to p minus 1 gets distributed to k minus 1 different object. The question is this p minus 1 things in how many ways we can distribute into k minus 1 different objects; that will count the special case namely where this p comes as a single ton set, this comes as a single ton set.

So, this is actually p minus 1 objects are to be distributed to k minus 1 indistinguishable boxes. So, this number is s of p minus 1 k minus 1, right. So, that counts the kind of partitions where this comes as a single ton set. Now look this may not be the case, so the remaining partitions are such that this p always comes with something else. So, we never have p alone. So, p may come with something or may be with 1 or may be 1 and 2 or p may be coming with 3 and 4 or p may be coming with 1 and 5. So, something like this, right.

So, but p never comes alone. So, what we do is we just remove p and then we see k non-empty boxes again, because p was never single ton. So, when you remove p we do not make that box empty, right. So, what we can do is to get the number of partitions we can ask how many possible ways are there to put this 1 to p minus 1 into k boxes, such that no boxes empty and then put p back into the system, right. This can be done in S of p minus 1, k ways clearly but now how will you put back p? p can go into any of the non-empty boxes, right, available but there are k non-empty boxes. So, there are k possible ways to insert p, right. So, the total number of partition is this plus this.

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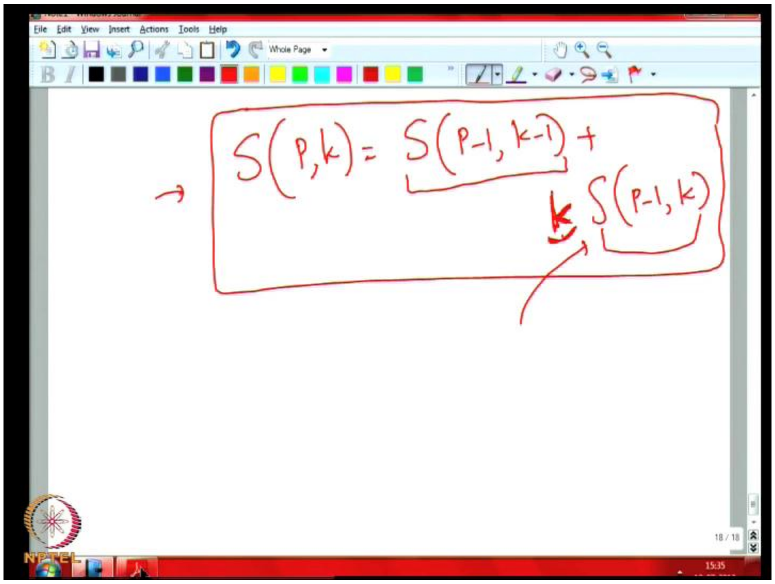

So, we get S p, k is equal to S p, k equal to S of p minus 1, k minus 1 plus k times S of p minus 1, k, right. So, if you have difficulty in remembering this, one thing you can observe is that if this k is not there it is just like the recurrence relation for binomial coefficients. For instance it will be like it is like this, so p choose k binomial coefficients, right, p choose k is equal to p minus 1 choose k plus p minus 1 choose k minus 1, but here this term with k as the lower index is getting multiplied by k here, right. So, here see this corresponds to this and this corresponds to this and then here this is getting a multiplier; that is the only difference. Otherwise, this is exactly like this. So, you can remember that when you want to write the recurrence relation S p, k you can write it like the binomial one and then insert that k there, or if you think about the proof it is very easy.

So, you just have to remember that we are partitioning it into two ways, the partitions where p comes as a single ton sets and the partitions where p never comes as a single ton set. And if p comes as a single ton set without that we have this many possible ways of putting the remaining p minus 1 things into k minus 1 boxes where none of them are empty, and if p is always has to be with something else then we can take p minus 1 things and put them into k different boxes k boxes which are indistinguishable. Now that can be done in this way and insert p in k ways; this is what it is and the next thing we want to discuss is yeah.

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So, the next thing we want to discuss is something called difference sequences. So, this we are discussing because of the reason is that this is somehow connected with the Sterling numbers of the second kind, right. So, we will build-up the concept first and then we will bring out the connection.

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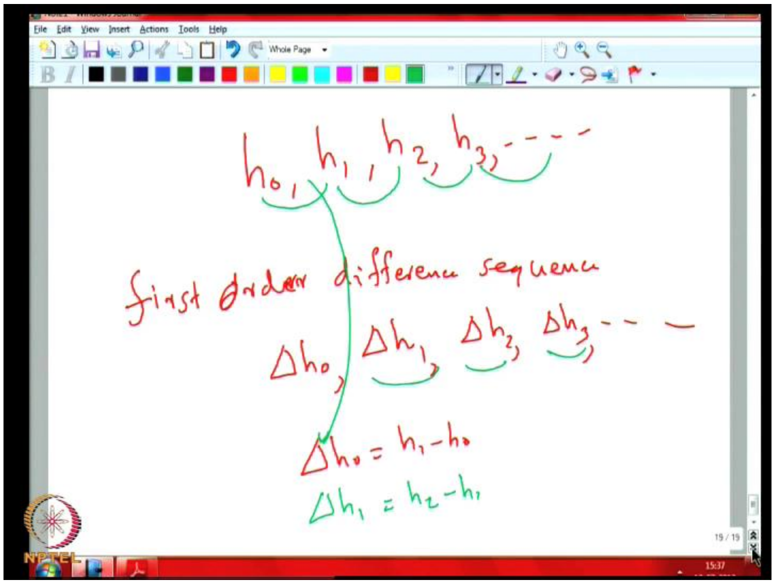

So, now consider a sequence like h 0, h 1, h 2, h 3, so this is sequence. Now we will see that the first order difference sequence for this given sequence h 0, h 1, h 2, h 3 is given by delta of h 0, delta of h 1, delta of h 2, delta of h 3 and so on where this delta of h 0 is what? That is h 1 minus h 0. So, actually we are taking the difference between this and this; that is what this delta of h 0 is and difference between this and this will be the delta of h 1 is h 2 minus h 1, right, that is what here, and this difference will be this, and this difference will be delta of h 3 and so on, right, and for instance I can consider some example.

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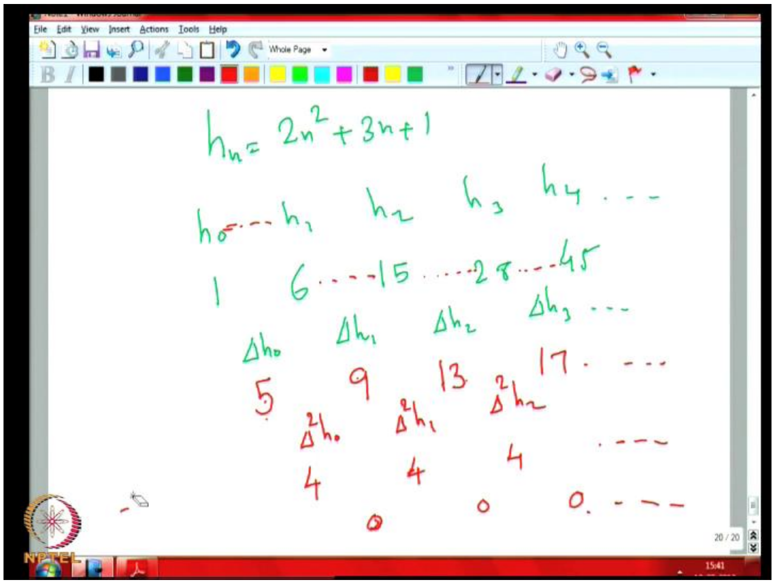

So, let us consider ah say h n equal to 2 n square plus 3 n plus 1, right. Now what would be the sequence put n equal to, so this sequence I will write like this h 1 h 2 h 3. So, put n equal to 0 we will get 1; put n equal to 1 we will get 6, right; put n equal to 2 we will get 8 plus 6, 14 plus 1, 15, right. So, if you put 3, so 9 into 2, 18 plus 9, 27 plus 1, 28 and so on. The next one h 4 will be equal to 45 and so you can work it out and we will write delta h 0 here, we will write delta h 1 here, we will write delta h 2 here, this delta h 3 here and so on, why? Because delta h 0 is actually the gap here the difference here; that means 5 here, 6 minus 1, delta h 1 is the difference here that is 9 and delta h 2 is this 15. So, 13 and this is 45 minus 28 namely 17, this is what.

Now we can define the difference sequence of this sequence, right, this 5 9 13 14 that will be the second order difference sequence for h n, because actually what is this second order delta square of h 0? This is delta of delta of h zero. So, in which case delta of h zero is 5 here, so delta of h zero is what? So, this will be delta of yeah this will be written as because this can be written as delta of h 1 minus delta of h 0 which is actually this h 2 minus h 1 minus h 1 minus h 0 which is h 2 minus 2 h 1 plus h 0, but when you write the table it is easy, because you just have to minus this from this, right. So, we have written 1 6 15 28 like that. So, we can just write we do not have to go to the first sequence, we can write it from the second sequence.

We will write delta of square of h 0 here, delta square of h 1 here, delta square of h 2 here and so on. This will be 4, 9 minus 5. This will be 13 minus 9 which is 4, right, and the third one 17 minus 13 is 4 and so on. Now we can define the third order difference sequence namely delta cube of h as delta of delta square of h; that means delta of cube of h 0 will be delta square of h 0 minus. So, delta square of delta of delta h 0 which is essentially delta square of h 1 minus delta square of h 0. Here 4 minus 4 is 0 that will be, this will be zero, this will be zero and so on. Now always in this way we can define the p th order difference sequence by defining it.

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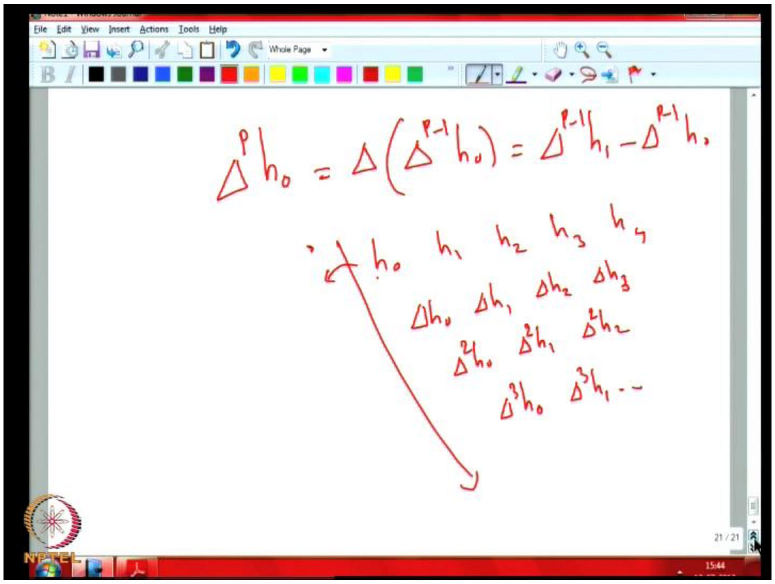

As see delta p of h 0 will be defined as delta of delta p minus 1 of h 0, which is actually delta p minus 1 of h 1 minus delta p minus 1 of h 0, right. So, if you keep on writing the tables first we have h 0, h 1, h 2, h 3, h 4, etcetera, the first sequence which can also be written as delta 0 of h 0 if you so prefer you can write this as delta 0 of h 0, delta 0 of h 1, delta 0 of h 2 and so on; just define delta 0 of h i is equal to just h i, right, so that is not difference. Now here we can write delta of h 0, this is delta of h 1, this is delta of h 2, this is delta h 3 and so on and here we can write delta of square of h 0, this is delta square of h 1, this is delta square of h 2 and so on. This is delta square of this is if you take the difference between these two you will get delta cube of h 0 this delta cube of h 1 and so on.

So, this the concept of p th order difference sequences. So, we can keep on writing this difference sequences, but we have seen that in the previous example when we reached this is the zeroth line where zeroth order difference equation namely the sequence itself is written as the zeroth line, and this is the first line, this is where the p th order difference sequence is written; that is the p th line, right. Here second order was written that becomes the constant, right, second order 4 4 4 4 like that. When the third order was taken it was already 0, everything was zero. So, when we started with a polynomial if H n was 2 n square plus 3 n plus 3 as a polynomial of degree two but the third order difference equation is becoming 0 0, what is the reason for that? In the next class we will see that if the general term is given by a polynomial of degree k then the k plus 1 th row onwards it will be zero. We will prove it in the next class.