Combinatorics
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Lecture - 37
Catalans Numbers - Part (2)

Welcome to the thirty seventh lecture of combinatorics.

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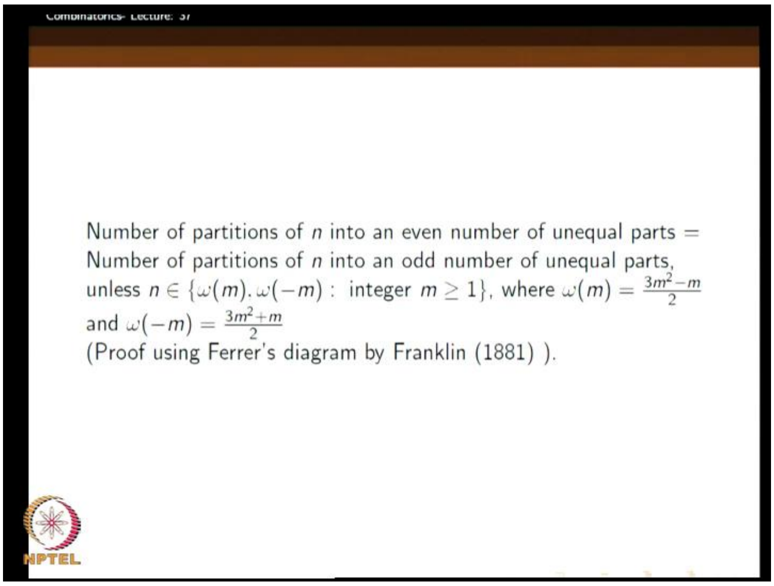

In the last class we were discussing about Catalan numbers and we will continue with that. So, here is another question; last class we considered some tree diagrams, so here is another question. Consider the sequences of the form a 1, a 2, a 2n, there are 2n terms in that and each a i here is either a plus one or a minus one. There are n plus one's and n minus one's equal equal. The requirement is that if you take a partial sum starting from the first to the kth term, say, a 1 plus a 2 plus up to a k that should be always greater than zero; you should never go to negative. a 1 should be greater than zero greater than equal to zero, a 1 plus a 2 should be greater than equal to zero, a 1 plus a 2 plus a 3 should be greater than equal to zero and so on for k equal to 1, 2 up to 2n it should satisfy this condition. Now we will show that the number of such possible sequences, how many such sequences are possible? This also is equal to the nth Catalan number.

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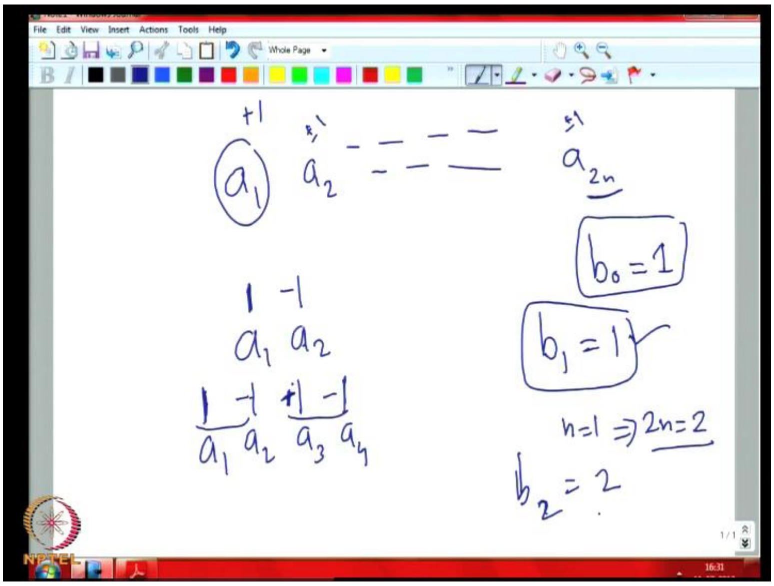

So, again we have this a 1, a 2, a 2n. There are this a i's this is either a plus 1 or a minus 1, right, all of them are plus or minus 1, right, but this a 1 has to be a plus 1 because you know our condition says a 1 greater than equal to zero; if a 1 is minus 1 how can a 1 be greater than equal to zero? So, a 1 plus a 2 is greater than equal to zero also, but a 1 is plus 1, a 2 can be either minus or plus 1, it does not matter, because if it is a plus 1 we will get 2 it is greater than equal to zero; if it is a minus 1 we will get 0 that is also fine, right. So, yeah we want to keep the partial sums to be greater than equal to zero; that is all we want. The question is how many such sequences can be formed? We can check again we will define b 0 equal to 1.

Let us say what b 1 is, b 0 means what? The empty sequence, right, so ends without there is one such sequence we will assume. So, we will look at b 1, b 1 is what? So, just one term, so that is only one sequence; you have to put a plus 1, b 1 is 1 and what about b 2, right? b 2 is two letter sequences, you have to have a plus 1 here. Now you can either have a plus 1 or you can have a plus minus 1, both are fine, but then this is not fine because you know, I am sorry; b 0 is fine, what is b 1? b 1 means actually I am taking n equal to 1; that means our 2 n is equal to 2, right. So, there is nothing like just one letter sequence, we have either zero letter sequence or two letter sequence or four letter sequence or six letter sequence, something like that, because we are always taking this as 2 n, right, so the b 1 equal to 1 case is actually a 1, a 2.

So, a 1 has to be 1, so this is 1 then this has to be minus 1; there is no other way, because there is only one sequence, right. What about a 1, a 2, a 3, a 4 that is b 2; that means n equal to two case. Here you can put a 1, then if you decide to put a plus 1 again here, then this both has to be minus 1, because that should be two one's then two negative one's this one is possible sequence. But on the other hand you can decide to put a minus 1 here then we again have a choice here. You can have a 1 or minus 1 here, but you cannot have minus 1. If you put a minus one what will happen? 1 plus minus 1 is zero, plus minus 1 will be negative. So, this is not allowed; this has to be a plus 1 and this has to be minus one right. So therefore, there are two sequences b 2 is 2, right. So, this looks like the Catalan sequence you can try the third one if you want.

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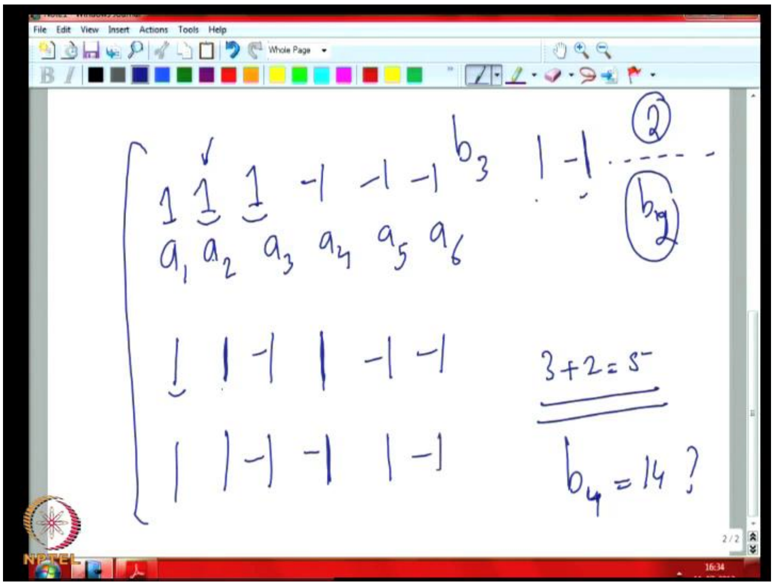

For instance b 3, right, b 3 means six letter sequences a 1, a 2, a 3, a 4, a 5, a 6, as usual this has to be plus 1 and this if you start with a plus 1 there is one choice here, right, plus 1. Here you again have a choice plus 1, right, then we have to put minus 1, minus 1, minus 1. So, here i executed a choice, here I also selected from 1 or minus 1, right. So, now we will keep the choice one and one as such initially. So, this first one and second is one. Here there is no choice; this has to be 1. The second one we have a choice; it can be plus 1 or minus 1, but we just fix it as 1 and then third one if you put 1 then is fully fixed because three plus one's means we have to have three minus one's, so this is a valid sequence.

Now this one you can try minus 1 here, then we get a choice here, right, we get a choice here. So, what should be that? We can have a plus 1 or minus 1. If you put a plus 1 then this two are fixed; that is fine, right. Then what about this one? We can try a minus 1 here, right, then we cannot put a minus 1 here because if you put a minus 1 here this partial sum will become negative. So, this has to be plus 1 right and this has to be negative. Now here these are the only choices when you put a minus 1 here, right. Now we can try 1 minus 1 here. So, if you put 1 minus 1 here then there are four sequences which have to have two equivalents, there should be two possibilities here.

We have already seen, this corresponds to our b 2, right. So therefore, two such sequences will come if you put minus 1 here. So, that is it, we do not have to count, right. So, the total 3 plus 2 is 5 sequences will come. So, this you could have thought like this, the total is 5 we can see. So, we can just check it once again if you want if you are not convinced. So, try to write it down; you can go one more and try for b 4 and see whether it is 14 or not, right, and what we are planning is if you look at the initial numbers it looks like Catalan numbers, right.

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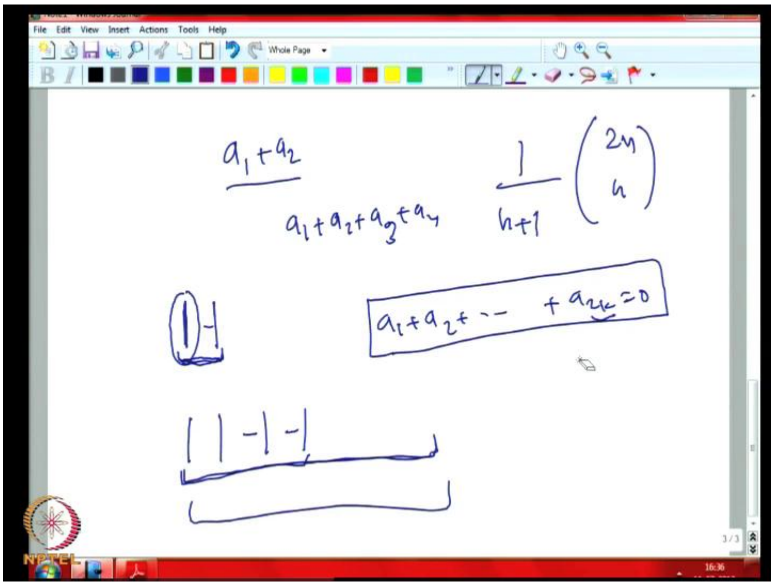

So, how will I show that this actually is the same 1 by n plus 1 into 2 n choose n as before. The method is this; you try with the sequence, you look at where it first becomes zero the partial sum first becomes, you start with the partial number first is 1. Then the next time if it is minus 1 then here it becomes 0, 1 plus the first partial sum, which becomes zero is a 1 plus a 2 itself or it can be like 1 plus 1. Then you have minus 1, minus 1, then here it becomes zero a 1 plus a 2 plus a 3 plus a 4 it becomes zero, right, or it can be after 2 k let us say.

So, that means this may be the a 1 plus a 2 plus a 2 k can be zero; that may be the first k for which it will become zero. Not that it will be always an even number, the total number of letters terms you require, so that it becomes zero when you add up because there should be an equal number of one's and minus one, then only it will become zero, right. So, we can split the total sequences; I mean the sequences it which satisfies our condition that partial sum should be greater than equal to zero condition.

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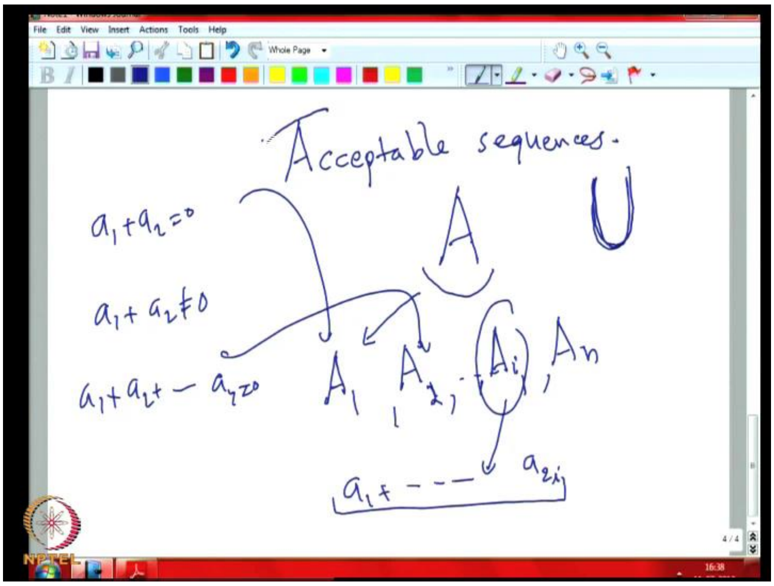

We will call them acceptable sequences; the set of such sequences will be denoted by A and the remaining sequences which are not acceptable are called unacceptable, right. Now if you want to look at the acceptable sequences, they can be categorized into n different groups A 1, A 2, A n and what is this A i? The i th group consist of those acceptable sequences where the first partial sum which becomes zero is this one a 1 plus up to a 2 i. So, that means if for particular sequence acceptable sequences if a 1 plus a 2 is equal to zero, they all go to this.

So, for those sequences for which a 1 plus a 2 is not equal to zero, but a 1 plus a 2 plus up to a 4 becomes zero those will go to this and so on. And those sequences for which the partial sums up to a 2 i minus is always remain greater than zero but a 1 to a 2 i sums to zero, that will go to A i. So, definitely this is a partition of the set of acceptable sequences, right.

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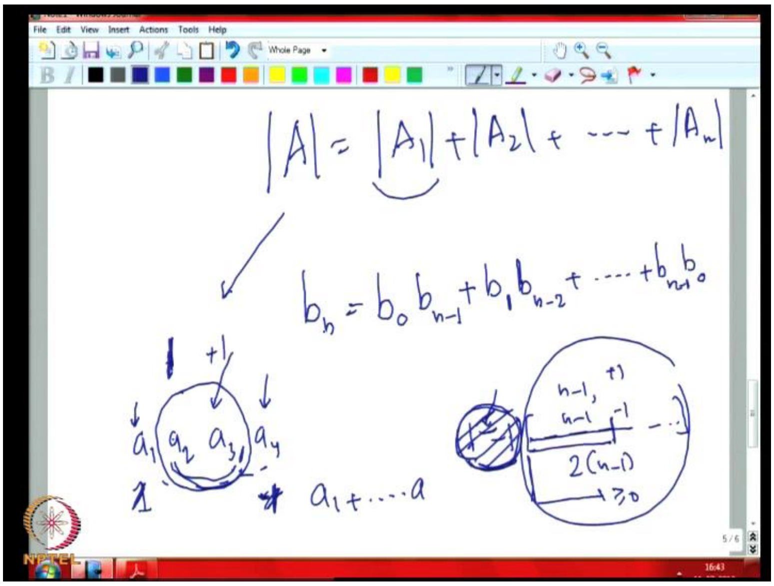

Now if you want to get the total the cardinality of A, we just have to sum up the cardinalities of A 1 and A 2 up to A n, right. Now there are n's two n letters, right. So, now if this a 1 you know there is a 1 and minus 1 initially, so it does not matter. There is nothing like there are 1 and minus 1 has to be first and minus 1 has to be zero, what is there inside, right, how many ways you can put one's and minus one inside, but there is nothing inside. So, we call it, say, b 0, right, the numbers of ways you can fill things here. It is an empty sequence, but from here to here we have 2 into n minus 1 letters so in whichever way you want to arrange, how many ways we have to arrange out of which n minus 1 is plus one's, n minus one of them is minus one's.

Because this is already zero we should have a partial sum condition here; starting from here if you find the partial sums they have to be always greater than equal to zero, because we cannot take any help from here. This is already cancelled one and minus one. So, now any partial sum up to here has to be greater than zero because from here to here it has to be greater than zero and this much itself is zero, right. So, this satisfies the condition here any acceptable sequence from here to here it will satisfy the condition of the acceptable sequence.

So, the number of ways we can put one's and minus one's with n minus one plus one's and n minus one minus one's here in 2 into n minus one positions is exactly b n minus 1, right. And definitely the next one we consider is when we have something here in a 1, something here in a 2 and something here in a 3 and something here in a 4, and they sum up to zero. This was plus 1 always; if this is the first time they sum up to zero, right; that means this is minus 1, right, because this is something greater than zero up to here; we have made it zero this has to be minus 1. So, you can cancel off this and this; you can see in how many ways you can put these things, right.

So, we know that this a 2 is now they have actually 1 and 1 minus 1, and then they should have the partial sum property also; suppose up to here this is 1 this n minus 1 if you take then so this is 1 and this is greater than zero and if this two together is more than. So, suppose this was plus 1 this will be 3, then one minus one cannot make it zero. So therefore, they should have again the partial sum property, right. So therefore, we can write it as b 1 because once you remove this thing this is b 1 into b n minus 2 because four is gone, then this is 2 into n minus 2 things. So, this recurrence relation can be written for b n minus 1 and finally b 0.

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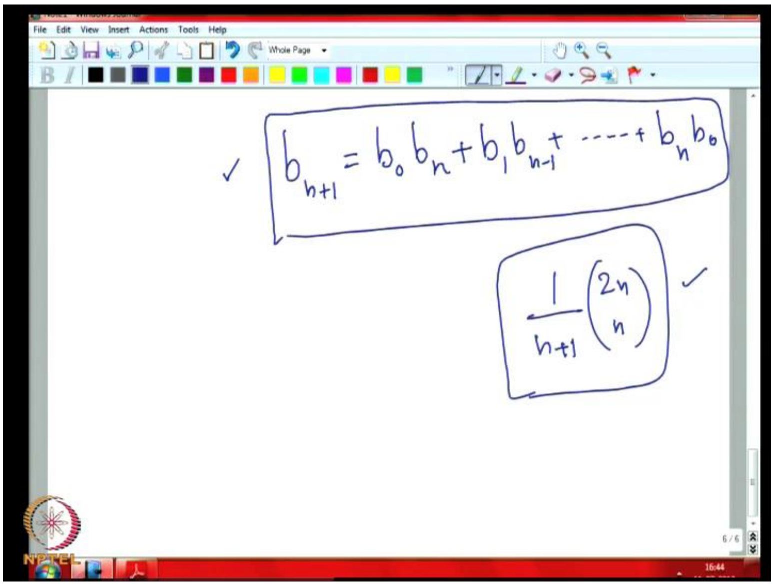

This is for b n or if you have expressed in terms of n plus 1 this will look like b n plus 1 equal to b 0 into b n plus b 1 b n minus 1 plus b n into b 0, right. So, it is the same recurrence relation. So therefore, it is giving the same, it should be giving and with the same initial conditions it should be the same sequence of numbers, right, that is the nth Catalan number I can see here, but our intention now is not to solve it with this way.

So, this method is essentially to show the same recurrence relation, so that the same recurrence relation is holding here and then say that it is the same as the previous problem because same initial condition and same recurrence relation means the same sequence. But now we will show a different method, I mean starting from this problem and then we will show that this is actually 1 by n plus 1 into 2 n choose n is the answer, right and therefore, it has to be the same as the previous count previous sequence, right, because this is the defining expression for the nth term of that sequence, right. This is what we want to do now rather than working with the recurrence relation. So, how will you do that?

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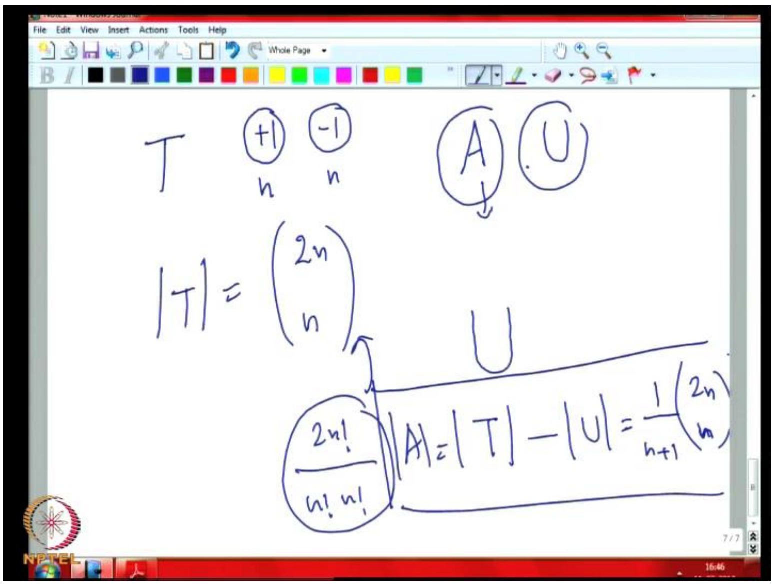

So, we have partitioned the sequences of plus one's and minus one's with n plus one's and n minus one's into acceptable and unacceptable, but the total number of sequence, say, T we know that that cardinality is equal to 2 n choose n, why is it so? Because there are 2 n positions we just have to select n position to place the plus one's, the remaining positions will automatically be given to minus one's, right, or we can say that there are 2 n objects of which n r of the same type minus the remaining n r of this different type.

So therefore, 2 n factorial divided by n factorial into n factorial is the possible ways to arrange them 2 n choose n. Now out of which how many are acceptable, how many are unacceptable? This is the question. The trick here is to count the unacceptable sequences and then we will show that T minus U is actually 1 by n plus 1 into 2 n choose n and definitely this is our cardinality of the acceptable sequences; this is what we are going to do, how will we do this?

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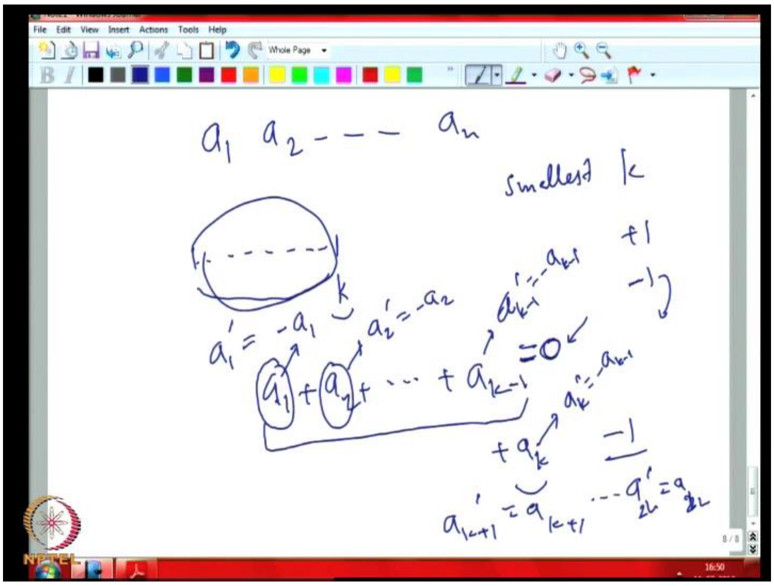

So, look at an unacceptable sequence. Because it is an unacceptable sequence the partial sum property is violated; that means there should be some place. So some k th place where if you consider the k positions, the first k positions; there should be some k, the smallest such k we will be interested such that if you sum up these things. That means a 1 plus a 2 plus a k that is strictly less than 0, that has become negative, but this is the first time it has happened. So, if you just remove this a k that is greater than zero or may be, sorry that has to be equal to zero, because you know with a k it has become negative. The only way it can become negative is because you have either plus 1 or minus 1; if it is plus one it only increase, if it is minus one it can decrease but only by 1.

So, that means up to a k minus 1 we had zero and when you added a k to that, it became minus one; that is what happened. This is all because we are assuming that this is the smallest k for which the partial sum is becoming negative; there should be such smallest k because we know that this is an unacceptable sequence; some partial sums are indeed negative. So, there should be a smallest k for which it happens, but when you sum up a 1 to a k minus 1 it definitely has to be 0, then only when you add a k also it can become minus 1. This a k which is a minus 1 happened to be in the wrong place, right; that is why it pulled down.

Now what we are going to do is to change the sequence we will get a different sequence from this given unacceptable sequence a 1, a 2, up to a 2 n, what we are going to do is this a 1 we will change the sign. If it is plus 1 I will make it minus 1, so this will be minus a 1; that is we define it as a 1 dash. This a 2 will be put a two dash which is minus a 2 and this a k minus 1 will be a k minus 1 dash is minus of a k minus 1 and also a k will be a k dash is equal to minus of a k. And from there onwards a k plus 1 dash will be a k plus 1 only and up to a n dash will be equal to a 2 n dash will be just a 2 n; that means up to the kth term, we will reverse the sign, we will negate it and from there onwards a k plus 1 onwards we would not do anything, we will keep as such.

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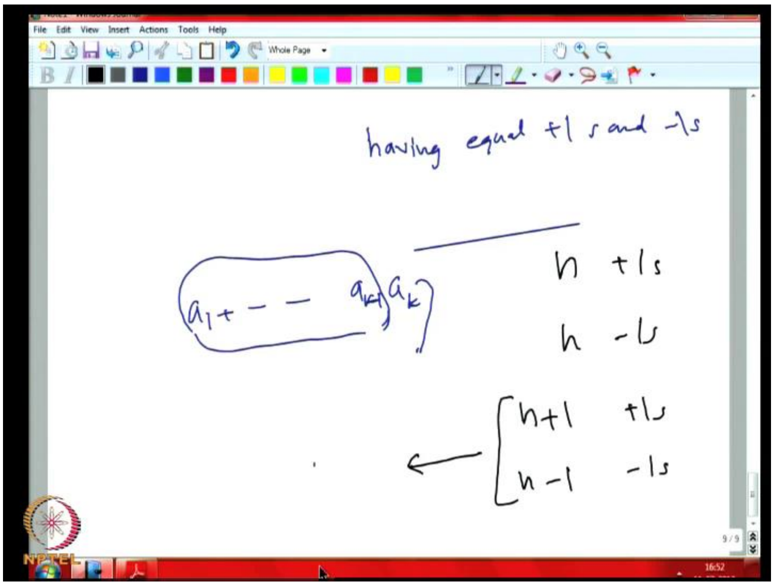

But now you should note that we have even lost the property of having equal number of plus one’s and minus one’s, why? Because you know when we consider a 1 to a k we got a negative one and up to a k minus 1 we were getting zero. So, that means up to here it was plus one’s and minus 1 equal up to k minus 1. We had k minus 1 by 2 plus one’s and k minus 1 by 2 minus one’s; k minus one by two has to be a integer, right; that is why they became zero and this a k is a negative one; that means we have one more negative one than positive ones. And if you sum them up together so they were getting negative ones that is why.

Now that I have changed the negative one to plus one and each positive one to minus one, we have one more plus 1 than the minus one's up to here. Earlier it was n minus, so n plus one's and n minus one's, but we changed some minus one's to plus one's and some plus one's to minus one but the number of minus one's so it became plus one. So, it is one more than the number of minus one's which became plus one's. So, now therefore we have n plus 1 plus one's now and n minus 1 minus one's now that is what has happened. So, we have got a sequence with n plus 1 plus one's and n minus 1 minus one's.

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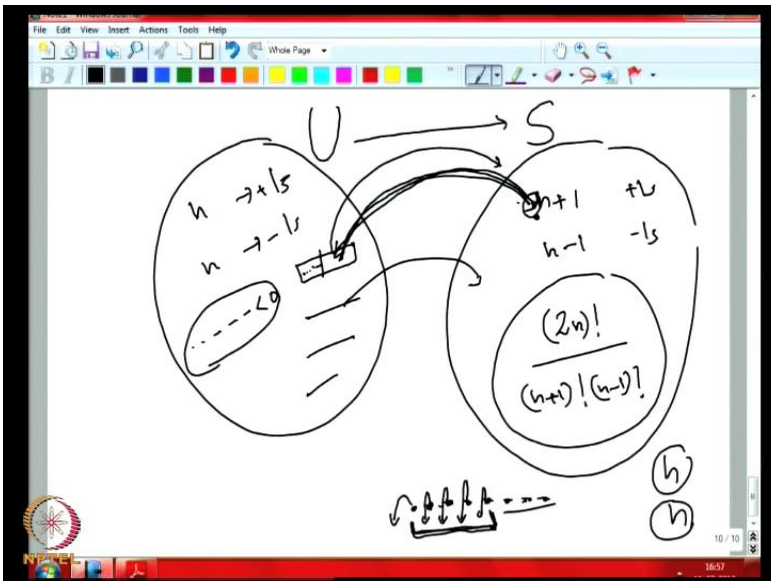

The point is we claim that if we consider the set of sequences one is the unacceptable sequences where there are n plus one's and n minus one's and this is where there is some partial sum is getting less than zero, right; that is why it is called an unacceptable sequence and on the other side we are considering the sequences with n plus 1 plus one's and n minus 1 minus one's all sequences; there are how many of them? There are 2 n factorial by n plus 1 factorial into n minus 1 factorial of them, why is it so? This is just 2 n choose n plus 1 because we just have to select the positions who are the plus one's out of the 2 n positions, right. So, this is the number. So, we claim that this is called say, S. We will find a bijection between U and S.

What now we have shown is that if you give me any acceptable string I can make a modification to that and then get a sequence in S; that means what I did is I found out the smallest k for which the partial sum becomes negative and then for all those initial k symbols I negated the sign and then now I got a string with a sequence with 2 n terms out of which n plus 1 are plus one's and n minus 1 are minus one's. This can be done for any of the unacceptable sequences and will get one here. Conversely, if you take any of this things here n plus one length 2 n length sequence where n plus 1 one's are there, n minus one minus one's are there, then what we can do is we know that there are n plus 1 plus one's you can always find the smallest k where the plus one's have exceeded the minus one's.

For initial parts we may allow minus one's to exceed the plus one's, but at some point of time plus one's number of plus one's has to exceed the minus one because overall the total number of plus one's is more than minus one's, right, at some point. It will find out the first place it happens and now there we do the negation, right, until there we do including that we negate each symbol and leave the remaining as such the same procedure, but now up to where we have to do is decided by looking at the first position where the plus one's have exceeded the minus one's. Now you know there is exceeded means it should be just one more minus one than one more plus one than the minus one because until then just previous position they should be equal, right, then only it can exceed.

So, now what will happen is so there are n plus 1 one's, now here because one more plus one was there it was made minus one. Now we will have n minus one's and then plus one's, right, because they were not equal equal, so one extra minus 1 will happen, n minus 1 was earlier it will become n and the earlier n plus 1 plus 1 it will become n. So, it will become n plus one's and n minus one's and the sequence is indeed unacceptable, why? Because if you go up to this k we will see more minus one's than plus one's, because there were more plus one's to begin with, right, so we negated them.

So, this is indeed an unacceptable sequence. We got a map back right, but we can see that this is the x-map back to the exactly that thing which for instance if you considered the original conversion from here to here, we look for the first time the minus one's exceeded the plus one's, right. So, now by after converting there exactly that position will have more plus one's than minus one's because if it was earlier then, right, because before converting we would have got more minus one in an earlier question itself, right. So, it is a map going from here and back, right.

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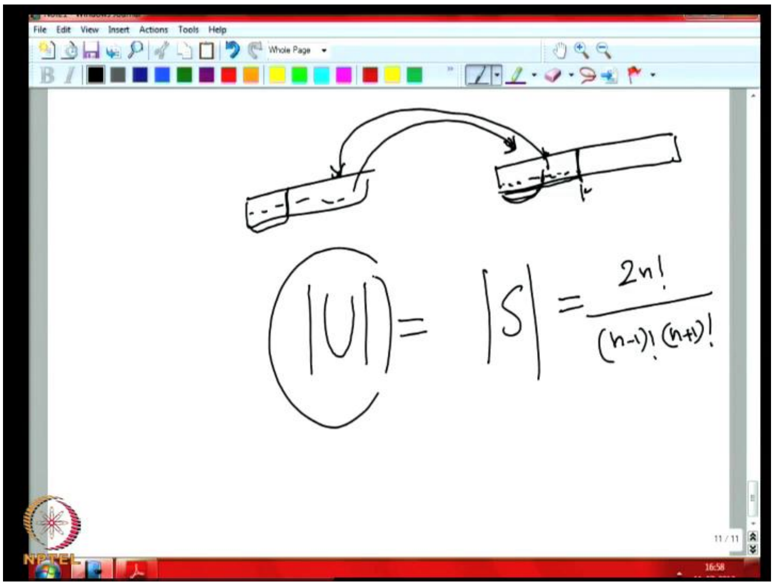

So, for instance what I am saying is when an unacceptable sequence was taken and we applied that operation of negating till the first k positions where k is the smallest integer where we have first the partial sum became the first time negative; that means there is one more minus one than the plus one's and here for the sequence which we got now and if you look for the first place where the plus one's are exceeding minus one's, that would be exactly that k, because it cannot be earlier; because if it was earlier then when we did the conversion and earlier itself you would have got a negative partial sum because up to their itself we will have more negative minus one's than plus one's, right.

So, therefore that will never happen. So, this is exactly the same k. So, we are doing exactly the reverse conversion, so we are getting back to this where we started, right. So it is a bijection between the two. So, if it is a bijection we can infer that this set of unacceptable sequences is equal to the set s whose cardinality we know which is essentially 2 n factorial by n minus 1 factorial by n minus 1 factorial into n plus 1 factorial.

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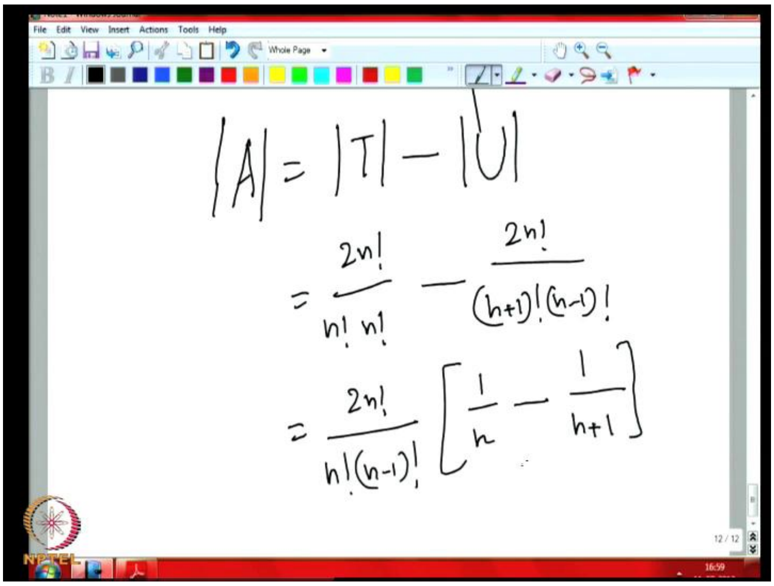

Now that I know what is the cardinality of the set of unacceptable sequences I know the cardinality of acceptable sequences also, how? This is actually as we have already observed T minus U. The only problem is what is T; T we know that is 2 n choose n, right minus 2 n factorial by n plus 1 factorial into n minus 1 factorial. Yeah, this 2 n choose n is actually 2 n factorial by n factorial into n factorial out of which we can take 2 n factorial by n factorial into n minus 1 factorial out. So, here we have a 1 by n left, right. Here we have a minus 1 by n plus 1 left. So now if you manipulate this n plus 1 minus n is 1.

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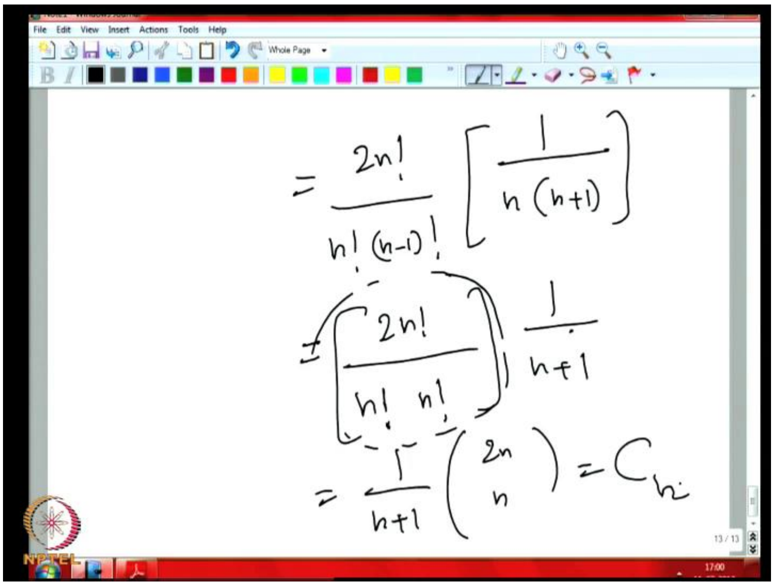

So, which is 2 n factorial by n factorial into n minus 1 factorial into 1 by n into n plus 1, right. So, we multiply by this; this is 2 n factorial by n factorial into n factorial into 1 by n plus 1 which is 1 by n plus 1, this 1 by n plus 1 into this part is what? This part is familiar that is 2 n choose n. So, this is our nth Catalan number. So, that is what. So, we have got a combinatorial proof for the problem directly; without using the generating functions we have found that 1 by n plus 1 choose 2 n choose n. So, any problem which can be mapped to this thing which has a bijection to this, right, will also have this 1 by n plus 1 into 2 n choose n as the answer, right.

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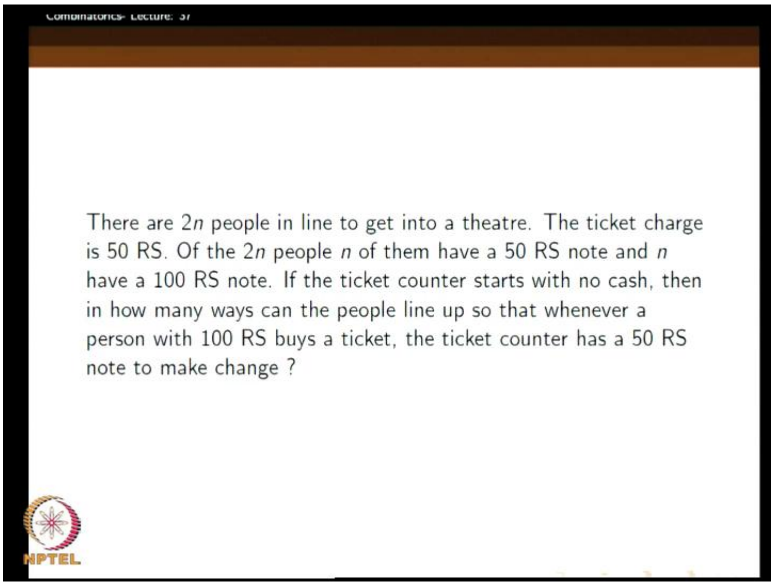

Now we will see one situation where this comes; see, I told that initially that once we get a problem like this, we can try to see whether that underlying recurrence relation is available or not. If the underlying recurrence relation is not available, one thing you can do is we can see whether that t n plus 1 equal to t 0 into t n plus t 1 into t n minus 1 plus t n into t 0. This kind of recurrence relation comes or not with the same initial condition, then we can infer that second. Another ways is to look at some of the non-problems which happen to have Catalan numbers as answers and try to get a bijection to that. For instance we can look at this question.

So, there are 2 n people in line to get, so there is a theatre. So, people have come up for watching a movie and there are 2 n people waiting outside. Then now the ticket counter start ticket office starts, they do not have any money with them. A ticket cost 50 rupees and all this people 2 n people, n of them have come with 50 rupees notes. They have one 50 rupee with them. They have the change for 50 rupees and the remaining people have just 100 rupee notes. So, the person with 100 rupee note will expect that he will get a 50 rupee note back, right, but then the counter does not have the ticket office does not have any money with them.

So, that means the first guy who goes to buy the ticket has to be one with the 50 rupee note with them, not one with 100 rupee, because when will not get the money back; there will be a problem, right. So, they will have to ask them to wait till they collect. So, maybe they ask the people to queue up in such way that this problem does not occur; that means any time some person with 100 rupee note goes to buy a ticket, in the office there is at least one 50 rupee note to give him back; may happen that again it becomes empty that is only 100 rupee notes now. But the next time then next guy will go with a 50 rupee note and then the problem should never occur that there are. When you consider the k th person in the queue, in the first k person when you consider there are more 100 rupee notes, people with 100 rupee notes than 50 rupee notes, right. So, how many ways you can queue up? This is the question; how many ways you can queue up?

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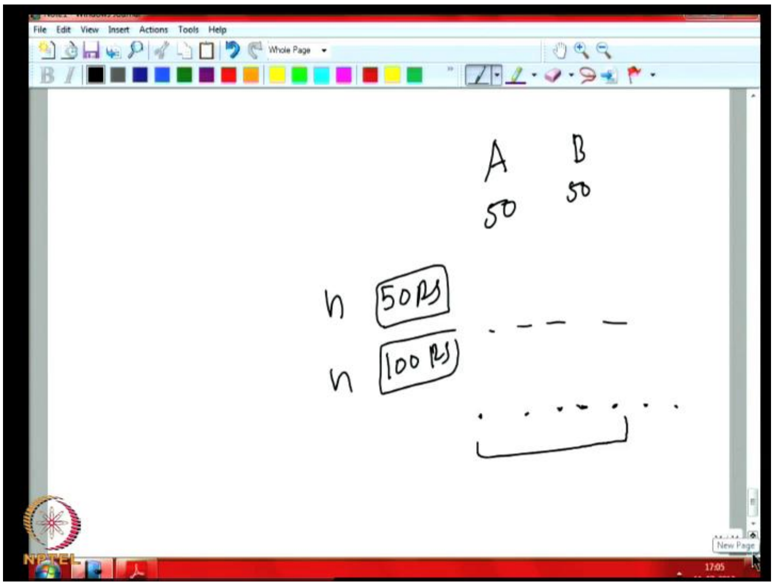

So, first what we think is there are now people I mean the people whether there are people A and B two different people A and B if he has 50 rupees and he has 50 rupees whether A is behind B or B is behind A; it is a different queue for us. But now that if you remember that these are people and just assume that, so I consider any person with a 50 rupee notes is identical. So, all those people with 50 rupee notes are identical, then we can say that we have n 50 rupee notes and 50 rupees and n 100 rupees, right, queued up, right. So, we have arranged them in a line in such a way that if you look at the first k positions in the line in the queue, there are always at least as many 100 rupees as possible; the number of 50 rupee notes is at least as much as 100 rupees.

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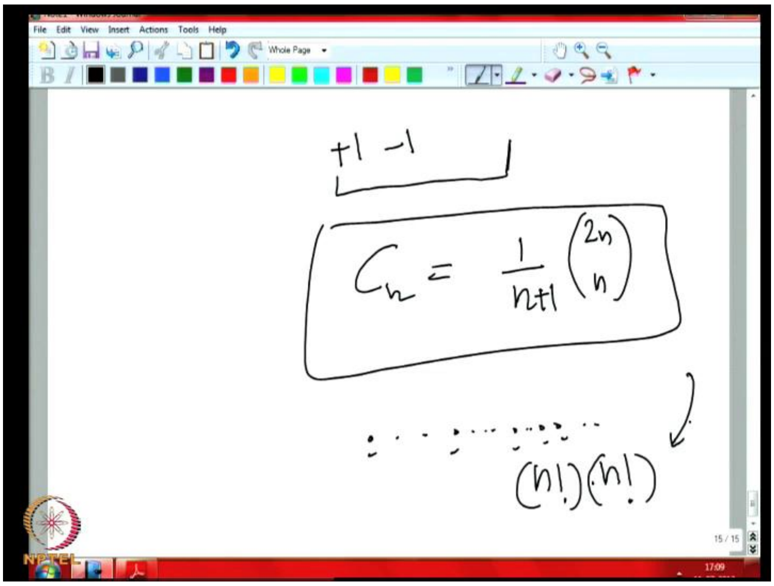

So, one way to model this thing is to put a plus 1 whenever we see a 50 rupee note and put a minus 1 whenever we see a 100 rupee notes. So, the partial sum till k th will give the number of 50 rupee notes left with the person whose sells the ticket, right, after when you consider k. So, it should never be negative that is the point, because negative means that means the previous guy came with a 100 rupees and he could not give it, right. So, it can be zero because that means he just balance, he gave off one 50 rupee note and took a 100 rupee, and there were no more 50 rupee notes there, only hundred rupees are left. That is why.

The other way it should not happen. So, that means no partial sums should become less that zero. So, this is exactly the previous problem; once we discarded the fact that there were people, and actually two different people with 50 rupee notes really make a difference whether whoever is standing nearer to the ticket office earlier in the queue, right, that will make it a define queue, right, that order, right. So, now once you assume that they suggest 50 rupee notes and 100 rupee notes and it does not make any difference, right, it is just that. So, we know that this is the nth Catalan number, because this is exactly the previous problem.

This is what i am saying; we have to identify the problem. This problem is mapped to the previous problem by converting a 50 rupee to a plus 1 and 100 rupee bill to a minus 1, right. So, we have the answer for 2 n people that is a 1 by n plus 1 into 2 n choose n is the answer, this is C n. This is the nth Catalan number, okay. The number of ways we can queue up the 50 rupee notes and 100 rupees. But the actual people if you want to queue up, then this n people who have 50 rupee notes in particular arrangement without changing the relative arrangement I mean with respect to the other people, we can permute them in n factorial ways.

We keep this position that 50 rupee note positions are kept as such, but those people can actually change positions among themselves. They all have 50 rupee notes, right. So, that can be done in n factorial ways and again the people with 100 rupee notes with them, there are n of them they can also permute themselves in n factorial ways. So therefore, n factorial into n factorial times n factorial ways of getting different valid queues from each of these arrangements of 50's and 100's, 50 rupees and 100 rupees, right, into the queues of people, right.

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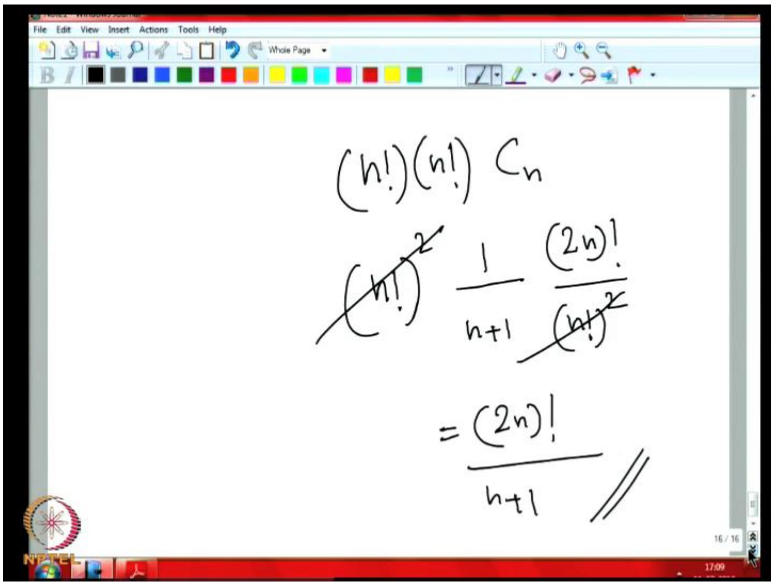

So, the final answer is nth factorial into n factorial into our C n namely n factorial square into 1 by n plus 1 into 2 n factorial by n factorial into n factorial. So, this cancels off; we get the answer as 2 n factorial by n plus 1.

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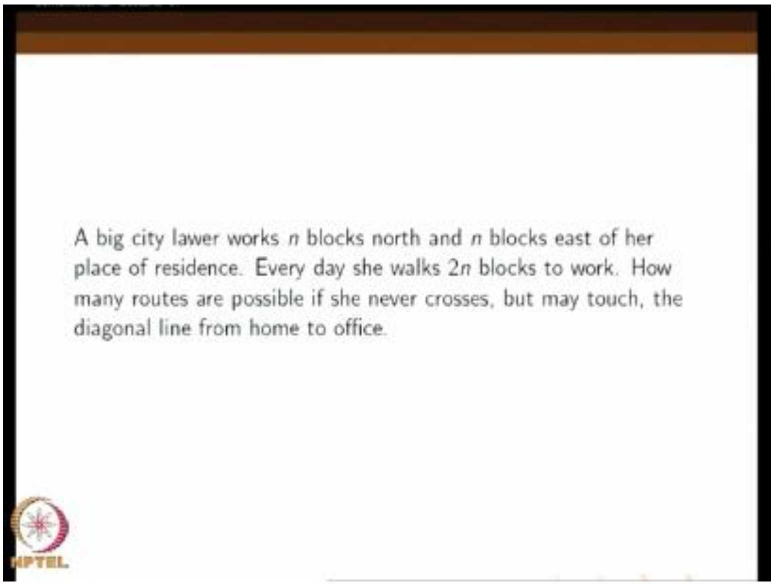

Now we consider another question. So, here there is a person who goes to his office by walking n blocks north and n blocks east of her place of residence. Every day she walks 2 n blocks to work. How many routes are possible if she never crosses but may touch the diagonal line from home to office?

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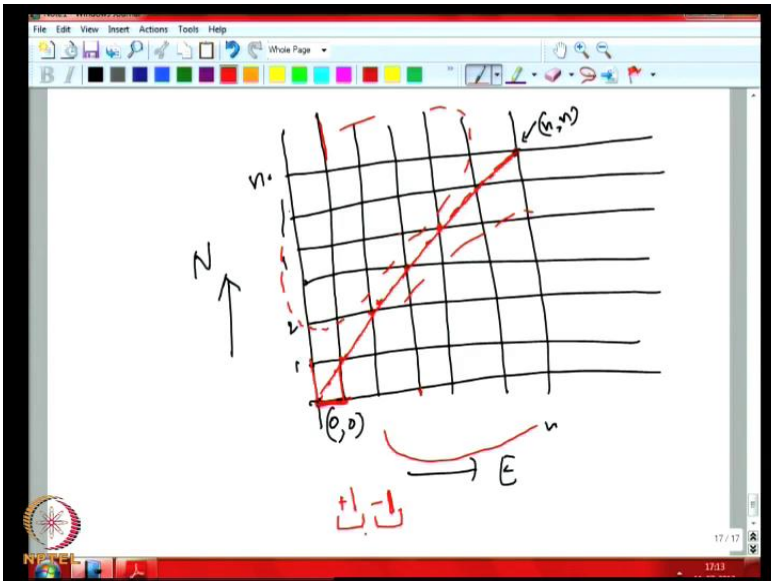

See, this kind of problems we have seen before. This is we called North-Eastern paths, right. There is this 0, 0 where she is staying. Now we have this kind of a grid, right. We have seen this diagrams before, say, this is n, right, this is 1, 2, up to n, and similarly we have this. Let us say this is n. So, that means this is n, n. Now I want to reach from 0, 0 to n, n and the condition is that we can either move like this or move like this. You never move backward, right. So, this is a familiar condition; we have already seen this, right, this kind of thing. So, we can only move either northwards one unit northwards one unit eastwards, sorry this is north upward, right; this is east, this is rightward.

So, at once step that is what he is doing one unit northward or one unit eastwards. Now to reach n to n he has to go n unit northward and n unit eastwards. How many routes are possible? Now when I talk of the diagonal that is this one, yeah, so it is the diagonal, right; diagonal means 0, 0; 1, 1; 2, 2; 3, 3; 4, 4; i, i; n, n, that is it. So, we have to finally reach a point in the diagonal. Now the condition is that he can touch the diagonal but he should never toes the diagonal. For instance if he starts walking like this then he should always be above this diagonal. He should not cross it and come down.

He can touch, but he should not cross. And similarly, if he goes like this then you should always keep down. So, by symmetry he can see that we only have to count either the paths above the diagonal or the paths below the diagonal and multiply by 2. Also therefore let us because we will look at the paths below the diagonal. So, below the diagonal means this will be, so for instance if you are moving eastward we will call it plus 1 and when you are moving northward we will call it a minus 1. Now what is the condition? So, we take so many northward steps.

At any point if you have taken more northward steps than eastward steps; that means more minus one’s than plus one’s, then we will cross the diagonal, right. Because at that point we will have i northward steps; that means on the axis, sorry i is towards steps on the x axis we are on i but i plus 1 or more northward steps, in the y axis it will be y plus 1 or more; that means it will be above the diagram, right. So, we are not allowed to do that. If eastward corresponds to plus 1 and if northward corresponds to minus 1.

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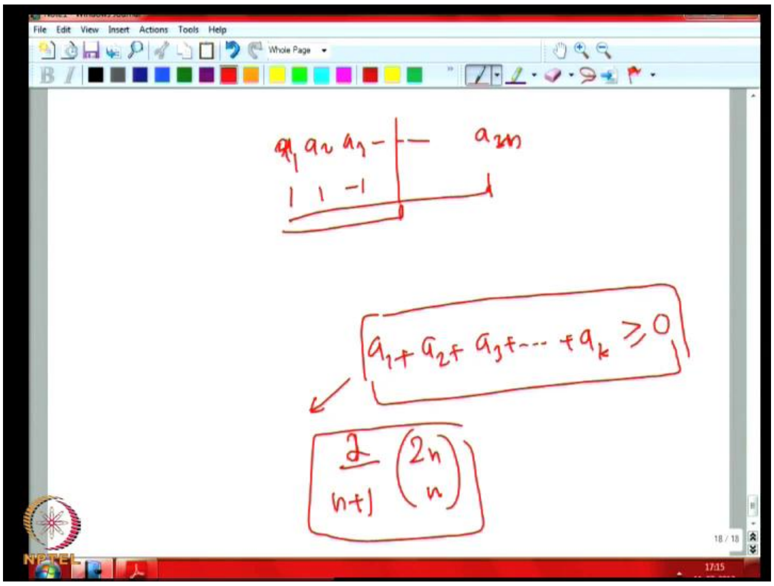

Then our condition is that, at any partial sum up to k so if you put this plus 1 plus 1 minus 1 like that a sequence; that means a sequence of a 1, a 2, a 3, up to a 2 n can be written because 2 n steps are to be taken to reach the point n, n, right. And each step whether she take a northward step or eastward step is denoted by either we put a minus 1 or plus 1 there. Now if you do not want to cross the diagonal, if you do not want to go above the diagonal; that means the partial sums up to any k here; that means a 1 plus a 2 plus a 3 plus a k has to be always greater than equal to 0. It can be equal to 0 in which case we will be hitting the diagonal, we will be touching the diagonal. If you go below 0; that means at that point we are above the diagonal, right. So, this is exactly the same problem as before.

So therefore, the answer has to be 2 n choose n into 1 by n plus 1, right, but now we have to multiply by 2; that is the only difference, because we can either go above the diagonal completely above the diagonal or we can be completely below the diagonal all the time in the routes whether the complete route can be below the diagonal or it can be above the diagonal. These are definitely equal, because you can see that there is absolute symmetry between the two cases; therefore, the answer is two times the nth Catalan number; that is 2 by n plus 1 into 2 n choose n, right.

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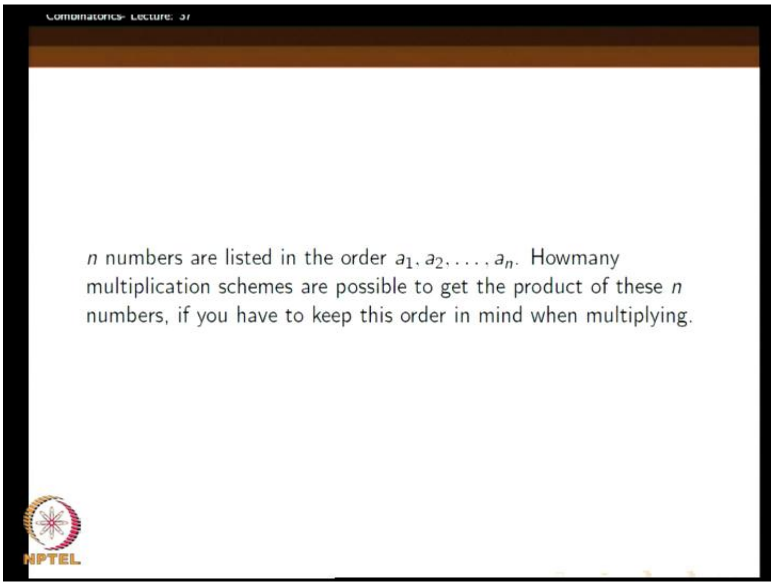

Now we will look at another question. So, may be this is one question of balanced parenthesizes.

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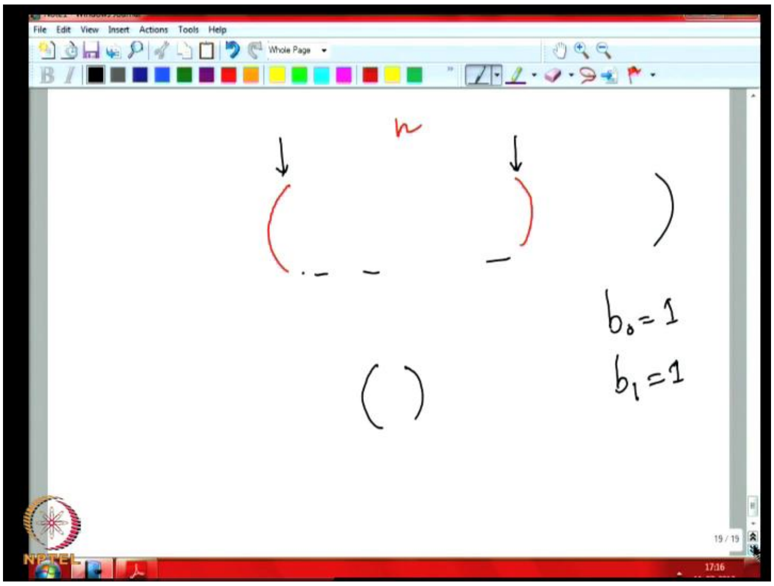

Suppose I want to put n parenthesizes, so this how can you do but balanced parenthesizes. You should not say that see this kind of parenthesis is not balanced, right, you reach here, right. If you have opened two parenthesizes to this thing then you have to close it also before ending like that, right. So, we can match for this thing this is the corresponding parenthesis and things like that. So, that is the way the balance parenthesizes can be put. For instance if we are talking about zero parenthesis you can say that b 0 the number of parenthesis is just b 0 is equal to 1, because yeah this is empty. So, b 1 this is 1, there is only one way of doing it pair of parenthesizes.

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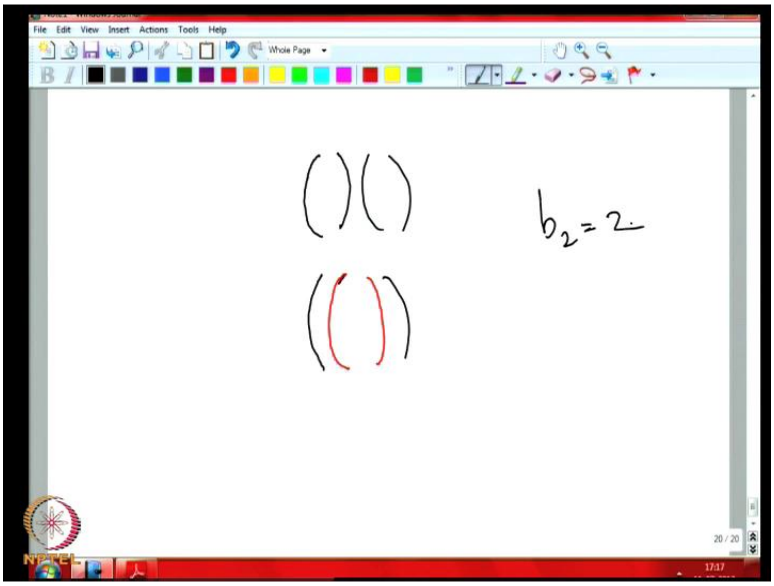

If there are two, so we can say we can start with this, we can close it and then we can start another, or we can start with this. Then we can start another one and then close the inner one and then close this one. So, there are either this way there are two ways of doing it, b 2 is equal to 2 and b 3?

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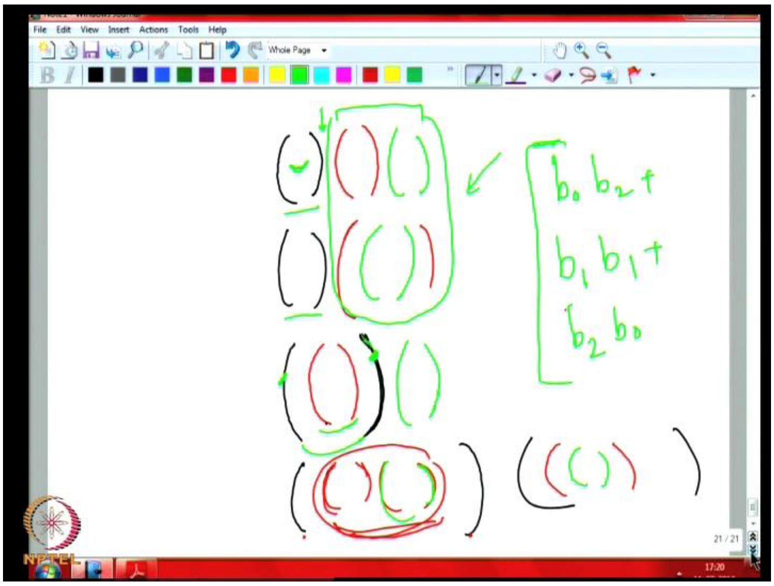

b 3 for instance you can start with 1 the black one, say, now maybe we can close it, and then we can start where with the red one, you know then we can start with a green one, and here we can either close the red one and then start with the green one, or we can have like this; start with the red one and then again open a green one and then close the green one and then close the red one, right, or otherwise what we can do is we can start with this one, but then do not close it, but rather deal with red one here. Then we can close the red one, and then we can close it, sorry we can close this black, and then we can start a green here, right. This is possible; that means here we are closing the black one the second position itself.

So, then the remaining you know this that recurrence relation is coming, right, remaining we have n minus 1 parenthesis to be put in all possible ways we are putting here. This will correspond to because now you have closed it; inside it we have b 0 ways of putting the parenthesizes, right, outside we have to put two parenthesis we will put b 2, right. And here we are closing the first parenthesis here, right, second so inside we have one another pair, so that is b 1, because once you have discarded this inside we have one pair and outside we have one pair b 1. Now what we can do is we could have done started this parenthesis and we have decided to end it the end; that means both red and green comes inside, so red and green comes inside.

So, there is another way of doing it these two parenthesizes. So, either we can start with red and end with a green, or we can do like this. There is a red and then we start with the green and end it and then we have red, this way of doing it. So, once we decided to put the black here there are two pair of parenthesizes which goes inside; that means one red and one green, so that means we have b 2 inside the black, right, b 2 ways of putting the two pairs of parenthesis inside the black, right, and outside we do not have any; that means just b 0. These are the sum, right. You can see that the same recurrence is working here, right, how is it shown?

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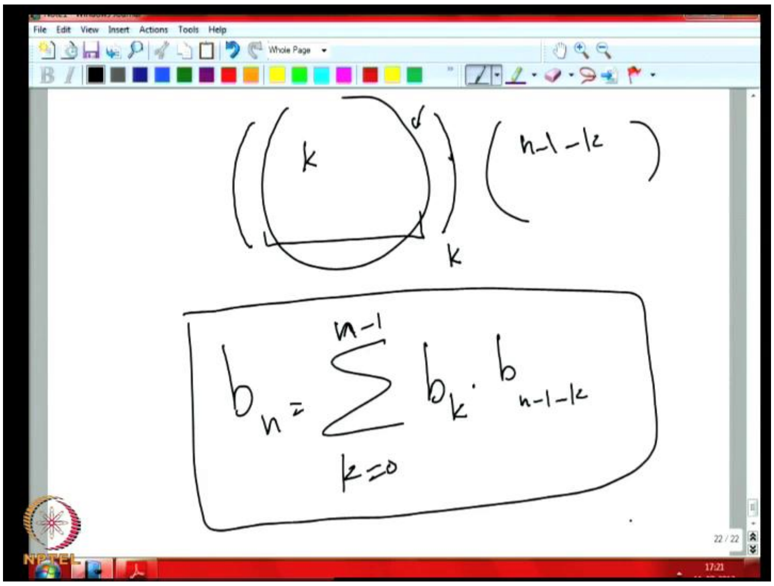

Because you know if there is n 1 of them, what will you do? We will just say the first is the black parenthesis starting and then it may be stopping at a position deciding that there are k minus 1 pairs also k pairs of parenthesis inside excluding this black and so, n minus 1 minus k pairs of parenthesis is outside it. So, it will look like b n is equal to b k sigma k equal to 0 to n, right, 0 to n minus 1, right, k equal to 0 to n minus 1 because the number of parenthesis which we are allowing inside can be either 0 up to n minus 1 because one is this black itself, right, and so that will be b k ways and outside we will have b n minus 1 minus k of them, right, then how many ways we can do this thing? So, this is this sum of the products from k equal to 0 to n minus 1. This will be the thing and because initial values are the same.

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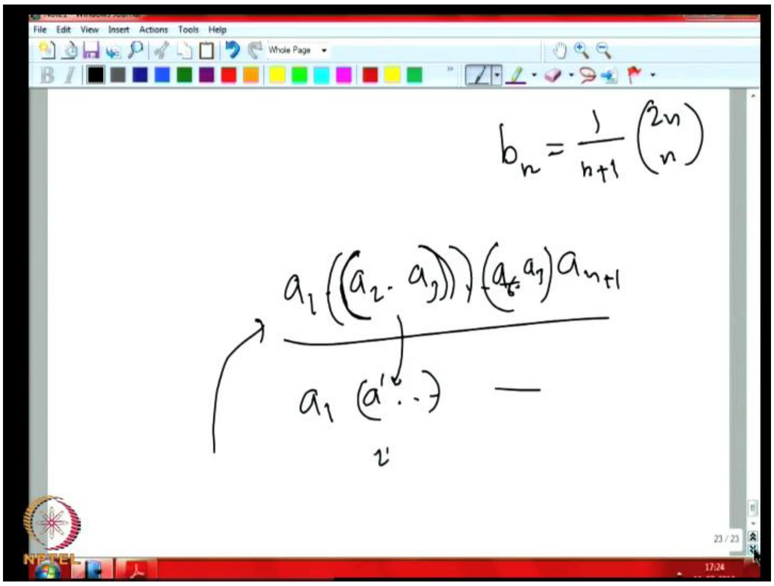

So, this will correspond to the nth Catalan number b n will be 1 by n plus 1 into 2 n choose n, right. Now you can also see that this parenthesis problem is like the plus one and minus one problem, because if you count for every left bracket we put a plus one and for every right bracket we put a minus one. At any time we are not going to put more right brackets than left brackets. The number of left bracket will be at least as much as the right brackets at any point, right. So, the partial some property will be satisfied. So, we can map it back to the earlier problem; that way also we can see that this is 2 n plus 2.

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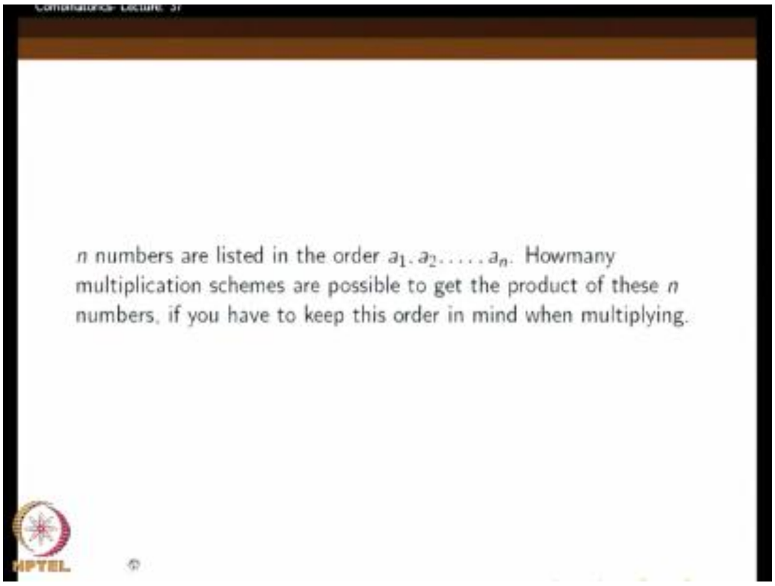

And then yeah so this balanced parenthesis can be interpreted as being used to evaluate a product of a 1, a 2, a n; you know if you want to evaluate a product of a 1 into a 2 into see a n. So, we usually have to first select two of them. If we keep this order of course so we are not taking some a 2 and then multiplying with a 4 and we can do that, but then in this case suppose we are supposed to keep this order and then put the parenthesis and then evaluate, right; that means we will decide to put first some parenthesis somewhere may be a 2 and to a 3 may be somewhere else we will put a 6 for the most basic multiplication.

Once you do this multiplication we convert it to an a 1 into some new number will come here, right, and then put another pairs; that means here the corresponding parenthesis will be put here and so on, right. So, the multiplication the system of parenthesizes using n pairs of parenthesis will be used to define the number of ways we can make this product get this product in different orders, right, in various ways where there are, say, n plus 1 things. If there are n plus 1 things we will need n multiplication, right, because every multiplication will reduce the number of terms by one, and finally we want just one term, because that is a final product, right. So, that we can think, right, yeah we will consider, maybe we will discuss it in the next class in more detail, right. As of now we will stop here. We will discuss a little more about Catalan numbers and one more example about Catalan numbers in the next class. We will make this also a little clear in the next class.