Combinatorics

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Lecture - 36
Partition Number - Part (4)
Catalan Numbers - Part (1)

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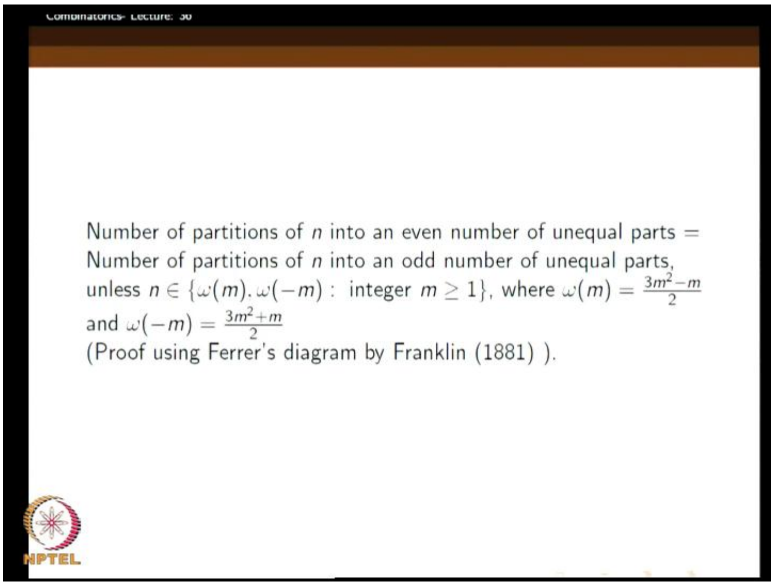

Welcome to the thirty sixth lecture of combinatorics. So, in the last class we were discussing on last theorem about the partitions. So, the theorem was about partitions of a number n whose parts are all different unequal parts. So, among that kind of partitions we have two groups; one is where the number of parts are number of parts is odd, the other is number of parts is e.

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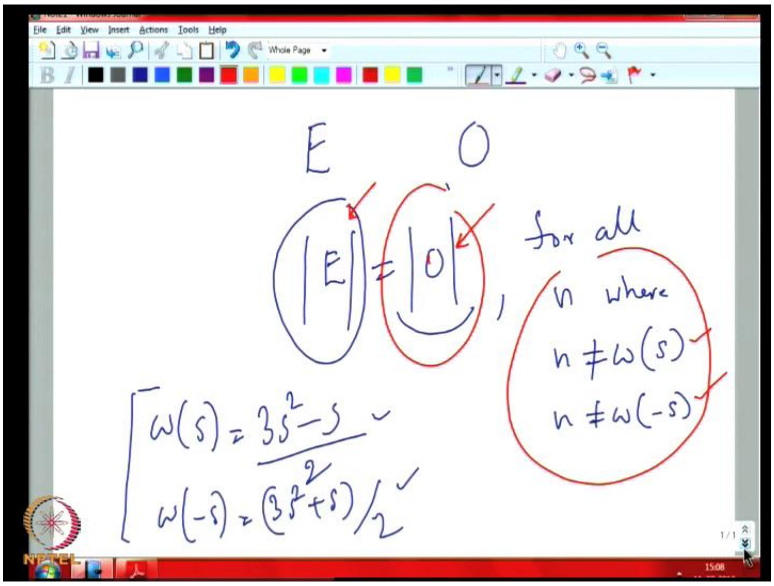

So, we used these two letters E and O to represent these two sets. We showed that for most n E is equal to O that is for all n positive integers n, where n is not equal to omega of s or n is not equal to omega of minus s where for s equal to 1, 2, 3 up to for all and there what is this omega of s? Omega of s is equal to 3 s square minus s by 2 and omega of minus s is 3 s square plus s by 2.

Then we showed that if n happens to be of this form or this form for some s, then either this cardinality may be one more than this cardinality, or this cardinality may be one more than this cardinality and which is bigger is decided by the parity of s. If s is an odd number then this cardinality will be bigger and if s is an even number this cardinality will be bigger. We have proved this statement by considering the Ferrer's diagrams of the partitions involved and then showed a bijection in the case in the first case this case and then bijection between, right. In the other case we removed one special partition from the set in which its contents and then we established a bijection. This is what we did in the last class.

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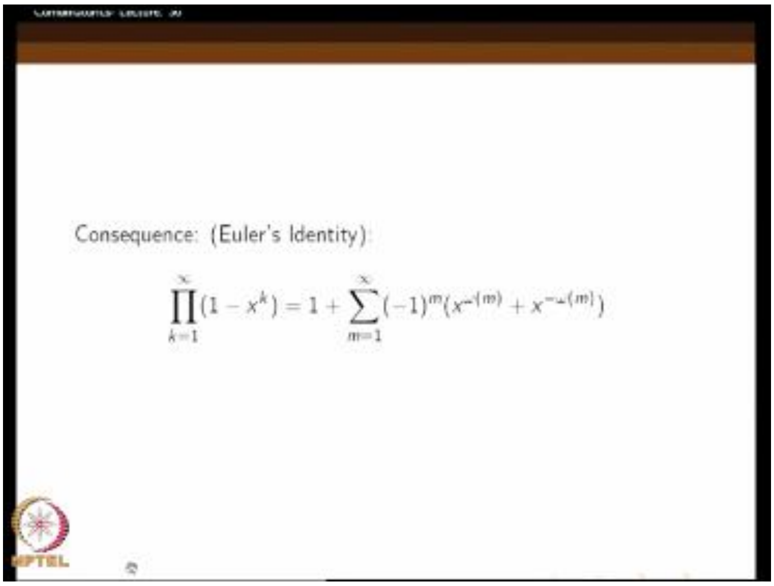

So, now today we will show a consequence of this. This was first discovered by Euler and Euler used it to infer this fact; what is the fact he inferred? It is called Euler's identity. So, k equal to 1 to infinity 1 minus x raise to k equal to 1 plus m equal to 1 to infinity minus 1 raise to m into x raise to omega of m plus x raise to minus omega of m.

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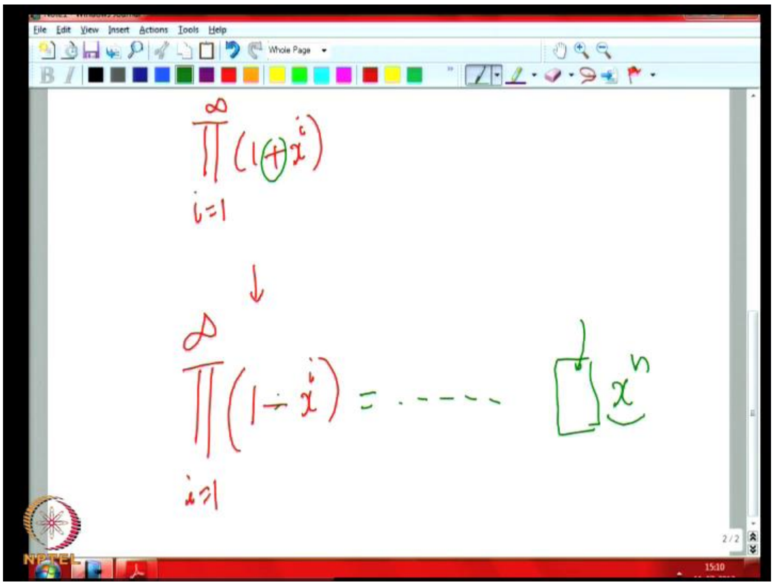

So, you remember this function i equal to 1 to infinity 1 plus x raise to i. This is the generating function for the sequence representing the number of partitions of n where all parts are distinct, but Euler considers the following functions following product 1 minus x raise to i i equal to 1 to infinity. So, instead of plus here he is taking minus that is the only difference. So, here minus, right; so how does it change?

Here you can see that because of the minus sign there will be some plus and minus signs in the terms, right, because there are negative one raise to something also. So, how do we decide which terms have negative sign, which terms have positive sign; what will happen to the terms? So, for instance let us take a general term x raise to n to the co-efficient of this thing you know it is again what is contributing to x raise to n is the partitions of n where each parts are unequal, right.

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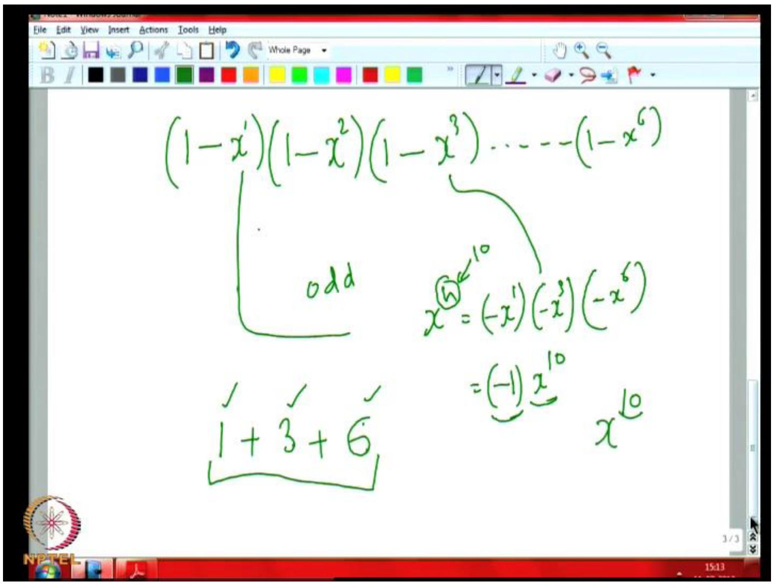

Because from the first for 1 minus x raise to 1, right. It will either take 1 x or it would not take an x but with a negative sign this time and from this thing from 1 minus x raise to 2, either we will not take any two or it will take a two with or without a minus one; sorry, with minus one always all the time, sorry, so it will either take or not, right. So, it is also like when I consider the co-efficient of x raise to n its contributions are coming from each partition of n where the parts summands are all unequal, but then there is a sign along with each of them; what is the sign? If the number of summands were odd then we will get a negative sign, right.

For instance I can see let us look x raise to 10. So, x raise to 10 can be made by 1 plus 3 plus, say, 6; yeah, this is fine, right. So, then that means this will be taken to form x raise to n like this. So, we have to get x raise to 1, x raise to 3 and x raise to 6. So, x raise to 1 will come from this but that is minus x raise to 1. So, then from this thing x raise to 3 will come minus x raise to 3 will come and from 1 minus x raise to 6 this minus x raise to 6 will come right. So, together this is n is equal to 10, but then we have there are three terms. This is minus 1 raise to 3 into x raise to 10 has come, right; it is not just x raise to 10. This is minus 1 raise to 3; 3 correspond to the number of terms here.

If this number of terms is odd this minus 1 will remain; if the number of terms were even this minus 1 will disappear, it will become plus 1, right. So therefore, what matters is in this partition of 10 into distinct parts, how many parts are there? What is the parity of that number of parts? If the parity is odd then we get minus 1; otherwise we will get plus 1, but we have seen that this number of see when we consider partitions of a number n where all parts are different unequal, the numbers of parts the number of partitions with even parts and number of partitions with odd parts are equal for most n, right, when n is not of certain form, right.

So, that means they will cancel off this minus 1 and plus 1's. So, sometimes we get a minus x raise to 10; sometimes we get a plus x raise to 10, another time we will get a minus x raise to 10, then we will get a plus x raise to 10. But then the number of times we get minus x raise to 10 is equal to the number of times we get plus x raise to 10 because the number of ways we can partition.

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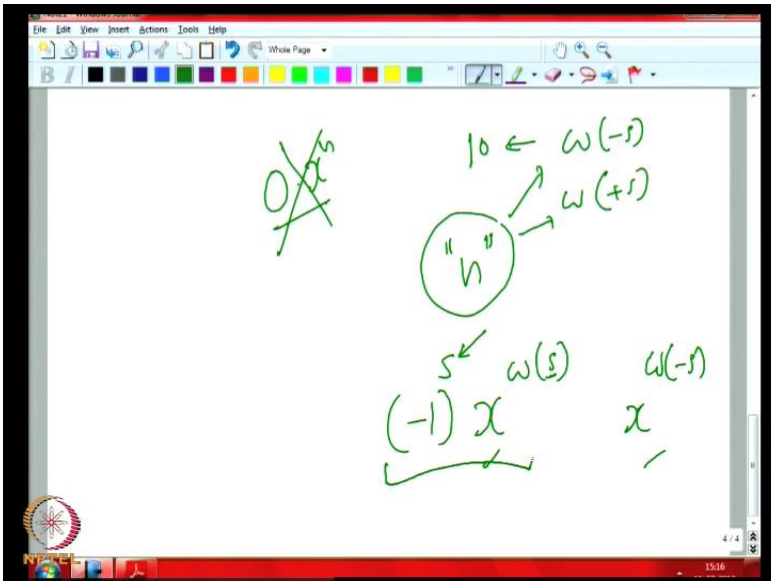

Yeah, so assuming that 10 is not of the form. I think n was not of the form omega of minus s or omega of plus s. So, let us take a take the case of a general n. So, if n was not of this form or this form we have equal number of partitions of distinct parts where the number of parts is even or number of parts is odd, right; therefore, they cancel off. So, they would not even contribute that x raise to n will not appear, because they will have a zero coefficient with them, so that will not even come. The only thing which appear are powers of x where n is of this form omega raise to minus s or omega raise to plus s x raise to omega of minus s, right. This kind of x raise to the powers of x's only will remain, but then also the coefficients are not at all complicated.

We know that the number of even summand partitions even the number of partitions with number of parts being equal to even is equal to the number of partitions with number of parts being odd; sorry, they are not equal but just one more either plus one or one, but then how will you decide which is more? If it is odd we should get a negative one, if it is even, even is more then we should get a plus one but we know that whether even is more or odd is more is decided by the parity of s.

So, we can just infer that it is minus 1 raise to s that will remain. If s was even the number of partitions contributing to the plus one's will; if s was even the plus one's will be more one more, so then that s being an even number minus 1 raise to even number will become a plus 1. But on the other hand if s was odd, so the number of terms which contributes to a minus x raise to omega of s, right, minus will be one more than the one's which is contributing to plus. So finally, you should have to get minus but it will come because s is an odd number minus 1 raise to s, so that is what will happen here.

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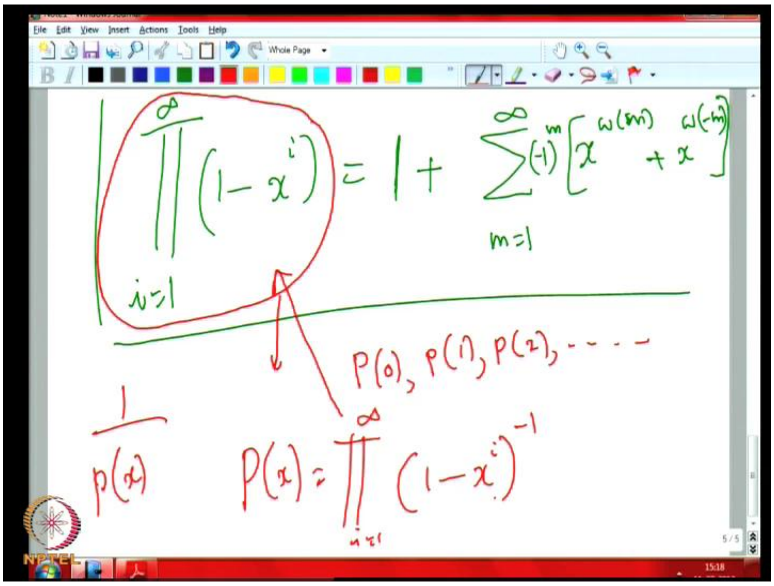

So that is why our final expression will look like this; when we take the product of i equal to 1 to infinity 1 minus x raise to i. So, what we will get is initially we will have just one that is okay because x raise to zero zero, we do not talk about any other partitions. Then we only have things of this form, right, x raise to omega of m and x raise to omega of minus m; other powers of x will disappear, but the co-efficient of x raise to omega of m and x raise to minus omega of m will depend on m that will be minus 1 raise to m whether it is even or odd, right. We will decide whether we should have a minus 1 here or not. So, m will go from 1 to infinity here. This is what is Euler's identity? Right. So, why do we get this thing? So, we get this thing because this stuff is familiar.

What is that? It is actually the inverse of the generating function for the partition number; remember the generating function for this function, the main partition function p of 1, p of 2, p of this generating this is the sequence corresponding to the number of partitions of an n p of n being the number of partitions of n. We had found that this is the generating function p of x for this thing is product of i equal to 1 to infinity 1 by 1 minus x raise to I, right, or we can write it as 1 minus x raise to i inverse, right, 1 by 1 minus x raise to I further. Now if i take 1 by p raise to p of x, then what will you get? We will get this one, right. So, this is actually the inverse of p of x, right, 1 by p of x is this thing.

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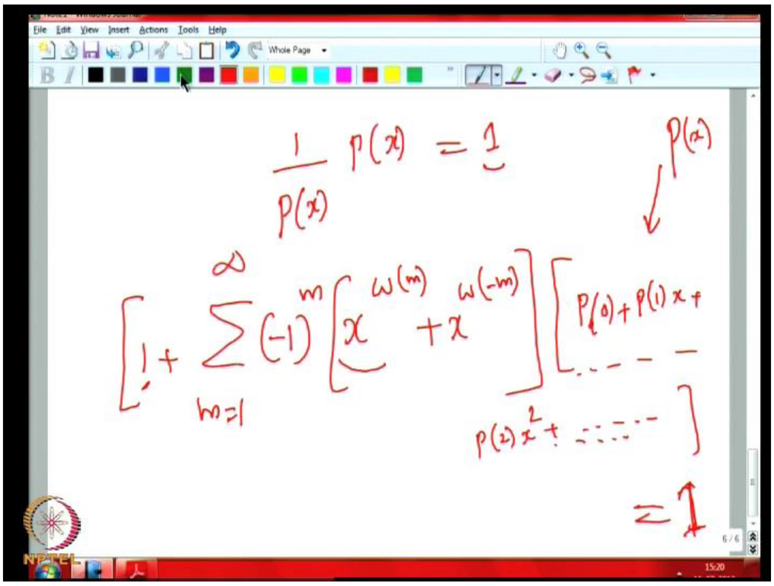

But we know that 1 by p of x into p of x has to be equal to 1, right. So, there would not be any powers of x on the right hand side when you multiply, right, only one will be there. So, we can try to express this this way, 1 by p of x is as we have expressed that is 1 plus sigma m equal to 1 to infinity minus 1 raise to m x raise to omega of m plus x raise to omega of minus m into p of x; p of x is what? p of x is p of zero plus p of 1 into x plus p of 2 into x square plus so on. This is what is p of x, right.

Now if you multiply this thing we have to get one. So, when we consider the coefficient of x raise to n from both sides, we can equate the coefficient of x raise to n from LHS and RHS from this thing, what will we get? So, if you do that because this any way is not contributing to the coefficient of x raise to n, because there are two types of terms. For each m we have x raise to omega of m. So, if you want to make an x raise to m then we have to select from the other part I mean from this p of x the x raise to n minus omega, right.

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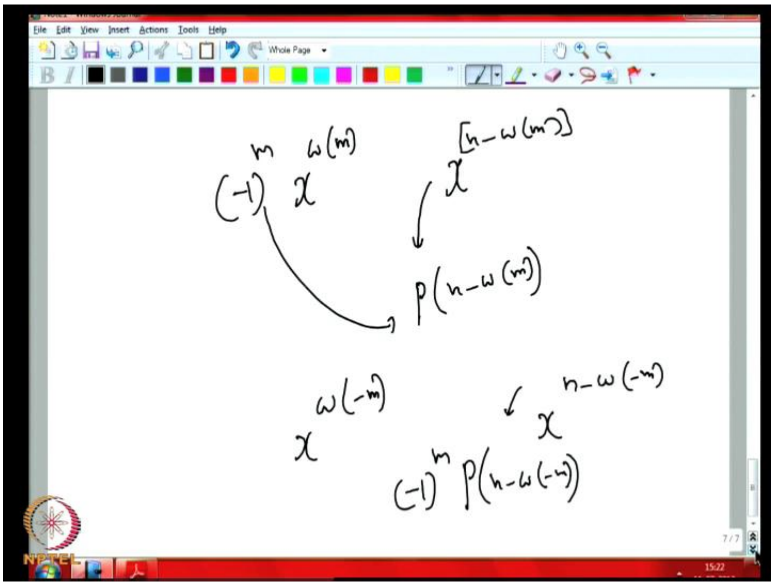

So, that means for corresponding to each x raise to omega of m when we select it from the first part of the product, from the other part we will have to take x raise to n minus omega of m, right, but we know that the coefficient of x raise to n minus omega of m, there will be p of n minus omega of m, right, but the coefficient here is just minus 1 raise to m, right. So, we will get minus 1 raise to m into p of n minus omega of m when we make that product.

And for x raise to omega of minus m we have to select from the other part x raise to n minus omega of minus m and we know that the coefficient for this thing will be p of n minus omega of minus m, right, because the coefficient of x raise to n minus minus omega of m in this generating function has to be by definition p of n minus what p of whatever is the power of x is there; that means n minus omega of minus m, right, so here but the coefficient is just minus 1 raise to m, right.

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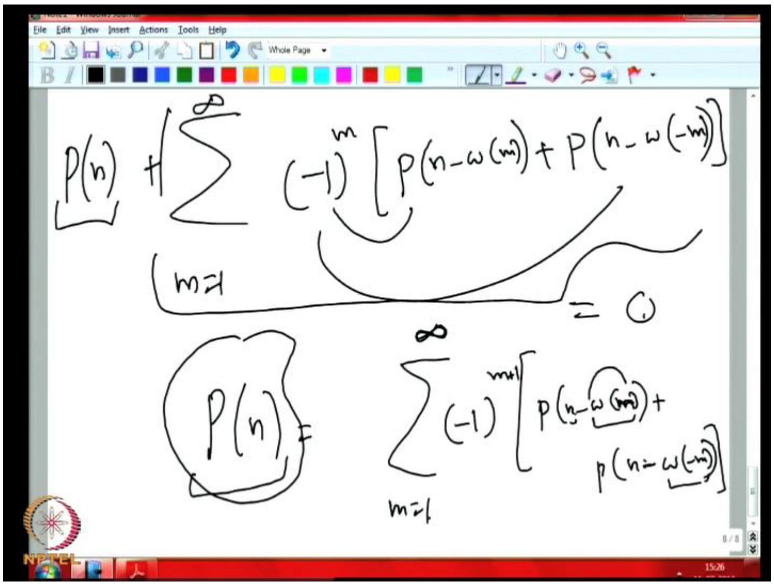

So therefore, for each m we will have two terms which contributes to the power of x raise to m on the other side; that is minus 1 raise to m into x raise to omega of m plus, sorry not p of n minus omega of m plus p of n plus omega of m, sorry p of n minus omega of minus m. These two terms will contribute along with this and this and this and this will be contributing to the x raise to m and this will happen for each m. So, m equal to 1 to infinity; this will happen and this will be giving the coefficient of x raise to n on the other side but the other side is just one, right. So, x raise to n has only zero coefficient there; this has to be equal to zero, right, and then there is also one term. This is not true; we have one more term. See we are finding this into this, right, it is not that this 1 also can contribute to x raise to n, why; because it multiplies directly this p of n into x raise to n.

So, p of n into x raise to n will come from there. So therefore, p of n into x raise to n will come from there, so that is p of n; p of n is contributed by that and for each other term, right, so for a term like x raise to omega of m will contribute minus 1 raise to m into p of n minus omega of m; a term like x raise to omega of minus m will contribute which is minus 1 raise to m into p of n minus omega of minus m, right. So, that together when you multiply we will get x raise to n. So, all these contributions are added; therefore, so the final contribution is this p of n plus m equal to 1 to infinity this thing this has to be zero. So, now we can take this entire portion to the other side. So, we get p of n is equal to minus of this one minus 1 raise to m into p of n minus omega of m plus p of n minus omega of minus m. This is what we will get but then this minus can go inside and then make it plus 1 here minus 1 raise to m plus 1 into p of m.

So, what is good about it? So, this goes from m equal to 1 to infinity. The good thing about this thing is that we are getting a formula of p of n in terms of smaller p of something which is smaller than m, right, this a recurrence relation for evaluating p of n, but then we will say that there are infinite number of terms here. This is going from 1 to infinity, so what is good about it at all? But then we can see that this omega of m or omega of minus m both of them are increasing as m goes bigger and bigger and bigger. So, for a fixed n after sometime when n becomes reasonably large this will become n minus some number which is bigger than n which will give you a p of minus something.

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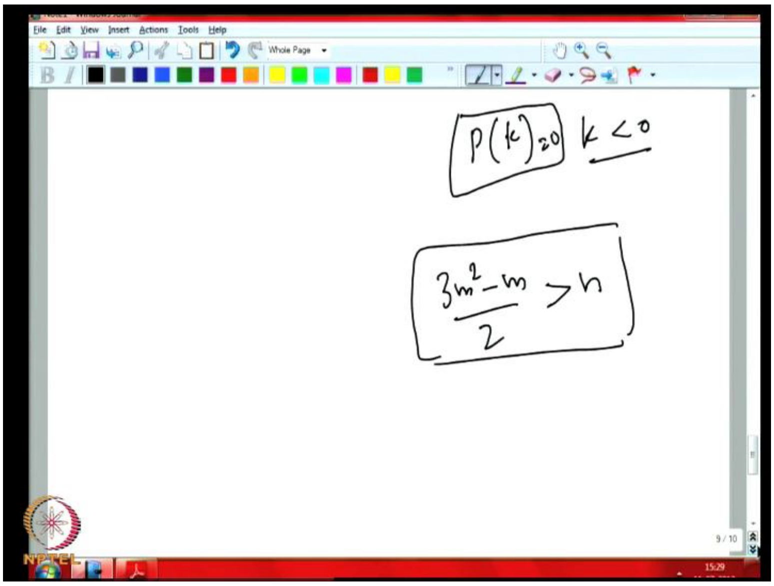

And we take p of something which is when k is less than zero as zero, right; that we have already mentioned. So therefore, these things will not have any contribution after sometime. So, actually it will contribute as long as this omega of m and omega of minus m both remain below n, right, strictly less than equal to n, right, until then only it can contribute. So, after that it would not contribute; therefore, it is a finite sum. So, you will ask how many terms will be there. So, when I put m equal to 1 there will be something n minus, yeah we have see that it is omega of m is 1, right. So, then m equal to 2 we will get something. So, n minus that and m equal to 2 we will get something. After sometime what will happen is see when this 3 m square minus m by 2 becomes greater than n for some m then onwards we can say that we do not have to consider. This will may be approximately square root of n values is to be so we can estimate exactly what some constant time square root of n values have to be considered.

So, there are for each of omega m and omega of minus m we will have to considers so many values, right. I do not estimate exactly. So, it is not a very nice recurrence relation in the sense that p of n is being expressed by some two or three terms which is smaller than n p of some number which is smaller than n, but we have consider about square root of n smaller values of n and then we have to add them together. When I am saying square root of n, some constants are assumed, some constant into square root of n number of such values we have to evaluate p and then add them together and minus by the a corresponding negative or positive number positive one's minus 1 raise to m plus 1 we will have to multiply and add them together, we will get p of n, right; this is good about it.

So, of course this is better than the earlier recurrence relation; we saw that we can actually find p of n by summing up p k of n's for each k and for p k we had given some recurrence relation, but you will see that we have to evaluate more values there, right. So, otherwise here we only have to consider about square root of n previous values; anyway this is better. So, this is one consequence of that theorem we have proved. So, I gave recurrence relation for p of n. Yeah, now that is all we want to discuss about partitions, but then there are more material on this topic.

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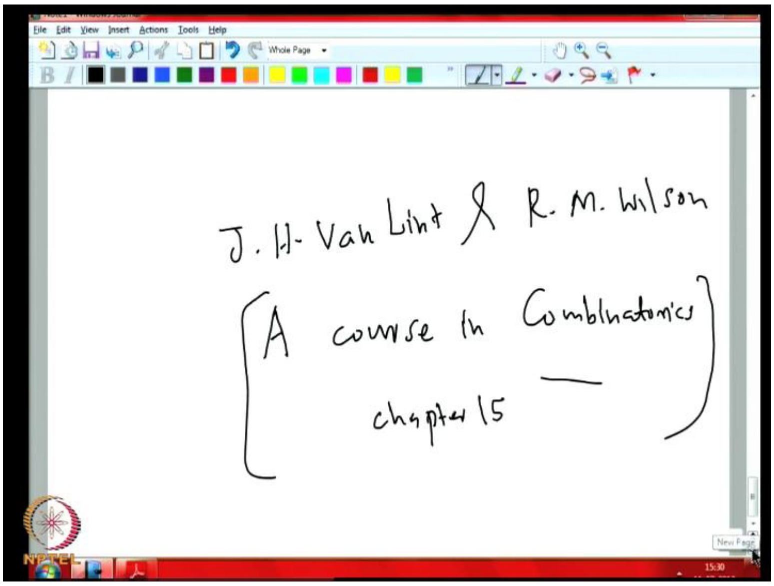

If you want you can read something more in this book J. H. Van Lint and R. M. Wilson, A Course in Combinatorics. So, this is the chapter is the fifteenth chapter which is the chapter fifteen. So, there is some more material on this if you want to read. So of course, there will be other books which will give but this is one reference I can give, but we do not want to discuss more on partitions, because we will want to see other things, right.

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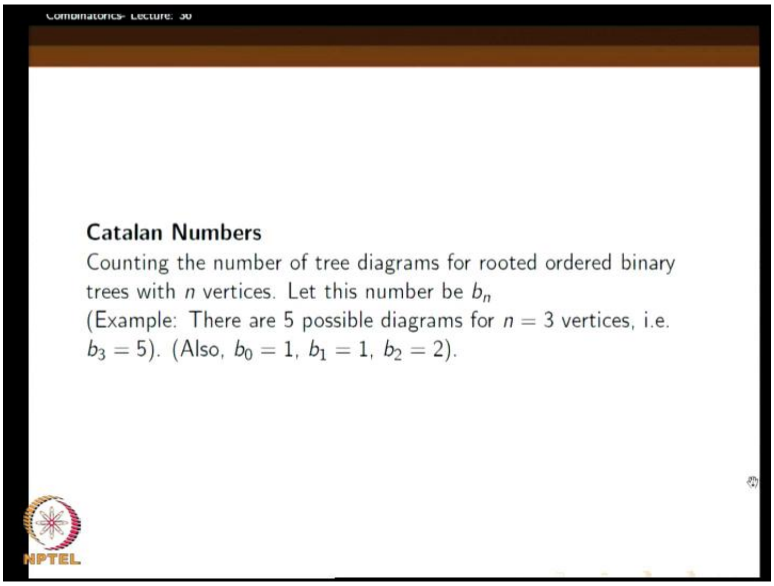

Now the next aim is to look at Catalan numbers. Here also we will have some use for generating functions but again this Catalan numbers are also quite interesting because they can capture or they happen to be the answers for several acute problems. See if you search in the internet you can see the listing of problems where Catalan numbers happens to be the answer first to those problems, right, and you can also see Stanley's book, 'Enumerative Combinatorics' for collection of such problems. Now let us say we have to take some typical problems where Catalan numbers happen to be the answer. Let us look at this one. Count the number of tree diagrams for rooted ordered binary trees with n vertices. This is not very common; therefore, I will try to explain.

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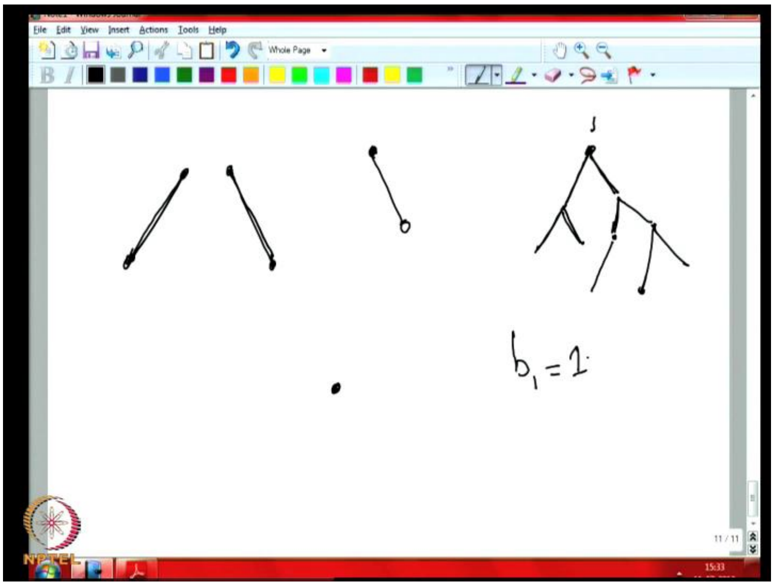

So, with one vertex, so we always talk about binary trees; binary tree means for each vertex it starts from top to bottom like it is a rooted tree like this. So, like to complete this trope or you could have drawn from below to upward, right, so we will decide to draw like this, right, but every time you may have one child or two children. It can be like this or it can be like this, right, or it can be just like without any children can stop. So, this is what the word binary means. It is a rooted tree because you will be starting from a root and growing downwards, then we also say that this is an ordered tree in the sense that you can draw this child or you can draw, say, this child.

We will consider them as two different trees; these are some diagrams we are drawing. So, this diagram is different from this diagram; otherwise, combinatorial both happen to be same tree, but in this context we will consider this as two different tree diagrams, right. So, now for instance with one vertex we have just one tree because as in a single vertex we will write it as b 1 equal to 1.

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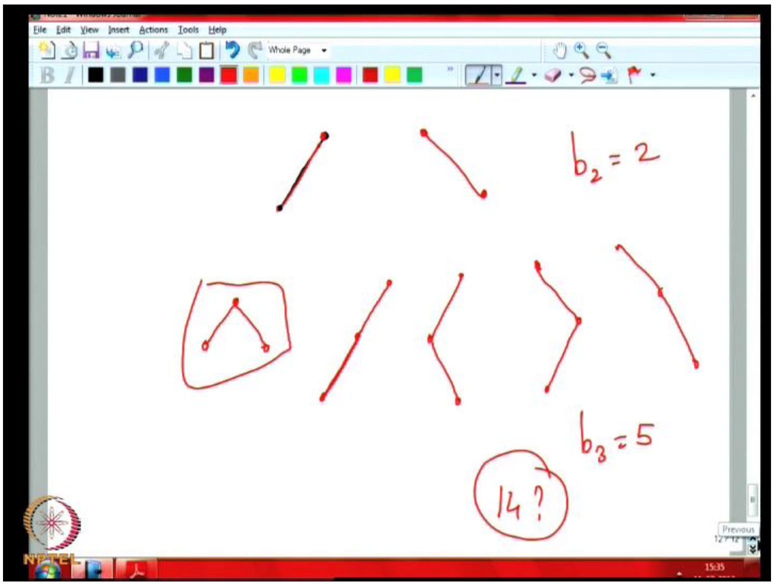

With two vertices we can either draw like this; we start with a root and then we can draw like this or we can start with the root we can draw like this; so, either to the left or to the right. So b 2 is 2; what about b 3? So, we can start with single vertex. First possibility is we can put both the children. So, this is one; another possibility is you can put a left child and the next child can be again left of it or you can put a left child here and then put a right child here. And now you could have given a right child to the root rather then left child and for the right child you can have a left child or you can have a right child itself.

There are four different possibilities; I mean in the usual sense all these things, this, this, all these four things are the same tree, it will look like a path on three vertices but when we are talking about this special type of tree diagrams, this order is important. Every time whether we draw towards the left or towards the right, the child is put towards the left or towards the right, it is important; that is different, right. So therefore, we get five different types of things here. So, b 3 is equal to 5. So, you can check how much is b 4, right. So, see whether it is 14 or not, right. So, that will give you some experience in drawing this thing, so next one.

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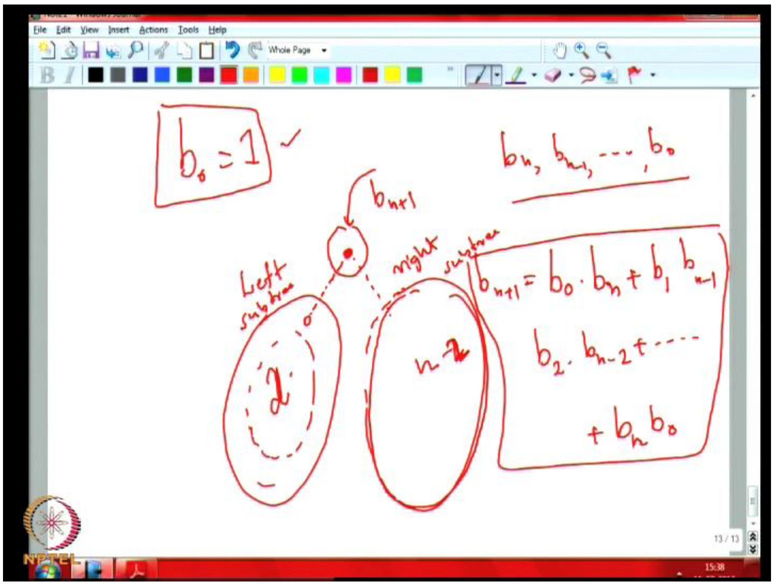

So, we want to get a general formula for this namely b n. So, before going further let us discuss what can be b 0; b 0 is when there are no vertices. So, we can either say it is 0 or 1, but we will decide to say 1 because we want write a recurrence relation. We will say that there is an empty tree, right. Now this is the recurrence relation we are going to write for this problem. So, once you put one vertex, once you add one vertex, so you are going to draw some tree towards the left. So, you are going to draw some tree towards the right. This is the left sub tree and this is the right sub tree. Now let us say we are trying to get a formula for b n plus1 in terms of, say, b n, b n minus 1 up to b 0, right. This is our aim.

So, once this root is taken we have n more vertices; out of the n more vertices, so we can decide to give zero things here in which case n thing should go here. Zero things are given then the left tree there is nothing. So, we will say that that is b 0, so there is a b 0 tree; that means there is an empty tree there. For corresponding to each tree possible here we can see how many trees are possible here; that is essentially b n by definition. So, b 0 into b n possibilities is like this. Now you could have decided to give one vertex here and one vertex in how many ways you can have? So, that is only one way b 1 base here let us say; b 1 into then here it is only n minus 1 vertices, this can be done in b n minus 1 base.

Now we can try giving two here and n minus 2 here, so that will be b 2 into b n minus always, n minus 2; that means this two vertices can be arranged in b 2 ways here and this n minus 2 things can be arranged in n minus 2 ways here, right; this is the thing and like that all the way to b n into b 0. This will be the b n plus 1. This will be equal to b n plus 1 equal to b 0 into b n plus b 1 into b n minus 1 plus b 2 into b n minus 1. This is the basic recurrence relation which works here.

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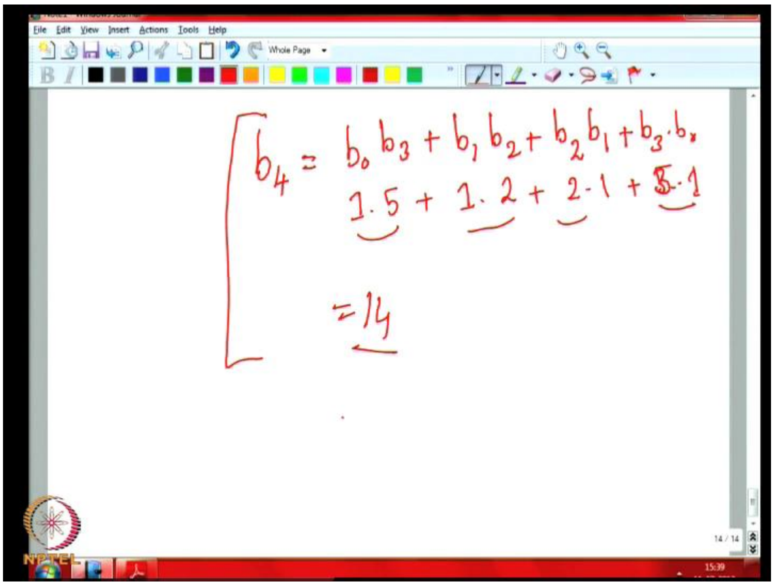

So, with this thing we can try to calculate our b 4, b 3 we have already calculated. What will be b 4? b 4 will be b 0 into b 3 plus b 1 into b 2 plus b 2 into b 1 plus b 3 into b 0 but b 0 is 1, b 3 is 5. This is b 1 is 1 and b 2 is 2, right, plus this is 2 into 1, this is 5 into 1. How much is this? So, we get 5 plus 5 is 10 plus 4 is 14. So, like that we can calculate b 5, b 6 like that, right, but our intension is not to compute it this way.

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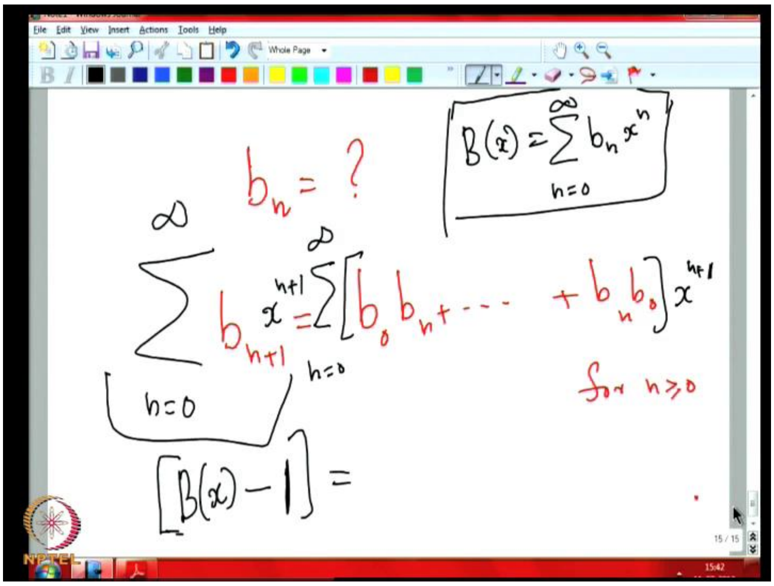

So, we want a general answer for b n. What is b n? So, we will use the method of generating functions. So, what recurrence relation we have is b n plus 1 is equal to b 0 b n plus b n b 0. Now let us say this will work for even n equal to 0, right, because b 1 is equal to b 0 into b 0 is what that would mean when I put n equal to 0. This is correct because this is 1 and then it is just like putting one root and saying that the there is an empty tree here, empty tree here. So, total number of trees is 1 into 1. So therefore, this is correct. So, even for b 2 also it will work b 2 is equal to b 0 into b 1 plus b 1 into b 0.

This is also one, this is also one; together it is two, right, which means that we have put a root; we can decide to make this empty and there is one here or we can decide to put one here and make this empty, right. So therefore, this is true for n greater than equal to 0 till infinity. So, as usual we can multiply by x raise to n plus 1 everywhere, x raise to n plus 1 on both sides and then sum from n equal to 0 to infinity. So, this will also be sum to n equal to 0 to infinity, but then what is this? This is because we are only starting from because n equal to 0 this is b 1; b 1 x raise to 1, b 2 x raise to 2 and so on.

So, this will be just b of x; if B of x is the generating function for this thing, say, let us define b of x equal to sigma n equal to 0 to infinity b n x raise to n. Suppose this is the generating function for the sequence b 0, b 1, b 2, b 3, etcetera. So, this will be B of x minus if you put n equal to 0, so this from b 1 on b 1 x raise to 1 onwards only we have, So therefore, we have introduce the b 0 into x raise to 0; x raise to 0 is anyway 1 b 0 and b 0 we have decided to be one b 1, right, b 0 is 1. So, B of x minus 1 is equal to what is this one, sigma b 0 b n into x raise to n plus 1. What we do is we will pull out x first.

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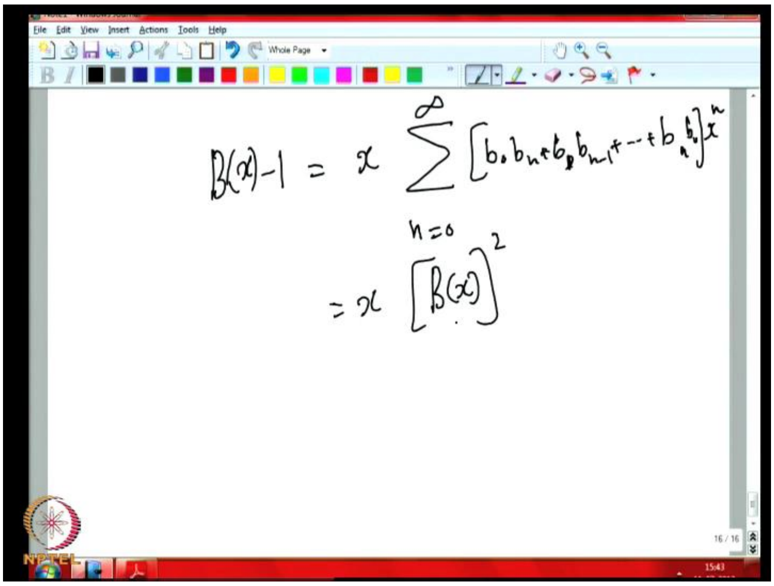

So, this is what we are getting B of x minus 1 is equal to x into sigma n equal to 0 to infinity, This is your b 0 b n plus b 1 b n minus 1 plus all the way to b n b 0 into x raise to n if you pull out x, right, and this is actually x into we can claim that this is B of x square, why is it so; because if you consider B of x square what would be the coefficient of x raise to n in it?

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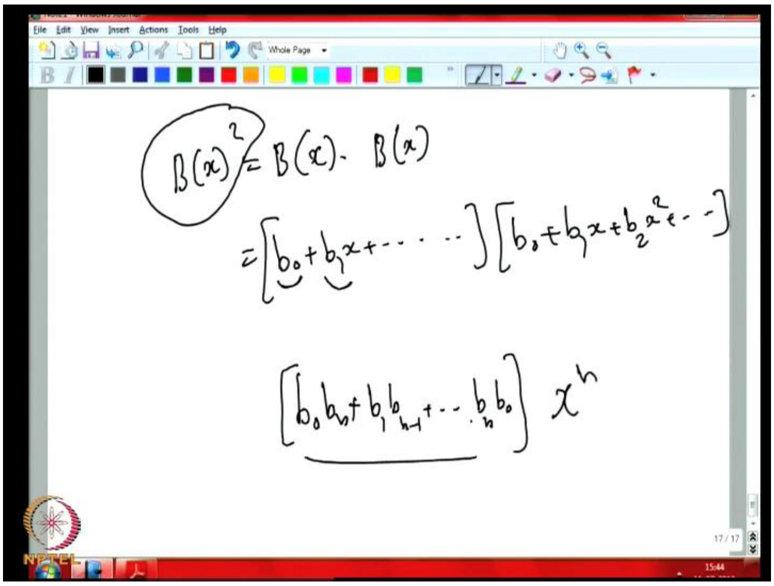

To see this thing you write B of x square as B of x into B of x and this B of x is actually b 0 plus b 1 x plus this thing, right, into b 0 plus b 1 x plus b 2 x square and so on. Now if you pick up the coefficient of x raise to n here that will be contributed by b 0 here, x raise to n should come from here; that is the coefficient is b n, right, and then this will b 1 into x the co-efficient we will multiply x raise to n minus 1 that is the coefficient of x raise to n minus 1 b n minus 1 we will get b 1 into b n minus 1 and so on, right. So, till b n x raise to n here b n will be taken and that will be multiplied by zero. This will be the coefficient of x raise to n. So, the coefficient of x raise to n b x in b of x square is actually this.

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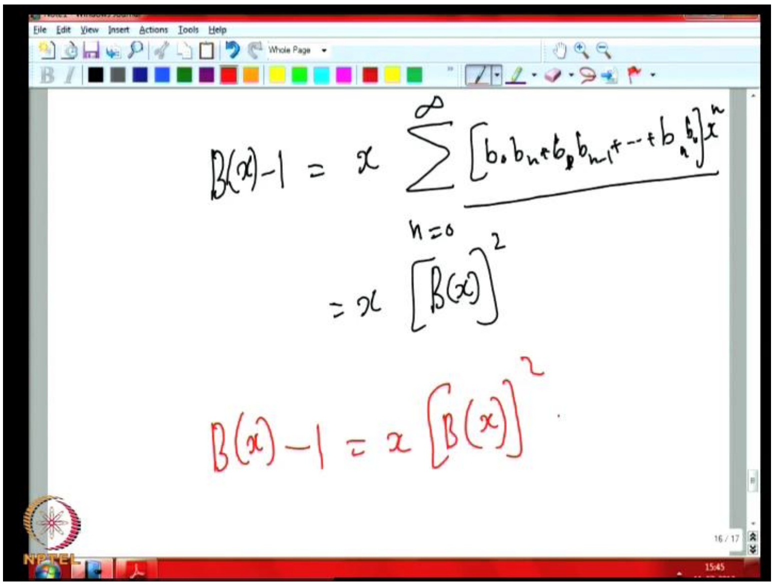

And then we can see that this stuff, see for n equal to 0 to infinity this is the way we are writing that is true for everything. So therefore, x into b of x square is this. So, again we repeat we get d of x minus 1 equal to x into b of x square. Now we can rearrange it, we pull it to the other side.

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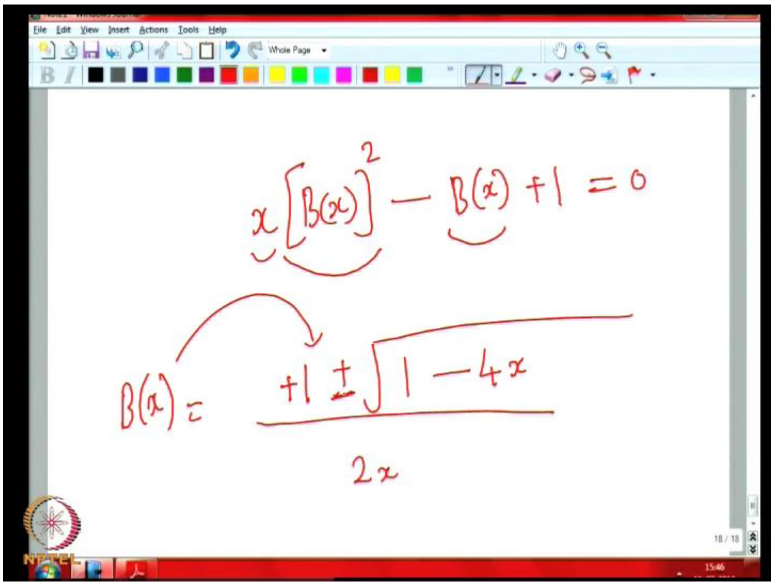

So, that is x into B of x whole square minus B of x plus 1. Yeah, this minus 1 will become plus 1 on the other side, so that plus 1 equal to 0. Now how will you find B of x? We use the method to solve quadratic equations because this B of x is quadratic, so this is B of x square and B of x here and this thing. So, we will use minus B that is plus 1 here, minus 1 will become plus 1 here, plus or minus b square minus 4 a c, that is 1 minus 4 x by 2 x; this is it. This will be the answer to B of x; B of x can be either this or this. So, let us see so before deciding what whether it should be plus or minus here; let us see what is this stuff 1 minus 4 x.

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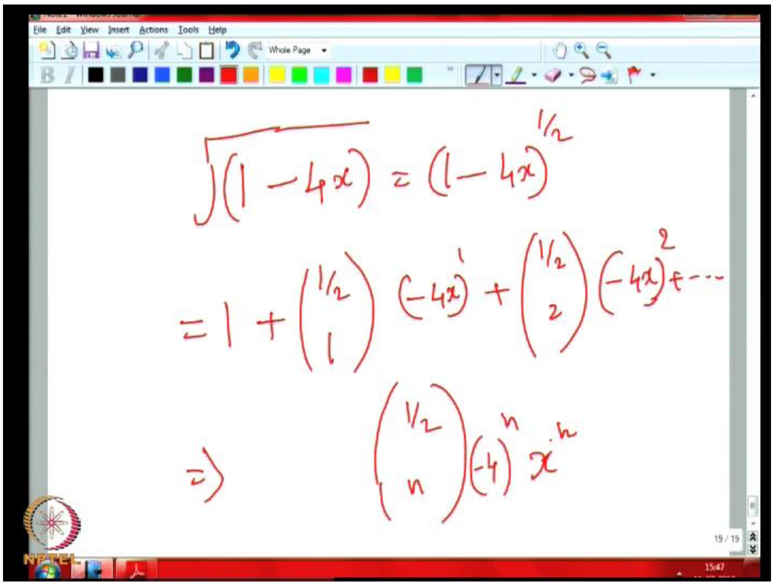

This 1 minus 4 x root is actually we know how to deal with this, 1 minus 4 x raise to half, right. This will be like 1 plus half choose 1 into minus 4 x to the power 1 plus half choose 2 into minus 4 x square and so on. So, n th term will be what? N th term will be so x raise to n's coefficient will be half choose n into minus 4 raise to n, right. So, we have to deal with this nth term now.

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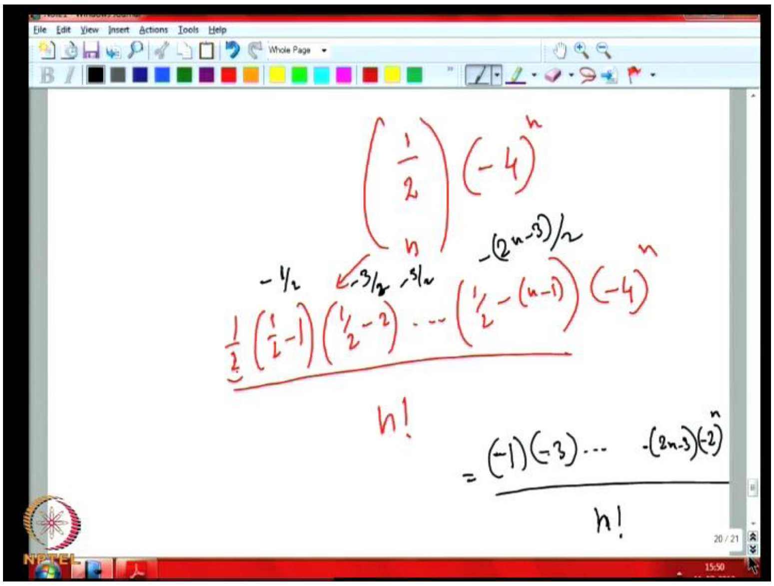

So, that will be like half choose n into minus 4 raise to n which is what is this? This is half into half minus 1 into half minus 2 into half minus n minus 1 because you know you started with one, so the second term is so we need total n terms. See here there is first term and there are n minus 1 more terms, right, and we have a minus 4 raise to n here and this is divided by n factorial as usual. If you have forgotten how to do this thing you have to go back to some classes where we discussed this kind of things and figure out what this was. And now we can see half is here, this can be converted to, say, this is half minus 1. This can be written as minus 1 by 2, right, and this can be written as half minus 2 that is 1 minus 4, minus 3 by 2, minus 1 by 2, minus 3 by 2.

The next will be minus 5 by 2, up to where will we go? This will be 1 minus 2 into n minus 1, right, 1 minus 2 n. See this is 2 n minus 2, right. So, this will be minus 2 n plus 3 that is minus of 2 n minus 3. This will be minus of 2 n minus 3. So, everywhere we have a by two of course. So, those by two's we can collect. There are n by two's, so half here, half here, half here, because here there is a half, from here onwards there n more half's, that will cancel with, say, one of the two raise to n's here.

So what we get is as a result we will get starting from see this is minus 1 into minus 3 into up to minus of 2 n minus 3 here, and we cancel off that. So, we will get a minus 2 raise to n left here; we just cancel off this half raise to n with 2 raise to n here, so minus 2 raise to n will remain and an n factorial will remain here. But then there is a minus 1 raise to n here, so everywhere we have minuses along with everything. So therefore, that there are n minus 1 of minus signs, so they can be collected and we can multiply with minus 1 raise to n here, so that will cancel off n minus one minus one's and just 1 minus one of the sign will remain.

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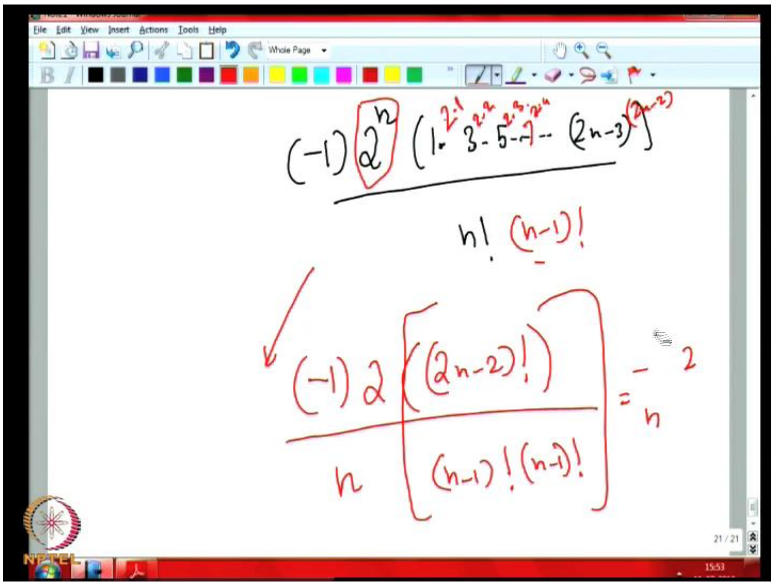

Minus 1 into we will have a 2 to the power n here, right, and then this is going from 1 to 3 up to 2 n minus 3, right, and we have n factorial below. Now what we can do is we can introduce n factorial. See what we can see is 1, 3, 5, 7, like this, right, odd numbers only we are going. I would like to introduce 2 here, 4 here, 6 here, 8 here and so on up to 2 n minus 2 here; that means total of n minus 1 even numbers I want to introduce, but this is actually what? This two is actually 2 into 1, right; we can write it as 2 into 1. This is this four is 2 into 2, this six is actually 2 into 3, this is sorry seven, this is actually 2 into 4 and so on, right. So, total 2 into n minus 1, so n minus 1 two sign in but then i have this stuff here, so that I can use up for that purpose.

So, we will get minus 1, just one, two will remain, because n minus two's I will give for that purpose and then but i also need to add 1, 2, 3 up to n minus 1; that means an n minus 1 factorial should be added that I will have to get below here n minus 1 factorial, so to balance this thing, right. So, I spend off this n minus 1 two's here, 2 to the power n minus 1 as this two's and this n minus 1 factorial I am just putting in the denominator. So therefore, we have above 1, 2, 3, 4, 5, 6, 7, till 2 n minus 2; that is 2 n minus 2 factorial above right, and below we have n factorial into n minus 1 factorial, right. So, this we can separate out. So, we can write it as n we can take here and write an n minus 1 factorial here, because this is familiar this is what. This is actually minus 2 divided by n into 2.

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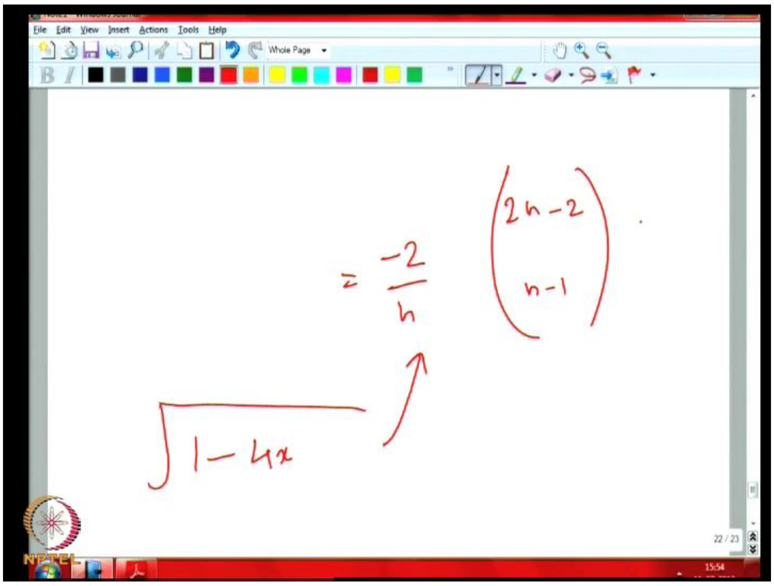

Sorry, I will write it as minus 2 divided by n into 2 n minus2 choose n minus 1; this is what it is. So, you can see that 2 n minus 2 choose n minus 1 is 2 n minus 2 factorial divided by n minus factorial into n minus factorial. This is exactly that term 2 n minus 2 choose n minus 1. So, in principle this is what the coefficient of x raise to n in the expansion of this 1 minus 4 x. This is what is the coefficient, this will be the coefficient minus 2 by n into 2 n minus 2 choose n minus 1 coefficient of x raise to n, but this is a negative term and it is a coefficient of x raise to n, but then this was not exactly the thing we were trying to do. We were trying to do something more than that actually our B of x was this, so we only explore this 1 minus 4 x. So, we can try to write this entire thing now.

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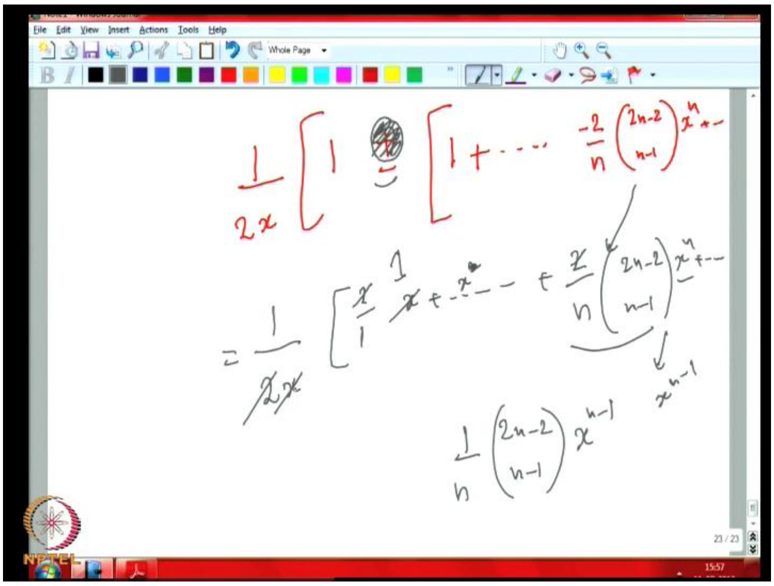

This will look like this. So, we have a 1 by 2 x and we have a 1 plus or minus and the rest is 1 plus something the x raise to n's term is minus 2 by n into 2 n minus 2 choose n minus 1 into x raise to n and so on. So, if you had selected this plus term here, then you can say that one plus one two plus, say, minus, then there is this minus terms will remain. So, the x raise to n coefficient sometime is going to be a negative term. So, that is not allowed because this b n is some count; it is only a positive integer or may be the best, the lowest it can become zero, right. So, this is not an option at all; this we can remove.

So, we have to take this other option. So, then what happens? One and one will cancel; this will read as 1 by 2 x into 1 minus 1. So, we will get x, right. So, here x coefficient you have to put this thing minus 2 by 1, right, that will look like minus 2 by 1 into 2 into n minus 2; that means when I put one here 2 minus 2 will become 0 and 1 minus 1 will become 0 choose 0 is 1; that will be x and so on, right, till this term will come, right, minus of minus will become. This minus will go because one minus right that will all become positive terms now 2 by n into 2 n minus 2 choose n minus 1 x raise to n and so on.

So, now we can cancel this two from everywhere, see every term has a two in this thing, it will go away. Now this x will go away, from the first term it was x raise to 1. Now it will become this one. So, the x square term will become x term and x cube term will become x square term and x raise to n term will become x raise to n minus 1 term. So this 2 by n into 2 n minus 2 choose n minus 1 will come with x raise to n minus 1 rather than x raise to n because this x canceled off one x.

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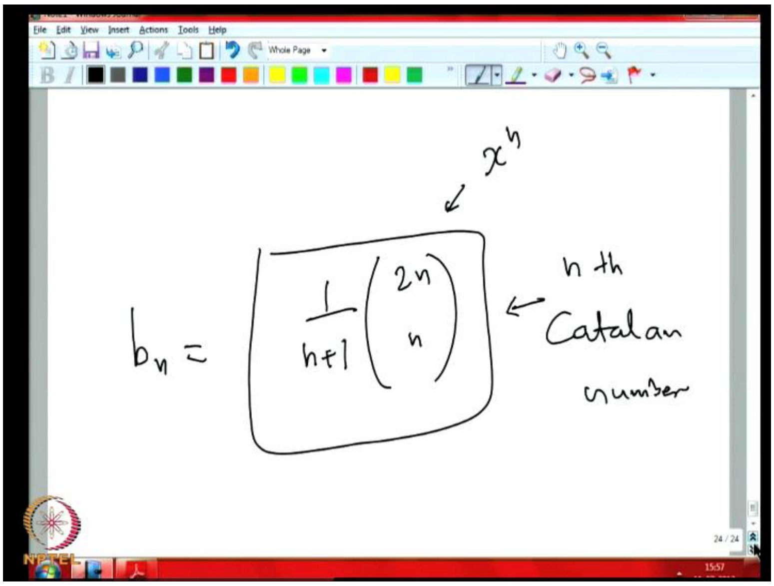

So, it follows that the coefficient of x raise to n will be 1 by n plus 1 because an x term we have to take 2 n choose n, and this happens to be that b n; b n is the co-efficient of x raise to n, right, and we will say that this is the nth Catalan number; let me note once again.

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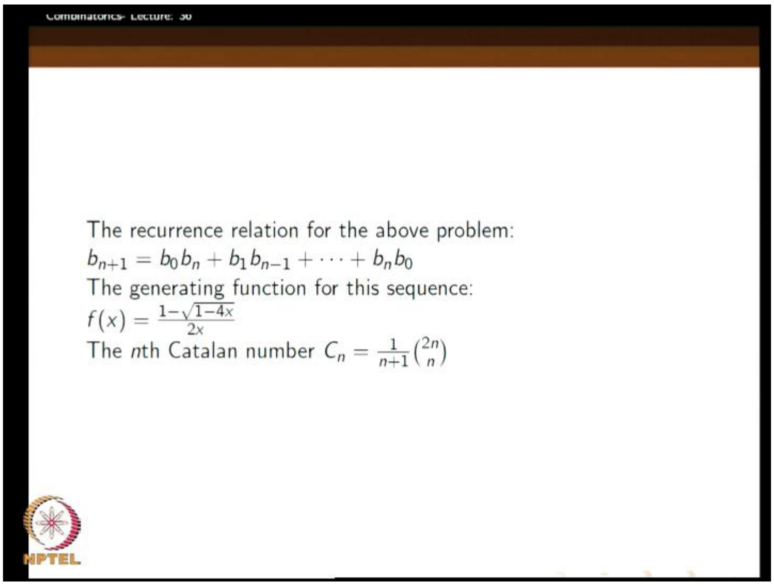

So, we started with this recurrence relation b n plus 1 is equal to b 0 b n plus b 1 b n minus 1 plus b n b 0 and noted that our initial conditions are b 0 is equal to 1, b 1 equal to 1 and so on; b 0 is equal to 1 is enough, because b 1 we can create from that for n equal to 1 onwards this recurrence relation works. The generating function is 1 minus 1 minus 4 x by 2 x; though we got two solutions when we solve the quadratic equation, it was 1 plus or minus root of 1 minus 4 x by 2 x, but we show that if we take 1 plus root of 1 minus 4 x by 2 x, then all this b i's will become negative numbers. That is not a reasonable thing to do, right, because we are counting something and then we want positive answers show positive integers as answers.

So, we selected this 1 minus root 1 minus 4 x by 2 x as the possible solution and that is the generating function. And now we evaluated the coefficient of the f of x raise to n, it happens to be 1 by n plus 1 into 2 n choose n. This is this is the n th Catalan number, this is the nth Catalan number and this was Catalan number; this is called Catalan number because it was first used by Eugene Charles Catalan in 1814 to 1894 is a Belgian mathematician, and he used it to find the number of balanced parenthesis to evaluate an expression of the form x 1 into x 2 into x 3 into x n; we will discuss it a little later. So, that is why it is called Catalan number.

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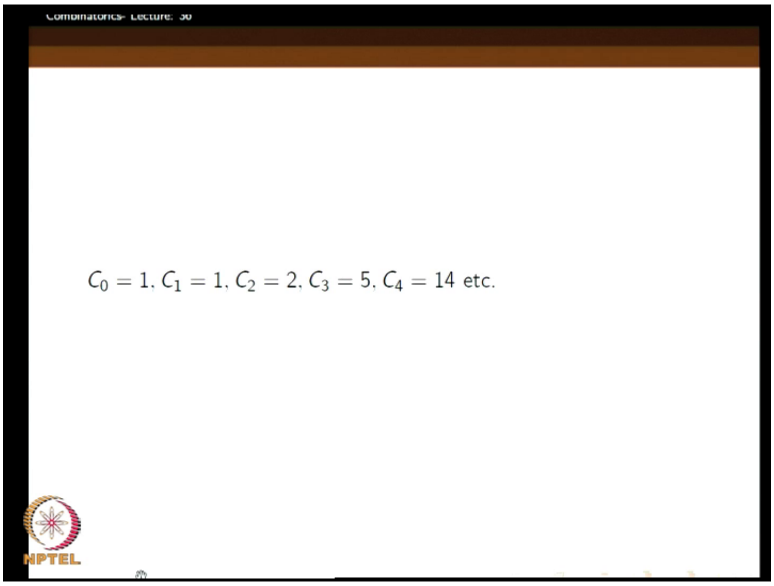

And then now for instance we can see a few examples initial values c 0 is 1, we want to check it.

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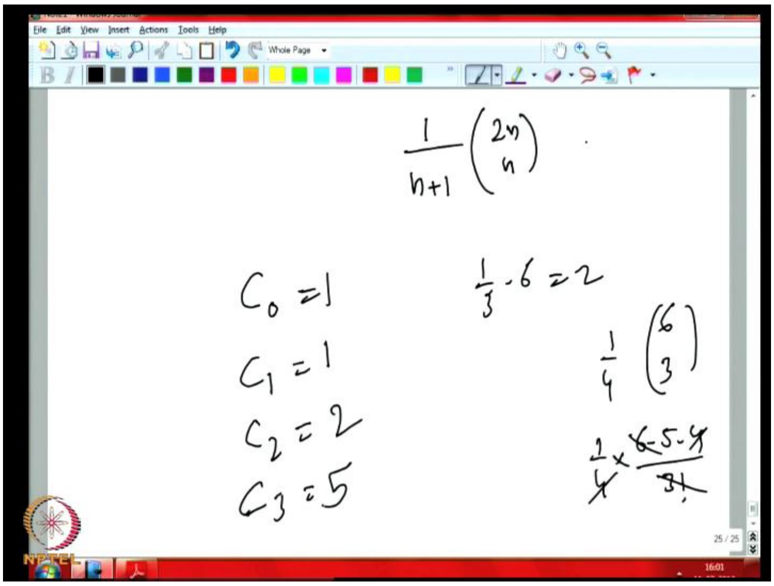

So, that should be right, because that is the way we have defined it choose n; if you put n equal to 0 this will be 1 by 1 into 0 choose 0 that is 1. So c 0, I can say c; c 0 is equal to 1 and c 1 when you put 1 that is 2 choose 1 into 1 by 2 is 1, c 2 that is 4 choose 2 that is 6 by 1 into 3, right, 2 plus 1 is 3 raise 1 by 3 into 6 is 2 and c 3 is what? Because this is six choose three and we have here 1 by 4. This is 1 by 4 into 6 into 5 into 4 divided by 3 factorial; this and this goes and this goes, this is just 5.

And again a few more numbers we can see c 4 is 14 and this exactly matches with the values we got for b 0, b 1, b 2, b 3, etcetera, so that number of tree diagrams the kind of tree diagrams we are considering, the binary trees rooted binary trees, which are ordered left and right being different, right, the way can actually that corresponds the Catalan numbers; the nth Catalan number gives the number of diagrams possible when there are n vertices in that way.

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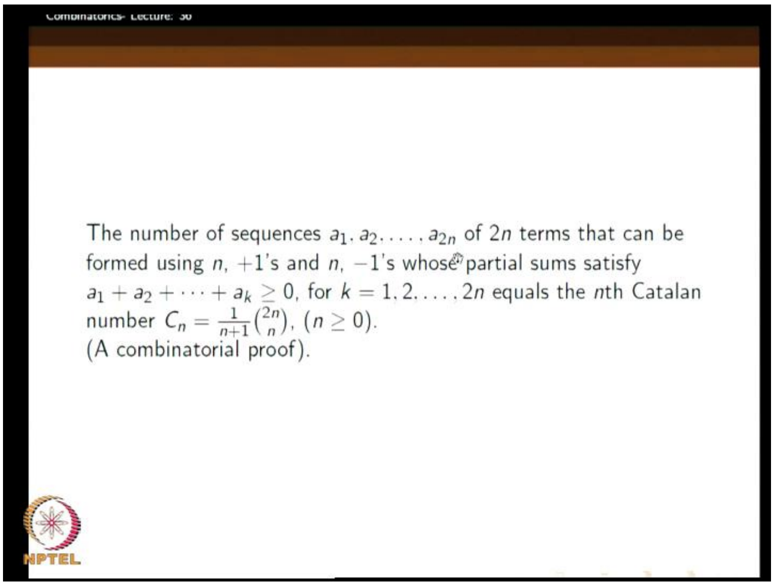

Now our next question is about some sequences. We will see that this Catalan numbers happens to be the answers of several other questions also. At first look it may look different but we will show that the underlying recurrence relation there is also the same as this; may be that is one way of figuring out the whether the Catalan number is hiding fact irrespective of the appearance of the problem. You just try to come up with the recurrence relation; if it happens to be the same recurrence relation as in the previous problem with the same initial conditions namely the zero th value being one, then the nth term will be nth Catalan number. So, we will some examples which will show how differently things can appear and then the same Catalan numbers can be the answer for those questions; in the next class will consider this.