Combinatorics
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Lecture - 34
Partition Number - Part (2)

Welcome to the 34th lecture of combinatorics. So, in the last class, we discussed about partition numbers p of n. It counts the number of partitions for a given positive integer n; number of partitions means, how many ways we can express it as a sum of positive integers, where the order does not matter. It does not care about, how we order the summands. So then after figuring out the generating function for the sequence p of 0, p of 1, p of 2, etcetera. So, we went ahead to study some special kind of partitions; special kind of partitions in the sense that we impose some extra conditions, we are talking about partitions of a second type. The first special type of partition we studied was where the number of summands was equal to k, exactly equal to k.

So, definitely if you want to get p of n we just have to add up the number of partitions where the number of summands is equal to 1, number of summands equal to 2, number of summands equal to 3, number of summands equal to n. Therefore, it makes sense to study this partition; of course this special type of partition and then we saw some recurrence relation which is valid for those partition numbers p k of n, right. We see involving, right, considering this as a family of sequences p k of n for each k there is one sequence and then using this p k can be expressed in terms of smaller and smaller k, right; that is what we discussed in the last class. Now we will just get some crude estimates for p k of n in terms of n and k. So, this is the next one.

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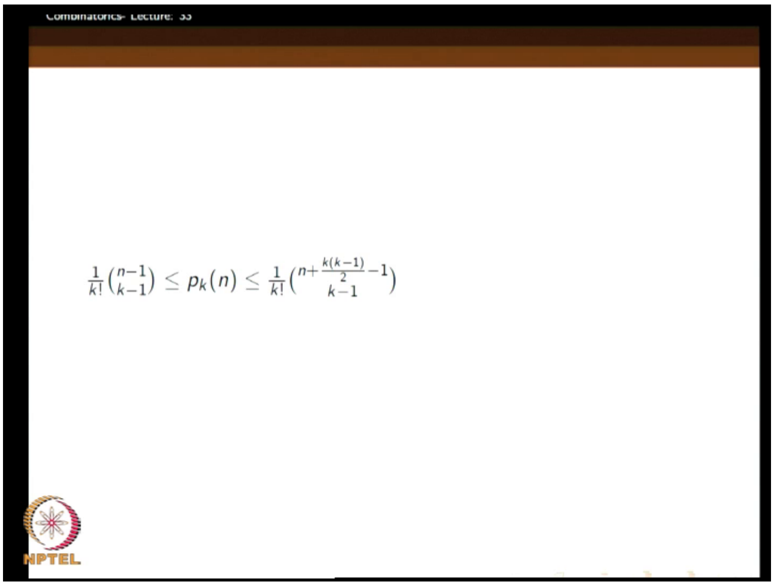

So, we are going to show now a lower bound and upper bound, right. So, first we will look at the lower bond; p k of n is at least 1 by k factorial into n minus 1 choose k minus 1.

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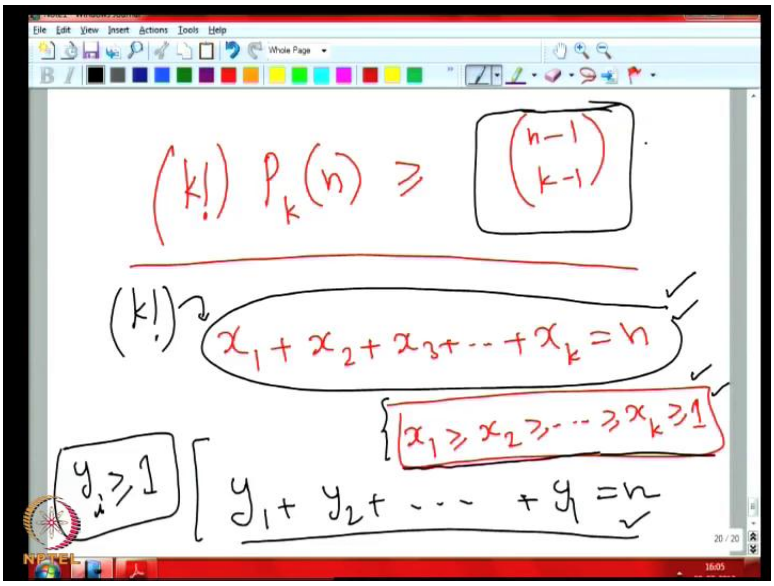

Greater than equal to1 by k factorial n minus 1 choose k minus 1. So, this we can always write like k factorial into p k of n is greater than equal to n minus 1 choose k minus 1. This comes from the fact that p k of n is the number of solutions for this equation namely x 1 plus x 2 plus x 3 plus x k equal to n, where the order does not matter. That means we have to make sure that the order does not, by changing the order, we will get a different solution. We also put the condition that this is greater than equal to x k and everything has to be at least one that is important because we want exactly k parts because if you put one of them 0, then we will get less than k parts only.

But on the other hand, we told that there was this old equation we studied y 1 plus y 2 plus y k is equal to n and without this condition we just have y i. So, the only condition we have is each y i greater than equal to 1 not 0, 1, right. So, what is the difference between this two? So, the only difference is that here we have to have x 1 strictly bigger than x 2 strictly greater than. So, what we can notice is if we get one solution for x 1, x 2, x k, etcetera, then we can try to permute that solution; for instance if x 1 is assigned a value a and x 2 is assigned a value b and so on. So, we can try to give a to here, b to here, right. So, permute the values assigned to x 1 x 2 x 3 x k to all possible ways; definitely it can be done in k factorial ways, right. So, what is assigned to x 1 can as well be assigned to x 2 or x 3 or x k and what is assign to x 2 can be assigned to their many of the remaining k minus 1 possibilities and so, on right.

So, actually there are k factorial maximum possible numbers of permutation set we can get; each of them may give a solution for the second problem, right, because there the order is not important second problem, right but you note that see when you permute. So, if they were repetition here x 1 x 2 x 3, etcetera where all equal. So, then there would not be k factorial permutations. There will be k factorial divided by some other something; that something will be, say, for instance if a certain value is repeating k 1 times then k 1 factorial will come in the denominator, we have seen it several times. But the maximum possible is when they are all distinct and that is k factorial. So, that means from a solution for the first equality satisfying this condition, we can get utmost k factorial possibilities for the second one; that covers all the possible solutions for the second one.

In other words if you get a solution for the second one, we can in fact see it as coming from a solution for the first one by some permutation of it, because what you do is we will rearrange the solution such that y 1 get the maximum, y 2 get the second maximum and so on, right. So, that will be a solution for the first equation, right, but then one solution for the first equation give raise to many solution for the second one but maximum k factorial; that is why k factorial in the p k of n is greater than equal to the number of solutions for the second one. We claim that the number of solutions for the second one is n minus 1 choose k minus 1, why is it so?

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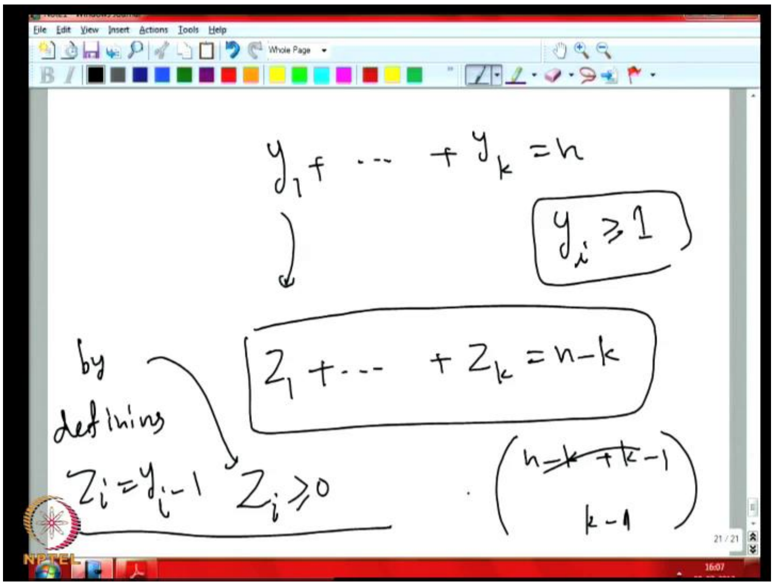

Because the second one is y 1 plus y k equal to n, but here we have the condition that y i is greater than equal to 1. So, now we can convert it to another system z 1 plus z k equal to n minus k by defining z i equal to y i minus 1. We just reduce one from each of them, because each y i is greater than equal to 1. Now z i will be greater than equal to 0, is it not, because that is what will happen by doing like that, because each y i was greater than equal to 0; you reduce one from each y i and get z z i. So, z I will remain greater than equal to 0. So, these are very familiar one. We know the solution for this thing is n minus k plus k minus 1 choose n minus k or maybe we can also set k minus 1 here, that is all same, right, so this cancels.

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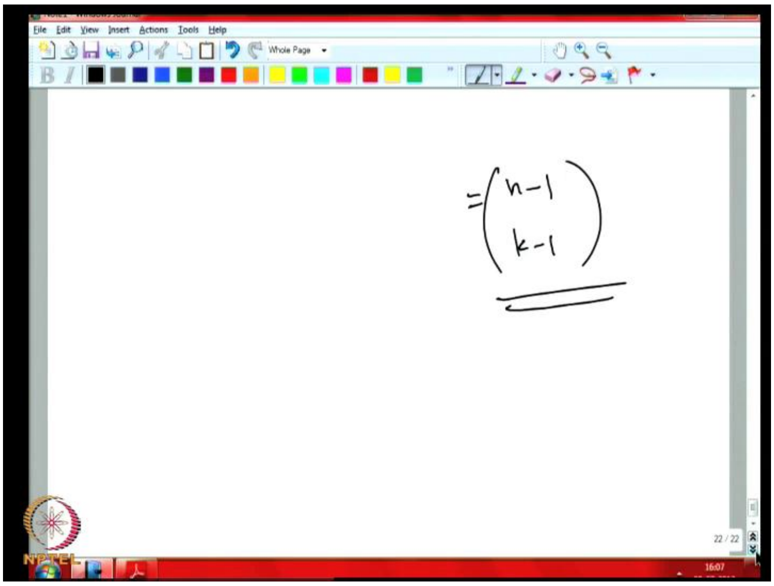

So, that is n minus 1 choose k minus 1. So, here I am saying that you see earlier we used to write n minus k here, but that does not make any difference because of the symmetry property of the combinatorial coefficients, right, whether we write n minus k here or k minus 1 here it is the same thing, right, because n minus 1 minus n minus k is actually k minus 1, right. So, this is where this is coming from, right. So, we get this thing. And now for the upper bond; the upper bond is 1 by k factorial into n plus k into k minus 1 by 2 minus 1 choose k minus 1, right. How do you do this?

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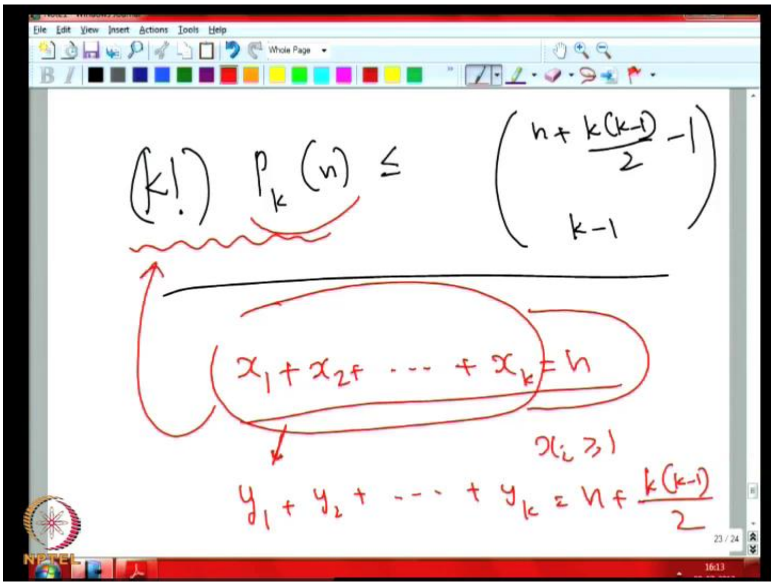

So, you want to show that p k of n is less than equal to 1 by k factorial into n plus k into k minus 1 by 2 minus 1 choose k minus 1, or in other words we can say we want to prove that k factorial into p k of n is less than equal to this. Now we will use the same argument as before; you remember this p k of n correspond to the solutions of this one x 1 plus x 2 plus x k equal to n, with this x 1 greater than equal to x k greater than equal to 1; this was the condition for this thing. Now given any solution for this thing, we can generate utmost k factorial solutions from this thing, right; for rotating solution for n but the only problem is there can be some. See suppose they were actually k factorial into p k of n solutions for this term. Then we can say that the number of solutions for this one when we drop this condition x k is great.

So, all these x I greater than equal to 1; this condition I have written, but the other condition that x 1 has to be the biggest, x 2 has to be the next biggest, that condition we dropped; that means we can have different orders, the values can be assigned to different. For instance whatever is assigned to the x 1 x 1 first can be assigned to x 2 later and so on and get a different solution. If that is the situation then we could have written, say, the number of solutions for these things namely n minus 1 choose k minus 1 in the upper bound. We cannot do it because actually we may not get k factorial into p k minus 1 different solution; that may not be there because many times the solutions for the corresponding to p k of n may have repetitions. So, k factorial may not occur. So, one thing we can try is to make these values distinct, how do we do it, right. So, this is the way we will do a transformation; we will define y 1 plus y 2 plus y k equal to something here n plus something here. How do I do it?

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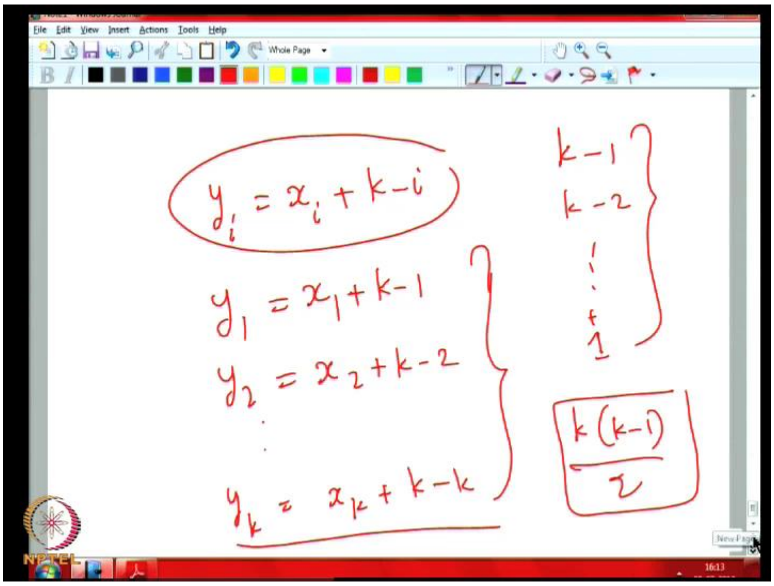

So, what I do is y i will be defined as x i plus k minus i. So, for instance y 1 will be equal to x 1 plus k, right; y 2 will be x 2 plus, sorry k minus 1, y 2 will be x 2 plus k minus 2 and so on, and y k will be equal to x k plus k minus k namely k x itself, right. So, we can try this conversion, right. So, convert each x i into y i by defining like this x i plus k minus i. Now see the total we are adding to the, so x 1 we are adding k minus 1, x 2 we are adding k minus 2; finally, y k minus 1 we are adding 1 and here we are adding only 0. So, how much we are adding? So, this is k into k minus 1 by 2 because there are the first term 1, last term k minus 1 and the number of terms k minus 1. So, k into k minus 1 by 2 is the total we are adding here. So, we just write like this. So, this is a conversion.

So, we still have y 1 greater than equal to y 2 greater than equal to this thing but then any solution for this thing, this first equation will give a solution to this equation where each value is distinct. Now each of the solutions gives k factorial permutations of the solutions, right. We can reassign these values; keeping the values same we can assign to y 1 y 2 y k in all possible ways. There are k factorial permutations possible coming from each such solution, right. So therefore, from this thing see of course the claim is that we get y 1 y 2 y k all distinct after doing this transformation. So, if they really happens to be distinct then this k factorial into p k into n will be the total number of solutions for this equation, right, for the original one. Total number of solutions with distinct values, right, because these are all distinct, sorry, what we say that corresponding to each solution of this thing we will actually get a distinct solution for this thing and that can be permuted in all possible ways that all corresponds to the solution for the equation of this sort.

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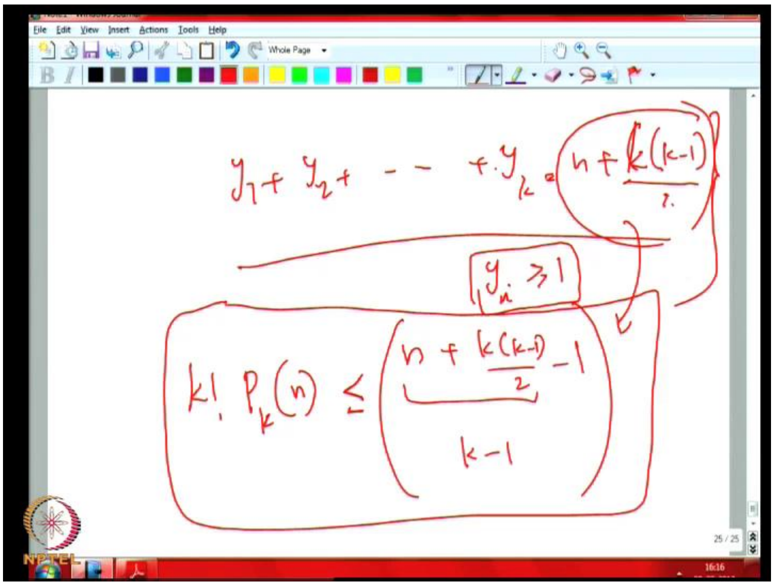

So, y 1 plus y 2 plus y k is equal to n plus k into k into k minus 1 by 2 where each y i has to be greater than equal to 1; the condition that y 1 has to be the biggest, y 2 has to be the second biggest, that is removed, right. Once again I am saying that you find a solution for this thing. Now make this transformation; now you get the solution for this equation where y 1 is biggest, y 2 is second biggest, y k is last and so on, and here they are adding up to n plus k into k minus 1 by 2, but now all the values for y 1 y 2 y k are distinct now you drop. Suppose you drop the restriction that y 1 has to be the biggest, y 2 has to be second biggest and all, then we will get k factorial times this number of solutions which are the solutions for the resulting equation namely this equation, right.

See here there can be more solutions because there is no restriction that their solution has to be distinct or something, but then this are all solution for this thing namely this k factorial into p k of n possible solutions are all solution for this last equality and we know this number actually has to be les8s than equal to total number of solutions for this thing which is actually n plus k into k minus 1 by 2 minus 1 choose k minus 1, right, because there k things; this takes all of the n here, right. So, n minus 1 choose k minus 1, right, because we have this extra conditions, y a is greater than equal to 1; that was always n minus 1 choose k minus 1, but n is replaced by n plus k into k minus 1 by 2 here minus 1. So, this will come; the only thing we have to now establish is that these are all distinct, right, why are they distinct?

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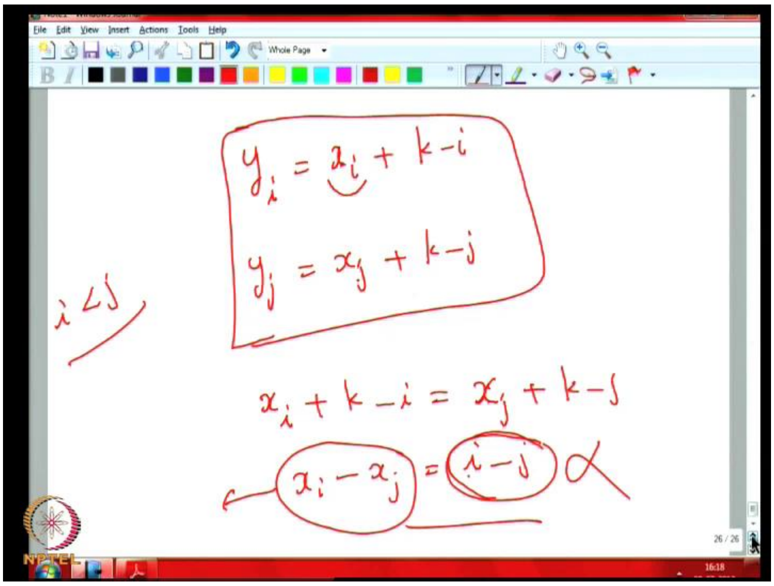

Because we defined y i is equal to x i plus k minus i. Suppose this was equal to some y j namely x j plus k minus j. So, without loss of generality we can assume that this x i this i was smaller than j; that means this x i is bigger that x j. Now when you take x i plus k minus i suppose this is equal x j plus k minus j. So, that is x i minus x j equal to i minus j, but i is smaller than j. So, this is a negative number, but the x i is bigger than x j, this is a positive number, alright positive number. Now positive or 0 does not matter, but this is strictly a negative number. So, therefore, this is wrong contradiction here. So therefore, y i and y j can never be equal, they are distinct. So, from this thing whatever we are trying to prove follows, right.

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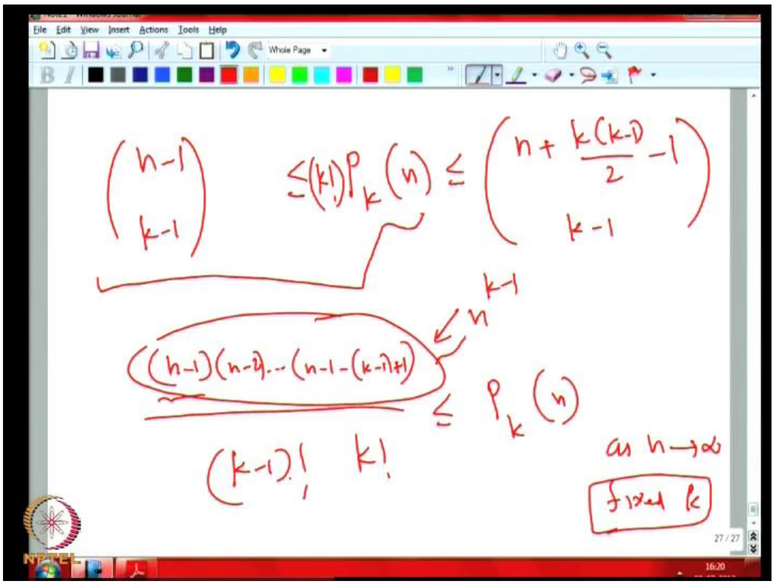

So, what we have got now is p k of n is less than equal to n plus k into k minus 1 by 2 minus 1 choose k minus 1 and the lower bound was, sorry k factorial into, lower bond was n minus 1 choose k minus 1, right. Now we can probably try to see how much is this? So, this n minus 1 k minus 1 will look like n minus 1 into n minus 2 into. So, k minus 1 terms n minus 1 minus k minus 1 plus 1, right, divided by k minus 1 factorial, and this k factorial I can take out, this is less than equal to p k of n.

So, now see suppose n is a very big number compared to k; that means you fix k and as n tends to infinity and fixed k, right, then what will happen? See we can say that these are all n minus 1 n minus 2 n minus 1 minus k minus 1 plus 1 and all which is n minus k plus 1. So, these are all about n, right, those small difference is there. We can say approximately this is n to the power k minus 1 because there are k minus 1 terms in the numerator; see this is approximate. We are assuming that n is very very big compared to k and as tending to infinity k is fixed. So, it is approximately n to the power k here below its k minus 1 factorial k factorial.

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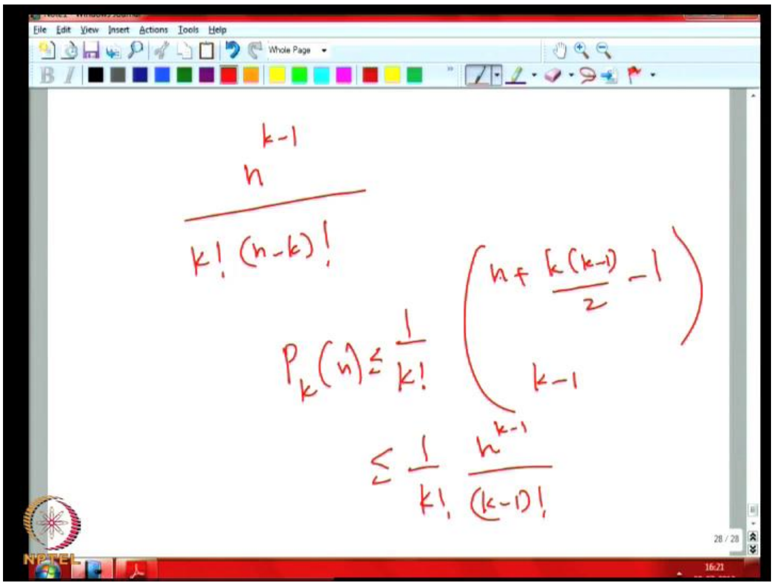

So, this lower bound is like approximately n raise to k minus 1 by k factorial into n minus k factorial. Similarly the upper bond, upper bond p k of n is less than equal to 1 by k factorial into, right, n plus k into k minus 1 by 2 minus 1 choose k minus 1, right. So, here you can see starting from n plus k minus k into k minus 1 by 2, we have k minus 1 term written downward; that is a following factorial starting from that n plus k. So, because k is considered to be much smaller than n we can just approximately each of them are n. So, this is again 1 by k factorial into approximately n raise to k minus 1 above and k minus 1 factorial below right. So, we again get that. So, approximately both lower bound and upper bound seems to be this.

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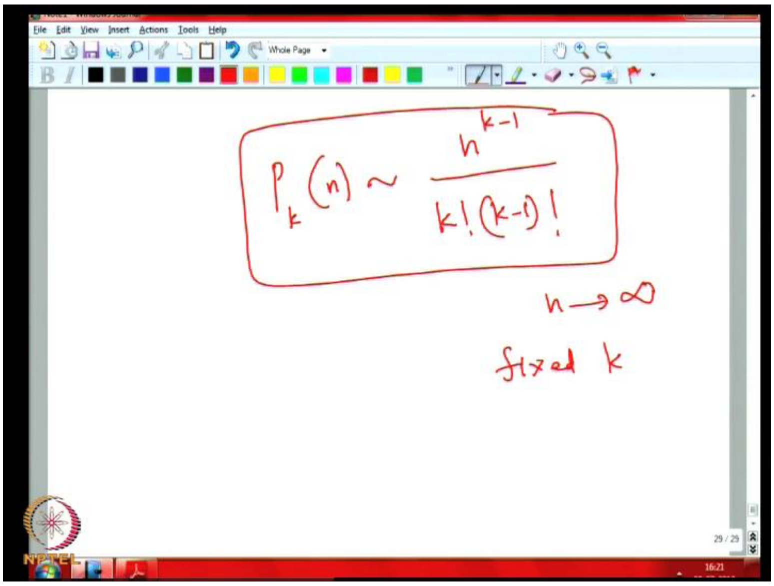

So, we can write p k of n is approximately n raise to k minus 1 by k factorial into k minus 1 factorial when n tends to infinity and fixed k. This is because we are assuming that n is too large compared to k, right; k will not much effect on these extra terms.

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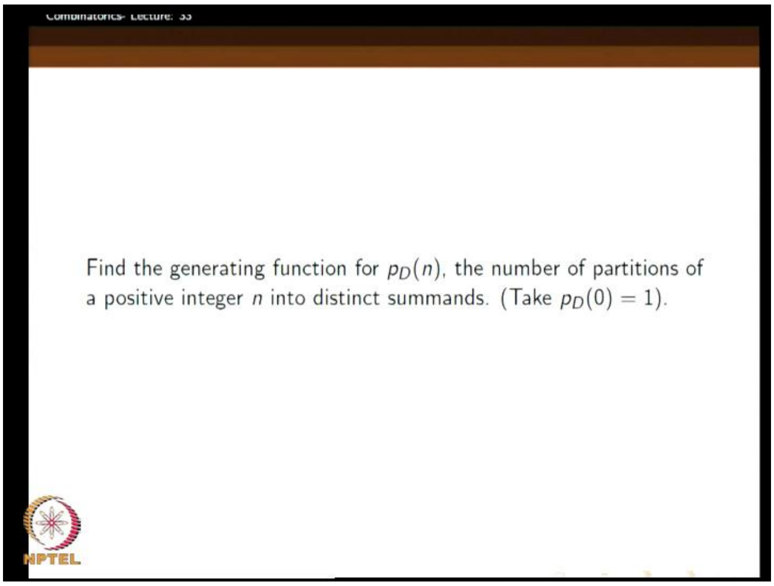

So, up to now we were discussing one special kind of partitions namely the partitions where the number of parts number of summands is equal to k fixed k, right, for a fixed k. now will consider a different kind of partition namely when all parts are distinct, what do I mean by that what, all parts are distinct.

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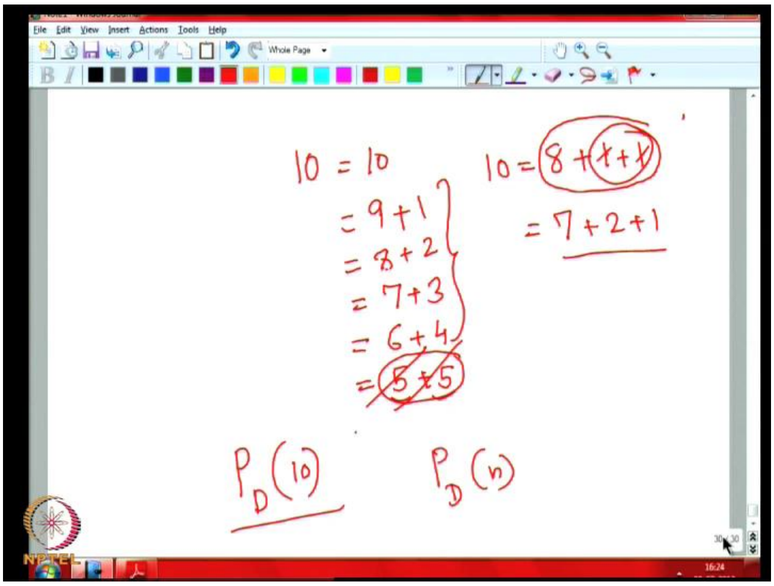

See for instance we can take 10. So, 10 equal to 10, this is a one summand partition; anyway there is nothing to discuss there, because all parts are distinct. Then for instance 10 can be written as 9 plus 1; here all parts are distinct 9 and 1. It can be written as 8 plus 2; all parts are distinct here. It can be written as 7 plus 3; all parts are distinct here. It can be written as 6 plus 4; all parts are distinct here. It can be written as 5 plus 5, but the parts are not distinct here 5 and 5, right. So, this is not a distinct partition where all parts are distinct.

So, for instance if you consider three summand partitions for 10. So, 10 equal to, say, 8 plus 1 plus 1, but this is not allowed for us, because all parts are not distinct here because this 1 and 1; 1 appears 2 times here, right. So therefore, this is not, but on the other hand 7 plus 2 plus 1 is allowed, parts are distinct here. Now we are interested in what is p D of 10 namely the number of partitions of 10 where all parts are distinct, p D of n in general, right. So, we want estimate the value, but we would rather see what is the generating function for that.

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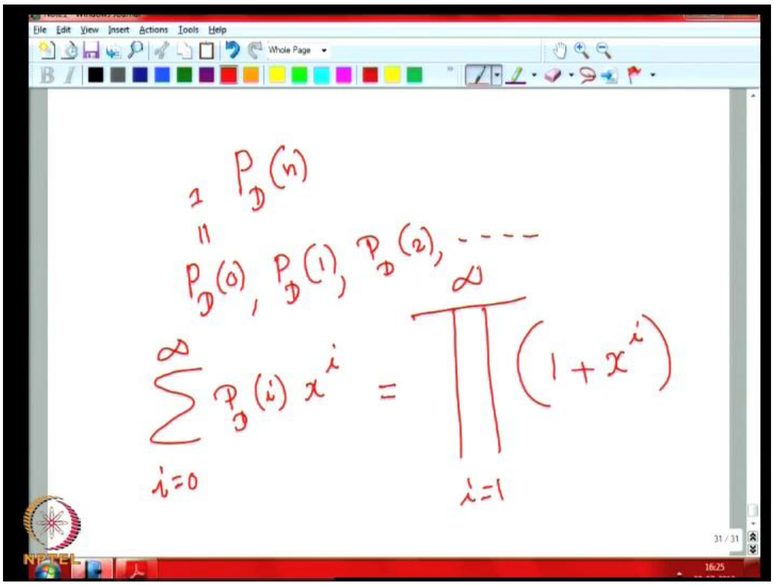

That is what our aim as of now p D of n, right. So, we have to worry what is p D of 0. As usual we will assume that it is 1 because yeah when we write down the generating function it has to be 1 because we are taking it as the co-efficient of x raise to 0 and that has to come as 1. So, p D of 1, p D of 2, this sequence; so, let us consider generating function for this sequence, right, so that means i equal to 0 to infinity p D of i into x raise to i. This is generating function; this generating function is actually if you gives some thought is equal to pi of 1 plus x raise to i i equal to 1 to infinity, why is it so?

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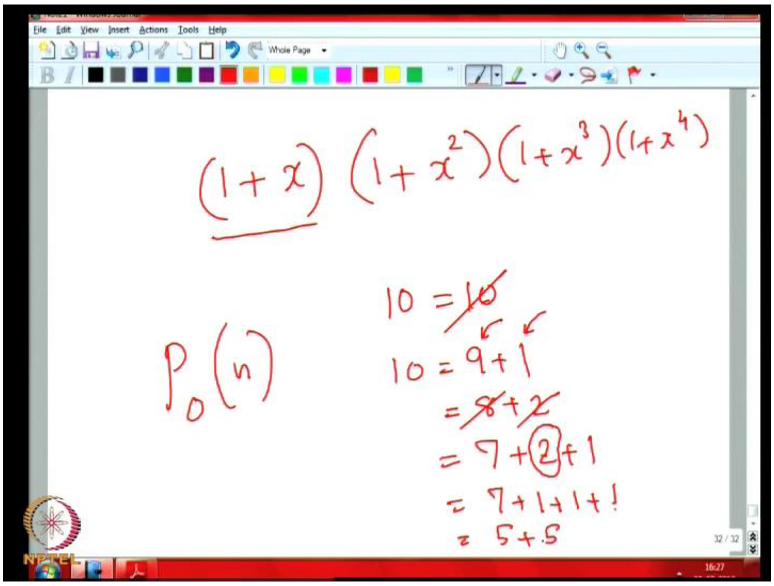

Because the first term will allow us to select the number of one's, but see now this is 1 plus x; earlier we used to write 1 plus x plus x square plus x cube plus like this an infinite series power series here, but that only meant that we are allowing to take 1 one or 2 one's or 3 one's or whatever number of one's we want, but now we say we will not allow 2 one's or 3 one's or more one's, right. So, we will just allow just 1 one or 0 one, that is it, right. So, 1 plus x is there. Similarly the next term will be 1 plus x square, what does it mean? We are allowing either 0 2 or 1 2, not more than that.

Earlier we used to write there this one, right, x raise to 4, x raise to 6 and so on; that means in general when you want to select k two's we would just picking up this x raise to 2 k from that, right. But then we are not allowing all those things now, because we are allowing only 1 2 or no 2, right, because we are talking about distinct part, no part should repeat, right. Similarly 1 plus x 3, 1 plus x 4, 1 3 will be allowed, 1 4 will be allowed, not more than that, right. So, this is why this generating function is like this, okay. Yes, p D of i into x raise to i.

Now the next thing next type of partition we want to consider is this one, the p o of n what is this? If the partitions of n where all the parts are odd numbers; for instance again 10 if you take 10 equal to 10, this is not a valid partition, why? Because 10 is not an odd number, but 10 equal to 9 plus 1 is a valid partition because both 9 and 1 are odd numbers, 8 plus 2 is not valid. Because these are not odd numbers, 7 plus 2 plus 1 is not a valid partition valid because 2 is an even number here, right. But on the other hand 7 plus 1 plus 1 plus 1 is valid, because these are all odd numbers, right, and similarly 5 plus 5 is allowed because both are odd numbers and so on, right.

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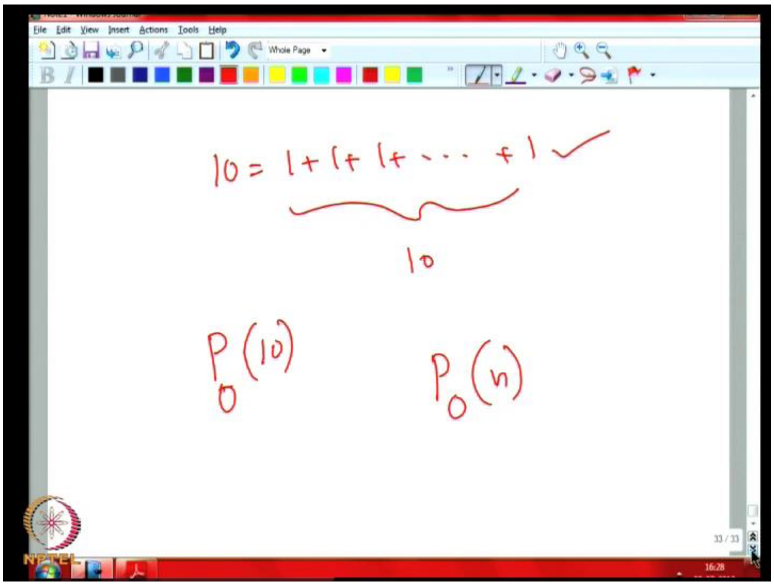

And similarly 10 equal to 1 plus 1 plus 1 plus 1; 10 one's, right, is a valid one because they are all odd numbers and the number of such partitions are called p D of 10, right, the number of odd partitions are p o of n, right, number of odd partitions of 10, right. So, we are in general interested p o of n.

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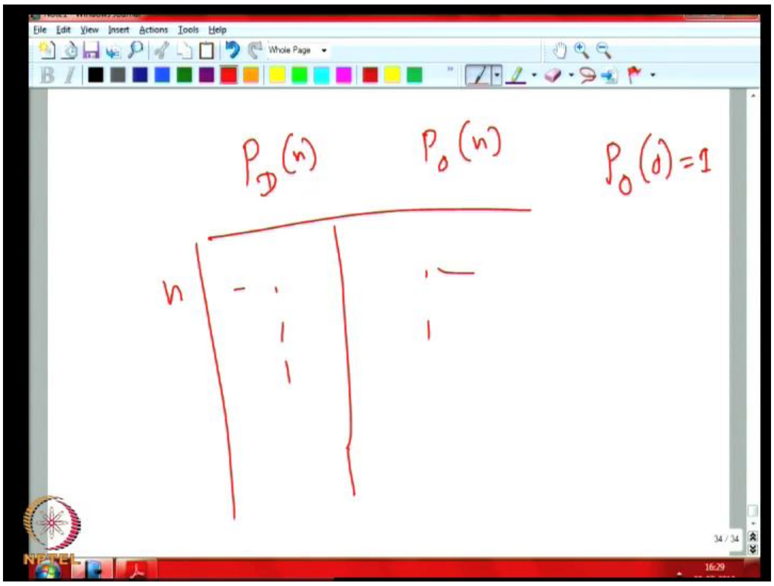

So, an interesting observation that can be made if you work with small examples; for instance try with some number 1, 2, 3, 4, like that up to 10 we try and count. Make a list of this values p D of n and p o of n make a table, right, for each n you just list down what is this value, what is this value. We will see that both are same interestingly. We will design p o of 0 is equal to 1 as usual, right. So, there also it will be equal. So, everywhere it will be equal is what we will see, but why is it so? We will give a proof of that, right.

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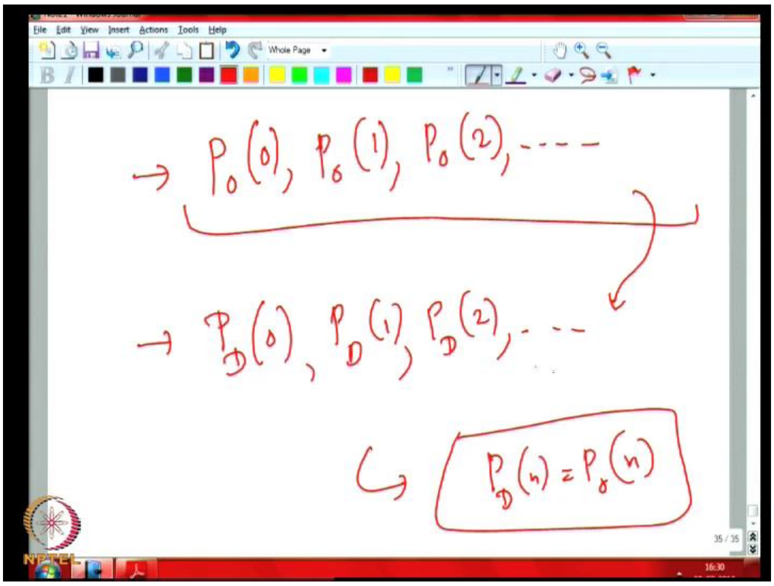

But before that we will see how I will write a generating function for the sequence p o of 0, p o of 1, p o of 2 this sequence, right. So, why am I writing the generating function because if I write down the generating function and somehow manipulate the generating function and show that this generating function is the same as the generating function I obtained for the other sequence namely p D of 0, p D of this sequence, right, what does it mean? It would mean that both sequences are same; for instance p D of n will be equal to p o of n; this is what we can infer from that, right, in particular this is what will come. So, now that is what our aim now.

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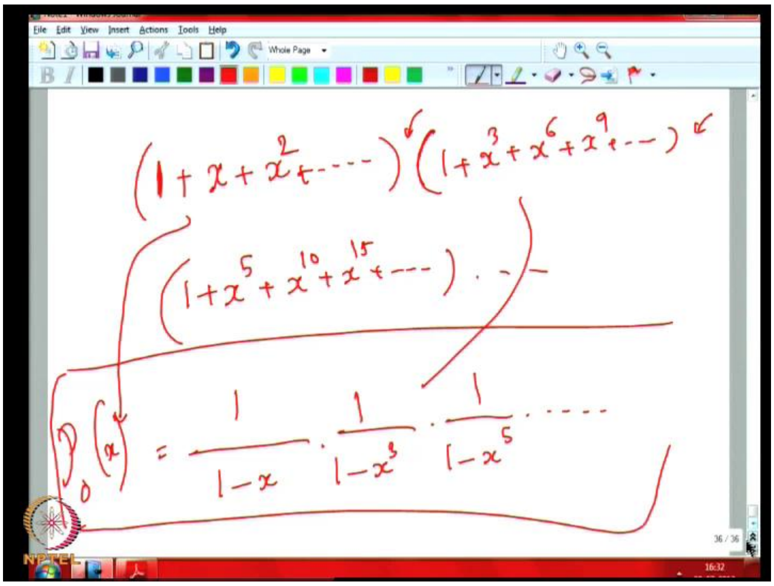

But then it is easy to find the generating function for this p of n sequence, because you know you are only allowed to take odd numbers, right. So, we should allow one. So, this is okay, this is the first one; that means 1 is allowed, right, but two is not allowed to take. So therefore, we would not add this stuff because how many two's are being selected; this was what this was determined, anyway we are not allowing this at all. So, you would not add it, but then three is allowed. Next term is there, right, 1 plus x cube plus x raise to 6 plus x raise to 9 plus so on. And now this term 1 plus x raise to 4 plus x raise to 8 plus x raise to 12. This was counting how many four's are being selected, but four's are not allowed to be selected at all.

So, we would not add that, but then next 5; yes, that is okay, 1 plus x raise to 5 plus x raise to 10 plus x raise to 15 plus, this is okay; like that we can write one term for each term. The intermediate terms which are missing namely the ones corresponding to the even terms that because you remember this is exactly the same generating function for the p of nm but only thing us that we are skipping the alternate one's because the one's written for two one's, terms written for 2, terms written for 4, terms written for 6, like that. But then you know this is what; this is 1 by 1 minus x, this is 1 by x cube and 1 by x raise to 5 and so on only odd numbers. This will be the generating function, say, p o of x will be like this. Now we will show that this generating function is the same as the generating function for p D of n.

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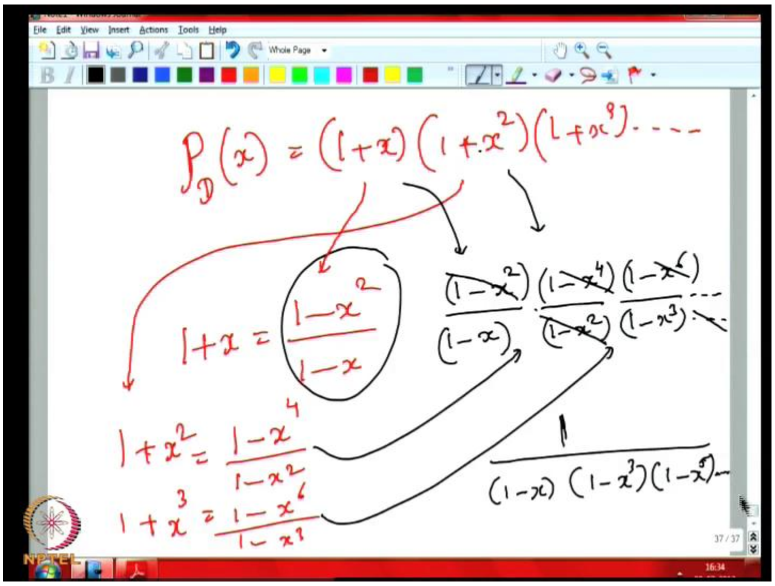

So, you remember the generating function for p D was this 1 plus x into 1 plus x square into 1 plus x cube into so on, right, but this 1 plus x is what? 1 plus x can be written as 1 minus x square by 1 minus x. Similarly 1 plus x square, the second term, this can be written as 1 minus x raise to 4 by 1 minus x square, right. If you multiply this and this you will get this 1 plus x square into 1 minus x square is 1 minus x raise to 4. Similarly, 1 plus x cube can be written as 1 minus x raise to 6 by 1 minus x cube and so on. Now you substitute here for this may be this 1 plus x I will take this and substitute.

So, that is 1 minus x square by 1 minus x into, for this one will substitute this, right. So, 1 minus x raise to 4 by 1 minus x square and this will come here 1 minus x raise to 6 by 1 minus x cube and so on. Now you know in the upper part we have 1 minus x square 1 minus x raise to 4 1 minus x raise to 6, all 1 minus x to the power and all even number. Below we have everything 1 minus x 1 minus x square 1 minus x cube 1 minus x raise to 4 and so on. So, all those things which appeared in the numerator we will cancel off from here like this, right. Now what will remain in the numerator will be 1 and bellow you will have 1 minus x because even things will go away 1 minus x cube 1 minus x raise to 5 and so on; if you remember this is exactly this, right, and that is p o of x.

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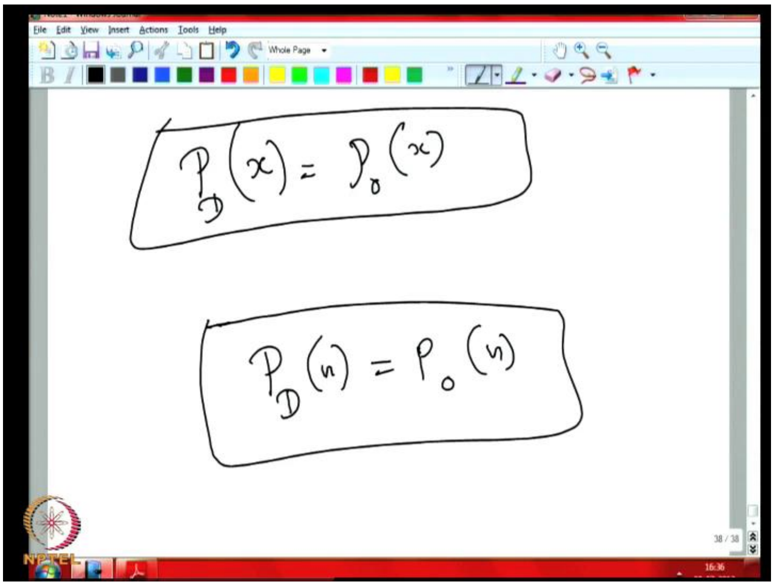

So, what we have seen now is just by manipulating the generating function while we have got p D of x is equal to p o of x, right, which means that the generating function for the sequence corresponding to the number partitions with distinct parts is the same as the generating function for the number of partitions with all odd parts. So, it follows that p D of n is equal to p o of n; it is a very cute proof using generating functions, alright. So, what one should notice is that even to make this observation it is a little difficult, because you have to play around with these partitions and then get a feel of these numbers then only we can numbers such a conjecture even, proof is different. So, one should try to come up with an accounting proof straight proof without using generating function. So, then this cute simple technique is in generating function, one may able to appreciate better.

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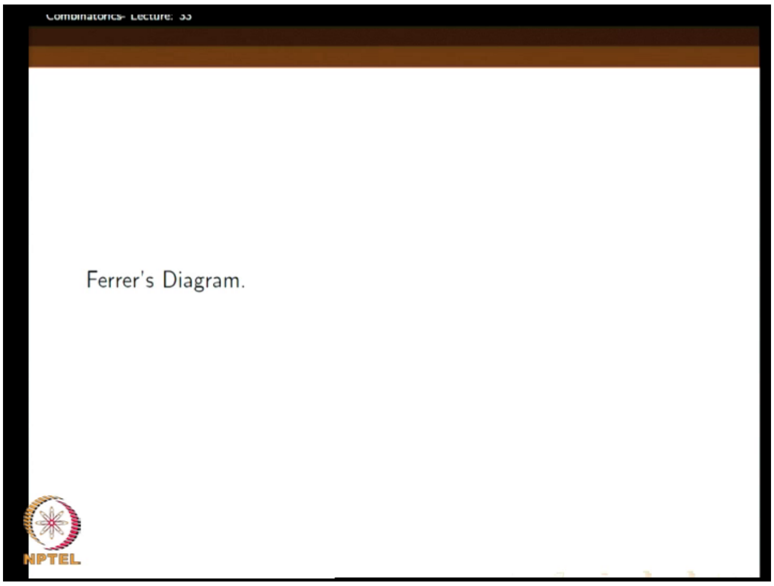

Now we will look at this. See because now that we have talked about proving it in a different way without using generating functions, what are the other techniques available? So, there is one technique by using a special type of diagrams called Ferrer's diagrams, right, what are this Ferrer's diagrams? In the partition literature this is very much used and given a partition we can pictorially represent the partition like this.

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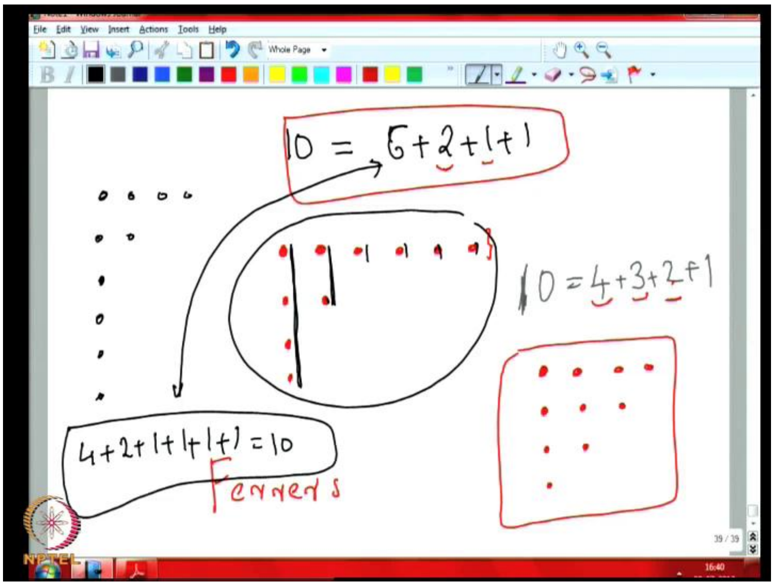

For instance let look at this partition of 10, 6 plus 2 plus 1 plus 1, right. So, what we do is to represent 6 in the first row we add 6 dots and and now the second part two, we just add two dots in the second and, then we add just one dot, and then we add just one dot. This is a representation of this partition of 10. If you count the total number of dots that is equal to 10 and the parts appear in the rows. So, there are this many rows; therefore, that is the first part. And you should know biggest part should not come in the first row, second biggest should come in the second row, third biggest should come in the third row and so on, right; this is the way.

We can take another example; for instance, say, 8 equal to 5 plus 3 plus or may be 9 equal to 4 plus 3 plus 2, right, maybe again I will write 10 equal to 4 plus 3 plus 2 plus 1 adjust to get 4 rows. So, the picture will be like this. So, I will put four rows, four dots in the first row; it correspond to this four and then I put 3 rows in the next on just below them correct in the upper way will starting from the left most and then two dots in the third row to match this and the one dot this thing; total number of dots is 10 and the parts of the partition appear in each rows. So, this is called Ferrer's diagram. So, each possible partition of 10 we get a Ferrer's diagram.

Now what is the conjugate of a Ferrer's diagram? So, you could have for instance given this Ferrer's diagram you could have, say, this Ferrer's diagram let us say, you could have read it like this as the first row, this is the second row, this as the third row, this is the fourth row, fifth row, sixth row. This will correspond to for instance if I rearrange it will look like four dots here, two dots here this one, right, this column will come here and this third column and then fourth, fifth, sixth. So, this is actually 4 plus 2 plus 1 plus 1 plus 1 plus 1 is equal to 10; it is a different partition of 10, right, this is the conjugate of this partition, right.

So, that means instead of reading the rows reading row-wise we read column-wise, that is all, right. So, it is obvious we will get a partition because the first column will be the longest possible, second column is next longest and so on, right. So, we can use this idea that by reading the columns we will get another partition of the same number and actually a Ferrer's diagram for another partition of the same number we can prove something.

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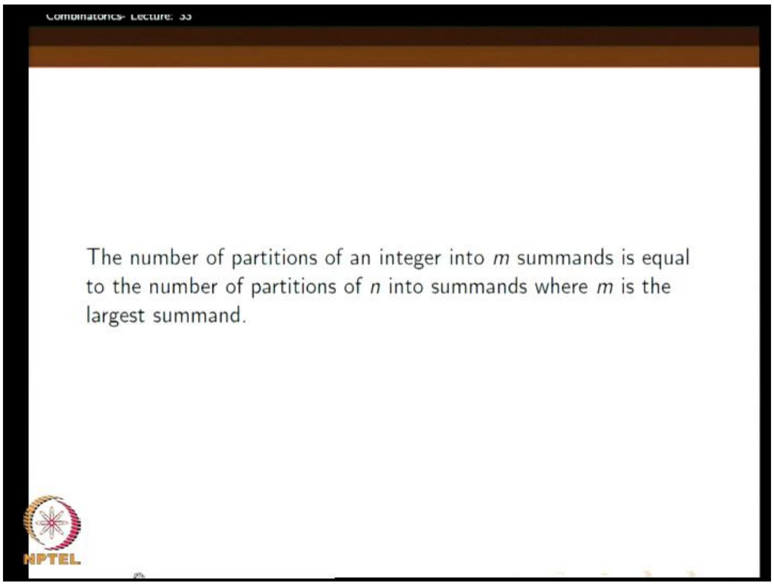

So, for instance look at this statement. The number of partitions of an integer into m summands is equal to the number of partitions of n into summands where m is the largest summands. So, again we were telling that we are considering different type of partitions, we still consider this thing.

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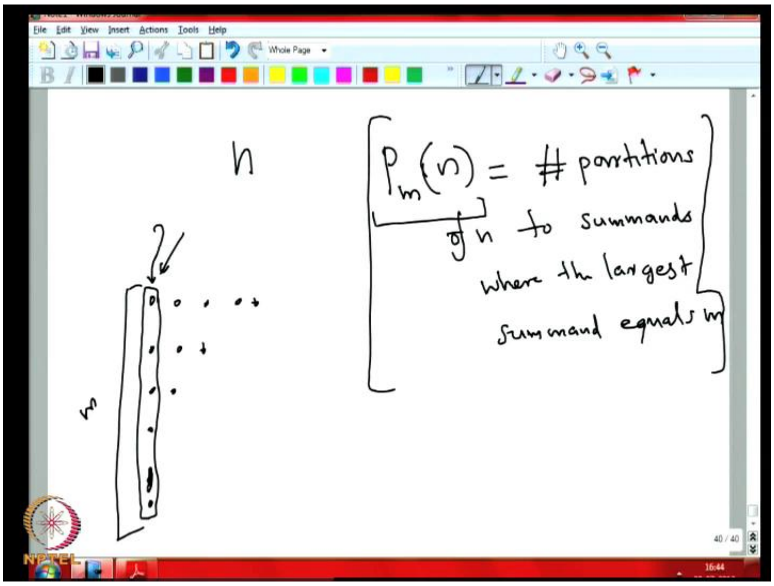

For instance first is familiar given n we are talking of p m of n; that means a number of partitions of n into m summands. Now we are saying this number is going to be equal to the number of partitions of n into summands where the largest summand equals m, why is it so? Because if you consider a partition of n into m summands it will look like this some row, first row, then another second row, and then third row, then fourth row. So, there will be m rows, right. This will be m rows because m summands are there; always such a partition will have m rows. Now we consider the conjugate partition, and now the first row will correspond to the first column, right.

This will be always exactly equal to m and obviously, this is the largest summand in it. So, consider the set of partitions of n where there are exactly m summands, and consider the set of partitions of n where the biggest summand is equal to m. There is a one-to-one correspondence between the two because take a Ferrer's diagram for one from these partitions set of partitions, take a partitions from the set of partitions where the number of parts are equal to m and then take the conjugate of that; that will be a partition of n where the biggest part equals m, because the column the first column will have m things in it, that becomes the first row; that is the biggest part.

And conversely see if you take a partition way of n where the biggest part is equal to m and consider the conjugate of that. It will become a partition of m where the number of parts is exactly equal to m, right, because the row becomes a column here, right. So, you can see that that is a one-to-one map; it is almost obvious from the picture, let us not waste too much on that. So, you should verify the bisection carefully, right. So, this is like from using the conjugate idea, right; the fact that given the Ferrer's diagram of partition of n and if you consider the conjugate of that partition, we will end up with another partition of n and the first look this is one theorem we can write now, right.

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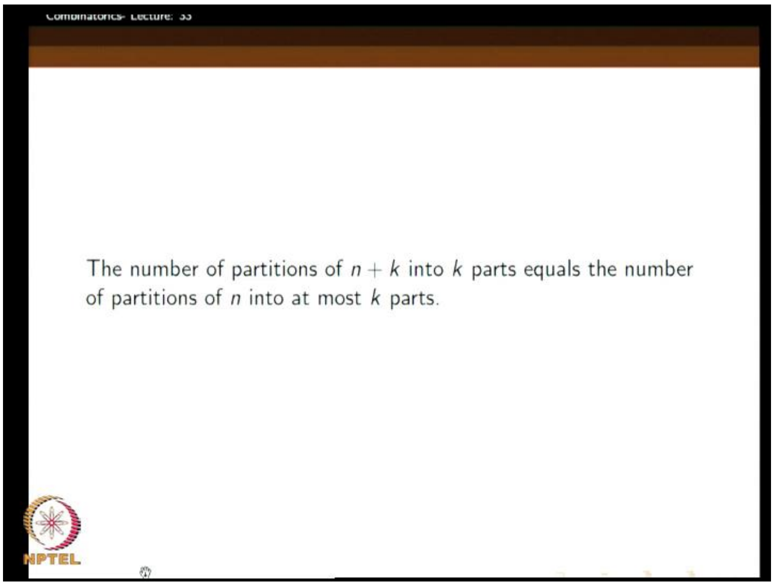

Now let us look at another one. The number of partitions of n plus k into k parts equals the number of partitions of n into at most k parts.

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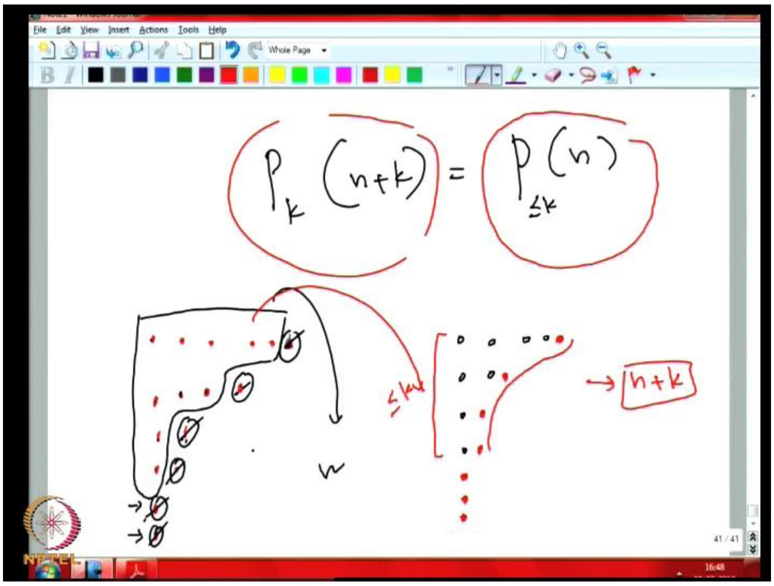

So, it is like p k of n plus k number of partitions of n plus k into exactly k parts; this is equal to number of partitions of n into at most k parts, how do we do that? So, we consider the set of partitions of n pus k first; we will show bisection to the set of partitions of n into at most k parts. So, if you have a partition of n plus k, say something like this, right, and it is exactly k parts; that means the number of rows are equal to k.

Now what we can do is we remove one from here, one from here, one from here, one from here, one from here, one from here. The last from each row will be there; not that some other rows may contain just on the other, the lowest rows may contain only one.

So, will just remove them like this, right. So, k things are gone. So, the remaining is a Ferrer's diagram; this is a Ferrer's diagram for n a partition of n, and how many parts will be there? It will never be more than k, it will at most k, right. So, given a partition of n plus k into exactly k parts by this transformation, we can get a partition of n into at most k parts. Conversely if you get a partition of n into at most k parts, considering Ferrer's diagram, say, something like this, right. For instance we can consider this particular partition.

Now what you can do is you add one; see add one, say, here at the end of every row. So, clearly it will be a new Ferrer's diagram which corresponds to see for instance what I do is suppose see I know that number of rows here is at most k less than equal to k. So, we will add one at each of them, but if this was strictly less than k; suppose this many more rows are required to make it k. So, then I will add one, one, one here; that means what I am doing is I am adding exactly k new dots to the Ferrer's diagram.

If there are not enough rows I will have to introduce this single dot rows in the lowest parts, right. So, total k things will be added; that means we will get a Ferrer's diagram for a n plus k by this operation, right, and it will have exactly k rows in it; it will have exactly k rows in it, and you can see that there is a bisection because if this got mapped into this by the reverse operation will get back this, right. So therefore, the number of partitions of n plus k into exactly k parts is equal to the number of partitions of n into at most k parts.

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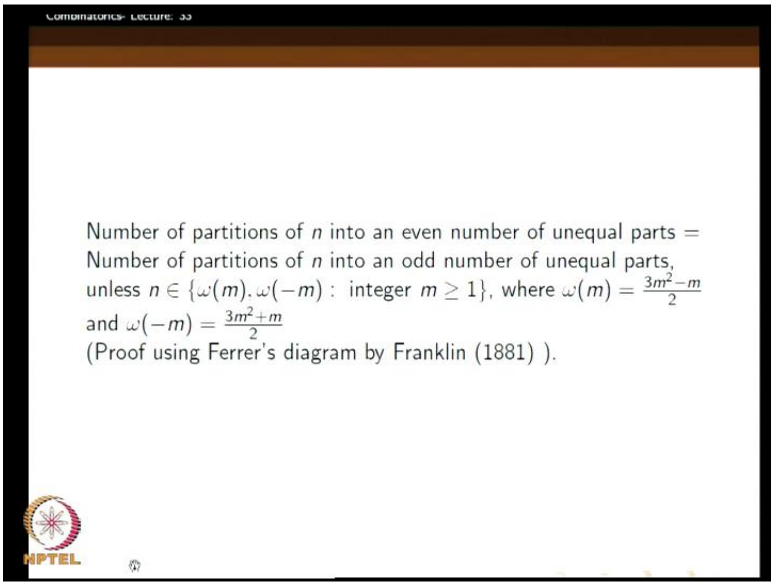

Now the next one is a little more nontrivial, let us look at it. So, this again we have to introduce what this about. The number of partitions of n into even number of unequal parts distinct parts; that is equal to number of partitions of n into an odd number of unequal part, unless n has some special form; this is what it says. So, let me introduce what we are looking for now.

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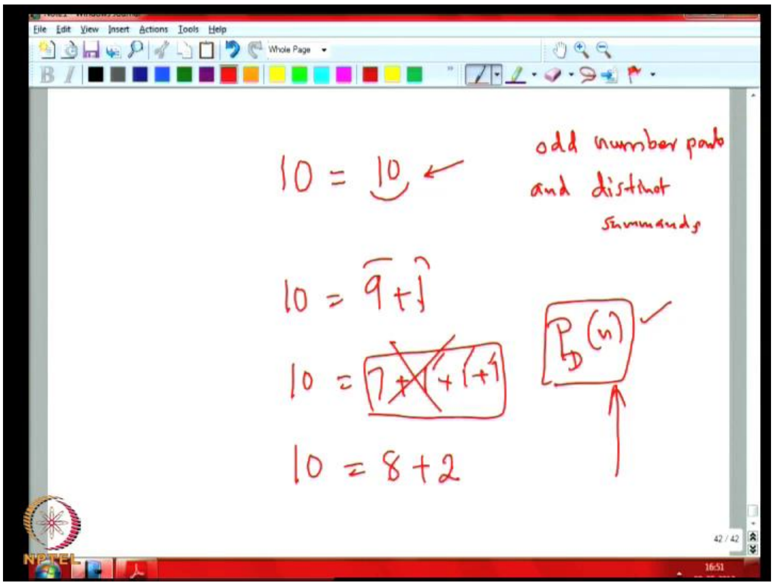

So, for instance, again take the example of 10 see 10. This is a partition of 10 into just one summand, and they all distinct of course one summand has to be distinct. So, this is an odd number of parts; there are two properties for this partition, odd number of parts and distinct summands, right. On the other hand if you consider 10 equal to 9 plus 1; this is again the summands are distinct, but the number of parts is even. So, that goes to the other kind, right. So, 10 equal to 7 plus 1 plus 1 plus 1; this is not even distinct. So, here 1 1 1, these are all same. So, this we do not even consider, right.

We are not interested in it. 10 equal to 8 plus 2, this is fine. There are two distinct summands, and this will be even number of summands, right. So, you remember when we counted p D of n we were looking at the partitions of n where the summands are all different. Now we are grouping them into two categories where the number of. So, we are only interested in the partitions which will contribute to this, but then we can put them into two categories, namely those which of an odd number of summands, those which have an even number of summands. We are saying that they happen to be equal for most m.

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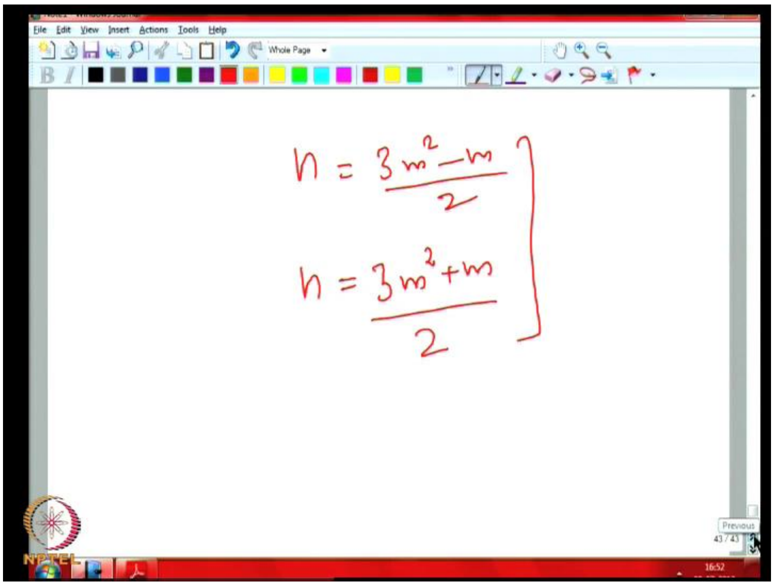

But it need not be equal in some cases; if n is of the form 3 m square minus m by 2, then it would not be equal. One of them will be bigger; that means the number of odd parts. So, the number of partitions distinct with distinct parts and odd number of summands can be more than the other type; that means the partitions with distinct summands are having an even number of summands, or the other way may be the number of partitions with distinct parts and even number of summands may be more.

But whichever is more it will be more only by one; the difference between the two numbers will be only one, that is what it is, and another type of number for instance here also this is true n equal to 3 m square plus m by 2. In this case also we can say that it would not be equal but whichever is bigger that will be only bigger by one. So, we can take small examples and see for instance.

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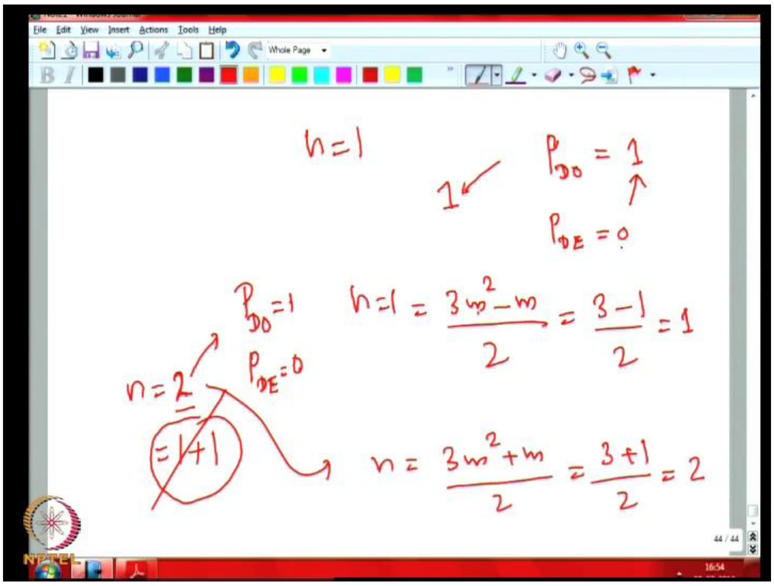

See for 1, n equal to 1. So, it is very difficult to try out big examples. So, n equal to 1, the number of partitions itself for instance it is only 1, right, just 1, but this is distinct summand, but then only odd number of summands. So, the first category the P D and O let us say P D and O, right; that is equal to 1. What about p D and E that is equal to 0? This is 1 more than 0. Now I will say that this n equal to 1 is of the type 3 m square minus m by 2, why? If you put m equal to 1 we will get that is 3 minus 1 by 2 is equal to 1; that is why it is happening.

Now we can take n equal to 2, what is happening? Distinct summands the odd number of sum, so the other partition is just 1 plus 1, right; here this is not even distinct. So, this is not useful; there is only one of them, but then this is again P D O equal to 1 and P D E equal to 0, but I am telling that n is again of the form 3 m square plus m by 2 this time, why? Because if we put m equal to 1 we get 3 plus 1 by 2 that is 4 by 2 is 2, see, but the difference between P D O and P D E is only one 1 here 0 here; here also its 1 and 0 right.

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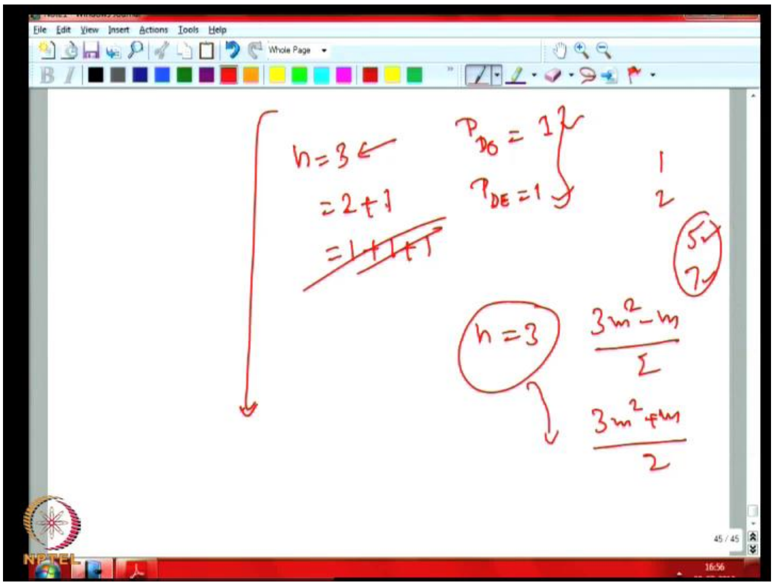

But on the other hand if you go to n equal to 3, now what are the parts? 2 plus 1 and 1 plus 1 plus 1 three summands, but then this is not distinct. So we do not have to consider this. So, here this distinct summand, this is distinct summand, but this contribute to our P D O, because even number of summands just one. Here this is odd number of summands the P D E equal to 1, 1 and 1 equal. So, you can see that n equal to 3 will not be of the form 3 m square minus m by 2 or 3 m square plus m by 2; it is not of the form.

We can try for instance when I put 1 here I got 1, when I put 1 here I got 2. If I get 2 here that will be 12 minus 2 that is 10 by 2, that is only 5, right; sorry 4 into 3 12 minus 2 that is 10 by 2 - 5. Here if I put 12 plus 2 by 2 7, right. So, n equal to 3 is not of that form, right, but then in that case, they are equal. Now we can try with 4, 5, 6, 7, etcetera and check whether what we are claiming is correct or not, right; that I will leave it to you. In the next class I will continue with proof of this statement using Ferrer's diagram. So, this is bit more nontrivial than the kind the proofs we have done earlier. I will continue in the next class.