Combinatorics

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Lecture - 33
Exponential Generating Functions - Part (2), Partition Number - Part (1)

Welcome to the thirty third lecture of combinatorics.

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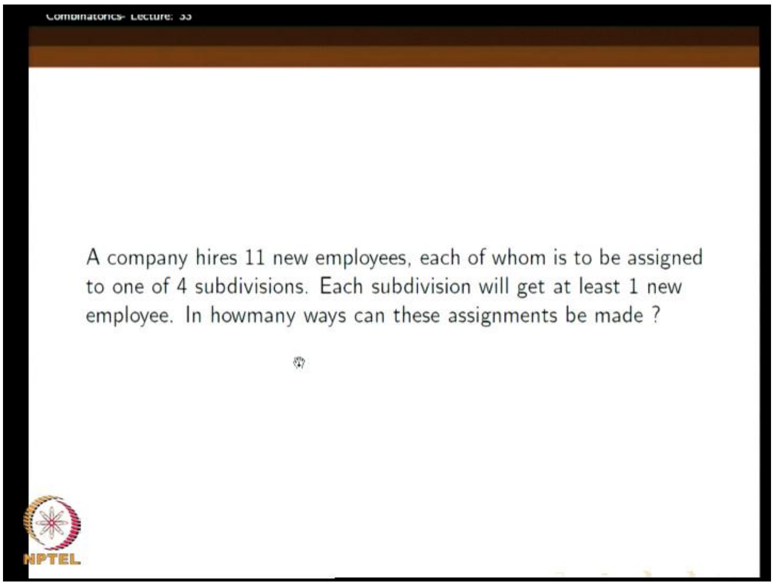

We continue with the last example for exponential generating functions we considered in the last class. This is the question – a company hires 11 new employees, each of whom is to be assigned to one of 4 subdivisions. Each subdivision will get at least 1 new employee – has to get. In how many ways can these assignments be made?

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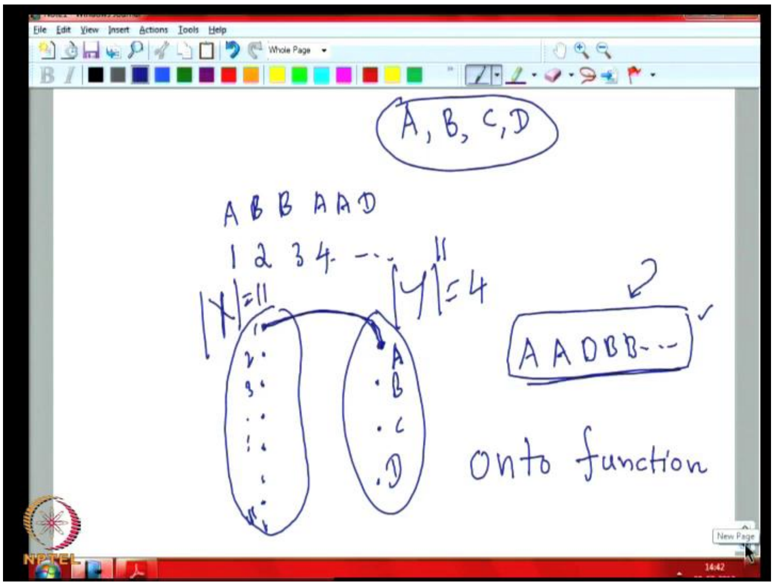

We just understood that, if you call the subdivisions as A, B, C and D and we say place… This is the first person we hired; the second, third and say 11. And here on top of the first person, we write down the subdivision to which he is assigned; say like this A B B A A D. So, that means first person goes to subdivision A; second person goes to subdivision B; third person goes to subdivision B and the fourth person goes to A and so on. Then we get a sequence of letters like A A D B B – something like that – eleven letters sequence I get. This corresponds to an assignment. And we are asked to count how many such strings using these four letters – A, B, C, D can be made such that each letter appears at least once. What does it mean? It means each subdivision gets at least one new person – newly hired person. So, this is the question.

We are familiar with another question of this type namely, the number of onto functions from the set say X to Y; where, the Y consists of A, B, C and D. And this X consists of the eleven people we hired 1, 2, 3, up to 11. Now, the condition is that, there should be a pre-image. So, I am… Saying that the assignment of persons to the subdivisions can be seen as a function defined on X to Y. Why? Because each person is assigned some subdivision and no person can be assigned to more than one subdivision. This is the property the function needs. So, this is just a matter of a function – defining a function from X to Y. But the extra property is that, each subdivision has to get one person; that means each of A, B, C and D should have a pre-image; that means we are talking about onto functions; we are familiar with these things; we have discussed the onto functions when we studied the inclusion-exclusion principle. So, we had used the principle of inclusion and exclusion to count the number of onto functions from X to Y given in the cardinality of X is equal to say n; cardinality of Y is equal to k. We know how to count it. If you have forgotten that, I would suggest you to go back and check it once again. So, it was… But then it was done using this idea of the principle of inclusion and exclusion. Now, we will use it. Use exponential generating functions to get the same idea. We will do it for this X equal to 11 and Y equal to 4 case. So, you can easily extend it to the general case. So, I would not waste much time on that. So, this is the thing.

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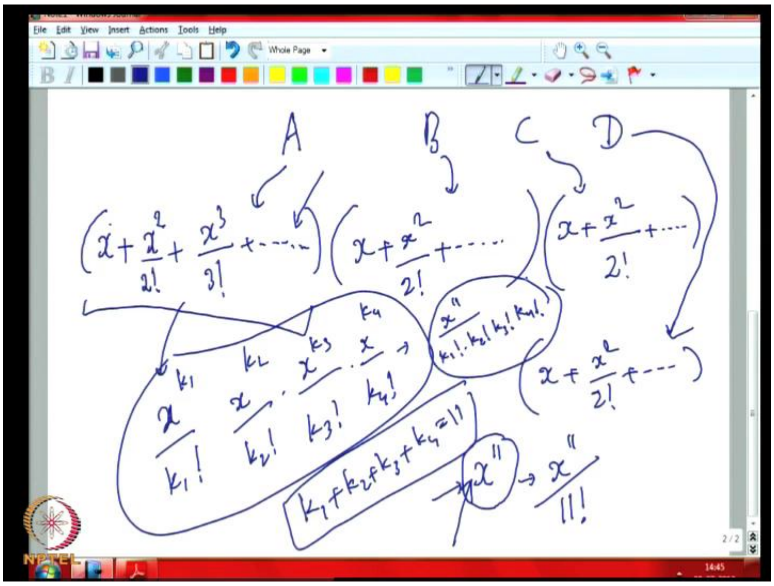

Now, what do I do? We will say for each subdivision – A, B, C and D, we will introduce a function like this x plus x square by 2 factorial plus x cube by 3 factorial plus… So, this is for A. And for B also, the same thing – x plus x square by 2 factorial plus… So, this thing. And C – x plus x square by 2 factorial… So, this is the way we do. And for D also, x plus x square by 2 factorial. So, for each one, we get one term.

So, what does this mean? This means to A – that to the subdivision A, we can either select 1 new person or 2 new persons, 3 new persons; up to 11 we can select. We can select more if possible, if there are, because what we are planning to do is to look at the coefficient of x raise to 11 in the product. So, if you are looking for the coefficient of x raise to 11, x raise to 12 onwards from each term is not going to contribute to that. Therefore, we can just add it; it is not going to affect the answer, because we are only interested in the coefficient of x raise to 11 in the product. So, we can put an infinite series rather than just stopping at x raise to 11 by 11 factorial.

Similarly, here x by x square by 2 factorial; everything. So, you see finally, we will look at x raise to 11. x raise to 11 can come from… Say you can take x raise to k 1 by k 1 factorial from the first term; say x raise to k 2 by k 2 factorial in the second term; and x raise to k 3 by k 3 factorial in the third term; and x raise to some k 4 by k 4 factorial in the fourth term; if you remember this discussion such that k 1 plus k 2 plus k 3 plus k 4 is equal to 11.

This is the way we can make an x raise to 11. This will correspond to the subdivision A getting k 1 of the 11 new employees; subdivision B getting k 2 of the 11 new employees; and subdivision C getting k 3 of the 11 new employees; and subdivision D getting k 4 of the 11 new employees. Now, we will look for the coefficient of x raise to… not actually the x raise to 11; we will look for the coefficient of x raise to 11 by 11 factorial. Then what will happen? In this product, we will have x raise to 11 by this k 1 factorial into k 2 factorial into k 3 factorial into k 4 factorial. This is what here will have.

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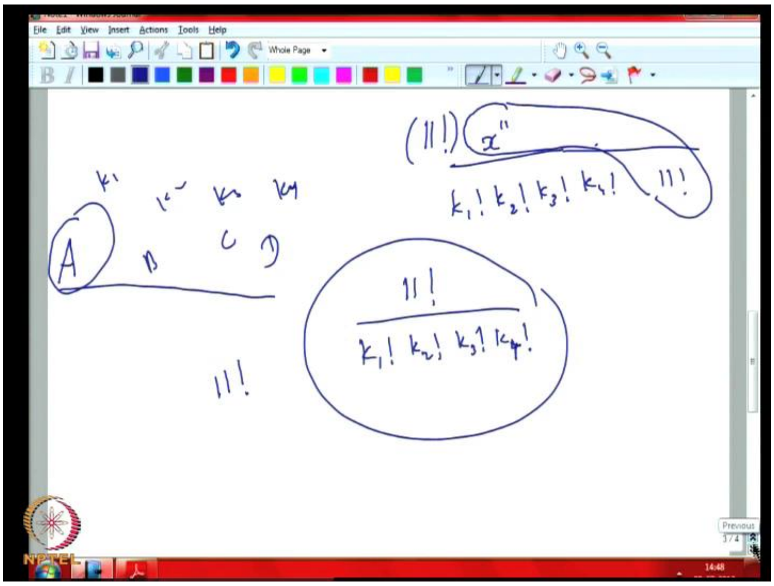

We will have x raise to 11 by k 1 factorial into k 2 factorial into k 3 factorial into k 4 factorial when we multiply it out. But since we are looking for the coefficient of x raise to 11 by 11 factorial, we will introduce 11 factorial in the denominator. So, to balance it, we will introduce 11 factorial in the numerator. So, the co-efficient of x raise to 11 by 11 factorial will be 11 factorial by this k 1 factorial k 2 factorial into k 3 factorial into k 4 factorial. So, this is actually the number of ways we can assign see k 1 of the newly employed people to A; and k 2 of the newly employed people to B; and k 3 of the newly employed people to C; and k 4 of the newly employed people to D. To understand this thing, what you can do is you can remember that, interpretation that we are actually looking for the strings of length 11; where… So, it is…

The string is composed of A’s, B’s, C’s and D’s. Here we are talking about how many such strings are there, where there are k 1 A’s in it and k 2 B’s in it; and then k 3 C’s in it and k 4 D’s in it. Because there are 11 factorial ways of permuting; but because there are k 1… k 1 of them are same; and k 2 of them are same; and k 3 of them are same; and k 4 of them are same; we will get 11 factorial by k 1 factorial into k 2 factorial into k 3 factorial into k 4 factorial as the final answer. So, this is the same argument as before.

Now, we have to consider all possible splitting of 11 into various ways. And then all those things will be added up to get the coefficient of x raise to 11 by 11 factorial in the end. That in fact gives us the possible ways of getting these strings such that there is at least one A, there is at least one B, there is at least one C, at least one d. Therefore, how do we get this coefficient of x raise to 11 is the only thing. So, we know that; we are actually looking for the coefficient of x raise to 11 in this product and this product again.

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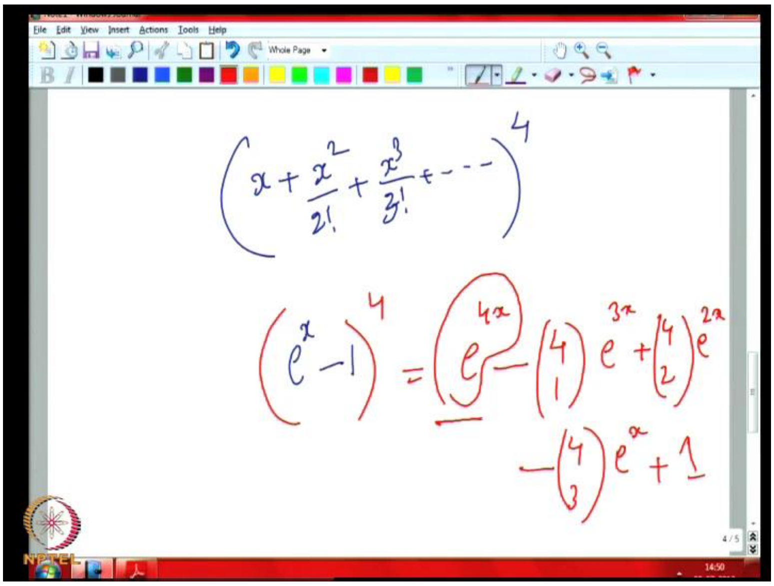

This again x plus x square by 2 factorial plus x cube by 3 factorial and so on whole power 4, because there are 4 of them. This is actually e raise to x minus 1, because if we had say 1 plus this; then it would be e raise to x. Therefore, this is e raise to x minus 1 to the power 4. This is what? e raise to 4x minus 4 choose 1 into e raise to power 3x plus 4 choose 2 into e raise to power 2x minus 4 choose 3 into e raise to power 1x. This is… And finally, plus 1. This is the way we just use the binomial theorem to expand it.

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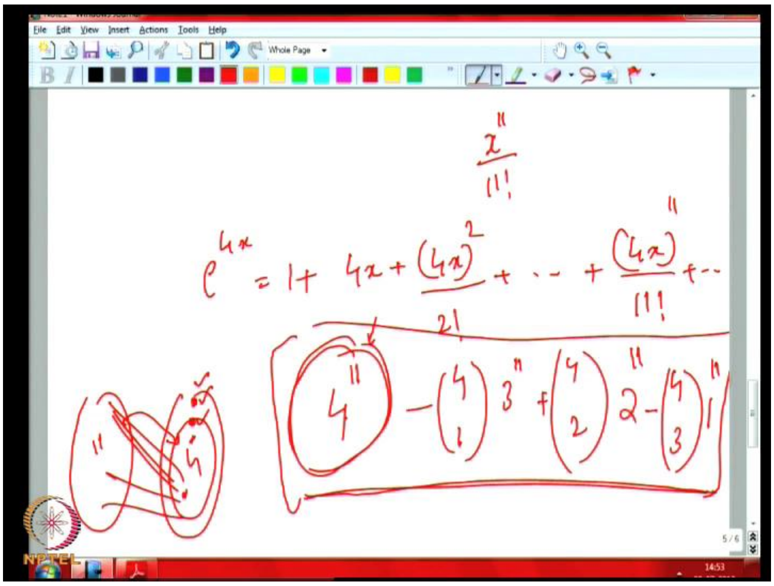

Now, what we are interested in is the co-efficient of x raise to 11 by 11 factorial in this. x raise to 11 by 11 factorial and this thing. Here we know the coefficient of x raise to… From this term, what will you get? Because that is like see e raise to 4x is like 1 plus 4x plus 4x whole square by 2 factorial plus like that. So, this will be 4x raise to 11 by 11 factorial for the…

For x raise to 11 by 11 factorial, the coefficient will be 4 raise to 11. So, that is one term. And then if you look for the next one namely, this one – minus 4 choose 1 into e raise to 3x. What would be the coefficient of x raise to 11 (( )) That would be 3 raise to 11 into 4 choose 1. That will be minus 4 choose 1 into 3 raise to 11. And then the next will be 4 choose 2 plus 4 choose 2 into 3 raise to 10, because this is what. Not 3 raise to 10; 2 raise to 11, because we are looking for this term – 4 choose 2 into e raise to 2x – 2 raise to 11. Here minus 4 choose 3 into just 1 raise to 11. That is just that – minus 4 choose 3 into 1 raise to 11.

And, here we will not get anything from the last term namely, this term. This does not have any power of x raise to 11. So, the eleventh power of x is not appearing here – from here at all. Therefore, this is the answer. You can remember that, this is exactly the number of onto functions also. When we are counting the onto functions what we use to do is we will count all the possible functions; that was 4 raise to 11, because from 11 sized set to a 4 sized set, how many possible functions can be there? This is this.

Then we will minus all those functions, which strictly avoid at least one element, so that one element can be picked up in 4 choose 1 ways and the remaining… So, that means 3 elements; we have to map to 3 elements. So, 3 is 3 raise to 11. But then we have to re-add. So, those things, which avoid 2 of them together; that means those things – 2 things to avoid are selected in 4 choose 2 each. And they are 2 raise to 11 possible functions.

Now, we have to minus again those functions, which avoid 3 things; that means they are the functions, which are coming… All of them are mapped to the same thing. That should be… It can be 4 choose 3 ways; that means 4 ways you can do this thing. And 1 raise to 11; that means once you select to which element you should map, there is only one function – 4 choose 3 into 1. So, this is the way it was done. But absolutely very different way, we have got the same answer here. By just writing down the generating function and then expanding it; noting that e raise to x minus 1; and then by expanding it, we got it.

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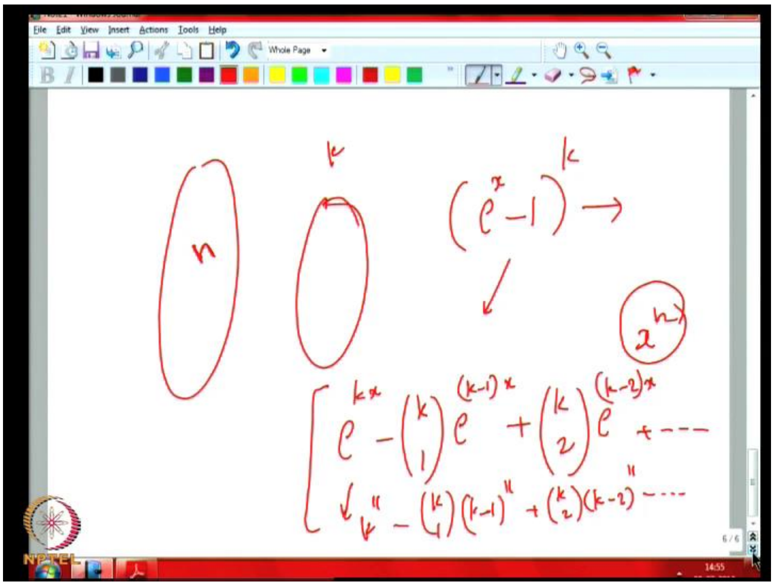

Note that if this was n and this was k; that means in our problem, there were n newly hired people and then k subdivisions. Then what will happen is that, we will have e raise to x minus 1 to the power k instead of e raise to x minus 1. In the expansion of this thing, we will be looking for the coefficient of x raise to n and then we know… So, this – when you expand it, this will be e raise to k x minus k choose 1 into e raise to k minus 1 x plus k choose 2 into e raise to k minus 2x and so on.

Now, we will get a similar… When you look for the coefficient of x raise to n; if from here you will get k raise to 11; here you will get k choose 1 into k minus 1 to the power 11 and so on; k minus 1 to the power 11. And then we will get k choose 2 into k minus 2 to the power 11 and so on. So, this alternating signs. This will be the answer for the general problem. But that I will leave to you to figure out. So, the question of counting the onto functions can be tackled this way also; that is what we are saying.

Now, we will move to a new topic, which we take up to study in the context of generating function, because the generating functions are used a lot in the study of this. But again this topic is an important topic in itself. Just because we are considering now, because we consider generating functions; and some generating functions will be discussed in this thing. But also other techniques will be discussed. This problem is about partition numbers.

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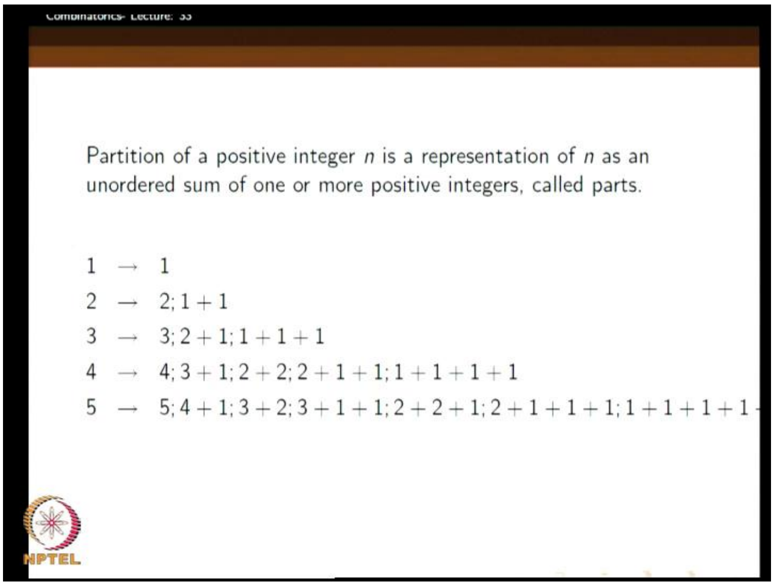

What is this partition numbers? Here we are given an integer – positive integer (( )) something greater than equal to 1. And then we want to split it into summands. But here the order of the summands is not important at all. We are asking this – how many ways we can split? For instance, 1 can be split in only one way – 1; 2 can be split as one summand – partition 2; or, you can write it as 1 plus 1. 3 for instance; 3 can be either one summand – partition 3; or, it can be a 2 summand partition – 2 plus 1. Note that, once you write 2 plus 1, we will not count 1 plus 2. See when we were looking for compositions; we did like that – 1 plus 2 and 2 plus 1; where, two different compositions of 3, if you remember that. But here we are considering unaltered cases to 2 plus 1 and 1 plus 2 are same here.

Now, another partition for 3 is 1 plus 1 plus 1. There are three different partitions for 3. So, the partition… So, we will say that, partition number is the number of different partitions for the given number in n. So, that is p of 1 is equal to 1; p of 2 is equal to 2. There are two here. And p of 3 equal to 3, because there are three partitions. What about p of 4? We can have one summand partition; it is just 4. How many two summand partitions? 4 can be written as 3 plus 1; then 1 plus 3 is not considered, because the order is unimportant. So, 2 plus 2 – it can be 2 plus 1 plus 1.

So, there are two summand partitions. And then 2 plus 1 plus 1. So, that is three summand partition. Now, it can be 1 plus 1 plus 1. There is a total 1, 2, 3, 4; 5 partitions are there; p of 4 is equal to 5. And what about 5? We can have one summand partition – 5 itself; two summand partition is 4 plus 1, 3 plus 2. Then the two of them. And then we have 3 summand partition – 3 plus 1 plus 1; 2 plus 2 plus 1. So, that is it. And then we have four summand partitions – 2 plus 1 plus 1 plus 1 plus 1. And then five summand partitions – 1 plus 1 plus 1 plus 1 plus 1. So, how many partitions are there? 1, 2, 3, 4, 5, 6; 7 partitions we can see here. 1, 2, 3, 4, 5, 6; 7 partitions; p of 5. Now, note that why this is different from the compensations; because the order is unimportant.

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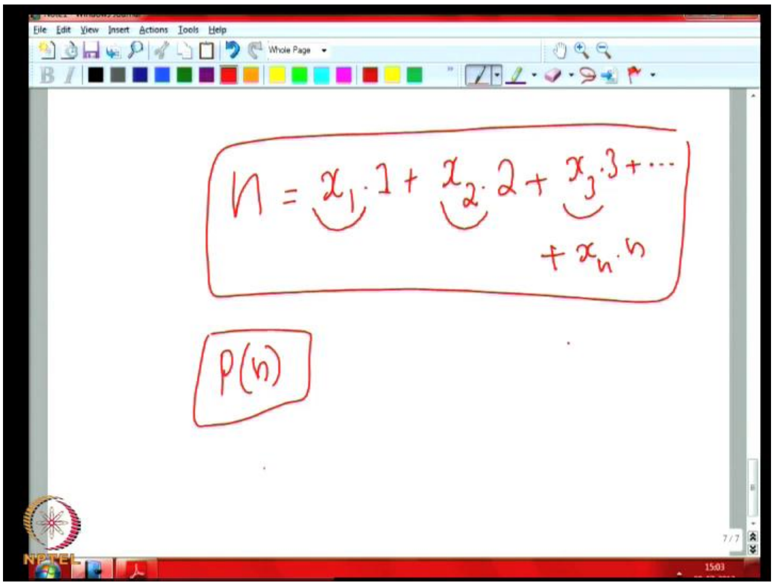

And, you see you can express this question like this – for the given n, we want to write it as x 1 plus x 2 plus x 3 plus something; even x n you can write. So, you can have n parts. We will try to see, because earlier, we had studied this kind of a question. This is not the question; I will just consider like this. We want to write n as a sum of so many partitions. So, we can write it… We are interested in knowing how many 1's are there. Say x 1 1's are there. So, x 1 into 1. Then how many 2's are there? x 2 into 2; x 2 2's are there. And how many 3's are there? x 3 into 3; like this. So, this x 4 into n. So, we want to split n like this. And we want to decide how many 1's are there; how many 2's are there; how many 3's are there. x n into n; how many n's are there? This will completely define the partition.

For instance, in the previous case; here in the case of 5, when I look at 5; that means there are zero 1's, zero 2's, zero 3's, zero 4's, but one 5. So, here there is zero 1's, zero 2's, zero 3's and one 4 and one 1. In this partition – 2 plus 1 plus 1 plus 1, there are three 1's and one 2 and zero 3's and zero 4's and zero 5's. Therefore, we have so many… Now, the solutions for x 1, x 2, x 3, x n, will give you a partition. And how many such solutions are possible? That would be the partition number p of n.

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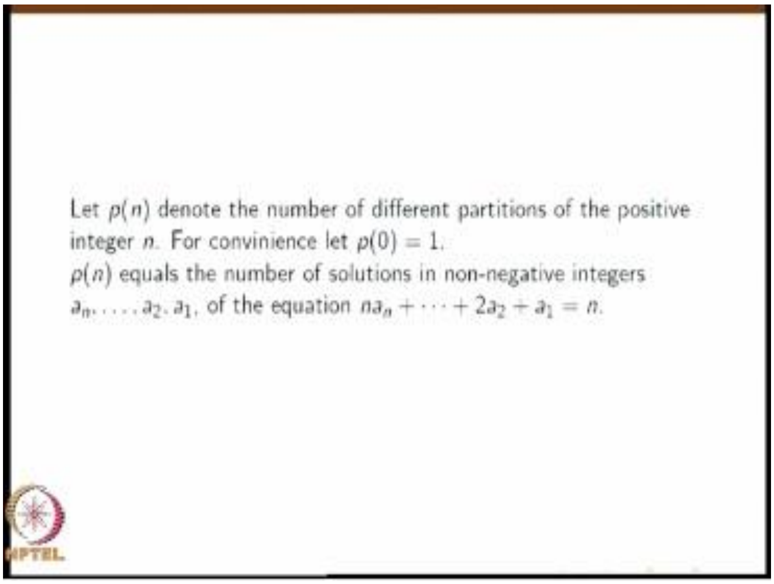

Now, let us see; p of n is the notation for partition number; denote the number of different partitions of the positive integer n. For convenience, let p of 0 is equal to 1, because we are actually not defining the partition; we do not know the partition number for 0, because we want 0 to be… We do not know means there is no meaning for that, because 0 and how many part… We cannot actually express it as a sum of positive integers. Should we write it as 0? No, because for convenience, we will say the p of 0 is equal to 1; p of n equals the number of solutions in non-negative integers. So, this is what I told – a 1… So, n times a n plus n minus 1 times a n minus 1 plus 2 times a 2 plus a 1 equal to 1.

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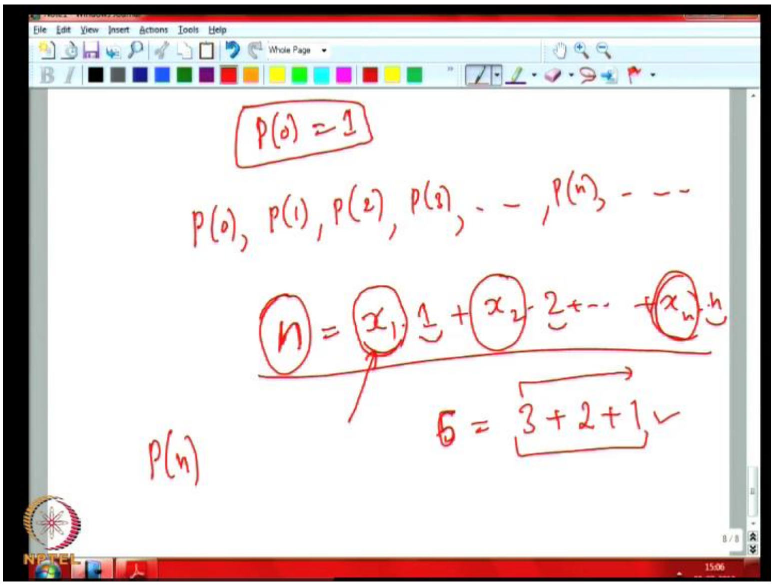

Again, now… We are asking, what is the generating function of this? We have a sequence here – p of 0 equal to 1; this is we are just defining it for convenience. Then p of 0, p of 1; p of 1 is the partition number for 1; p of 2, p of 3. This is the sequence; where, p of n is the partition number for n. How many partitions are possible for the given integer n? And as we have seen, any partition of n can be expressed by this equation x 1 into 1 plus x 2 into 2 plus x n into n.

And if you give non-negative integer values to x 1, x 2, x n; that will define a partition, because this many 1's, this many 2's, this many n's together forms n such that it together forms n. That we can write. We can always write the members of the partition in non-decreasing order. For instance, if 5 is equal to 3 plus say 2; say six is equal to 3 plus 2 plus 1. So, we can… Instead… Because the order is unimportant, we can always make sure that, we write it in decreasing order – non-increasing order.

First, all these x n n's will be listed; then x n minus 1, n minus 1's will be listed. See it is possible that x n is 0; that means no n's may be there. So, we can… Whatever is there… If 0… If a particular number is appearing only 0 times; that means not appearing, we would not list it. Now, if this is the equation and what… The number of partition that, p of n for a first two; the number of non-negative integer solutions for this x 1, x 2 and x n. How do we find it out? So, this kind of questions we have tackled before, because this is like making… For instance, we have say 1 rupee note, 2 rupee note and n rupee note.

Now, we want to make change for n. How many ways we can make change? That is the question using this 1 rupee note, 2 rupee note up to n rupee note.

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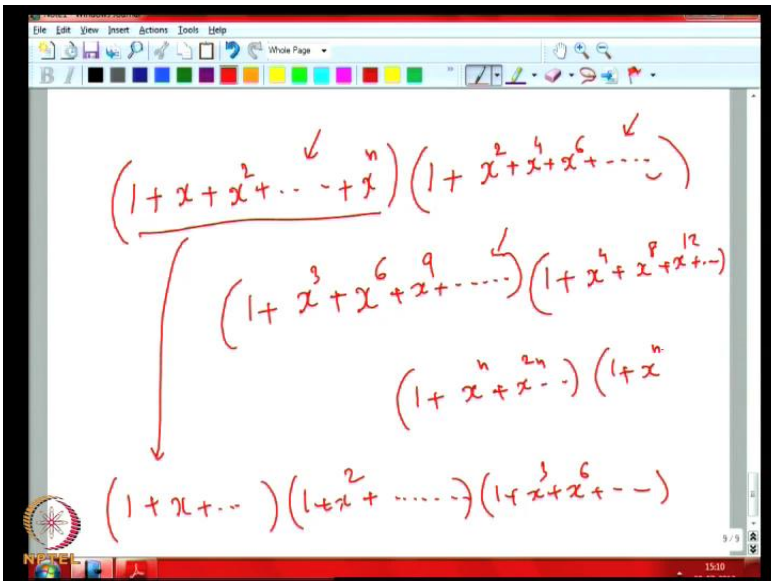

This we have seen using generating functions. We will have to decide first, how many 1's I will take. For that, we will write something like this – 1 plus x plus x square plus infinite series we will take from… If you plan to take k 1's, then you have to pickup x raise to k from this thing. The second term will tell us how many 2's I will take. This will be 1 plus x square plus x raise to 4 plus x raise to 6 plus and so on. And then this is 1 plus x cube plus x raise to 6 plus x raise to 9 plus… This will define the third term, which will decide how many 3's we will select. And this term – 1 plus x raise to 4 plus x raise to 8 plus x raise to 12 – multiples of 4, will decide how many 4's we will select and so on.

See in principle, we cannot… we can stop at this one – 1 plus x raise to n plus x raise to 2n plus x raise to 3n plus this one. We can stop that, because if you are only interested in the coefficient of x raise to n. Or, actually, how many partitions of n? And we are going to take the coefficient of x raise to n in the product. So, next term onwards, it is not going to make any contribution. So, we can as well add that. So, in principle, what we can do is we can just go… This can be taken up to x raise to n; (( )) then putting an infinite series. This can be taken up to (( )) whichever is the smallest; the multiple of 2, which is just below x raise to n. We can stop there. And everywhere we can do that.

Then, in this case, we can just stop at this itself, because that is all we need, because we are only interested in how many ways we can produce the x raise to n in the product by taking something from here, something from here and something here; because if we are taking from x raise to n plus 1 from this product, we are not going to contribute to the coefficient of x raise to n in the product at all.

But then that is the same reason. That reason tells us that, we can actually make the full term, because it is convenient to take the power series instead of this polynomial, because the power series is easier to handle, because the infinite series has a close form, which is much more easier to handle than this. So, we can write it as this. And then 1 plus x square – instead of stopping at just before n, this biggest multiple of 2 below n, we will take the full terms like this – 1 plus x cube plus x raise to 6 and so on. And then here also, we need not… See in this thing, instead of stopping at x raise to n, we can actually add all the things – x raise 2n and so on. And also, we can go for the next terms; where x raise to n plus 1 and so on.

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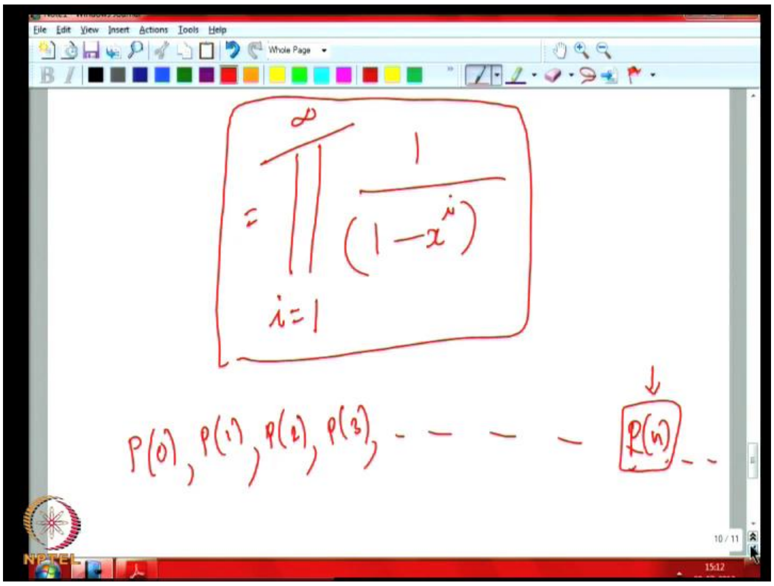

What we get is this happens to be the product of pi equal to 1 to infinity 1 by 1 minus x raise to i. Why is it so? Because 1 by… Put i equal to 1. So, 1 by 1 minus x. That will happen to be (Refer Slide Time: 32:33) this – 1 plus x plus x square plus this thing. This we know. So, this is infinite series, if you do not stop it here. This is the infinite series. This will correspond to this one – 1 plus 1 minus x square. And then the next one will correspond to 1 by 1 minus x cube and so on. And here 1… We can go on. We do not have to stop with this last term – 1 plus x raise to n plus x raise to 2n and so on; we can also go beyond n, because these are anyway not going to contribute to the coefficient of x raise to n, because from here onwards, we will have to take this first time – 1.

There is no other way, because otherwise, if once you take x raise to n plus 1, we are above x raise to n; and that is not going to contribute to the coefficient of x raise to n. Therefore, this is the generating function for the sequence p of 0, p of 1, p of 2, p of 3, p of 4, because now, in this product, we can decide any n. And then we can read out the coefficient of x raise to n in it. That will correspond to p n. That is what we have seen. But unfortunately we can just write the generating function, but we cannot do much more than that. We cannot really infer much about the coefficient of x raise to n in it, because this is a difficult one. But still knowing this thing will help us to tackle some questions about partitions; where, when we assume some properties for the partitions.

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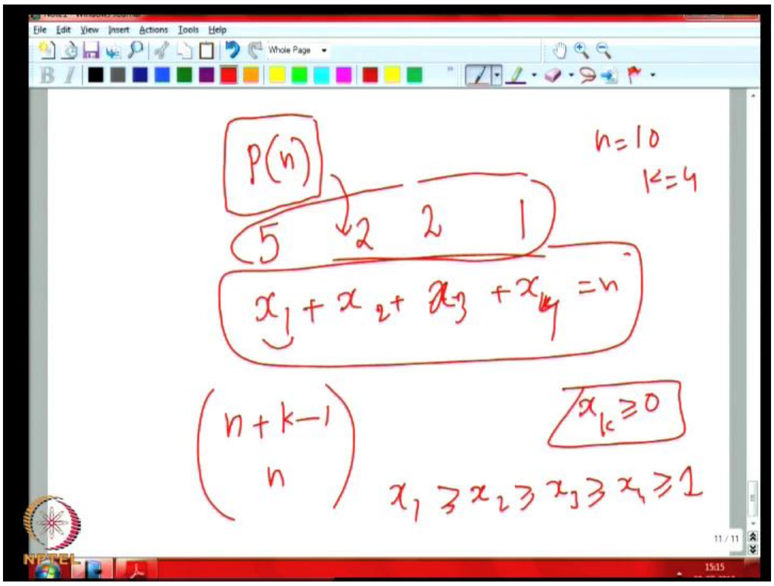

This p of n refers to how many partitions are possible for n without any restrictions. We just allow all kinds of partitions. But now, we will consider special type of partitions, where we have some extra restrictions, which are the possible restrictions?

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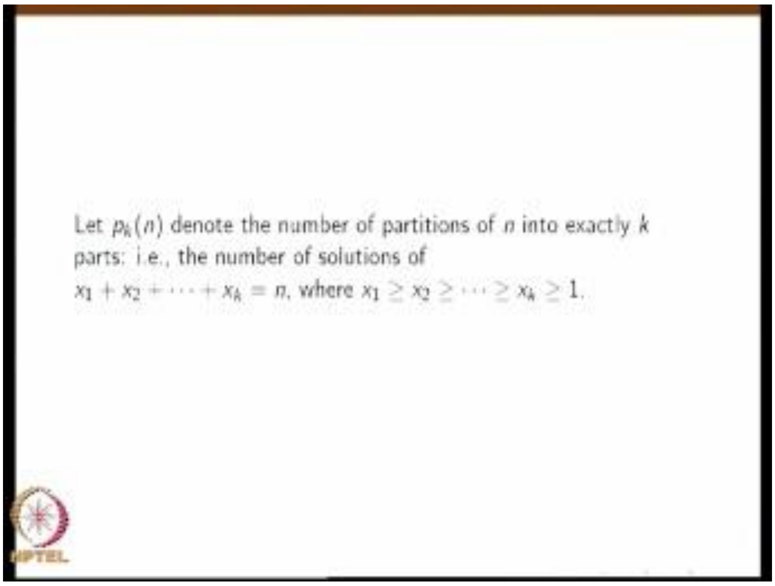

First natural, most natural restriction we are considering is when we insist that we need exactly k parts in it; exactly k parts in it. So, it denotes the number of partitions of n into exactly k parts. The number of solutions of… So, definitely, it is the number of solutions of x 1 plus x 2 plus x k equal to n. But here we have this condition that x 1 is greater than equal to x 2 is greater… So, we cannot consider x 1 then. The same numbers we cannot permute here in this thing. So, if you remember, this kind of an equation we had tackled before; for instance, x 1 plus x 2 plus x k equal to n. How many solutions are there for this thing? With x k greater than equal to 0, we had n; we had answered. So, we had seen that this is n plus k minus 1 choose n; k minus 1 choose n.

But, there the difference was that, we were not bothered about the order. For instance, if n was equal to 10 and k equal to say 4; we can split 1, 2, say… 5, 7 plus 1 plus 2. But then if I had given 1 here and 5 here, this would be considered a different solution. But in the case of partition, we cannot do that. So, this has to give 5 to get the highest number. This has to give the next lowest and this has to get the next lowest and this has to give… In the sense, we are saying that, permuting the same values are not allowed. We have to consider… The order is unimportant here.

Therefore, we will always assume that, x 1 is the greatest; x 2 is the next greatest; and x 3 is the next greatest; and x 4 is the next greater. On top of that, we had this condition here that, it is greater than equal to 1, because we want exactly k parts in n; we are not allowing some of them to be 0, because if it is zero, that means that number of parts have become k minus 1, because when you write it down, the 0's are not written down. So, we only see three parts though we have considered the solution of this thing.

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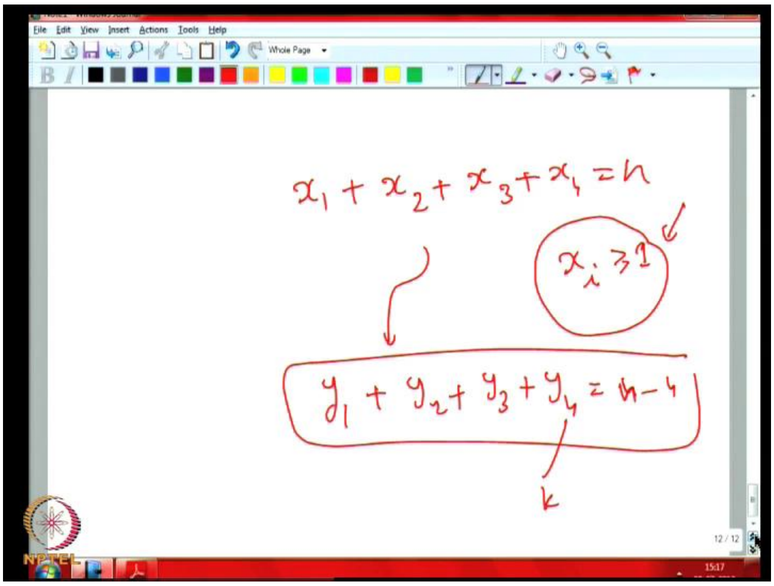

But, that would not be a problem, because we could have formulated our earlier problem also like this – x 1 plus x 2 plus x 3 equal to x 4 equal to n. We could have told that, I want x i greater than equal to 1. That would have converted the problem to y 1 plus y 2 plus y 3 plus y 4 equal to n minus 4 kind of problem. And then we could have solved it to get some n minus 1 choose k minus 1; where, this is k and this is n.

But that is not the issue here; the order is more important. Whether we are not allowing 0; that is ok; we could have been included that condition in this earlier problem also. But in the earlier problem, we were giving importance to the order whether x 1 gets some value and x 2 gets another value. If we stop the value, that gave us a different solution. Here stopping the values will not give us a different solution. So, to make sure that, that does not happen, we impose the extra condition that x 1 is greater than equal to x 2 is greater than equal to x 3 is greater than equal to x 4; and each x i is greater than equal to 1. So, that is the difference between the earlier problem and this one.

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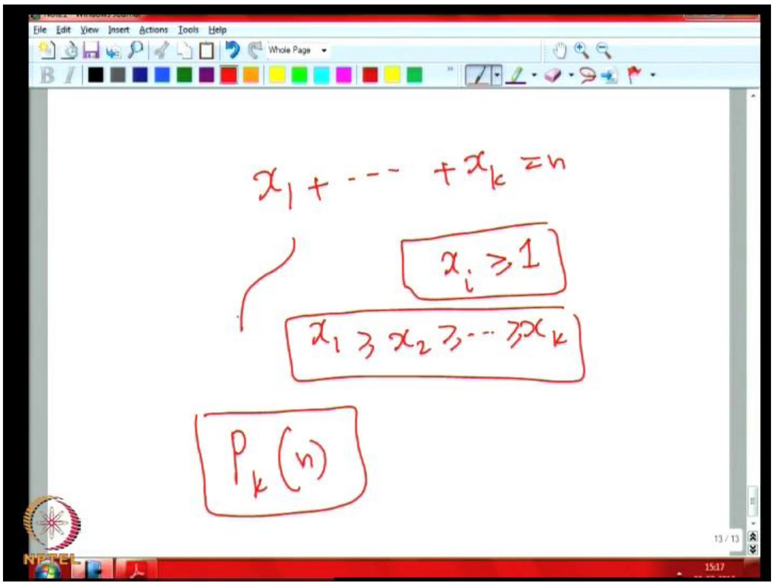

Now, the number of solutions for this equation with x k equal to n and each x i greater than equal to 1. So, this is… And x 1 greater than equal to x 2 greater than equal to x k. This number of solution is written as p k of n. But it is just that we want to get k parts in the partition. That is extra restriction we have put. Now, if you want to consider… Or else, if you cannot to get some idea about this thing; we can try to get a lower bond and upper bond for this thing.

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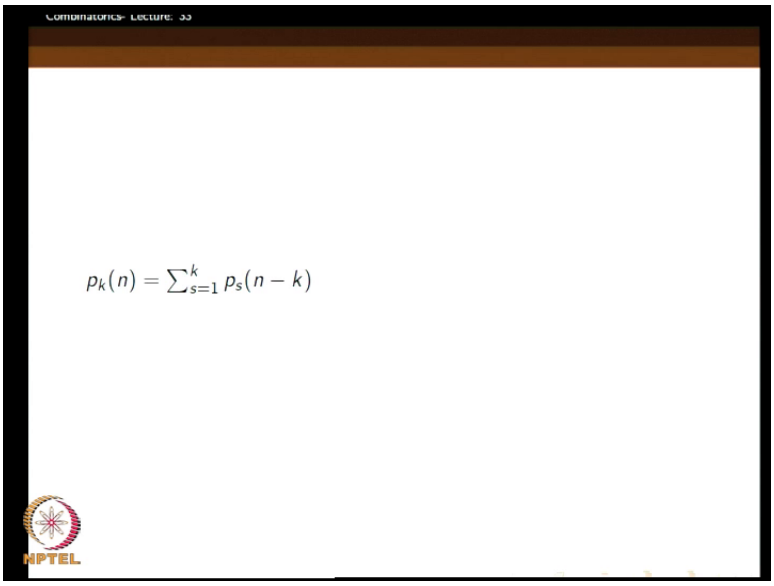

Before that, we will just consider some recurrence for this p k of n. First one is p k of n is equal to s equal to 1 to k p s of n minus k.

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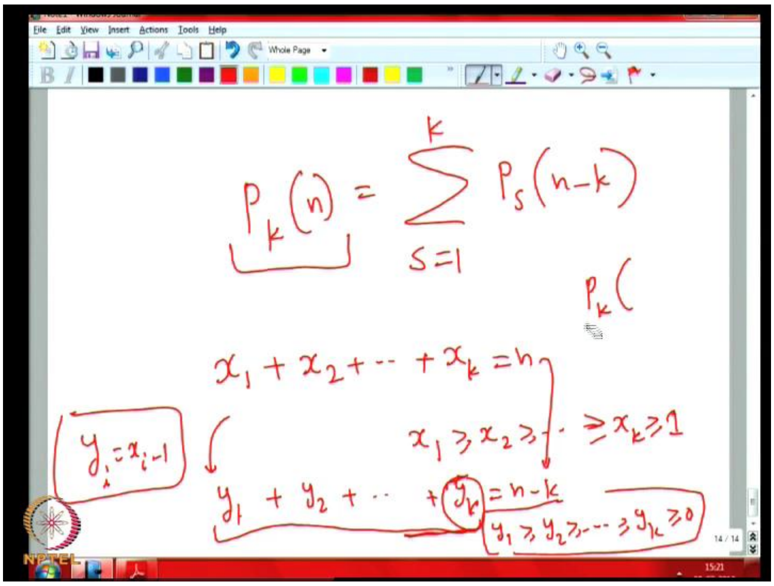

p k of n is equal to sigma of s equal to 1 to k p s of n minus k. How does this happen? This is because as we have mentioned, we are looking for the solution of x 1 plus x 2 plus x k equal to n with the condition that, x 1 greater than equal to x 2 greater than equal to x k greater than equal to 1. Now, we could have converted it to another equation – y 1 plus y 2 plus y k is equal to n minus k, if you define y i is equal to x i minus 1. The only thing is this will still be valid – y 1 greater than equal to y 2 greater than equal to y k. But now, we have to put this as greater than equal to 0; that is the thing. What is the good thing about it? The good thing is that, now, n has become n minus k. So, that is a smaller number than n.

But then if I cannot claim that this is the number of solutions for the second equation – that y 1 plus y 2 plus y k equal to n minus k is actually p k of n minus k; that is not. Why? Because some of this y i's, the last ones – y k, y k minus 1, y k minus 2, etcetera, can be 0. You remember this is the order; y 1 is the greatest; y 2 is the greatest; but then some 0's can be appear in the end. Earlier, we were not allowing 0. If you are allowing 0's, then we cannot say that, the number of solutions are p k of n. But then we can collect each set of solution and see how many 0's are in it. If there are no 0's, then we know that it is this solution of…The number of solutions of that sort is p k of n minus k.

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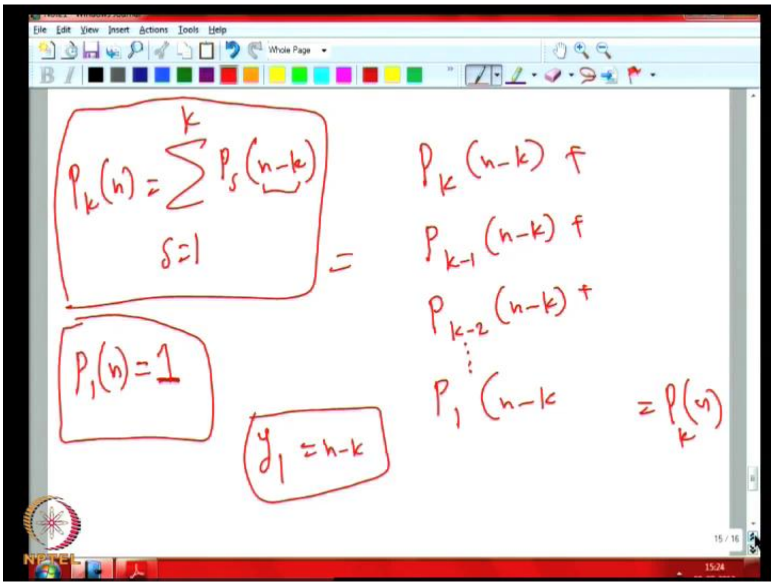

p k of n minus k; but then there can be one 0 in the end; that means this y k can be 0; that means our equation is y 1 plus y 2 plus y k minus 1 equal to n minus k and up to… From y 1 to y k minus 1, all of them are greater than equal to 1. Therefore, that corresponds to… Such number of solutions correspond to p k minus 1 of n minus k. If there were two 0's, that means y k and y k minus 1 were both zeroes and all others are nonzeroes. The numbers of such solutions corresponds to p k minus 2 of n minus k, because y 1 plus up to y k minus 2 is adding up to n minus k here with each of them being at least 1.

And y 1 greater than equal to y 2 greater than equal to… Similarly… All except the first one can be zero; that means y 1 equal to n minus k may be our equation. So, this corresponds to p 1 of n minus k among all these things. We sum up all these things. This is what we have written. When you sum these things, we will get what? p k of n. Therefore, p k of n is equal to sigma s equal to 1 to k p s of n minus k. This was the thing.

You should note that, this recurrence relation is useful, because here we are using n minus k as the parameter; that is the smaller number than n. Though we have several terms, we have actually k terms here including p 1 of n minus k, p 2 of n minus k up to p k of n minus k. But for n minus k, assuming that we know all the values, then we can sum up and get the value for p k of n. We cannot use this formula to evaluate p 1 of n. p one of n will be just… p 1 of n from this formula will come to be p 1 of n minus… p 1 of n has to be, because it is like how many partitions are there for n. If we look at the meaning of p 1 of n; it says how many partitions are there for n with just one summand in it. That we know the answer is 1. But in this formula if you put; and what happens is it will look like this.

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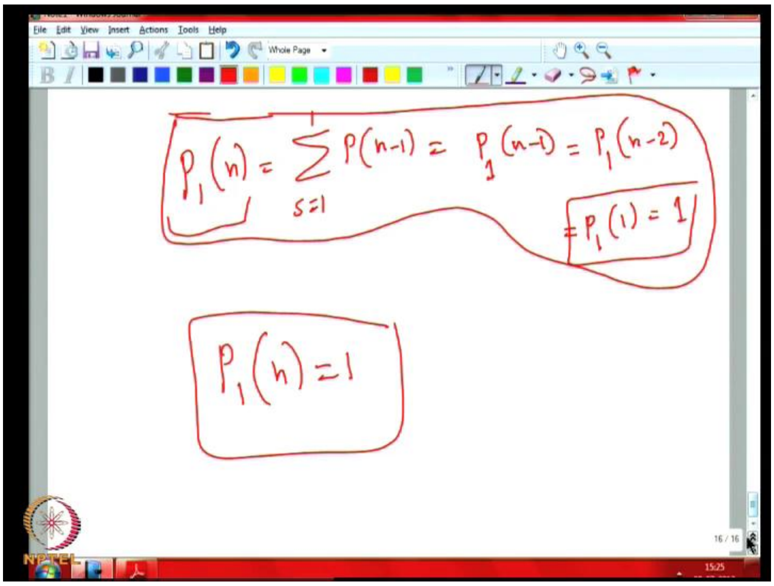

p 1 of n is equal to sigma s equal to 1 to 1 only – p of n minus 1; which is essentially just p 1 of n minus 1. So, this will… Again, we can do. So, that is p 1 of n minus 2 and so on. At some point of time, we have to reach p 1 of 1. So, we have to use some initial condition like this first if you want to solve it like this; or otherwise, we can just notice that, p 1 of n equal to 1. Do not apply the recurrence relation for the case k equal to 1. So, use it for (Refer Slide Time: 47:08) k equal to 2 onwards.

So, this is one recurrence relation we can use for p k of n. So, the… Actually, we have not written down any such recurrence relation for p of n. But if you know p k of n for values from k equal to 1 to n; then definitely, we can sum up all of them and get p of n, because p of n is all the partitions of n. Now, any partition if you take; it can have one summand or two summands or up to summands. So, we can group them according to the number of summands and we can sum up the cardinality of each such collection and we get the total number. So, in that way, we can use p k of n to get the final value of p of n.

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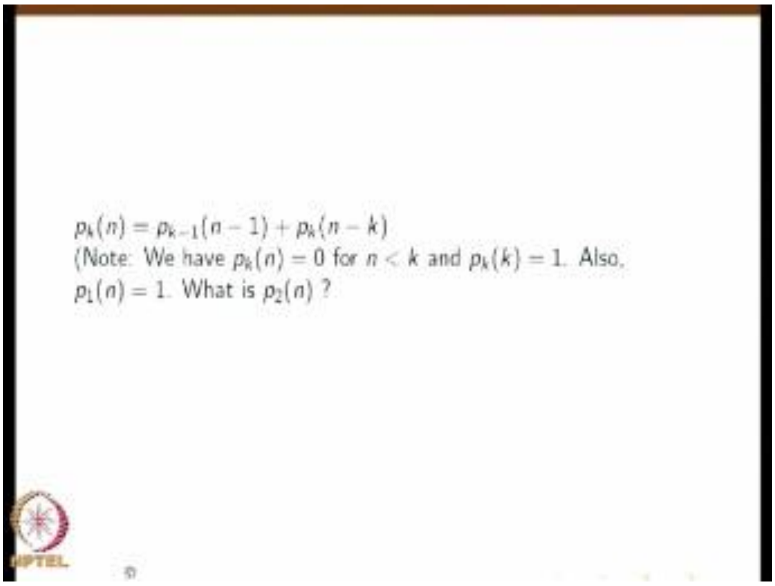

Now, let us look at another recurrence relation for p k of n. p k of n is equal to p k minus 1 of n minus 1 plus p k of n minus k.

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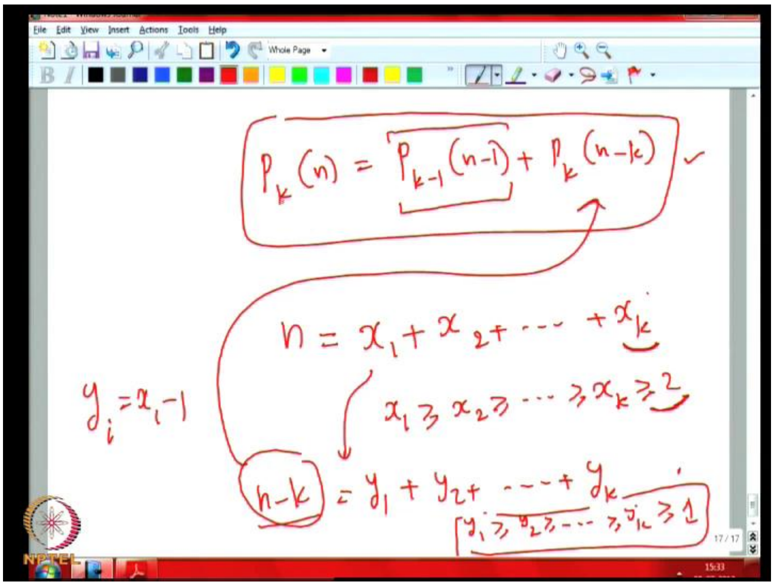

p k of n is equal to p k minus 1 of n minus 1 plus p k of n minus k. And some initial conditions are to be noted while… When n is less than k, we have to have p k of n equal to 0, because if you are asking for number of partitions of n into more than n parts; it has to be definitely 0. We have mentioned before p k of k is equal to 1, because if k has to be split into k parts, then there is only one way namely, 1 plus 1 plus 1 plus 1 plus k times;

that is only 1. And p 1 of n is 1, because one summand partition for n is only n; we just write n. And do we know p 2 of n? p 2 of n. So, we can try to use this thing; write formula. But maybe we can try to prove this first and then try to use this to get p 2 of n. So, this p k of n is equal to p k minus 1 of n minus 1 plus p k of n minus k is what we want to do. Now, we can see whether 1… For n, when we take the partition; whether 1 is there at all and (( )) 1 plus. So, we write may be x 1 plus x 2 plus x k. This is the last smallest partition and we just… So, we know x 1 is greater than equal to x 2 is greater than equal to x k greater than equal to 1.

Now, we consider this case, where x k is equal to 1 or not. This is the question we ask. If x k equal to 1; then what happens is such partitions – number of such… We can group the number of k summand partitions of n into two categories. One category is where x k is equal to 1; one category is x k is not equal to 1. If x k equal to 1, such partitions are actually equal to p k minus 1 of n minus 1, because once 1 is removed, where there is n minus 1; the number has become n minus 1; and then we have to get k minus 1 part partitions – summand partitions for n minus 1.

Therefore, this will count such partitions. Now, suppose 1 is not there; that means s k is not equal to 1; that means x k is greater than equal to 2. So, that means we can change it to x k is greater than equal to 2; that means we can translate this to say y 1 plus y 2 plus y k; where, each y i is x i minus 1. I will just… minus 1 from this thing; that means this will become n minus k, because I am minusing from each of the terms. So, this is n minus k equal to y 1 plus y 2 plus y k.

And off case, we have the conditions like y 1 is greater than y 2 is greater than y k. But instead of this greater than equal to 2, we will have just greater than equal to 1, because we have reduced 1 from each. This can drop by 1; that means this is… But this is like the earlier problem – the partition problem. But we are only considering the partitions of n minus k. So, that answer will correspond to this – p k of n minus k. How many partitions are there for n minus k with k summands? What we have done is this recurrence relation comes by considering this thing in the… Consider the partitions having k summands for n.

Now, group them into two categories: one category consists of those partitions, where 1 is a summand. And because 1 is a summand, we can remove 1; and then now, we have n minus 1. So, we are looking for partitions of n minus 1 using k minus 1 summands; that is, p k minus 1 of n minus 1. And then the other type is when where 1 is not a summand; then that means we can reduce. So, we can convert this equation to y 1 plus y 2 plus y k; where, each y i is actually 1 less than the corresponding x i. So, we are looking for an equation like n minus k equal to y 1 plus y 2 plus y k; where, each y k is greater than equal to 1. Actually, it corresponds to the number of partitions of n minus k into k summands. So, that is p k of n minus k. That is where this is coming from. Now, we are coming back to where we are stuck. What will be p 2 of n? We can try to get it from here; p 2 of n is equal to p 1 of n minus 1 plus p 2 of n minus 2 is what I say.

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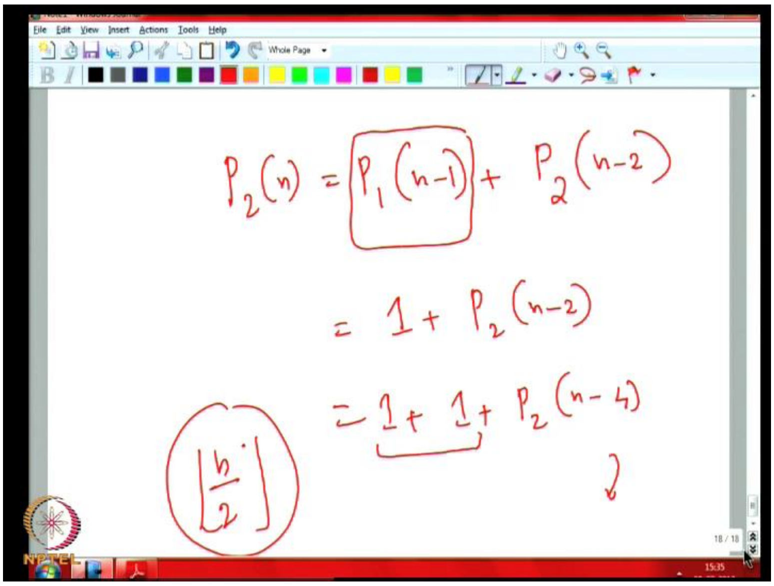

I will write it like this – p 2 of n equal to p 1 of n minus 1 plus p 2 of n minus 2. But then this is 1. Because of one summand partitions of n minus 1, this is equal to just 1 plus p 2 of n minus 2. Now, this will be what? 1 plus 1 plus p 2 of n minus 4. So, we will be reducing by 2 at each step. At each step, we will be accumulating 1. So, we can see that, the answer will be n by 2 floor. Why? Because when we have done n by 2 floor, we are actually in p 2 of 1. If it was either p 2 of 1 or p 2 of 0; p 2 of 1 is where 2 is bigger than 1. Therefore, we cannot have two summand partitions for 1; that will become 0. So, this will be the answer. But we can also get the answer for p 2 of n directly if you think what it means. p 2 of n means how many two summand partitions are there.

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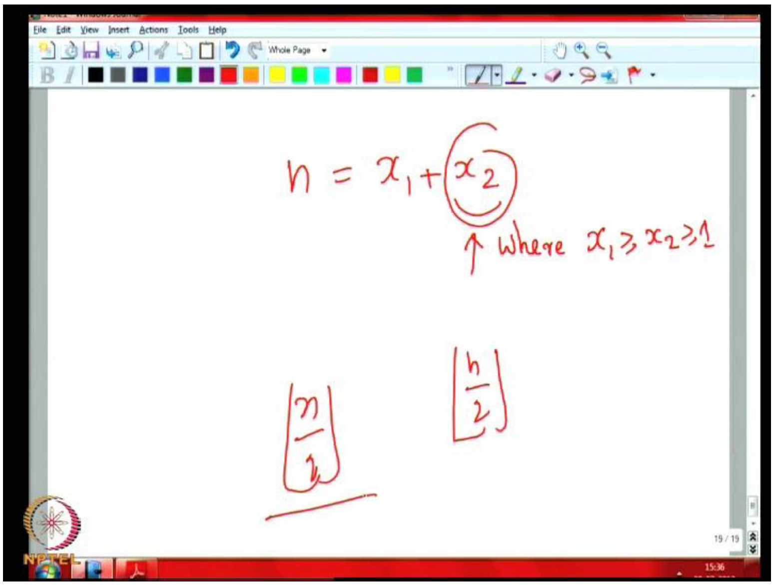

We want to consider this equation – n equal to x 1 plus x 2; where, x 1 is greater than equal to x 2 and this is greater than equal to 1. Now, we will see how many times… We can count x 2's. Once you select x 2, x 1 is automatically determined. But then this has to be smaller than x 1. So, we can go up to n by 2; if n is an odd number, up to n by 2 minus 1 for selection for this thing, so that number of partitions will be n by 2 floor. So, we will continue in the next class.