Combinatorics
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Lecture - 32
Exponential Generating Functions - Part (1)

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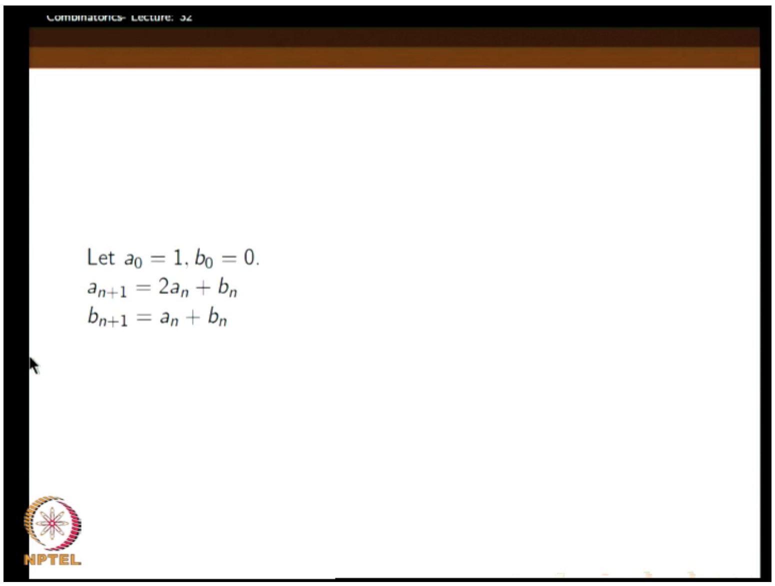

Welcome to the thirty second lecture of combinatorics. So, we continue with what we stopped in the last class. We were describing recurrence relation involving two sequences of numbers; a 1, a 0, a 1, a 2, a 3, etcetera and another sequence - b 0, b 1, b 2, b 3, etcetera.

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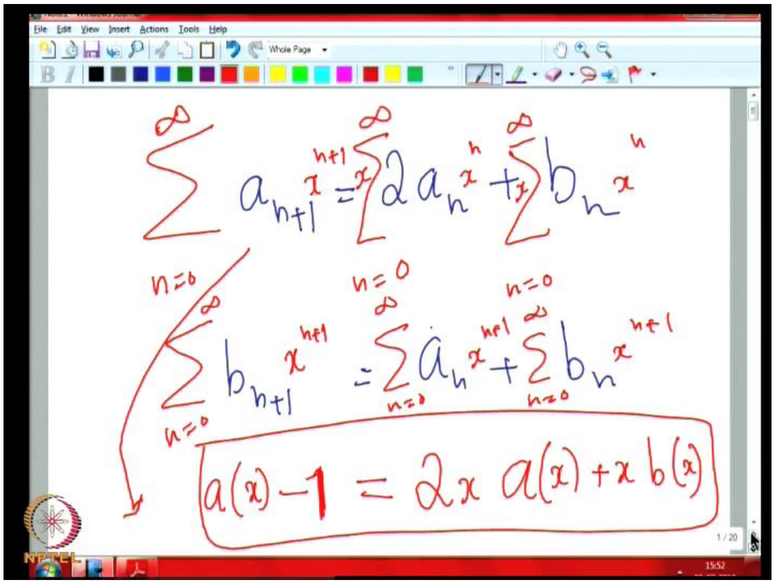

The recurrence relation for a, so is given say n plus 1. Not only in terms of some previous number in the a sequence, but also some number in the b, something this is one and then… And the another thing is, this one b n plus 1 is equal to a n plus b n. Now, b n plus 1 is x plus both in terms of a n and b n. Right. Now, how do we solve this kind of recurrence relation? So, to illustrate that we have taken this example, so we summed it from n equal to 0 to infinity here. That means, everywhere we have to sum. Sorry, first, one minute, first, we multiply this by x raise to n plus 1 everywhere. x raise to n plus 1 everywhere and then we sum it n equal to 0 to infinity.

So, that means, here also we sum n equal to 0 to infinity, and then this becomes, because this n equal to 0 to infinity, that means, n plus 1 starts from 1 and goes to infinity. So that therefore, this is not really a of x, but it is a of x minus a 0 into x raise to 0, but a 0 is given to be 1. So, 1 into x raise to 0. That is just 1, right? So, we get 1 here is equal to, here what happens is, 2 we can take out; here n equal to 0 to infinity, but x raise to n plus 1 is multiplying. So, we can just remove this and take x out, right?

So, 2 x comes out and then it is from n equal to 0 to infinity a n into x raise to n. This is a of x, right, plus, here we have we can take again x out here. x can be taken out. So, x into b of x. This is one thing we can get. Now, we can rearrange this as we mentioned yesterday to get, see, the 1e you take this side and a x and b x to this side and that means, 1 minus 2 x times a x.

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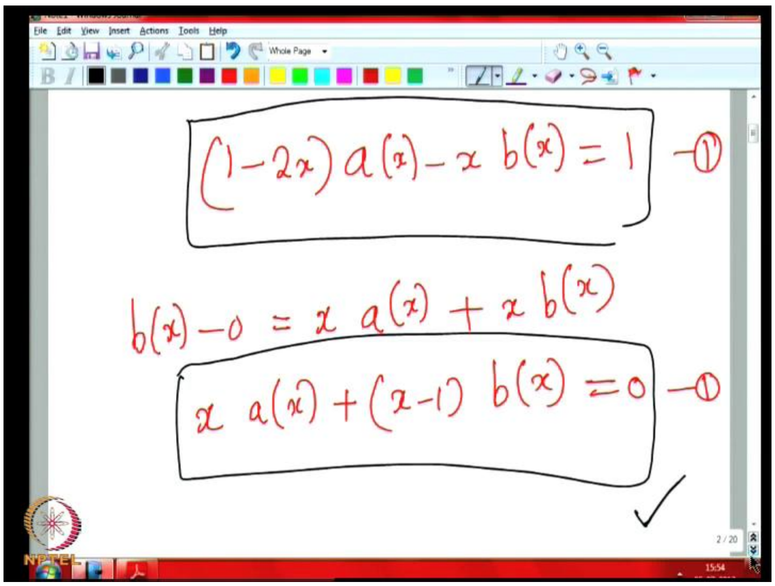

1 minus 2 x times a x minus x times b x is equal to 1. This is the first equation we get, right? Same day, we did for the first recurrence relation and we can do it for the second one, b n plus 1 equal to this one, b n plus 1 is equal to a n plus b n. So, what we do, multiply by x raise to n plus 1 everywhere and then sum from n equal to 0 infinity. Same arguments, so because now this is b of x minus b 0, so that will be b of x minus b 0; b 0 was equal to 0, so therefore, this is just 0. Then here, we can take x out and that means, x times a of x plus, here we can again take x out, x times b of x will come. Now, we can rearrange. So, we get x times a of x plus x minus 1 times b of x equal to 0. This is the second equation.

So, this and this together, so we have to solve. So, we have to get an expression for a of x and expression for b of x. For example, first you can try to get rid of b of x. Say, you can multiply this by x minus 1, this by x and then add them together. Then, b x will disappear and then we can solve for a of x. Similarly, then after getting a of x, we can solve for b of x. So, I will leave it to you to figure it out. So, that way, we get the generating function for a of x, as well as b of x, right? So, this is, this solves that.

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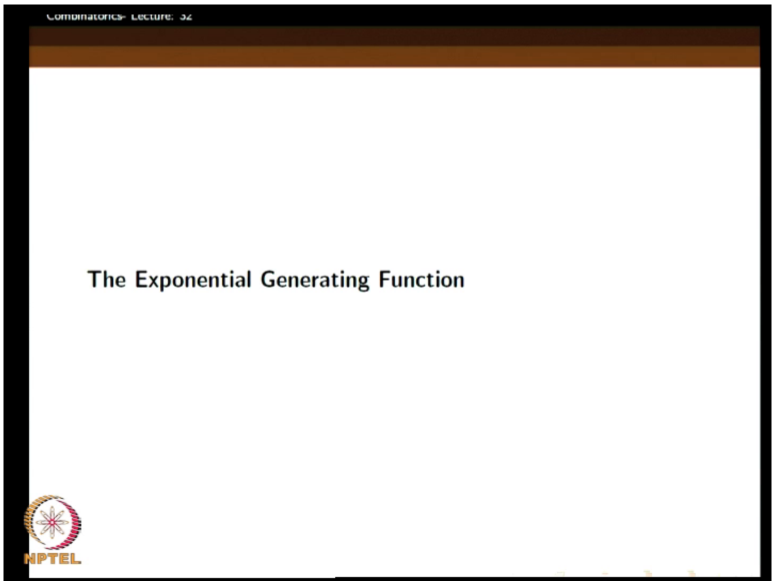

Now, we will go to a different kind of generating functions called the Exponential Generating Functions. The orientary generating functions that we were considering up to now are suitable for counting problems, where involving just selection. While here, these exponential generating functions will be useful as we will soon see in cases, where we are counting things, where order matters, right? Order matters.

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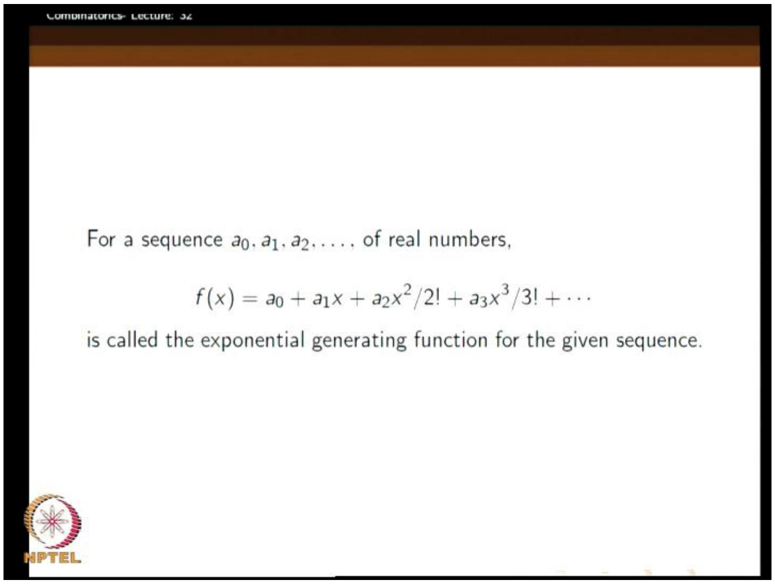

So, let us introduce the exponential generating function like this. So, suppose here sequence is a 0, a 1, a 2 etcetera etcetera and then we write the generating function f of x equal to a 0 plus a 1 times x plus a 2 times x square by 2 factorial plus a 3 times x cube by 3 factorial and so on. This generating function is called the exponential generating function for the given sequence.

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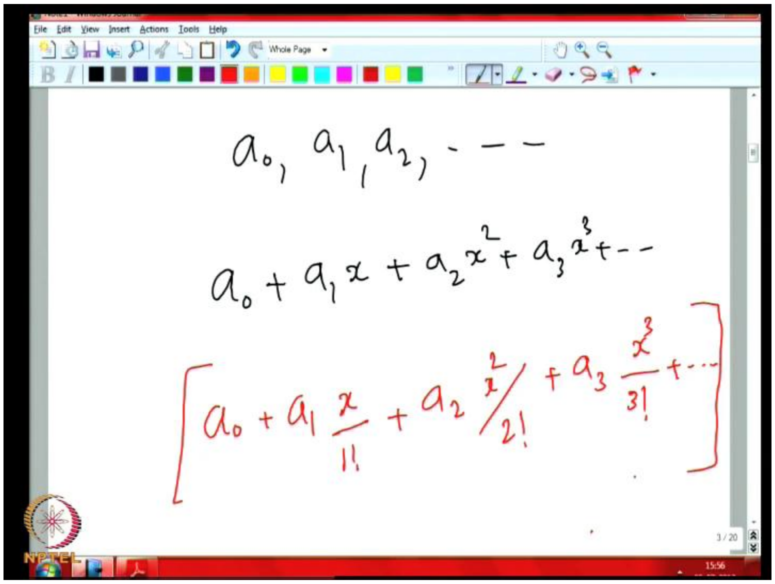

Once again, see again, our sequence is a 0 a 1 a 2 and so on. See, note that our orientary generating functions will be a 0 plus a 1 x plus a 2 x square plus a 3 x cube and so on and the co-efficient of x raise to n will be n. Here, the difference is that exponential generating function a 0 plus a 1 into x by actually 1 factorial plus a 2 into x square by 2 factorial plus a 3 into x cube by 3 factorial and so on, right? So, for a n; a n will be the co-efficient of x raise to n by n factorial here. So, that is the difference. So, this looks a little strange to begin with. But, you will see soon that this is useful in solving certain problems. But some examples we can easily see.

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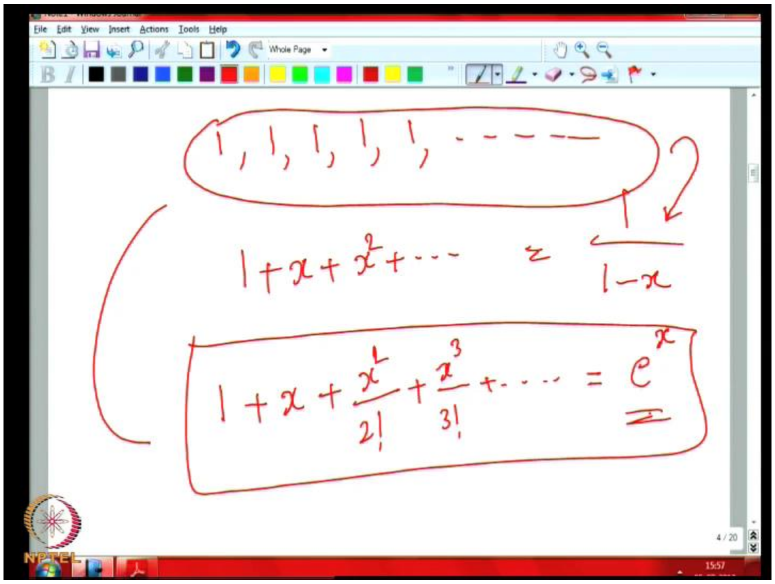

Suppose, the sequence is 1 1 1 1 and this is the most familiar sequence for us. So, a 0 equal to 1, a 1 equal to 1, a 2 equal to 1 and a n equal to 1, right? For this sequence, the orientary generating equation is what? It is just 1 plus x plus x square plus, this one right, 1 by 1 minus x, right. We have seen this before. But, the exponential generating function will look like this; 1 plus x plus x square by 2 factorial plus x cube by 3 factorial and so on. From calculus, we know that this is e to the power x, right?

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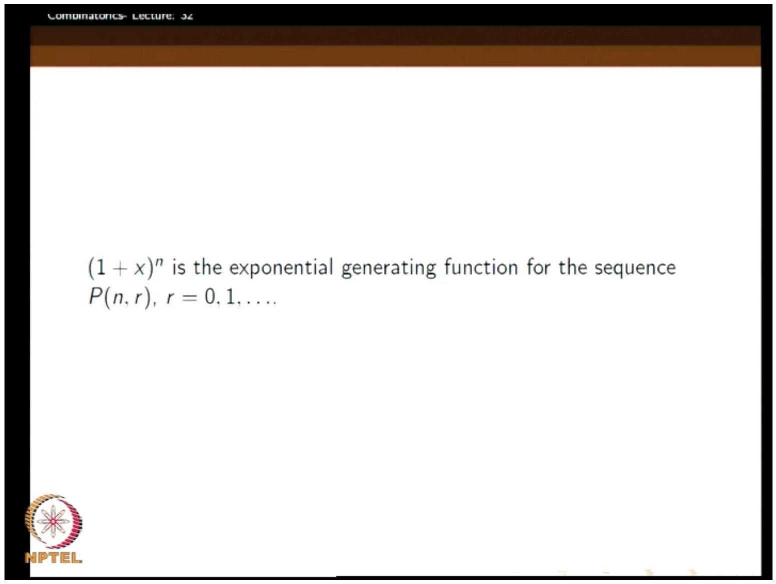

So, the exponential generating function for this sequence, whose orientary generating function was this, 1 by 1 minus x is going to be like this, e raise to x. Another example is, what about 1 plus x raise to n, which is, what is that? 1 plus x raise to n is another familiar generating function for us.

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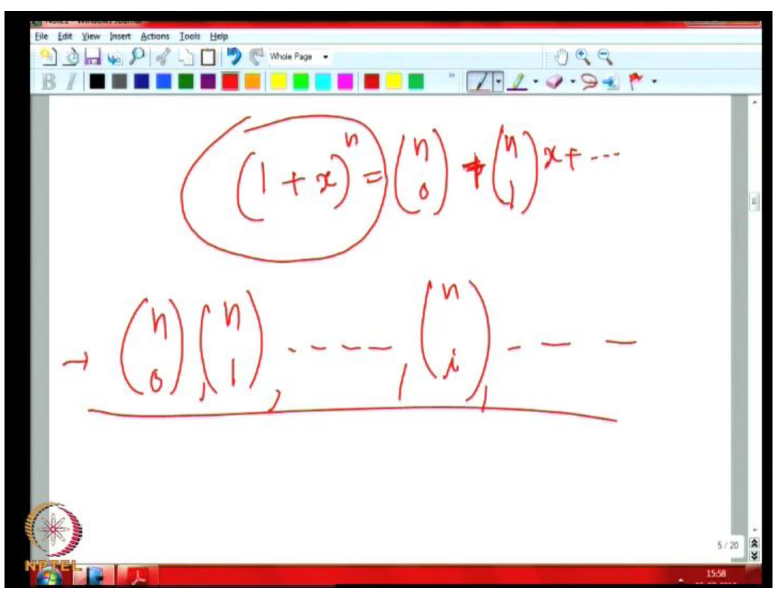

1 plus x raise to n, this is, n choose 0 into x raise to 0 plus n choose 1 into x and so on, right. So, this is the generating function for this numbers n choose 0, n choose 1 and n choose i, right. This sequence of numbers, say n choose 1 and after that m n choose n plus 1 and from there onwards it is 0. So, what is this? In other words, this 1 plus x raise to n is the orientary generating function for this sequence of numbers. So then it is the exponential generating function for another sequence of numbers, which is that sequence of numbers.

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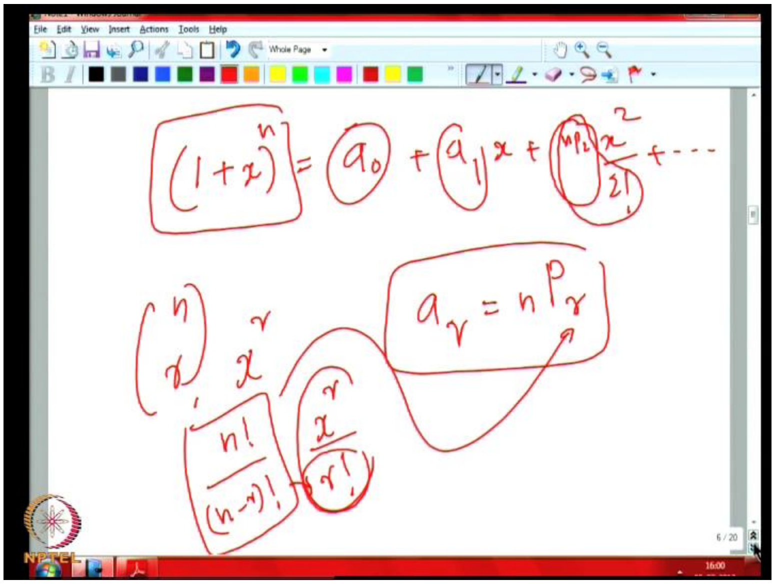

So, in other words, if I want to write 1 plus x raise to n as some a 0 plus a 1 x plus a 2 x square by 2 factorial and so on, then what should be this a 1 a 2 a 0 etcetera? That, what would be a i for a n say, right. See, you can easily see that, it is n p r, right. n p r. Why is it so on? It is because you put here, if you put n p 2, n p 2 by 2 factorial is going to be n c 2, which is the actual number, which should be the co-efficient of x square here. For x raise to n, the actual co-efficient is n. Sorry, not again; a r, I am saying a r; a r should be n p r, right. So, for instance, when I consider the co-efficient of x raise to r, that is n choose r when you expand this stuff, right. Now, but I am only interested in the co-efficient of x raise to r by r factorial. So, that means, this is n factorial into n minus r factorial into r factorial. This r factorial goes here, right. I am only interested in this part and this actually n p r, right.

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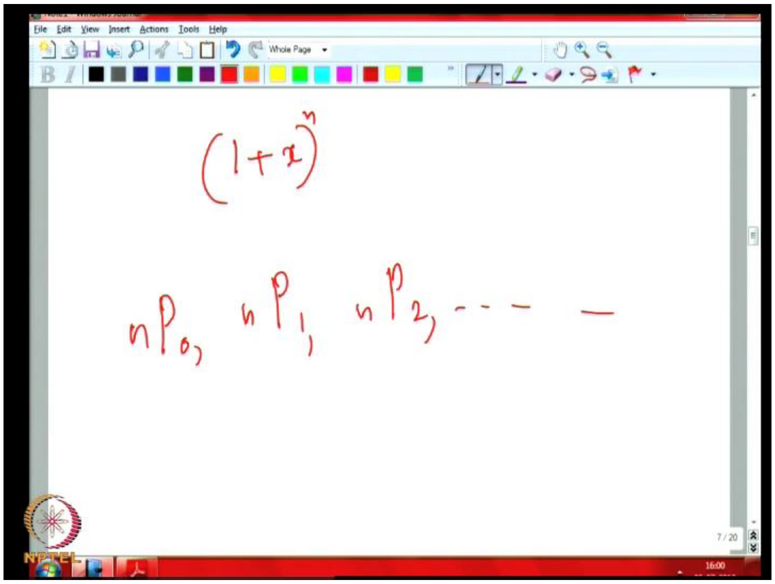

So therefore, we see that 1 plus x raise to n is the exponential generating function for this sequence, n p o, n p 1, n p 2 and so on. While it was the orientary generating function for n c 0, n c 1, n c 2 and so on.

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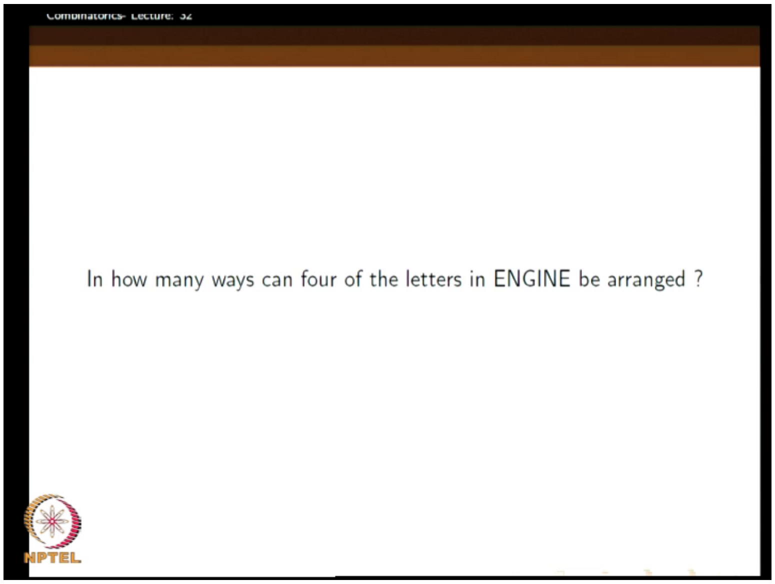

Then, now we will look at how we can use this stuff. So, look at this question. How many ways can 4 of the letters in engine be rearranged.

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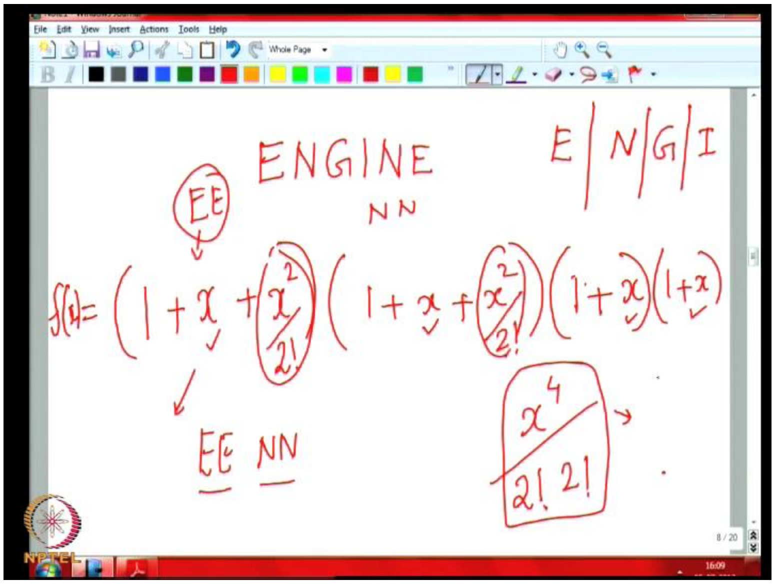

4 of the letters. There are 6 letters here. So, you have to pick up any 4 letters you like from this thing and then you want to permute them, rearrange them in various ways; order them in various ways; permute them. The point is, you can pick up possibly these 4 letters and then you can do it in 4 factorial ways or you can do e e and n n, these 4 letters you can take out of that. Now, you do not have 4 factorial. Actually, 4 factorial by 2 factorial into 2 factorial ways are there to permute these things, because 2 letters are here are same and 2 letters are there. So, we have studied how to count this thing. Initially, we will think that all are different. So, this is e 1 and e 2 and this is n 1 and 2 and then we have 4 factorial ways. But then now e 1 and e 2 are actually one and the same.

Therefore, we divide by 2 factorial because among them, you can permute into 2 factorial ways. Similarly, n and n. So, n 1 and n 2, it is just n. So therefore, we will divide again by 2 factorial, right. So, and again, there are many ways of selecting 4 letters from in these 6 letters. So, it could have taken e n, and n i. So, now this is 4 factorial by 2 factorial, because this repeats 2 times. So, now finally the answer we see is the sum of all these things. First, select 4 letters and then you consider how many ways they can be permuted and then add them together, how many will be there. So, this is where we are going to use exponential generating function for counting.

So, what we do is, we will introduce 1 term for each of the letters. There are this many letters. One letter is e, and another letter is n, e type of letters I can say and another is g and another i, engine. So, since there are 2 e's, we can either not take of any of the e's in our; we are picking 4 letters out of those 6 letters. We may not select any of the e's, right. That means 1. That means, x raise to 0. So, or you may select just 1 e or we may select 2 e's. This is the term introduced for e. This corresponds to the terms 2 e's. Now, for 2 n's also we can have one such term, 1 plus x plus x square because you may not select any of the n's in the 4 letters we pickup from this thing or may select 1 n or we may select 2 n's. For other letters, g and I, which appears only once, we have only 2 choices. Either, we may select or may not select this way.

Now, we claim that, so let us say, this is f of x. We claim that this f of x is indeed the exponential generating function for the count we are looking for, which essentially means, we are looking for how many ways 4 letters can be picked from these 6 letters available here and permuted; arranged, right. So, we saying that, ok, if you are looking for picking the 4 letters k's, then you just have to consider this thing and you just have to consider the co-efficient of x raise to 4 by 4 factorial, the co-efficient of this in f of x. Then, you will get the answer for it. If you are only talking about picking 3 letters from this thing, then you can, probably you just have to look at ways. So then we mean that we have to look at the co-efficient of x cube by 3 factorial. x cube by 3 factorial, in a f of x.

In general, if you are looking for picking k letters, then x raise to k by k factorial co-efficient is what we are looking for. So, that way, f of x is going to be the generating function for this count, the sequence of numbers defined by the k th term. k term of the sequence being the number of ways of permuting the letter after selecting k letters from this, right. So now, how is it true? That is what you are asking, right? Now, I will try to explain this, why is it so. Because, see for instance, in the case of 4 x raise to 4, how can you get x raise to 4 from this thing. Sorry, here I have to because exponential. I have to put this. Sorry, so mistake. So, because we are talking about exponential generating functions, when we take this thing 1 plus x plus x square by 2 factorial, so that is all.

Now, we have to find the co-efficient of x raise to 4 in it. So, how many, what of the possible ways to get x raise to 4? Definitely, we have to select something from here, something from here, something from here and something from here to make x raise to 4. For instance, if you take x square by 2 factorial from this thing and x square by 2 factorial from here, then you have to take 1 and 1 from here because x square plus x square is already x raise to 4, right. So, that would correspond to selecting 2 e’s and 2 n’s and no i’s and no g’s. On the other hand, if you have picked up x from here and x from here and x from here and x from here, it would correspond to selecting 1 e, 1 n and 1 i and 1 g.

So, that is all various possibilities, all possibilities of making x raise to 4. We will correspond to how you will select on this thing. Now, you have to also incorporate the fact that we have to permute them, right. So, that is where we are looking for, for instance, let us look at one selection. For instance, 2 e’s are selected. That means, this terms comes from here and 2 n’s are selected. That means, this term comes here and here, it is a 0. It is a 0 i and 0 g from here; 2 e’s and 2 n’s. So, in the product, how we get is, not directly x raise to 4. We get x raise to 4 by 2 factorial into 2 factorial because here we are getting x square by 2 factorial and here, we are getting x square by 2 factorial. This what we get.

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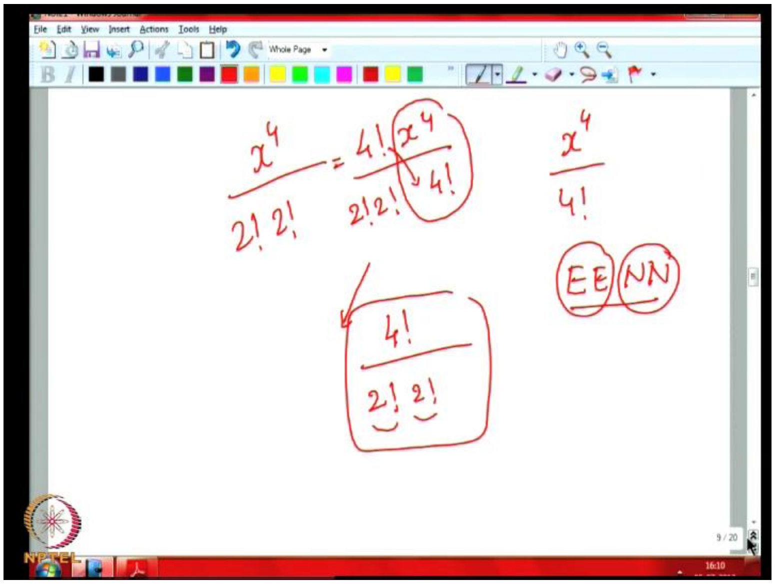

But then what we looking for is the co-efficient of x raise to 4 by 4 factorial. What we have got is x raise to 4 by 2 factorial into 2 factorial. But, where is this 4 factorial? So, what we do is, convert into that form. So, we write this as x raise to 4. So, this has, a 4 factorial is introduced here and 2 factorial and 2 factorial is kept here. But, I cancel this and will put a 4 factorial here. So, we have a x raise to 4 by 4 factorial. So, co-efficient is 4 factorial by 2 factorial into 2 factorial. But, we know that this is exactly the number of ways to permute these 2 e’s and 2 n’s; whether our 4 letters that corresponds to 4 factorial. These two factorial comes twice because there are 2 e’s and 2 n’s here. Now, the same way for any possible selection, so to make x raise to 4.

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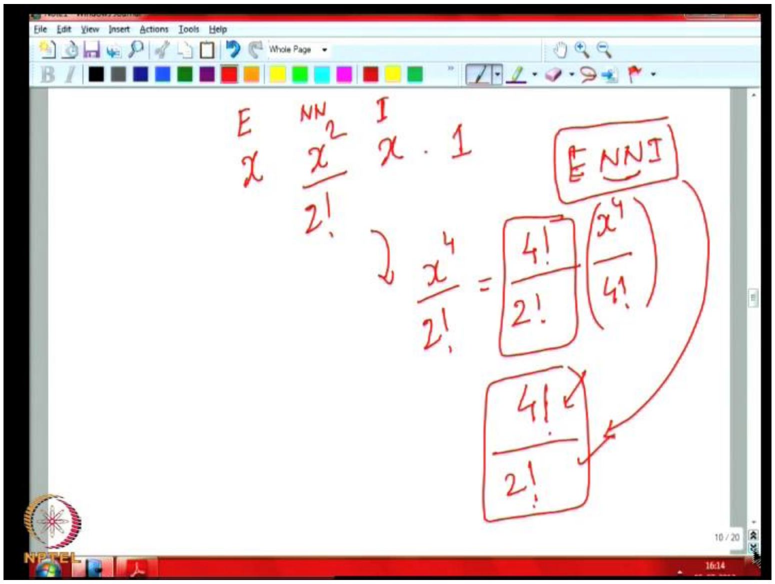

If you take one from here and two from here and one from here, say and nothing from here, that means, an x comes from here and an x comes from that first term and then x square by 2 factorial comes from the second term and x comes from the third terms and one comes from the last term. This corresponds to 1 e, this corresponds to 2 n's, this corresponds to n i and this correspond to g. Sorry, this corresponds to no g's. There is nothing because here 1 means x raise to 0. So, the selected letters are e n n and i. Now, we get not x raise to 4, we get x raise to 4 by 2 factorial. But, we are looking for the co-efficient of x raise to 4 by 4 factorial. So therefore, we introduce 4 factorial above and below. That means, in the numerator and denominator. So, then this will look like this, right. The co-efficient of this thing is 4 factorial by 2 factorial.

We know this is exactly the number of ways we can permute this thing because there are 4 letters. That is 4 factorial here and 2 n's, therefore, 2 factorial in the denominator, 4 factorial by 2 factorial. Now, for any possible way we see, selection of 4 letters will correspond to a selection of some term from here, some term from here to make of x raise to 4. So, reverse. Conversely, if you consider any breakup of x raise to 4, some term from here, some term from here and some term from here and here that will correspond to a selection of 4 letters from here, right. Somehow, by looking, by putting, see this 2 factorial in the denominator, whenever we are selecting 2 letters, so we also have the 2 factorial coming in the denominator in the product, right.

Now, because we are we are interested in the co-efficient of x raise to 4 by 4 factorial, we can introduce a 4 factorial in the numerator, so that, 4 factorial numerator and in the corresponding 2 factorial; how many we will say? Depending on how many times that x square, that was taken, we get so many 2 factorials in the numerator. That counts exactly how many ways you can permute those things; selected letters. We add them of all and that will be the final co-efficient of x raise to 4 by 4 factorial.

The final co-efficient of x raise to 4 by 4 factorial will be this plus, for each of them, say this plus, so this was, sorry, this was one and this was another. Like that, for each selection, we got some term and we are adding up, right. So therefore, in the exponential generating function for f of x, the co-efficient of x raise to 4 by 4 factorial will turn out to the number of ways for selecting 4 letters and then permuting them. How many ways you can permute after selecting 4 letters, right. Select any 4 letters and permute them and count the total orderings on selected some 4 letters selected from these 6 available. So, this is how the exponential generating function works, right.

So, you should note that the fact that we are looking for the co-efficient of x raise to 4 by 4 factorial was very crucial here. That is how we got that 4 factorial in the numerator of that time. So, that is how we could ensure that we are actually getting that count of the permutations of those 4 letters that we selected each time. Now, let us take a new example, which will make things clearer.

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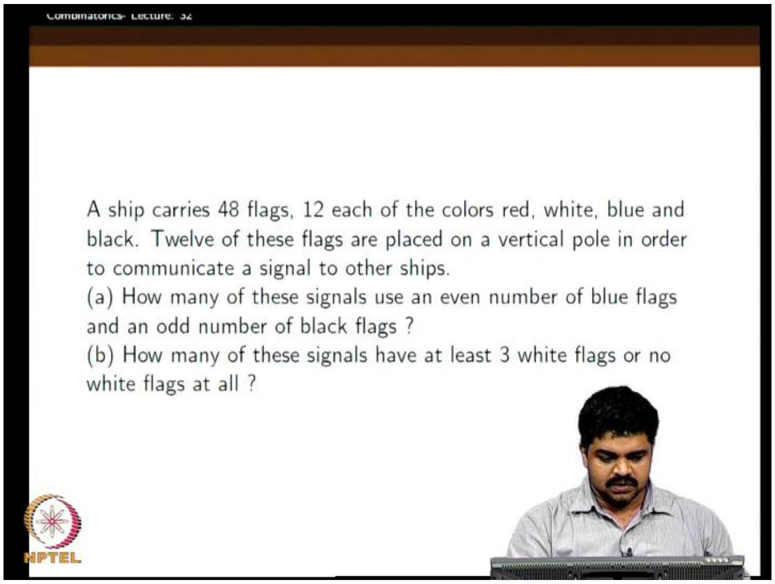

So, a ship carries 48 flags, 12 each of the colors red, white, blue and black. That means, we have 12 red flags, 12 white flags and 12 blue flags and 12 black flags. What they do when they want to give a signal to another ship, which is coming near? They will use 12 of these flags. Out of these 48 flags, they will take some 12 flags and arrange them in the flag post.

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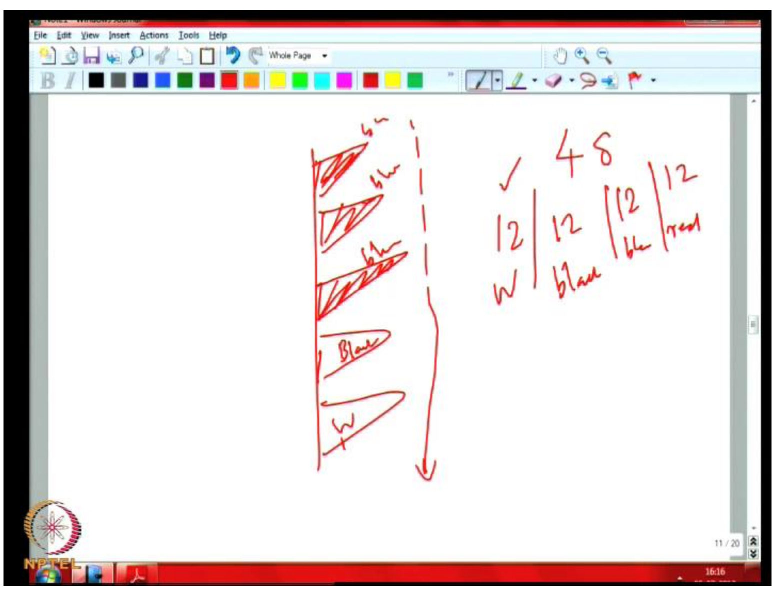

Some, may be they will select. Suppose, they will select the flags and they will tie like this here, right. 12 flags will be tied. So, this order is important here. For instance, this was a blue. These are 3 blue ones and then 1 black here and then 1 white here, right, and then 1 white here. So, white. This is a black and these are blues. If you reorder them in some other way, it will correspond to another sequence. For instance, you can put white on top and black later and 3 blue later, below that. Then, it will give a different message. That is a different signal. So, the question is, how many possible signals can be made this way?

Remember, we have total 48 flags, 12 from each color. 12 white, 12 black, 12 blue and 12 red. Note that, you can make a signal using only 1 color because 12 flags are there of each color. Full signal can be of the same color, and that is also possible, right. Now, how many possible signals are there? Always remember, that we are only using 12 flags in the signal, though we have 48 flags total, right.

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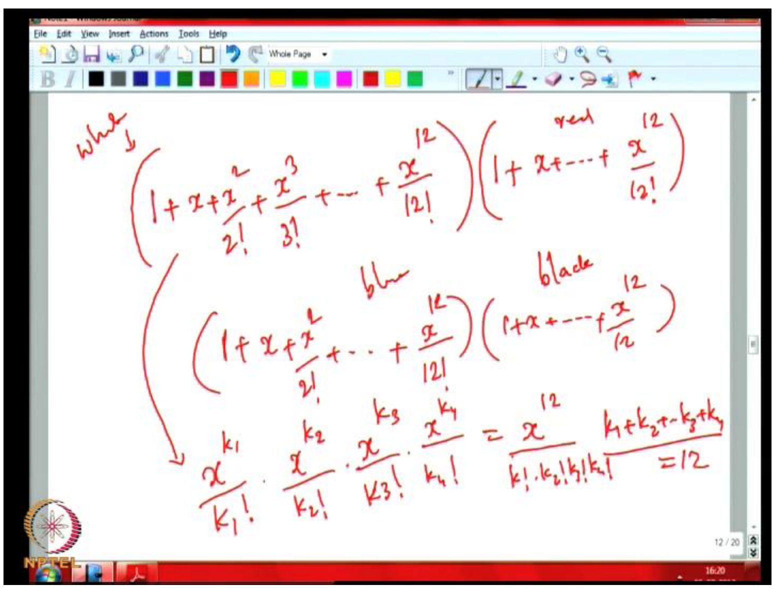

Now, to solve this thing, what we do is, if we had, we will say, we have a sequence of numbers, say a 0 a 1 a 2 a 3 etcetera, and a i will correspond to the number of sequence that can be made, provided we are using i flags in the signal, right. So, we will be interested in a 12, right. So, we will get somehow a exponential generating function for this sequence; so that, we can look for the co-efficient of the x raise to 12 by 12 factorial for the answer. This is our plan. So, what we do is, introduce one term for each color. First color, say white will get this term. What is this now? This will be 1 plus x plus x square by 2 factorial x cube by 3 factorial and so on, till x raise to 12 by 12 factorial.

Now, the next one also has the same kind of term, 1 plus x plus till x raise to 12 by 12 factorial. See, this is the same problem like the earlier. We can create something like this, right. 12 factorial x raise to 12 by 12 factorial, multiply it out. Now, you look for the co-efficient of x raise to 12. So, x raise to 12 by 12 factorial. That is going to give your answer what we are saying, because x raise to 12 by 12 factorial, so this x raise to 12 should be formed by selecting some k 1 from this. x raise to k 1 from this, and k 1 by k 1 factorial, from this.

So, first x raise to k 1 by k 1 factorial will be selected from the first term, which means that we are going to select k 1 white flags. Then, from the second term, we will take x raise to k 2 by k 2 factorial and then from third term we will be selecting x raise to k 3 by k 3 factorial. Then, we are going to select x raise to k 4 by k 4 factorial. So, this will be x raise to 12, where k 1 plus k 2 plus k 3 plus k 4 is equal to 12, right. Because, we are selecting some white flags, some red flags, k 1 white flags cum k 2 black flags, and k 3 blue flags. This is red, this is blue, this is red, sorry, this is white red blue and finally, black, right. So therefore, the total has to be 12. That is how we are splitting this k 1 k 2 k 3. The only thing is, so we will end up; the answer will be not just x raise to 12. We will get k 1 factorial into k 2 factorial into k 3 factorial into k 4 factorial, the denominator.

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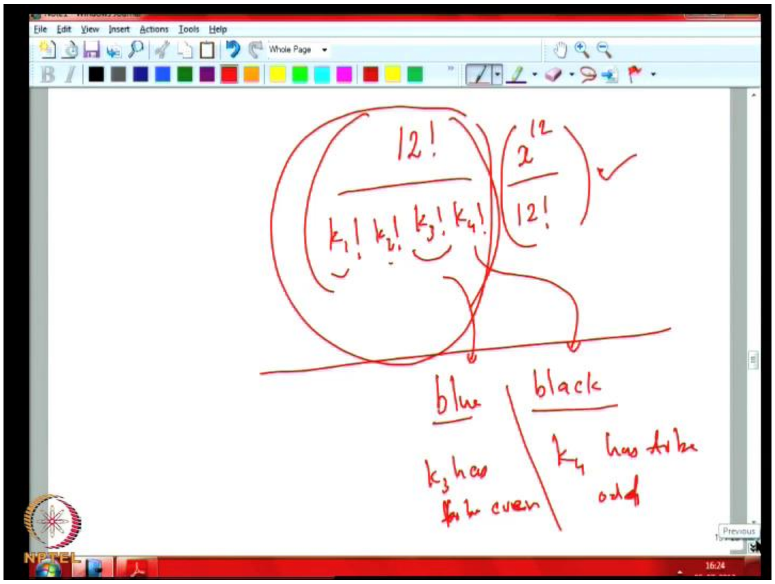

Now, we are not looking for the co-efficient of x raise to 12. We are rather looking for the co-efficient of x raise to 12 by 12 factorial. To get that, we will introduce a 12 factorial in the denominator. To balance it, we will introduce a 12 factorial and the numerator, so we already have this k 1 factorial into k 2 factorial into k, sorry, k 3 factorial into k 4 factorial, right. This clearly is the number of ways to get the signal. Provided, we have selected, we had decided to select k 1 white flags, k 2 red flags and k 3 blue flags and k 4 black bags flags, right. Because, there are 12 of them and 12 factorial comes from that. We are dividing the k 1 factorial, because there are k 1e white flags, etcetera, right.

So now, all possible splitting, so splittings' of 12 right into white, red and blue and black flags, will correspond to some k 1 k 2 k 3 k 4, right, coming from each different term and every time, you will get the number of permutations as the co-efficient of x raise to 12 by 12. That contribution to that, right and then finally the sum of all these things will give the final co-efficient of x raise to 12 by 12. Clearly, the 6 th number of signals we can make because the number of signals we can make is actually, first you have to decide how we are going to distribute this 12 as to how many white flags should be there and how many red flags should be there and then how many blue and how many black should be there, right? Then, we should permute them, right and then sum of over all possibilities of splitting that one. That is exactly what we are getting here is the co-efficient of x raise to 12 by 12 factorial. So, we will be ending up with the correct answer**.**

Instead of 12, if you are looking for something more, then also we can get the answer. For instance, some other number we can see. That also will give you the crack. Now, we have a different question. How many of the signals use an even number of blue flags and an odd number of black flags. So, little petty question. How many of signal even number of blue flags and odd number of black flags. Now, we are not allowing, now we are not allowing for blue and black, we are not allowing all possible numbers. For instance, this k 3, which has the number of blue flags earlier and this k 4, which has a number of black flags has some restrictions. This k 3 has to be even. Similarly, k 4 has to be odd. This is the thing. k 3 has to be even and k 4 has to be odd. How do we find it out?

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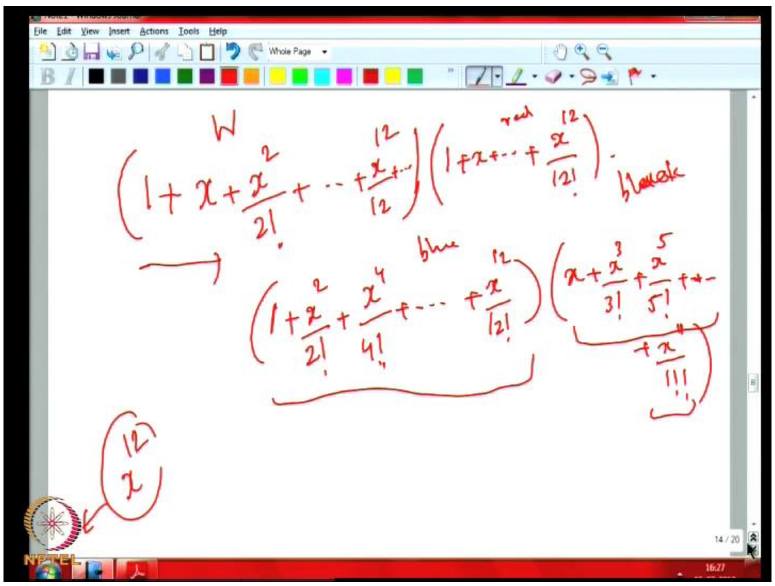

Now, we can, for the class case, this is blue, and this was red. This, for the black, what we can do is, we can take x plus x cube by 3 factorial because only odd numbers are allowed. x raise to 5 by 5 factorial and so on, till x raise to 11 by 11 factorial like that, right. Till x raise to 11 we can go. Now you know, because we are looking for the co-efficient of x raise to 12 in the product. You know, whenever a term is coming from this, it will be some x raise to some even number divided by that even number factorial. Similarly, whenever something is coming from here, x raise to some odd number is divided by odd factorial. This will be the thing. The other case of other two first terms, it is ok. Any number is ok. Now, we can, for that ease of calculation, what we can do is, we can extend it further. Instead of using up to here, we can add all the remaining terms.

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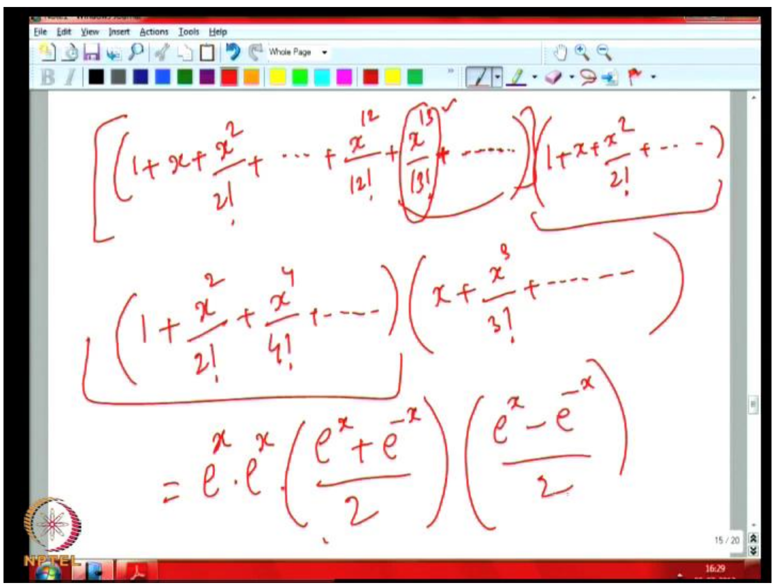

First term can be added as 1 plus x plus x square by 2 factorial into, instead of stopping at x raise to 12 by 12 factorial, so we can actually add x raise to 13 by 13 factorial and so on, till infinity. Why because anyway this is not going to, these new terms we are adding, are not going to contribute to the co-efficient of x raise to 12. Suppose, I pick up x raise to 13 by 13 factorial. We are already above x raise to 12. So then whatever that term we are going to produce by picking this and multiplying something from the other terms is not going to contribute to the co-efficient of x raise to 12 at all.

So before, we can take it as a power series till infinity, right. We do not have to stop. Similarly, for the red flags, 1 plus x plus x square by 2 factorial plus, like that, till infinity we can take. Similar case for the blue flags. But, only thing is, we are only allowing even powers, right. x raise to 4 by 4 factorial and so on, till infinity and this last one, this is x plus x cube by 3 factorial plus and so on till infinity. This we know is e to the power x. This is again e to the power x. So, e to the power x and what is this? This is actually e to the power x plus e to the power minus x divided by 2. Why is it so? The last one, I am claiming, that is e to the power x minus e to the power minus x by 2. Why is it so? These two things, I have to substantiate.

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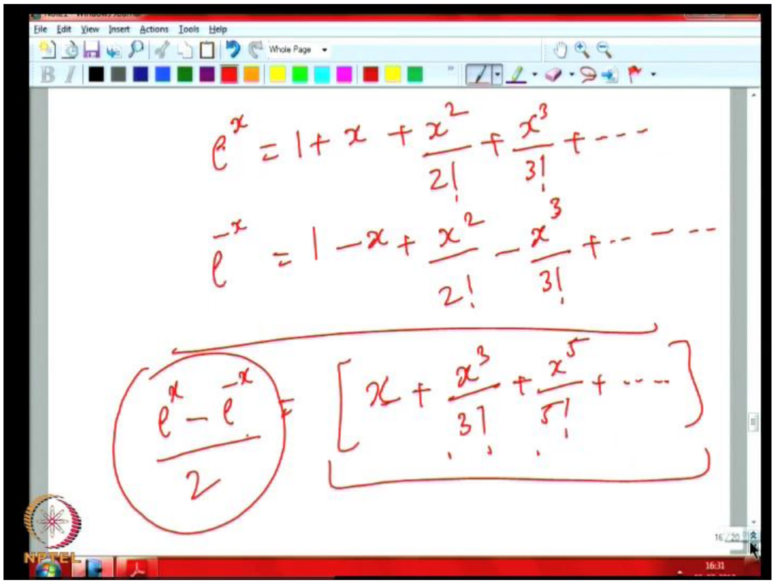

I will just write once again, what is e to the power x. That is, 1 plus x plus x square by 2 factorial plus x cube by 3 factorial and so on. Now, what is e to the power minus x? This is 1 minus x. Just put, instead of wherever you see x, put minus x. This will be again x square by 2 factorial because that squaring keeps the minus negative sign and this will be minus, right. Minus because this is an odd power, x cube by 3 factorial and then plus and then minus, like that. Now, we add up these two things, e raise to x plus e raise to minus x. What we get? We get 2 into 1 plus. This will go away. But here, this will remain x square by 2 factorial x raise to 4 by 4 factorial. All the even powers will remain and all powers will cancel of. So, we can take this 2 here. So, this is what we are looking at, 1 plus x square by 2 factorial plus x raise to 4 by 4 factorial. So, this is this. So, that is the third term for the blue, right.

Now, on the other hand, if we had tried to take minus l, this minus this, then what we get? 2 into, this one will go away, but here we will get a different thing. So, 1 will go away. But rather, you will get 2 minus x 2 times x. Then, here this will go away, but this will remain. This x cube by 3 factorial; 2 times has come here. So, x raise to 5 by 5 factorial and so on. So, this was the term, which we got for the black flag, right. So, if we can take this 2 below and that is why this term. So, that is how we got this thing, right.

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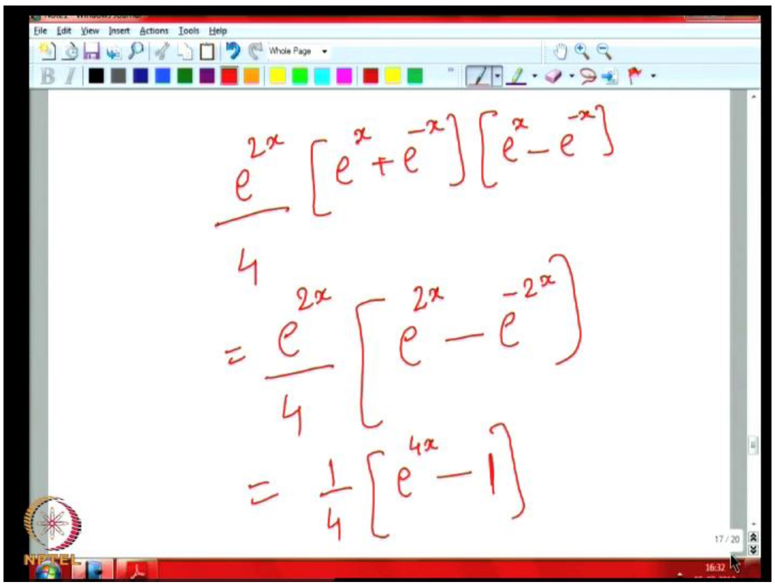

So, this is e to the power 2 x. So, this is, sorry, e to the power 2 x by 4 into e to the power x plus e to the power minus x into e to the power x minus e to the power minus x, which is e to the power 2 x by 4 into e to the power 2 x minus e to the power minus 2 x. This is x plus y from x plus y and x minus y. So, x square minus y square is what we are getting. So, e to the power 2 x e to the power minus 2 x, right. So, when you multiply this out, this will become e to the power 4 x by 4 or maybe, we can say 1 by 4 into e to the power 4 x minus 1. Guess what?

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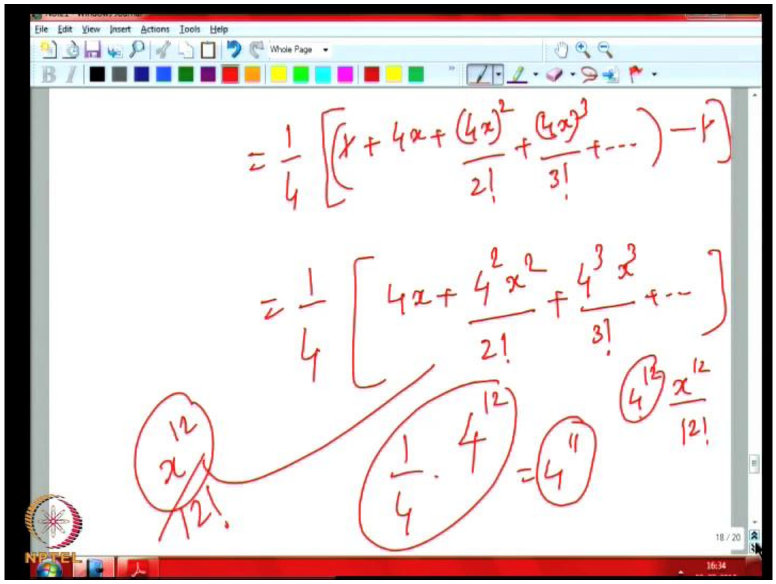

So, this is, now, this is what? 1 by 4 into e to the power x 4 x expansion 1 plus 4 x plus 4 x whole square by 2 factorial plus 4 x whole cube by 3 factorial plus like this and finally, you have to minus 1, right. So, this one will cancel of, right. So, you will get this 1 by 4 into 4 x plus 4 square x square by 2 factorial plus 4 cube x cube by 3 factorial plus, so on, right. So, what would be the co-efficient of x raise to 12? In this x raise to 12 by 12 factorial, in this thing right, that will be 1 by 4 into, because x raise to 12 will have 4 raise to 4 raise to 12, right. Because here, when we look for some point of time, we will get some x raise to 12 by 12 factorial. Here, 4 raise to 12 will be there, right. This 1 by 4. This is the thing. So, this is 4 raise to 11. This is the answer. 4 raise to 11 will be the answer. So, that is the number of ways of getting the signal.

So, number of signals, such that there are even number of blue flags used and there are odd number of black flags used and no restriction on white or red flags. Total number of flags, we can use is 12, right. See, of course we are using n flags. We can look for the co-efficient of x raise to n, right. This is the way. Now, we can look at the other part. How many of these signals have at least 3 white flags or no white flags at all. How many of these signals have at least 3 white flags or no white flags at all. So, there is a restriction on white flag now, right. All other flags are ok.

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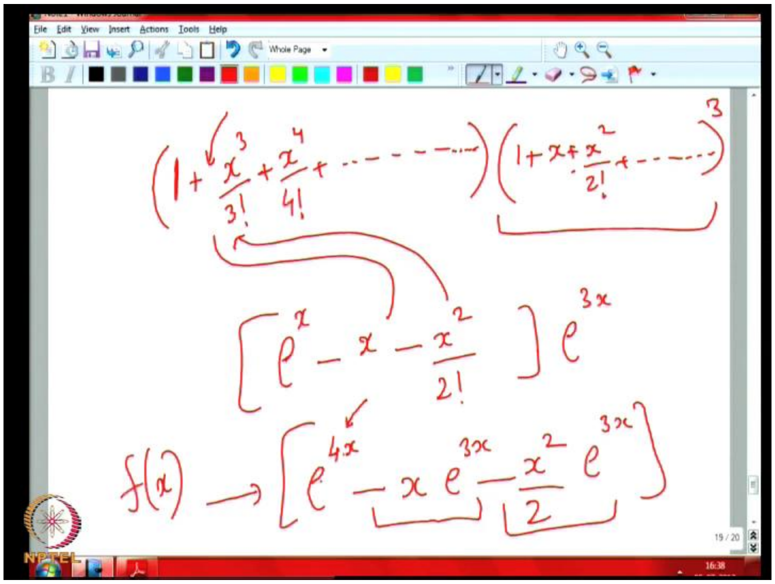

So, we only have to worry about the white flags. Say as first term. So, we allow no white flags. That means, one is fine. Now, this is not fine. x raise to 1 by x by 1 factorial x, that is because that 1 white flag. That is not allowed. We need at least 3. No flag is fine or at least 3. So, x square by 2 factorial also we can output because 2 white flags are not allowed. So, from here onwards we can put and so on.

The other things we do not have any restriction. We allow all possibilities. So, that 3 terms are like this. One for red, one for blue, and one for black. You see, we are not restricting up to x raise to 12 alone because it does not matter. We can take x cube onwards also, x 13 and x 14 etcetera. Because, we know we are looking for the co-efficient of x raise to 12, the contributions will be from only the first 12 terms for each term because first 13 terms, till x raise to 12, right. Because, x raise to 13, once you take from any term, then it is already more than x raise to 12. Then, how can it contribute to the co-efficient of x raise to 12. So, we can always, we can without any problem, we can take the infinite series. The power series we can take, rather than just restricting it up to x raise to 12 by 12 factorial, right.

So, we know all these things. This is e raise to x cube. That means, e raise to 3 x, and this is e raise to 3 x. But, what is this? This is something less than e raise to 3 x, sorry, e raise to x. This is ah e raise to x minus first term. That is x minus x square by 2 factorial. That is what we removed from here. Here these two terms are missing. This term and this term was missing, right. So therefore, we minus it. So now, we multiply and we get e raise to 4 x minus x into e raise to 3 x minus x square by 2 into e raise to 3 x. Now, we are looking for the co-efficient of x raise to 12 in this exponential. Then, x raise to 12 by 12 factorial in this exponential generating function, right. So, how do we know that? We have to take the co-efficient of x raise to 12 by 12 factorial and the expansion of this. Then, we have to take the co-efficient of x raise to 12 by 12 factorial from the expansion of this and then from this and add them up. How will you do that?

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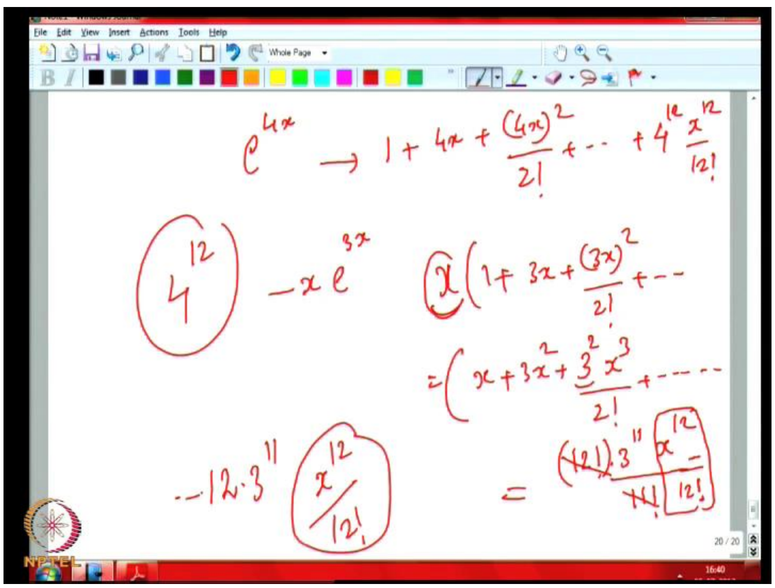

For this e raise to 4 x, it is easy. We have already seen that, right. Because, this will expand to 1 plus 4 x plus 4 x square by 2 factorial and so on. Now, if you want x raise to 12 by 12 factorial, right, this will be 4 x, sorry, 4 to the power 12. That is coming extra here, right. That will be becoming extra here. So, we have to take that 4 to the power l2. This will be the first contribution. Contribution of this, this first term. Then, this minus x 3 x, what will be this minus x into e raise to 3 x, right? e raise to 3 x. Now, note that when you expand it, this will look like 1. See, this is x into 1 plus 3 x plus 3 x whole square by 2 factorial and so on. Now, multiply by this thing. This will be x plus 3 square plus 3 square x cube by 2 factorial plus and so on, right.

Now, if you looking for x raise to 12 by 12 factorial, the co-efficient of x raise to 12 by 12 factorial, so you know that this 3 power will be see somewhere here and x raise to 12 will come here. Here, will see 3 raise to 11 right, because at x, it has taken the power of x by 1 etcetera. But, below we will have a 11 factorial. Not 12 factorial because 12 became, actually x raise to 11 became x raise to 12, because of this multiplication by x. But, what we want is only the co-efficient of x raise to 12 by 12 factorial.

So, we introduce the 12 factorial in the denominator. But then we have to balance it by putting the same 12 factorial in the numerator. Cancel this off. So, that will give you 12 into this 12 here, 11 factorial and 12 factorial cancelling that, you will get 12 into 3 raise to 11, right. That will be the co-efficient of x raise to 12 by 12 factorial here. This is the contribution from this term. So, we have a minus also of case. It is minus of this, right. Now, last term is this x square by 2 into e raise to 3 x, right.

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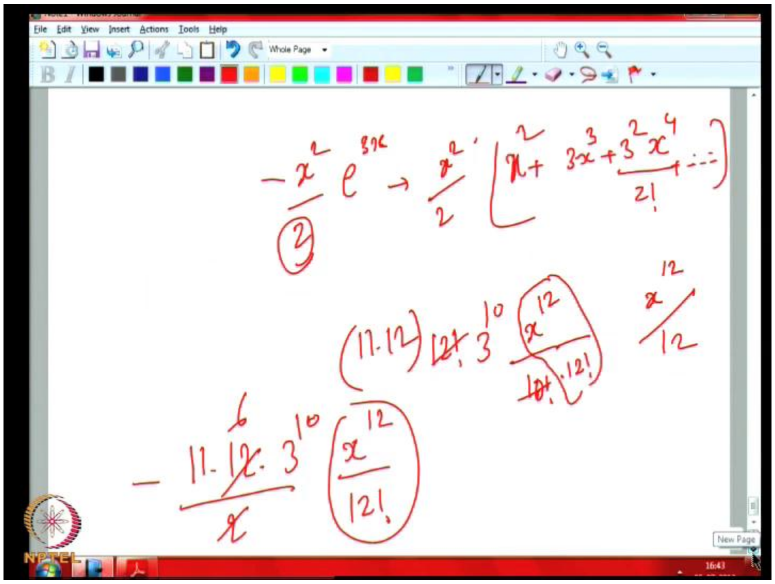

So now, as usual this minus x square by 2 into e raise to 3 x. If you want to find the contribution of the co-efficient of x raise to 12 from this, so now you know e raise to 3 x will be 1 plus 3 x plus 3 x whole square by 2 factorial and so on as usual. But because we have to multiply this by x square by 2, right. So, let us remember 2 separately. x square goes and multiply here everything. So, here this will become a x square. This will become x cube and this will become g. This will become 3 square into, 3 square will come out, so x raise to 4 and so on. So, when you are looking for the co-efficient of x raise to 12 by 12 factorial, you should look at x raise to 12.

So, actually we are looking at x raise to 10 by 10 factorial. That we have the 3 raise to that here, right, which became 12 because of this multiplication, right, this x square, right. Now, to get the co-efficient of, so this is now, this has become x raise to 12. x raise to 12 by 12 factorial, what we do is, we introduce a 12 factorial in the denominator and introduce something here. Now, cancelling off, you will have 11 into 12 left here, right. So, we have 11 into 12 into 3 raise to 10, x raise to 12 by 12 factorial. But remember, we initially had a 2 here.

So, this will cancel off 6, right. 11 into 6 into 3 raise to 12, 3 raise to 10 into x raise to 12 by 12 factorial. Now, adding the contribution from each of this minus case, the terms, this, this and this, we will get the final answer, ok. This was, I mean, I am not really evaluating the final answer. It does not matter. Just wanted to illustrate how we can pull out the co-efficient from each of these terms.

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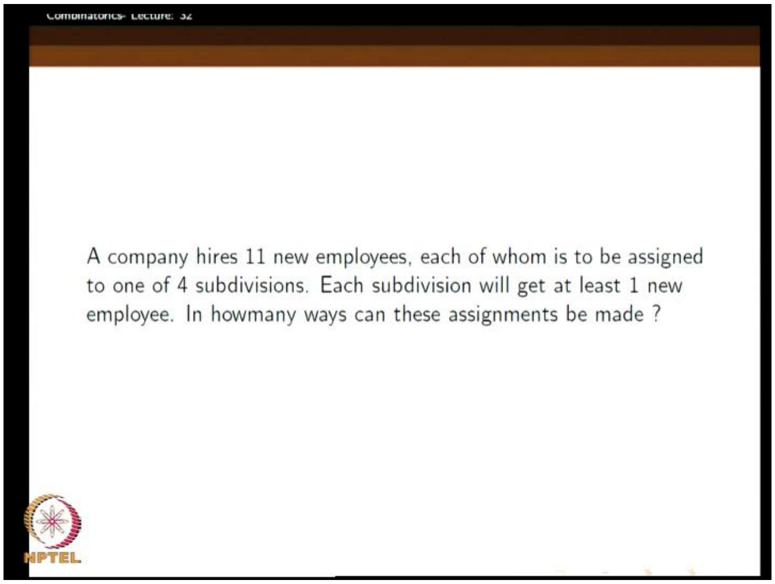

Now, that is all about. Next question is this. A company hires 11 new employees, each of whom is to be assigned to one of 4 subdivisions. Each of division will get at least 1 new employee. In how many ways can be assignments be made, right.

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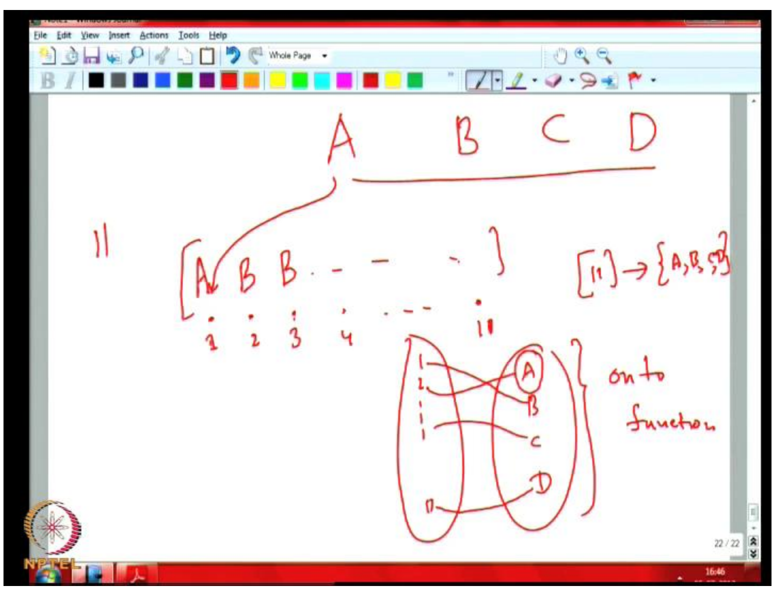

So the, not that, so let subdivisions are called a b c d. 11 people are coming. So, there are different people, of case. So, now we can assume that we have positions. So, the first one, this is the second one and the new coming, the newly selected people, right, for the company. So, these are the 11 people, right. Now, we can write on top of them A, if the first guy is going to the subdivision A. We can write A there. If 2, the second person is going to the subdivision b, can write here. Third position can be written b and if the third person is going to b and so on.

So, that is a sequence of letters; 11 letter word we have produced. The question is, how many 11 letter words can be produced, right. Using these letters a b c d, such that, every letter appears at least once. So, in other words, in the function terminology, it will look like this. The people are written here, 1 2 3 4 5 6, sorry,1 2 3 up to 11 and here, the subdivisions a b c d. Each person is assigned to a subdivision, right. Clearly this is a function because each person is assigned to subdivision and same person is not assigned to more than one subdivision. One person can be assigned to only one subdivision and each person has to go to a subdivision, right.

So, this is a function from 11 to a b c d ok and it not just a function. We also have the constrain that each subdivision has to get at least one person. So, there is nothing like a does not get any free image in this function. So, it is in d onto function. So, onto function from 11 to 4. 11 size set to a 4 size set. How many are there? We have used the inclusion exclusion principle to count this. We can always ask this in a more general setting. But, to illustrate the point, we can take the example of 11 and 4, right. So here, we are trying to use this concept of exponential generating functions to get the answer.

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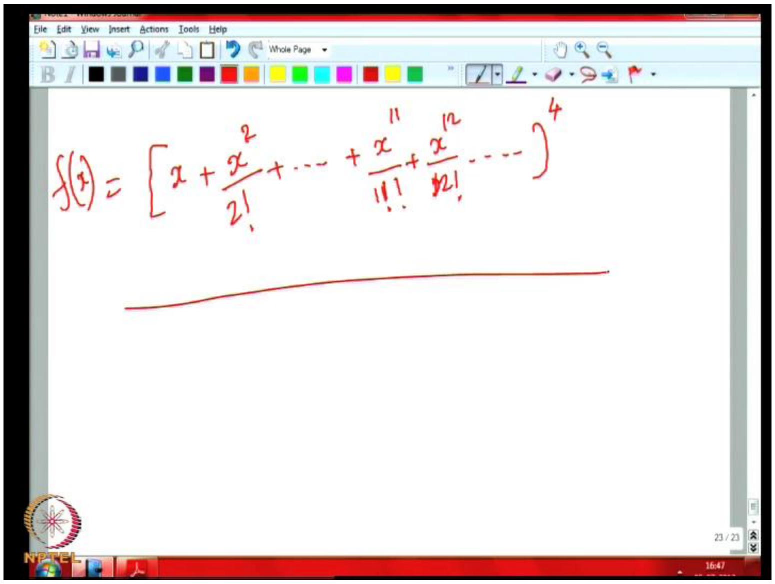

How will you do this thing? So, the first thing for subdivision a, we can give either one person or two persons. So, we write x for one person, x square by 2 factorial for second person, until we can have all the 11 persons, sorry not possible, right. So, 10 up to 10 person, we can give it. Not even that. But we do not mind. We will put it full, because it will be automatically taken care of. So, we write the full infinite sequence here, for each term and we introduce a term for each subdivision. So therefore, this raise to 4. We will say that this is the exponential can writing for the answer. So, I will continue this in the next class.