Combinatorics
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Lecture - 31
Solving Recurrence Relations using Generating Functions - Part (2)

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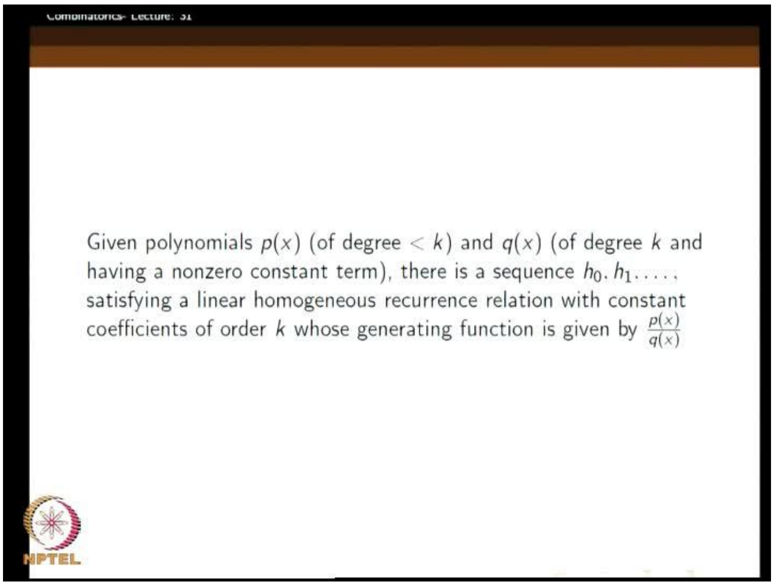

Welcome to the thirty first lecture of combinatorics. In the last class, so we have seen that given sequence h 0, h 1, h 2 etcetera satisfying a linear homogeneous recurrence realtion with constant coefficients and of order k. We can find a generating function of the form p of x by q of x, where p of x is a polynomial of degree strictly less than k and q of x is a polynomial of degree k, and the constant term being equal to 1.

Now, we will show that the converse is also true, suppose you are given a polynomial some, some generating function of the form p of x by q of x, where p of x is a polynomial of degree less than k and q of x is a polynomial of degree k. And having an non-zero constant term, we are not insisting that the constant term be 1. So, we just wanted to be non-zero. And then there will always be a sequence h 0, h 1, h 2, h 3 etcetera, such that this is the generating function… this is the, not only that, the sequence will be there, such that it will satisfy a linear homogeneous recurrence relation with constant coefficients of order k. And the generating function for that sequence will be this one; this is our aim today, aim now.

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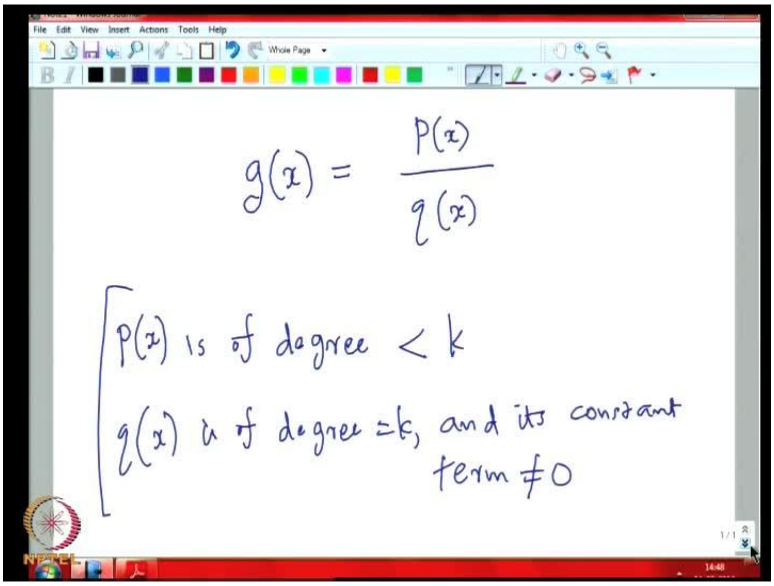

Again I will repeat, so we are given a generating functions something some g of x of the form p of x by q of x. And what are the properties of p of x, p of x is of degree less than k and q of x is of degree equal to k and it is constant term not equal to 0. This is what we have two conditions about p of x and q of x is given.

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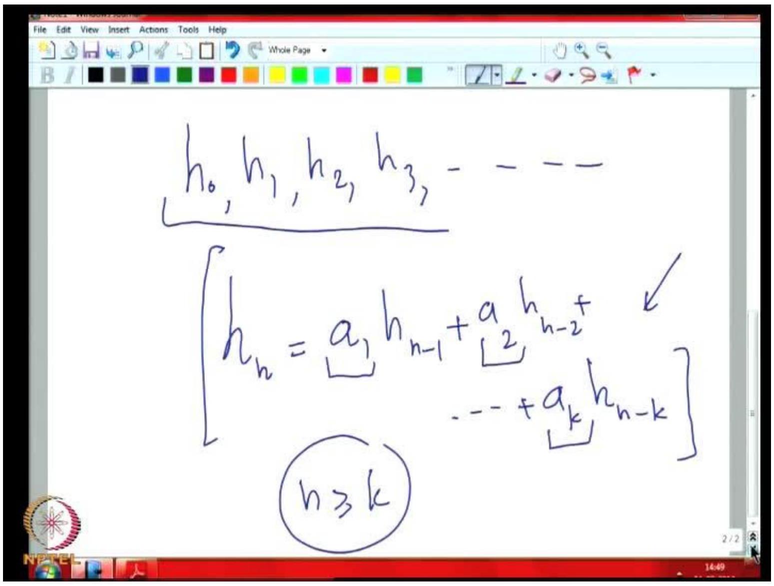

Now, we say that we will show that there exists a sequence h 0, h 1, h 2, h 3 etcetera. Such that it will satisfy a relation of this form h n equal to a 1 into h n minus 1 plus a 2 into h n minus 2 plus a k into h n minus k. This is a order k recurrence solution and is co-efficient of constant and definitely this is a homogenous recurrence relation, linear homogenous recurrence relations, this is what we are going to show. So, we have to show that show the initial values h 0, h 1, h 2, h 3, h k minus 1 can be found out and also show that for the terms h n where n is greater than equal to k, so this relation will be to this equation will be satisfied, this is what our aim.

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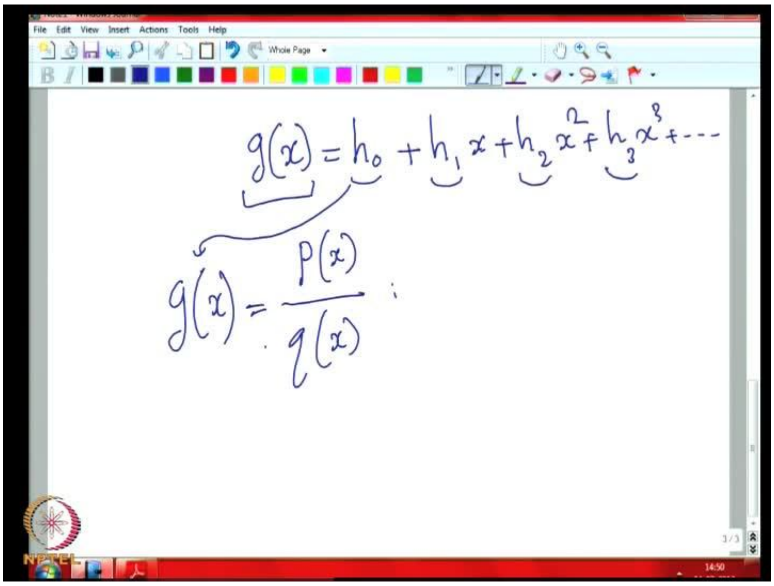

So, what we do is we just write, so the, so the g, suppose there is one such sequence then g of x has to be like this right h 0 plus h 1 x plus h 2 x square plus h 3 x cube and so on. So, we have to we are talking about this h i's, if at all such a such a sequence exists g of x has to be like that. So, our claim is that this g of x is equal to p of x by q of x, where p of x and q of x are given and have the properties mentioned before. That means we have so, we can cross multiply this is a g of x. So, we can cross multiply.

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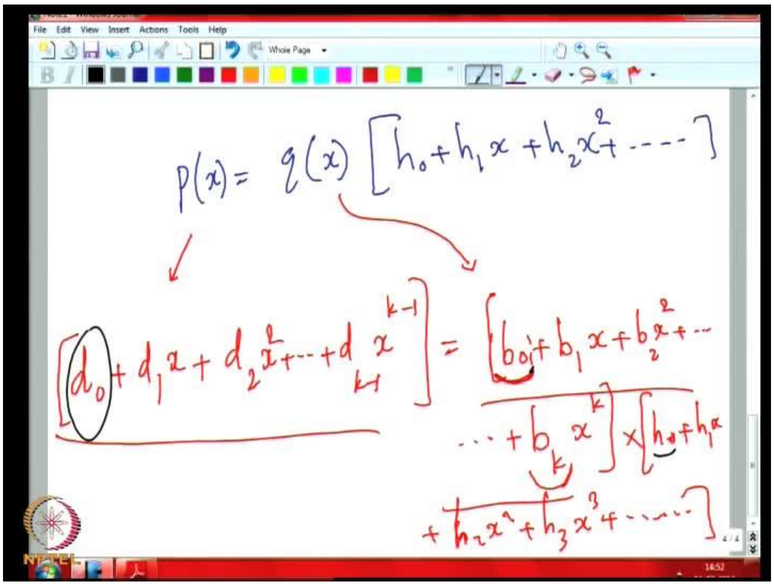

We get p of x equal to q of x into h 0 plus h 1 x plus h 2 x square plus and so on. Now, let us say this p of x is say d 0 plus d 1 x plus d 2 x square plus d h minus 1 x raise to k minus 1, because we are saying that p of x is a polynomial of degree strictly less than k minus 1. So, then let q of x be equal to say b 0 plus b 1 x plus b 2 x square plus, like that so b k x raise to k. So, here you know the q of x is a polynomial of degree k. So, b of this b k is non-zero and also we have the assumptions that this is this, b 0 is a non-zero term.

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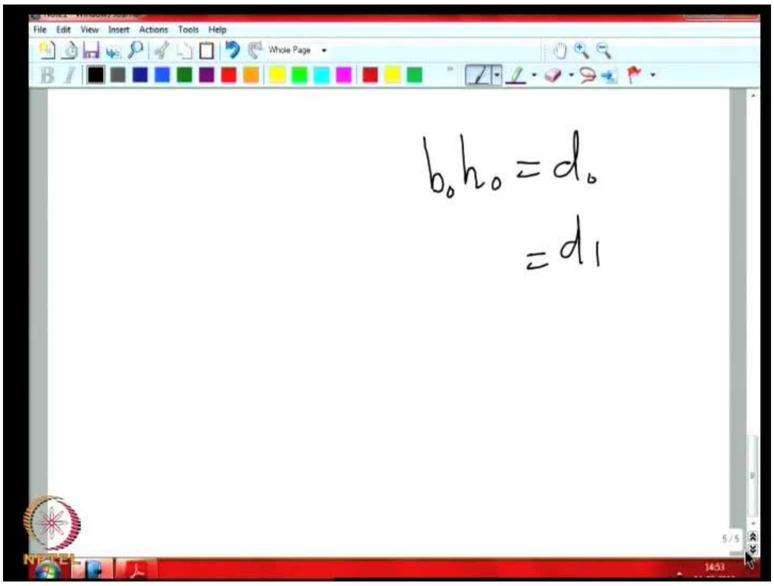

This is non-zero b 0 is non-zero and also b k is non-zero this is what we know. Now, so this is the equation this is p of x this is q of x into we have the h 0 plus h 1 x plus h 2 x square h 2 x square plus h 3 x cube and so on. Now, what we do is we compare the, the like terms that means the first we compare the constant terms. So, the constant term here is this d 0 and here the constant term will be b 0 into h 0 right there is no other constant term here. So, b 0 there is no other contribution to the constant term. So, we get b 0 h 0 is equal to d 0 this is first or we can write b 0 h 0 is equal to d 0. Now, we can look for the co-efficient of x, so the co-efficient of x here is d 1.

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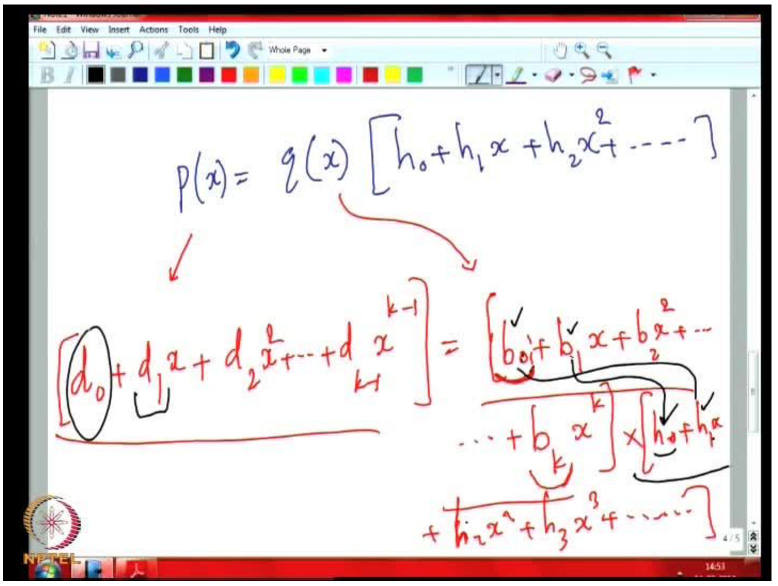

So, we write d 1 equal to what ,the co-efficient of x from this side the from the RHS the right hand side is b 0 into h 1 plus b 1 into h 0, d 0 into h 1 plus d 1 into h 0. Because, here b 0 is a constant term here h 1 has an x with it h 1 x, so b 0 h 1 x will come here b 1 has an x b 1 x, b 1 has an x with it. So, we have to multiply the constant terms of the, the second part, b 1 into h 0 that is that is what we get.

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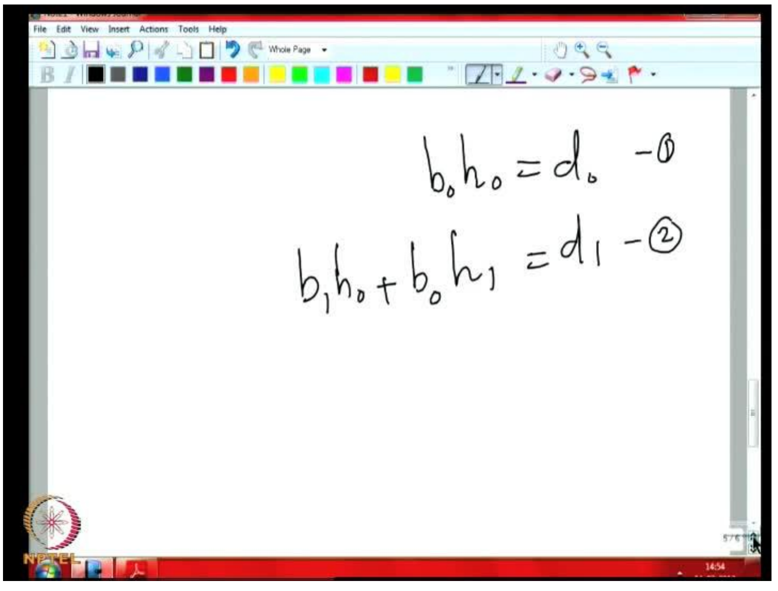

So, b 1 into h 0 plus b 0 into h 1will be d 1. So, these are the equations one, this are the equation two. Now, we can compare the terms corresponding to sorry the co-efficient of x square. So, d 2 is the co-efficient of x square so, d 2 will be equal to what? d 2 will be equal to b 0 into h 2 because, b 0 h 2 will make it, and now, from here b 1 into h 1, b 2 into h 0.

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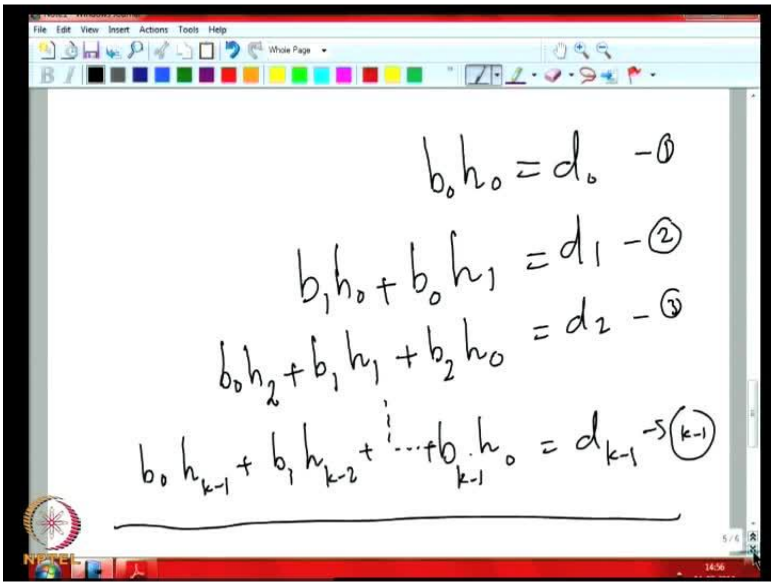

So, this will be b 0 into h 2 plus b 1 into h 1 plus b 2 into h 0 this will be d 2 and like that we can compare the co-efficient of each term and you see… So, we can write for the d k minus 1 that is the co-efficient of x raise to k minus 1. This as b 0 into h k minus 1 plus b 1 into h k minus 2 plus like that, when we say b k minus 1 into h 0 this will be the 2. So, this is the k minus 1 th so, this are so many equations this is k minus 1 th equation.

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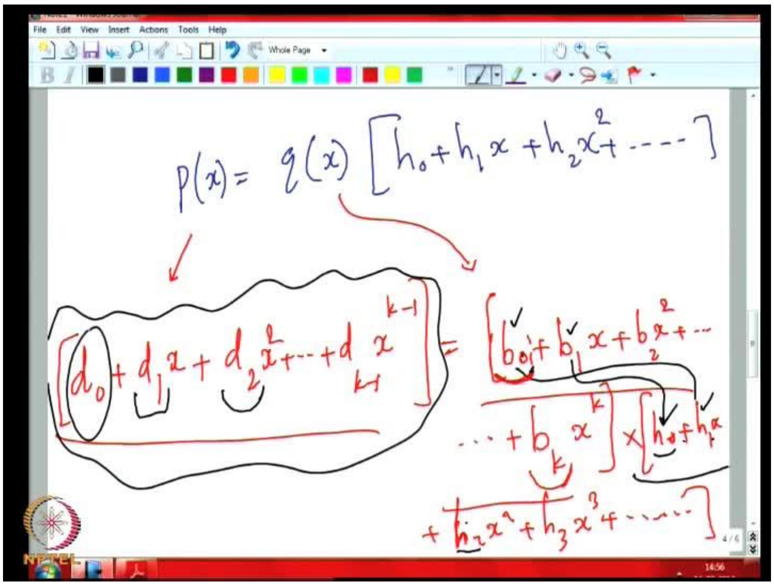

Now, from now on what we get is, so for instance when you consider the the co-efficient of x raise to k, what will happen. The co-efficient of x raise to k is d k that will be equal to see on the other hand we should not that this d k is 0, why because here this is p of x, it is a polynomial of degree at most k minus 1. So, d k has to be 0, so this is 0 and then now, you look for the co-efficient of x raise to k from the RHS. So, from the RHS we will see that b 0 will multiply h k and b 1 will multiply h k minus 1 and so on.

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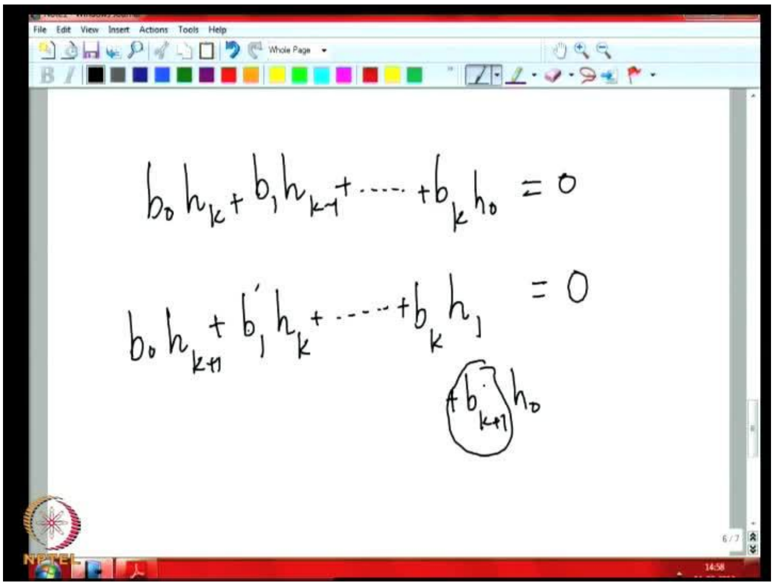

That is the way we get this thing b 0, h k plus b 1, h k minus 1 plus, like that. So b k, h 0 this will b equal to 0, this is what we get. Now, from now on everything will be like this for instance if you want to consider the co-efficient of x raise to k plus 1 again it is 0 from the left hand side because, p of x does not have any x raise to k in it because, it is a polynomial of degree at most k minus 1. So, that will be again 0, but from the other side we will get b 1 into h k… what will get is b 0 into h k plus 1 because, we are talking about k plus 1.

And then plus b 1 into h k because, you know this will in the first term b 1 x will come h k into x raise to k that will provide x raise to k and like that. So, all the way to b k into b k has x raise to k already so, we have to take it 1, b k has h raise to k we have to take h 1. But you see you cannot get the next term for there is nothing like b k plus 1 into h 0, why because this b k plus 1 is anyway is 0.

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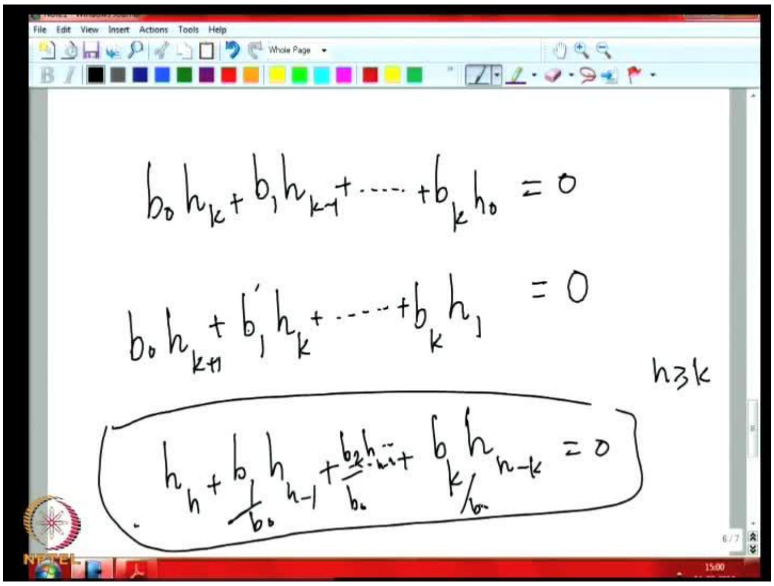

Because, our polynomial q k this polynomial, q of x, q of x this q of x was a polynomial of degree k therefore, up to b k only we have here. So, b k plus 1 onwards is 0 therefore, this term will not be there. So, we will stop here and this is equal to 0 is what. So, like that for an arbitrary term, when you take the co-efficient of x raise to n for n greater than equal to k what we get is b 0 into h n plus b 1 into h n minus 1 plus.

So, b k into h n minus k that is what we will get, because each term will give a contribution to the co-efficient of x raise to k. But from b k plus onwards we have 0 therefore, no more contribution this is always 0. So, we can see that b 0 is a non-zero quantity therefore; we can divide each term by b 0. So, that means this b 0 will disappear and divided by b 0 we can put here and say here we have b 2 into h n minus 2. So, divided by b 0 we can put here, so here also. So, everything will become something like b.

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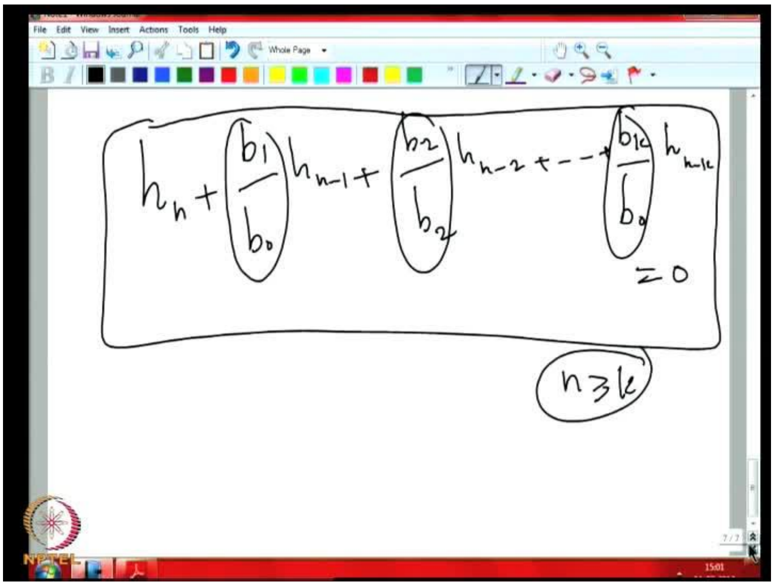

So, 1 plus b 1 by b 0 into h sorry… Once again, h n plus b 1 by b 0 into h n minus 1 plus b 2 by b 0 into h n minus 2 and so on up to b k by b 0 into h n minus k equal to 0 is what we get. So, this is the recurrence relation, this is you can see that this is the recurrence solution of order k and these are constant coefficients and this is a linear homogeneous recurrence relation. The only thing is how the only thing is left is how we are going to get the initial values. So, this recurrence solution is valid for n greater than equal to k we have seen. How to get the initial values?

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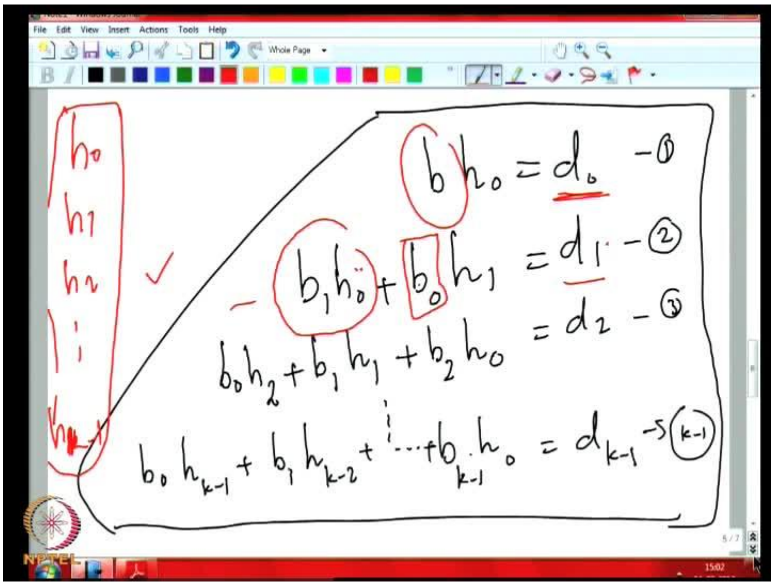

This initial values are easily available from the first k minus 1 equations we have wrote first here, it is a triangular system of equations for instance we look at this. We know that this b 0 is non zero and d 0 is given also. So, we can divide d 0 by b 0 and get h 0. Now, once you get h 0, so we know b 1 so, b 1 into h 0 can be calculated this one, now b 1 is known. So, we take this part to the other side that means b 0 into h 1 is equal to d 1 minus b 1 times h 0 but, this again this b 0 is non-zero. So, we can divide by b 0 and then we will get h 1, like that we can get the values for h 0, h 1, h 2 up to h k minus 1. So, from there onwards do not need the values because, the recurrence solution is valid, so as we have seen before.

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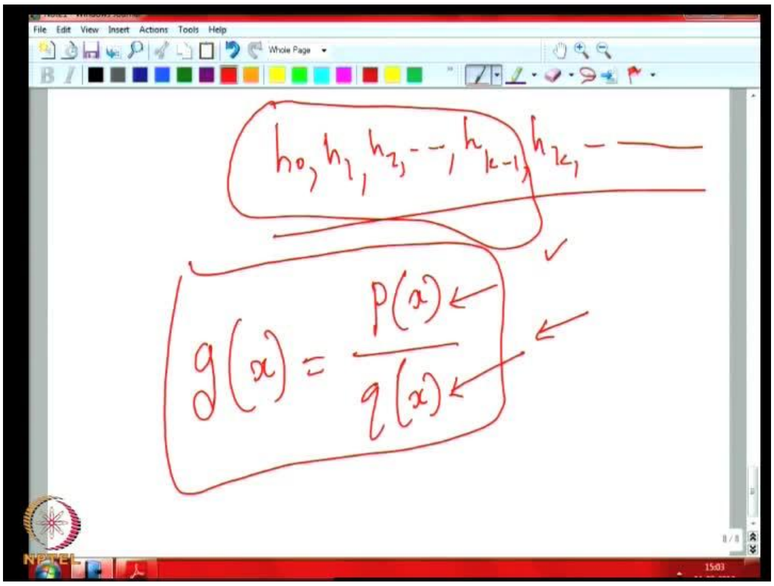

So therefore, we have obtained a sequence h 0, h 1, h 2, h k minus 1 h k like that, satisfying a linear homogenous recurrence solution with constant coefficients and of order k and we found the initial values. So, that means what we have proved is so, if g of x is given p of x by q of x, for p of x any polynomial of degree less than k. q of x polynomial of degree k and constant term not be in equal to 0. Then we can get the corresponding sequence for this g of x this is what we have shown. Now, we will…

So, we having one too many examples, but for this we do not need too many examples because, the method is very clear it is always. So, we have done the general cases and we have done a little some analysis of those the method. And we showed that how we can get generating functions always for a sequence satisfying non-linear, linear homogeneous recurrence solution constant co-efficient and of order k.

And reversely for certain kind of g of x is how we can make sure that it is very general that g of x, we can make sure that there is a sequence. So, that that is the generating function. So, we just need p of x by q of x 1. q of x is only the condition, but it is of degree k and it is constant terms it is not 0, p of x is of degree less than k that is all. So, we do get a generate sequence satisfying a linear homogeneous recurrence solution with constant coefficients and of order k. And such that this is the recurrence generating function for that. Now, we will now our plan is to consider some slightly different kind of recurrence relations.

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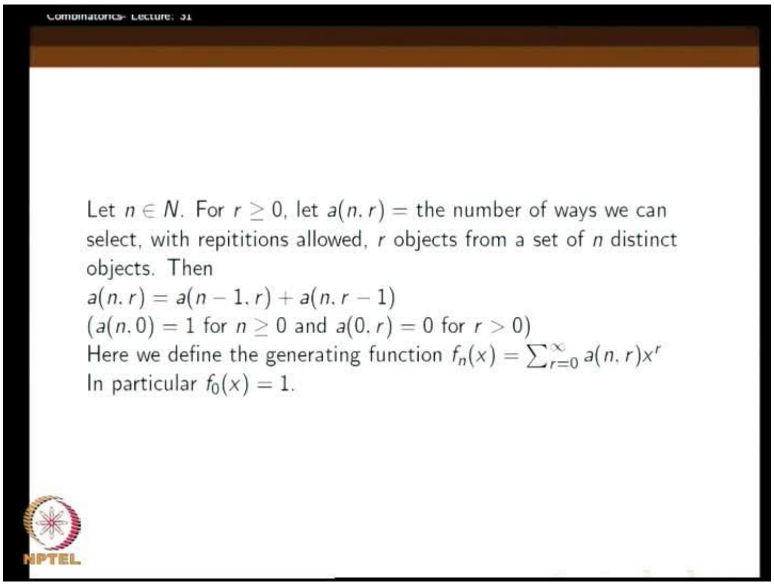

Then what we were considering up to now and we will solve it, this for being familiar with it. So, these problems themselves are not very unfamiliar for instance, we let us look at the problem. Let us define a of n coma r be the number of ways we can select with repetitions allowed or objects from a set of n distinct objects. So, we have already seen this problem we know that a of n coma r is n plus r minus 1 choose r. So, n plus r plus n minus 1, choose r that is what we have seen. So and then now, we want to get the same thing using the method of generating functions, just for a practice. So, we have though we have already know it. So, this is, but then to use generating functions we have to mean the way we were studying generating functions, as I told to solve the recurrence relations we would first have to come up with a recurrence solution for this a of n coma r. So, let us think about it.

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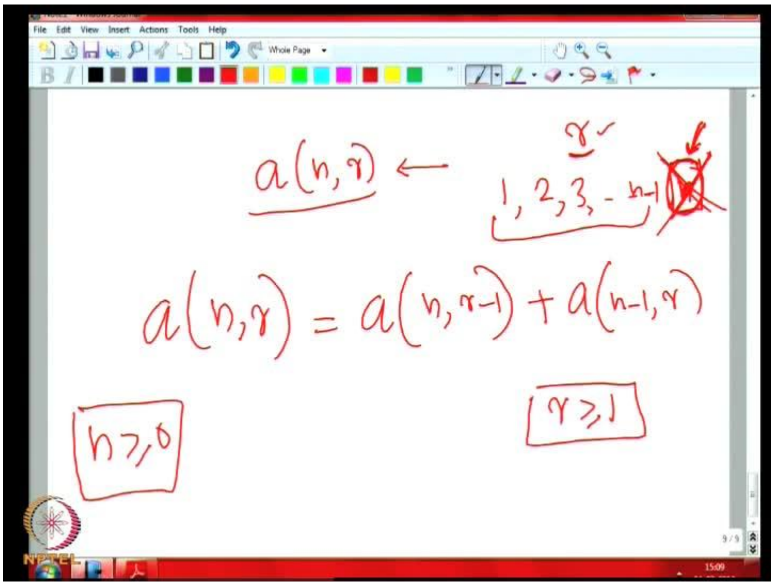

Again, a of n coma r if the number of ways to select r things from n different objects different types of things or something different objects, we want to select r things, but repetitions are allowed. So, we have studied this in detail before. Now, what we can do is we can write a recurrence solution like this a of n coma r is equal to… to write the recurrence solution we consider like this. So, they have n objects write 1 2 3 up to n, out of this n objects let us consider this n th object and if we decide to pick this n th object once at least once.

Then you know, that means that we have to select then r minus 1 more objects from the n objects. The repetitions are allowed because the objects will not disappear because I took this once because the n th object is taken once it is again available, because the repetitions are available repetitions are allowed. So therefore, once we decide to take n, the only we have to seek r take r minus 1 objects, but all the n objects are available. So, that is a of n coma r minus 1, the other cases when we decide not to take this at all, this object is not at all being used, that means we have only n minus 1 objects available we have to take all the r objects from that.

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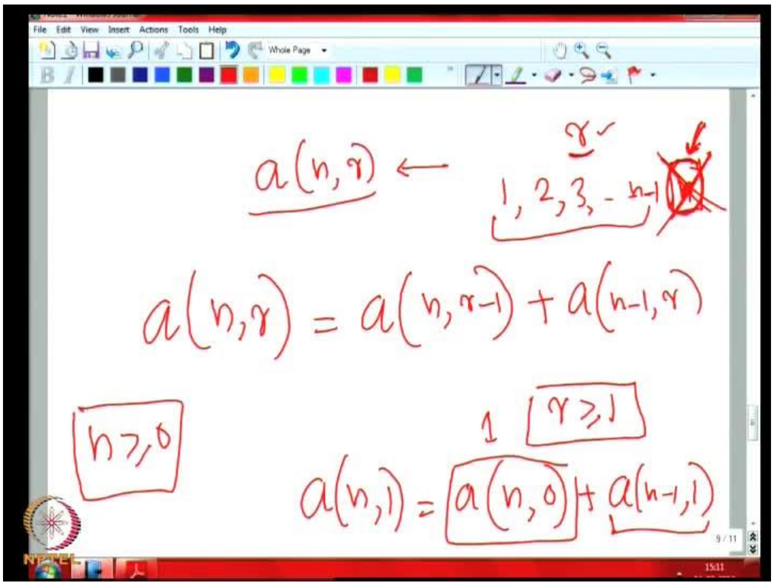

So, that is a of n minus 1 coma r, this is the recurrence relation. And now, for what values so, fixing an n, so what values of r this is valid. So, we can say that this is this is valid for r greater than equal to 1 because, so here let us say n is will assume n is greater than 0 of case. Now, if r is greater than… so, one thing is very clear if a of n coma r if r is right, so 1 a of n coma r. So, we are assuming n is greater than equal to 0. Now, a of n coma 1, that means I want to take 1 object from n objects.

So, if I decide not to take it, so we will get a of n coma 0. So, that has to be different… So, if you decide to take that particular object one and n equal to 1 here, then that means we want to pick up zero objects from n objects. So, we will decide so, we will write a n 1 equal to a n 0 plus the other thing is and I decide not to take it, then that is a n minus 1, 1. So, this is we know that n minus 1, 1. So, one object from out of n minus 1 you can take in n minus 1 ways, this has to be defined as 1 that.

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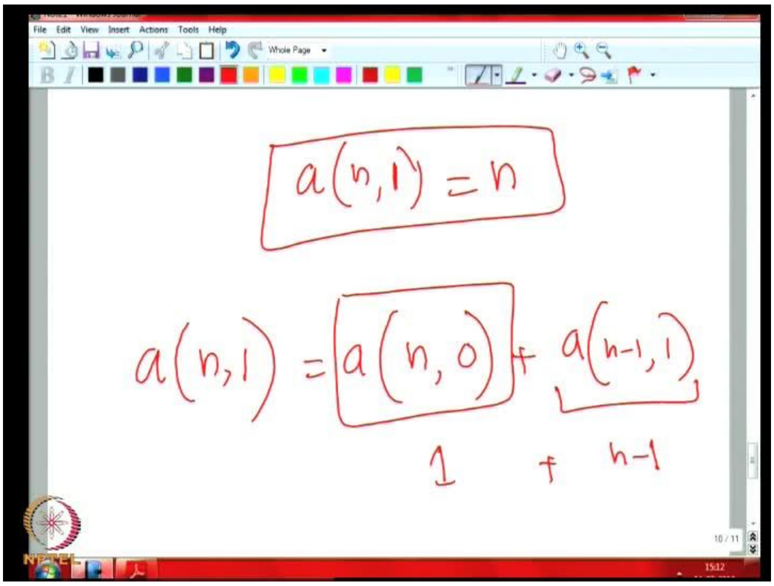

So, see intuitively what I am trying to tell is a n 1 a n 1 has to be equal to 1, we know that because if you want to select one thing, one object out of available n things. Repetition allowed because, anyway repetitions does not make much sense because, we are taking only one object. So, the answer has to be n here right because, there are n ways of selecting it. Now, if I want to make the recurrence solution true, so a n 1 is equal to a n 0 plus a n minus 1, 1. So, this should be n minus 1 plus this has to be 1, so n a n 0 should be defined as 1. That means when I am thinking of selecting zero objects with repetitions allowed from out of n, we should define it as 1.

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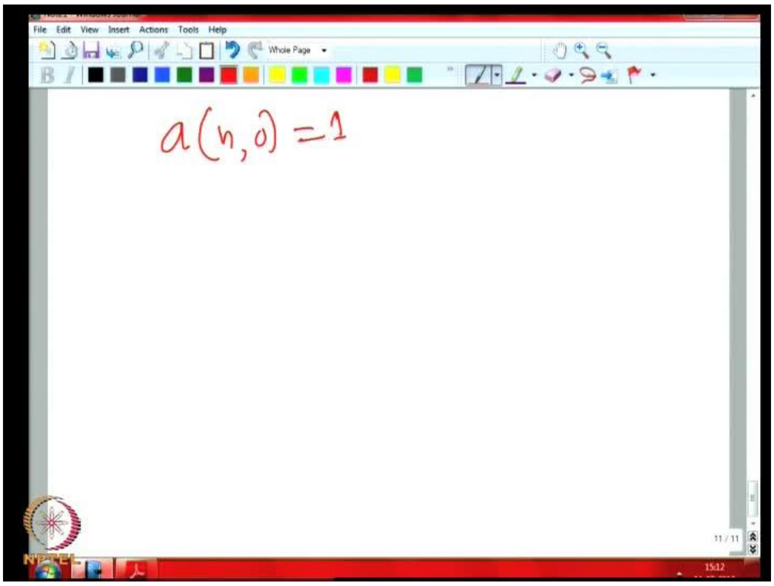

So, we have a n 0 is equal to 1, this is one thing.

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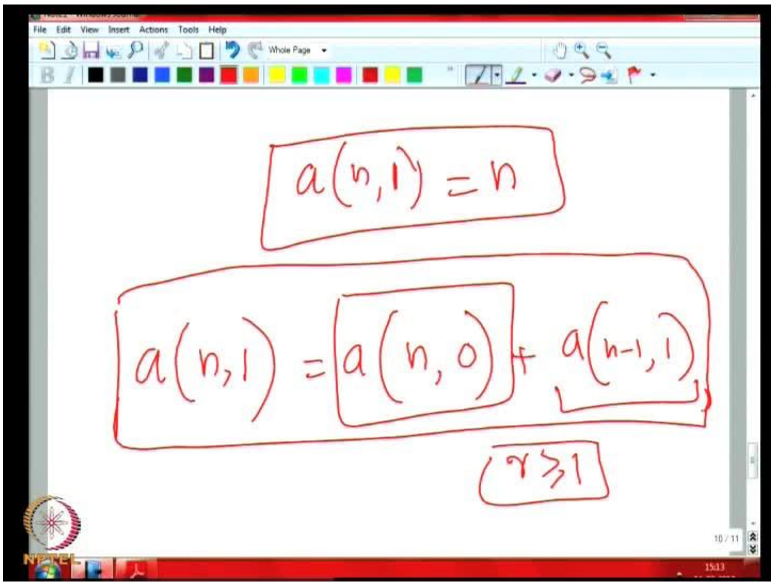

Now, where will we start for instance what about... So, we will design as decide that this recurrence solution is true for all n greater than equal to 1, for n equal to 1, sorry for r all r greater than equal to 1. For r, all r greater than equal to 1 this is true or if you put a n 0 than we will have to define something for a n minus 1 also, that we are not going to do. So therefore, we will say that this is valid only for r greater than equal to 1.

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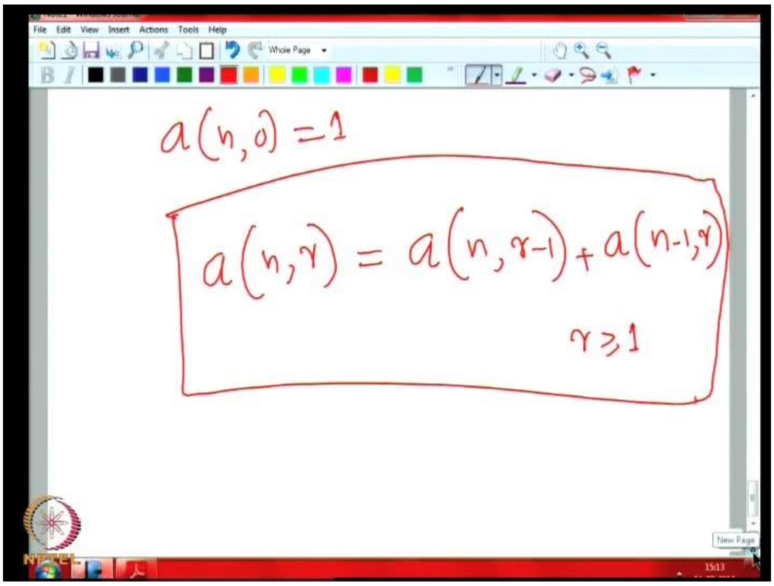

So, once again I will write the recurrence solution a n r is equal to a n r minus 1 plus a n minus 1 r, this is true for r greater than equal to 1. Now, we will define some generating functions for this sequence.

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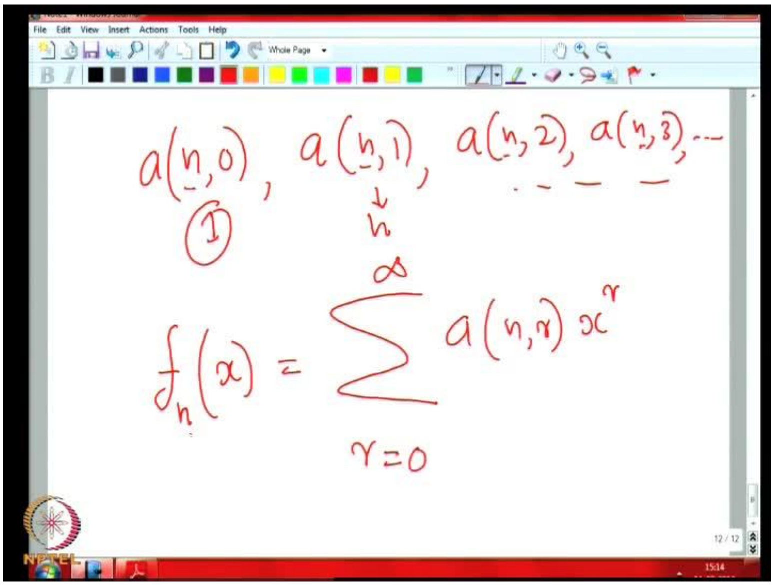

So, what about a n 0, so consider this sequence a n 1, a n 2, a n 3 and so on, you remember this has defined to be 1 and this will be n and like this. This sequence will give raise to that generating function. So, let us write it as f n because this n becomes the parameter in the generating function f n of x is equal to sigma r equal to 0 to infinity a n r into x raise to r. So this is a generating function will start from 0 onwards.

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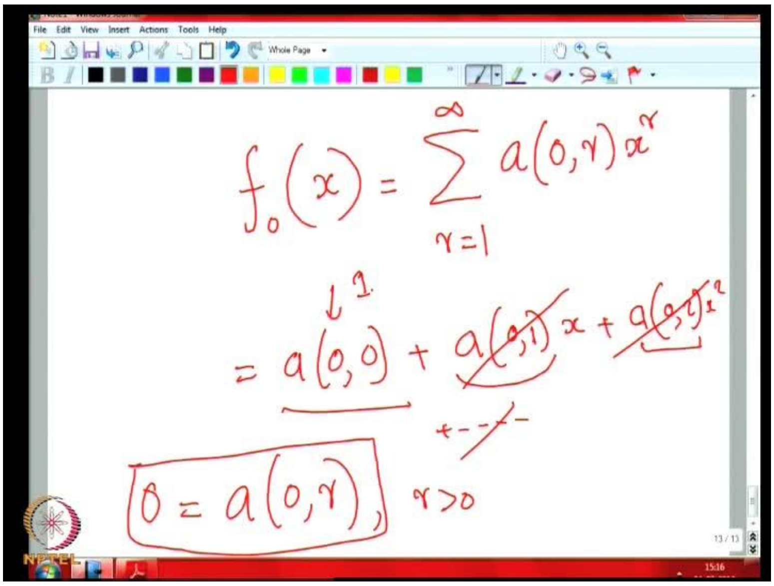

So, we have to so, we know what happened is defined for n equal to 0 onwards. So, for instance what will be f 0 of x f 0 of x, f 0 of x has to be sigma of a of 0 coma r, r equal to1 to infinity x raise to r. Here we have the first term as a 0, 0 plus next term is a 0, 1 x raise to 1 plus next term is a 0, 2 x square and so on. But here we are talking about selecting one objects from 0 objects, this should be 0, this should be 0 and so on. So, all this things will be considered 0. So, that means we will define a 0 coma r, where r greater than 0 as 0 you will define this way right. So therefore, this will this will not be with this will be this f 0 of x will be just a 0, 0 which is 1. So, we note that f 0 of x is just 1.

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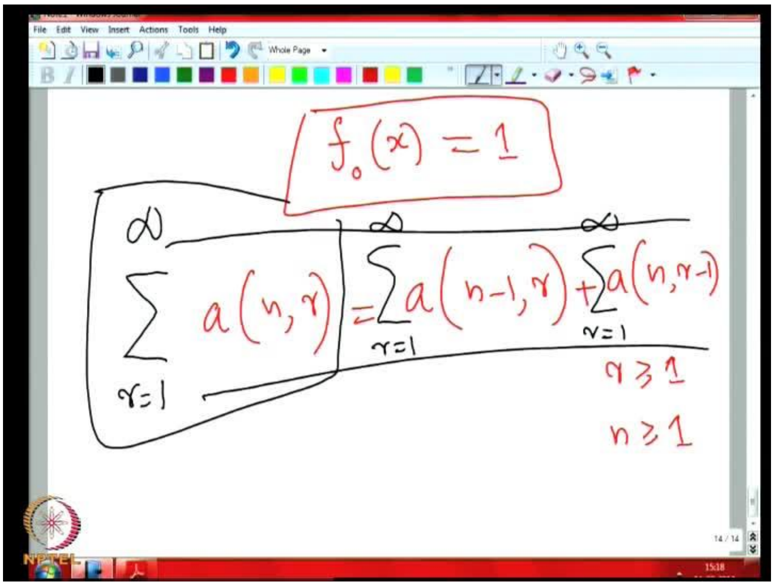

Now, recurrence solution we told that is a of n coma r equal to a of n minus 1 coma r plus a of n coma r minus 1. This is valid for r greater than equal to 1 only, also we will make that this is valid for n greater than or equal to 1 only. Because, because otherwise we will have to again think about what will be a minus 1 r. So, we will think that n is at least 1, so that we can get this thing. Now, what we are going to do is from this recurrence relation, I will again write it once again. Because this is what this recurrence relation, so this is the thing. I will sum it from r equal to… this is remember this recurrence solution itself is valid only from r equal to 1 onwards, r equal to 1 to infinity, I will sum. So, this will become again r equal to 1 to infinity, this side also will r equal to 1 to infinity then what is happening.

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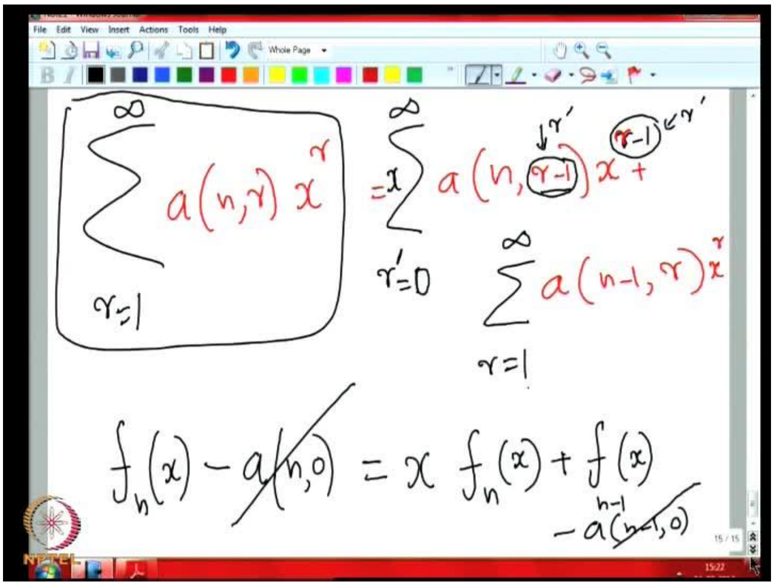

So, this is as we have defined if we had sum from r equal to 0 to infinity this, so what we have to do is each term should be multiplied by. So, I will, I will make it clear into fine, so before summing this, what is should do is, so we write like this. So, this is equal to this I will go back again once again, a n coma r is equal to a n coma r minus 1 plus a n minus 1 coma r, this is the equation. Now, we can multiply this by x raise to r that is right. Here also we multiply by x raise to r, here also we multiply by x raise to r. Now, we sum, sum r equal to 1 to infinity and so, this should be multiplied by r equal to 1 into infinity and this should be multiplied by from r sum r equal to 1 to infinity.

So, if you had summed if the sum was from r equal to 0 to infinity that would be just f n of x, but then we are taking from r equal to 1 to infinity. So, we get f n of x minus the first term a n 0. This is equal to here also the same thing just that we have to do some adjustment here this is r minus 1 that is x raise to r. So, what we can do is we can change this to r minus 1 and take an x out here, I can take an x out here. So, x into x raise to r minus 1. Now, this variable can be changed this is r minus 1 let us call it r dash.

So, this will also become r dash and now we can say that this r dash goes from 0 to infinity, because r goes from 1 to infinity is an r dash equal to r minus 1 will go from 0 to infinity. So, this is x into r dash equal to 0 to infinity a of n coma r dash into x raise to r dash, which is essentially the generating function f n x and what about this term. This term is just r equal to 1 to infinity n minus 1 r, x raise to r and you know if had sum from r equal to 0 to infinity this would have become just f n minus 1 of x and minus a, this would become minus a minus a n minus 1 0, because the first term should be minused. So, but a n 0 is 1 a n minus 1 0 is also 1. So, we are remembered we are only talking about this entire thing n is assumed to be at least 1. So, n minus 1 0, n minus 1 0 is a 0, 0 that is all our case a f n minus 1 0 is already defined, we know that just 1, in that if n equal to 1.

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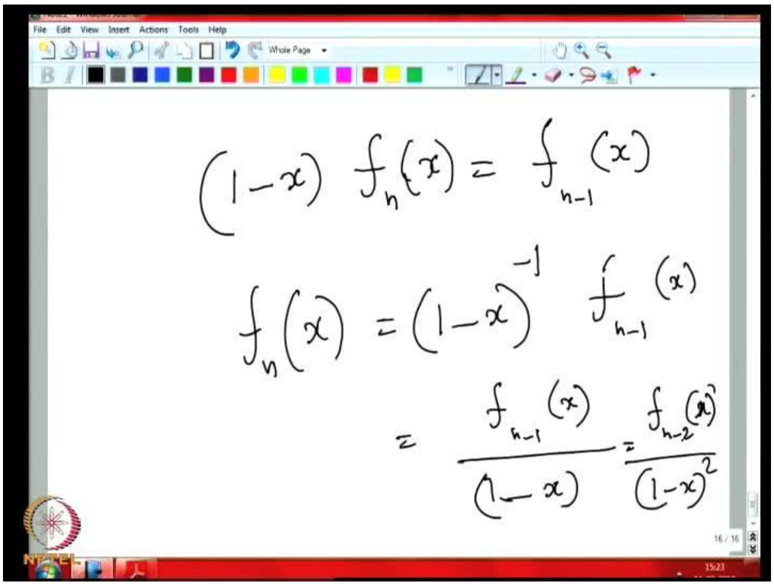

Now, we can rearrange the terms and get that 1 minus x into f n of x equal to f n minus 1 of x. So, that is f n of x is equal to 1 minus x to the power minus 1 into f n minus 1 of x or we can write it as this way also f n minus 1 of x by 1 minus x. Now, you see you keep unrolling it f n of x is equal to f n minus 1 of x is equal to f n minus 2 of x by 1 minus x square and so on.

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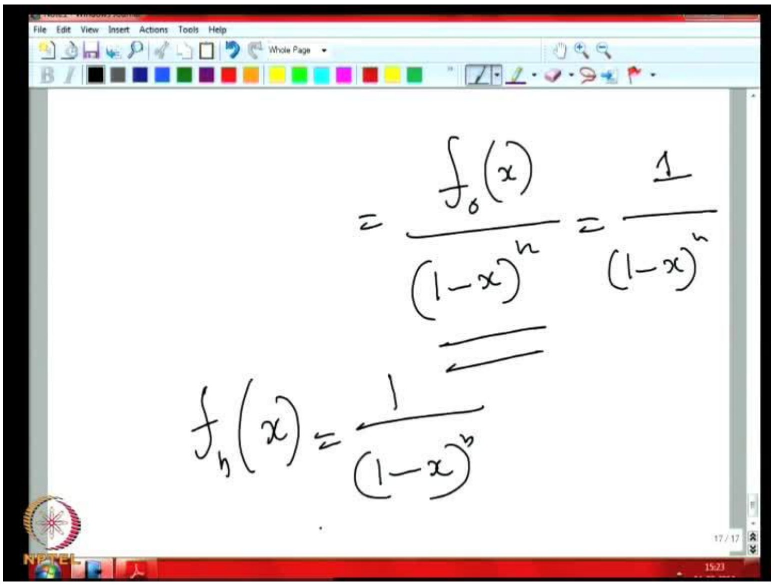

Now, when it goes to 0 this will become f 0 of x that means n times we have done it. So, this is the, this is the answer, but then this f 0 of x is just 1 as you have seen, 1 by 1 minus x raise to n. So, we get f n of x is equal to 1 by 1 minus x whole raise to n. So, this is generating function.

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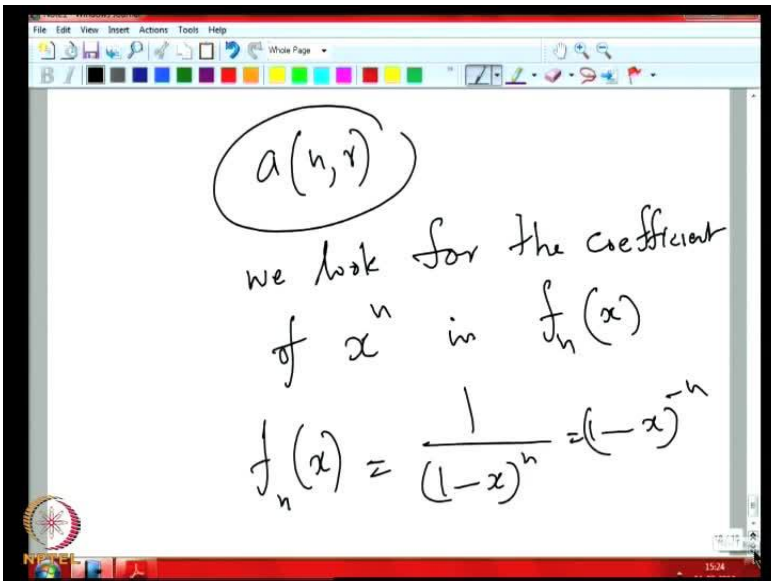

Now, if you want to find a n coma r, we want to we look for the co-efficient of x raise to n in the expansion of f n of x. This will be f n of x, what will be that because f n of x is equal to 1 by 1 minus x raise to n power minus n.

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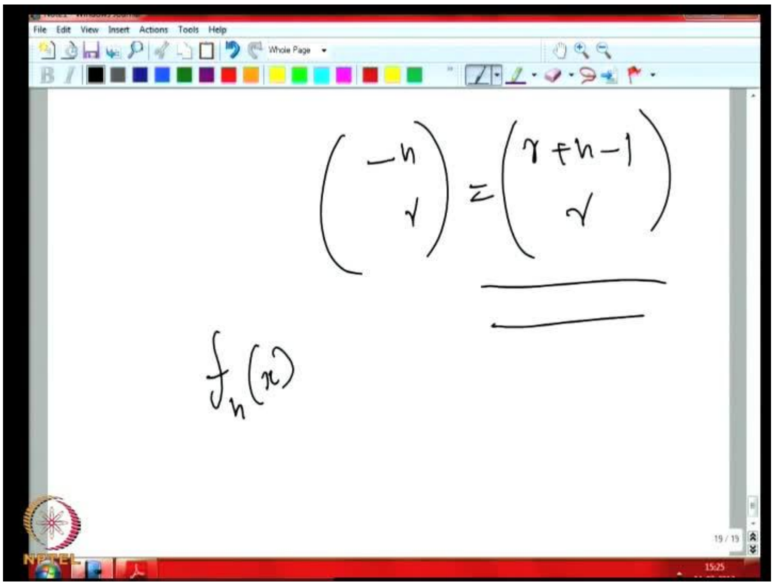

We know the co-efficient is minus n choose r, which is what r plus n minus 1 choose r. This we already know the answer, but we can derive it this way of also using the method of generating functions. So, that is, that is the derivation for this thing, this is a slightly different thing. So, in the sense that we had a sequence corresponding to each n for a fixed an n and then we had sequence corresponding to that. And corresponding to that we could get one generating function f n of x. So, when we are not dealing with just one generating function we had a family of generating functions here. Therefore, this example was interesting though the problem was already familiar to us and we knew the answer already.

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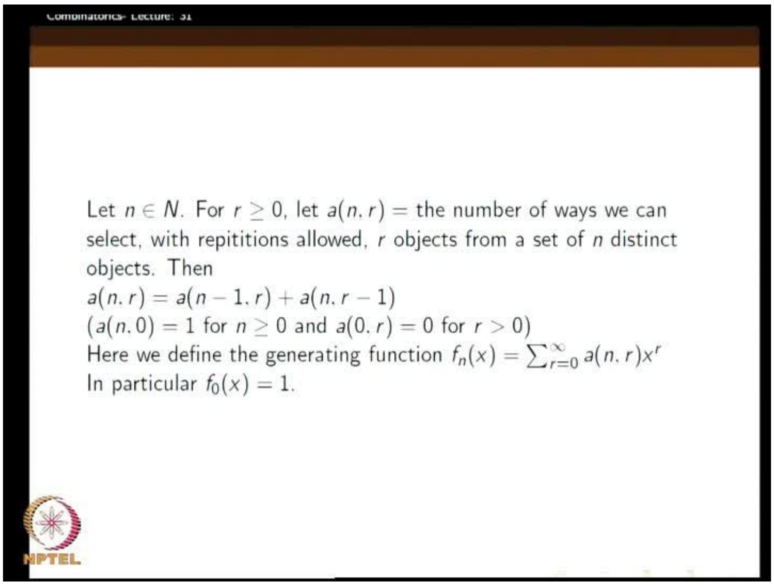

Now, we will take a very similar example.

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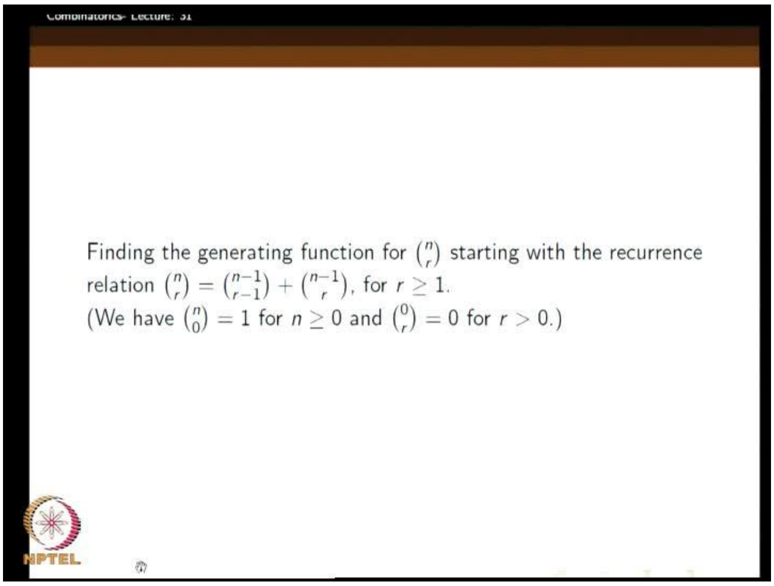

And we already know all these things for instance n choose r, what is a recurrence solution known for that. The n choose r being the number of ways of selecting r objects out of n, without allowing repetition this time, more familiar.

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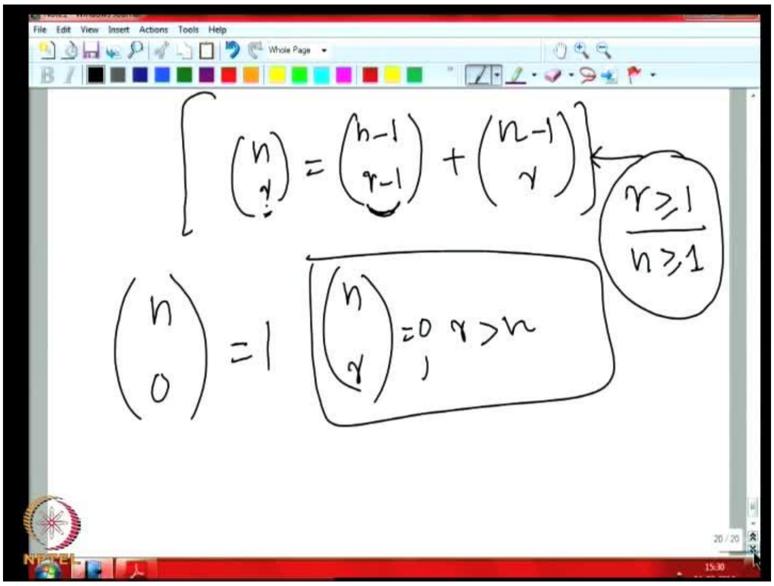

So, n choose r is equal to what, the same argument if you remember. So, you can select one object and then that will be that will make the selection like this or you can decide not to take it at all. So, that means you have to select from n minus 1 objects r objects this is the recurrence relation. This definitely works for n greater than equal to 1, but we have to be careful for instance n equal to…

So, this for instance we are only talking about… So, we already discussed the entire thing of right we had very elaborately discussed when we are allowing r equal to negative n all so on. But here let us just remind you or n choose 0 is always equal to 1. So, if you consider n choose r where r is bigger than n then we will get this is equal to 0, for n choose r is bigger than n, you will get this is equal to 0. So now, what about this recurrence relation, this recurrence solution is it for instance n choose 1. So, we want to take one object out of n, so that means you can select that object. So, we want to select r minus 0 objects out of n minus 1 object this is 1, another way of doing it is you can decide not to select it. So, you want to select one object from n minus 1 object this was fine.

But if you put r equal to 1 the problem is we have negative n minus 1 choose minus 1 we are so, this recurrence solution is not valid. Because, we do not have any counting interpretation of n minus 1 choose minus 1 etcetera, we had discussed it earlier. So therefore, we will only say that this recurrence solution is valid for r greater than equal to

1 and similarly, we can say that for n equal to 0 out of zero objects we want to select r objects. So, then this recurrence solution will not be because we will be talking about selecting out of minus 1 objects something. So, that also will not assume, will assume that n is greater than equal to 1. So, if n equal to 1 that is fine, because you want to select r objects out of one objects.

So, we will be thinking about a selecting zero objects out of 0 and so on. So, for instance if it is bigger it should be 0 r is bigger than 1 then it will be 0. So, what happens is the will be taking something bigger here from zero objects and then both are 0. On the other hand if r was equal to 1 then we will be talking about selecting 0. So, this will be 1 0, choose 0 here and this will be 0, choose 1 which is 0 which is also correct. We have a very hided before again, when in the case of n equal to so, n 1 choose 0 that should be 1. So, that means 0 choose… r is not 0 r is 1 maximum therefore, that is fine.

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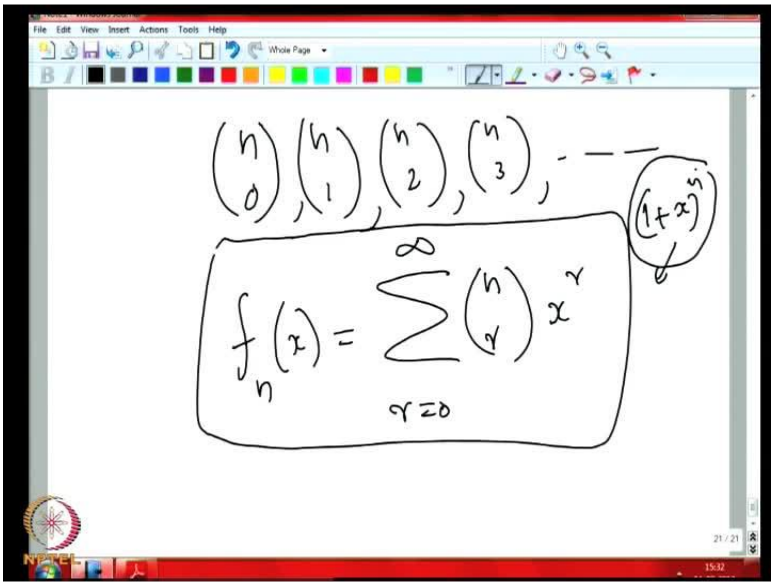

So, we are assuming that r is greater than equal to 1 n is greater than equal to 1 should verify it be carefully whether we can make this assumption. And then now, that we have the recurrence solution and we will define some generating functions for that, because we need like we did in the earlier case in the last case. We will define sequences like this fix an n and then we can define the sequence n choose 0, n choose 1 this is a sequence, n choose 2, n choose 3, n choose this is 1 sequence. And the generating function f n of x is defined to be sigma r equal to 0 to infinity n choose r x raise to r, this is familiar.

So, we know this thing is 1 plus x raise, but we will derive it. So, we will say that we are doing this thing, but is just to show how use the recurrence solution use our generating function methods some to derive the generating function see which already by binomial theorem. This is equal to 1 plus x raise to n, 1 plus x raise to n, but will get this thing by a different method just to illustrate.

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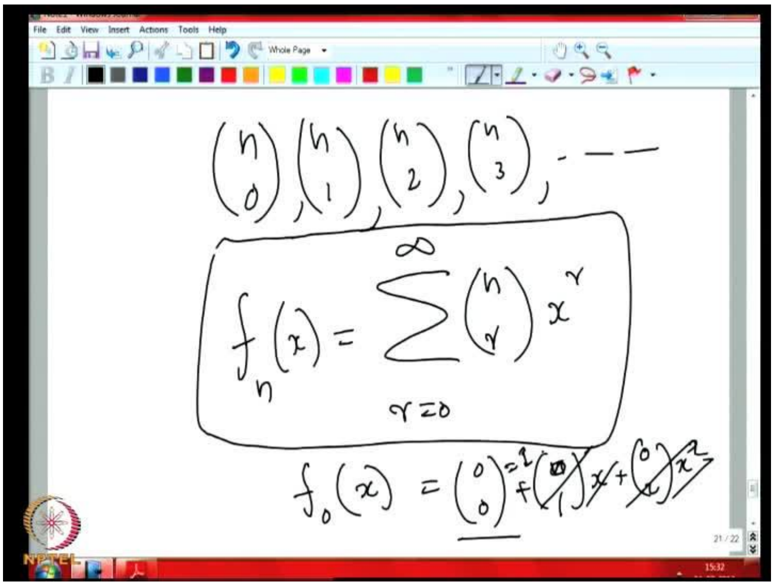

Now, just let us so, I will do one more thing, I will see what is because this recurrence solution was valid for n greater than equal to 1. So, I will see what is f 0 of x, f 0 of x will be n choose 0 plus… n 0 choose 0 plus 0 choose 1 x and so on, 0 choose 2 x square. But these are all 0’s therefore, they won’t be there, this is just 0 choose 0 and which is just equal to 1 as before, that 0 of x will be 1.

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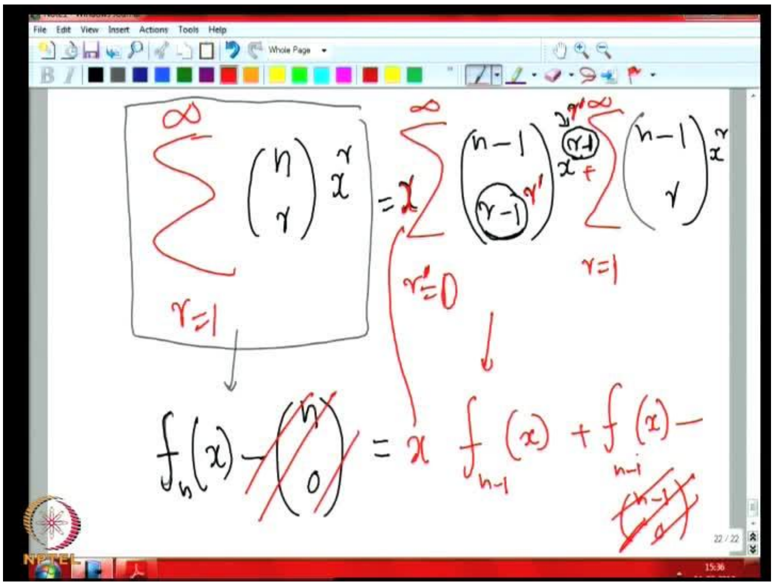

Now, let us look at our recurrence solution once again, it is n choose r equal to n minus 1 chooser minus 1 plus n minus 1 choose r. Now, we can multiply by x raise to r everywhere that is not a problem, then after that we sum. Here we sum only from r equal to 1 to infinity because, the recurrence solution itself is valid only for r greater than equal to 1. We cannot sum for n equal to 0 to because, r equal to 0 this equation we wrote is not correct.

So, this will also go from r equal to 1 to infinity and here this also will go from r equal to 1 to infinity. Now, what we do is this one if we had sum from r equal to 0 to infinity, we would just get f n of x, but then we just miss the first terms. So, we can write this as f n of x minus the first term is what n choose 0, x raise to 0 it is not necessary n choose 0. So, this is equal to what, here which is here this is r minus 1 here is r. So, what we do is we pull out 1 x 1 here and make it r minus 1. So, that means you will get an x here 1 x will come out 1 x will come out. Now, we can change the variable, so let us call it r dash now r minus 1 is called r dash, so this will be called r dash. So, we can use a different color to indicate that, we can call it r dash. So, this x was pulled out here.

Now, we see r dash is going from r dash is going from r dash is going from 0 to infinity because, r was going from 1 to infinity, r dash being r minus 1 that will go from 0 to infinity. So, this is perfect, so r dash is 0 to infinity and n minus 1 choose r dash into x raise to r dash. That we know this x anyway copied and this will become f n minus 1 of x because, this full summation is there r dash equal to 0 to infinity. Here we have n minus 1 choose r into x raise to r and that is going from 1 to infinity r is if r is going from 0 to infinity we could have just wrote f n minus 1 of x here. But then here we have to minus the first term because we dint take the first term here.

So, that is n minus 1 choose 0, x raise to n anyway that this 1, but n choose 0 and n minus 1 choose 0 both are 1 is not that n was at least 1. So therefore, n choose 0 this can be in the worst case n minus 1 0 can be 0 choose 0 therefore, we have it, both are 1. So, we can sell it of it is not necessary. Now, rearranging so we will see… no need to rearrange we can combine x f n minus 1 x plus this one.

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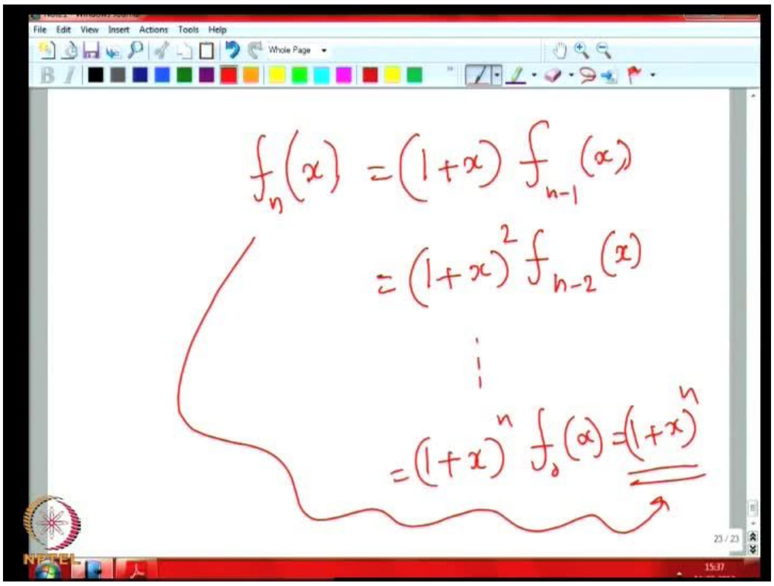

So, that is f n of x equal to1 plus x into f n minus 1 of x. Now, we can keep on unrolling this will be 1 plus x into f n minus 2 of x square and so on. So finally, this will become 1 plus x raise to n into f 0 of x, which is just 1 plus x whole raised to n. This is what the binomial theorem from binomial theorem we already knew that it should be 1 plus x raise to n. But we can get it in a roundabout way like this, so this is just for illustrating the technique.

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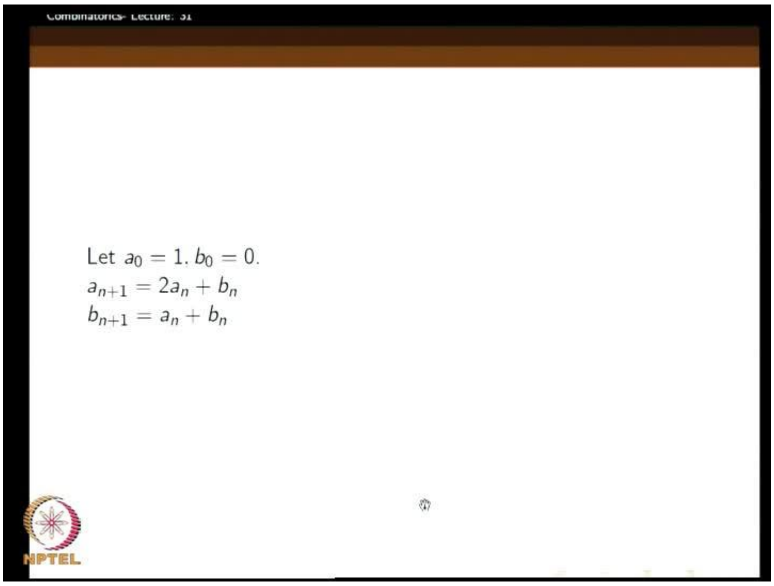

Now, we want to consider one more example which is slightly different, let us look at a system of recurrence relations we have a sequence a 0, a 1, a 2, a 3, a 4 etcetera and another sequence b 1, b 2, b 3, b 4 etcetera. They are given some initial conditions also a 0 equal to 1, b 0 equal to 0. So, how will you deal with this kind of a setup? So, I will copy this here a n plus 1 equal to 2 times a n plus b n.

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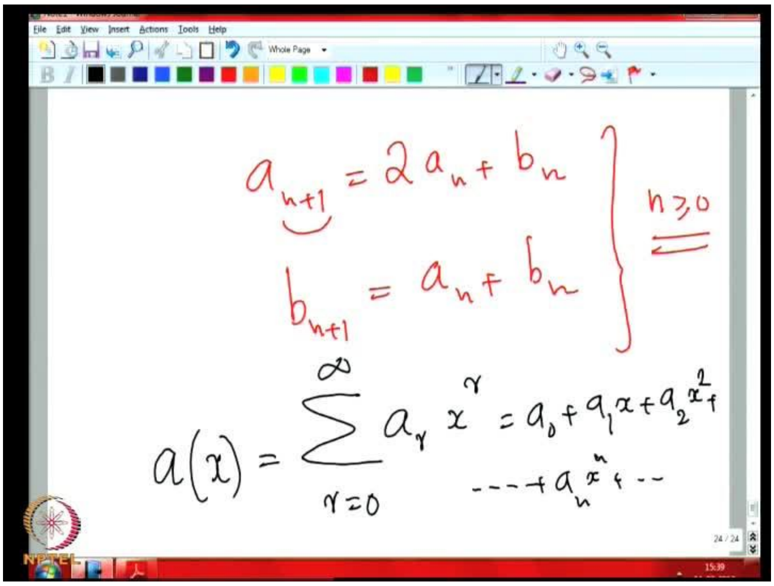

a n plus 1 equal to 2 times a n plus b n and b n plus 1 equal to a n plus b n and this is valid for say n equal to 0 onwards. So, both equations are true for n greater than equal to

0 onwards so, but case this starts with 1. So, what we can do is we can define two generating functions one is say for corresponding to this we can define a generating function a of x a of x, a of x can be defined as sigma r equal to 0 to infinity a r into x raise to r. So, that is like a 0 is a 1 x plus a 2 x square plus so on, a n x raise to n and so on.

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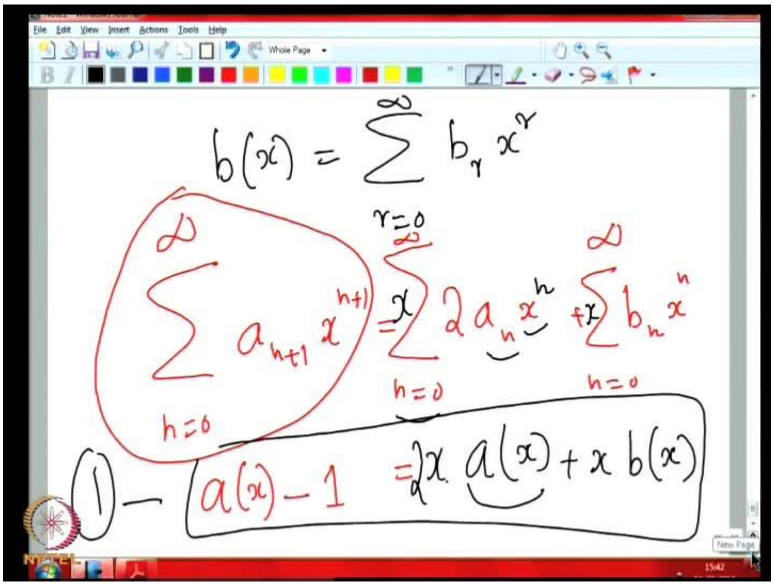

Similarly, we can define b of x, which is sigma r equal to 0 to infinity b r x raise to r. Now, we want to figure out what is this a of x and b of x. So, let us copy this two recurrence relations once again a n plus 1 equal to see a n plus 1 equal to 2 a n plus b n and b n plus 1 equal to… from this thing what we get let us see. So, what we can do is we sum this from n equal to 0 to infinity, n equal to 0 to infinity, before that before that we have to multiply this by x raise to n plus 1.

So, this here also multiply by x raise to n plus 1, so x raise to n plus 1, this is correct because, we multiplied a term by x raise to n plus 1. Now, we sum from n equal to 0 to infinity. So, when you sum it from 0 to infinity, so note that this is n plus 1 therefore, a 1 x raise to 1, a 2 x raise to 2 and so on is that. So, this is not really our a of x this is only a of x minus a 0 into x raise to 0 that is a 0, a 0 is known to be 1 as we have a given that a 0 is 1 we can write it a of x minus 1.

This is on the other hand so, this also will sum, this will be n equal to 0 to infinity, this will also from n equal to 0 to infinity. Here we can definitely take out take out say this x we can take out, so that means we will just keep x raise to n here. So, that means we have x into a of x… 2 x into a of x, here this 2 comes out, x also comes out. So, then this a n and x raise to n here and n is going from 0 to infinity therefore, that is a of x into 2 x. Similarly, here x comes out x comes out say this will remove and put an x here and then from n equal to 0 to infinity b n into x raise to n is b of x, this x into b of x. So, we get an equation like this, so this is what we get a of x minus 1 is equal to, so this is my first equation a of x minus 1 is equal to 2 x into a of x plus x into b of x.

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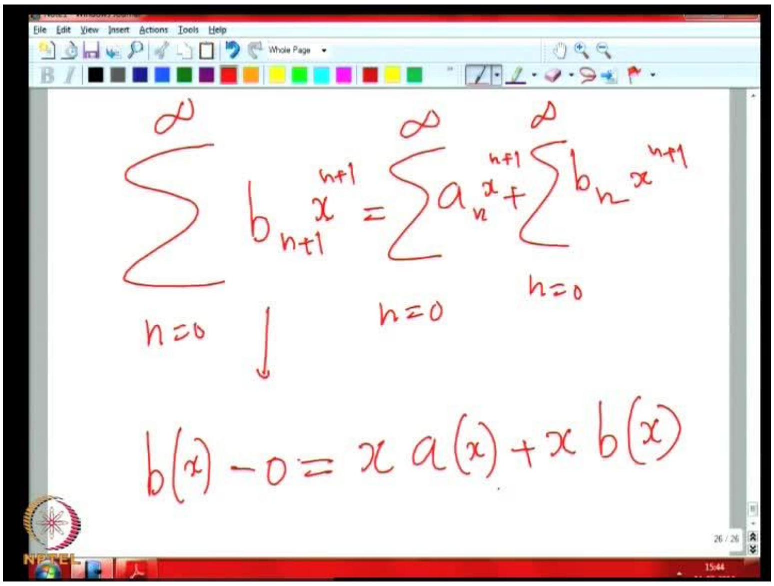

Secondly so, maybe we can use our, the recurrence relation, the other part of the recurrence solution b n plus 1 is equal to a n plus b n. Here also we can multiply by x raise to n plus 1 first, x raise to n plus 1 first, x raise to n plus 1 first and then now sum it from n equal to 0 to infinity everywhere. What we get this is because it is only going from n equal to 0 to infinity means n plus goes from 1 to infinity. So, this not really b x it is only b of x minus b 0, which is 0 we know, b of x b of x minus 0 equal to, here what happens x comes out. So, this x into then a n into x raise to n will come and will go from 0 to infinity. Therefore, that is a of x plus this is x comes out x into b of x will come. Now, we have two simultaneous equations in a of x and b of x, this I can try to rearrange what happens is this will be x into a of x plus x minus 1 into b of x.

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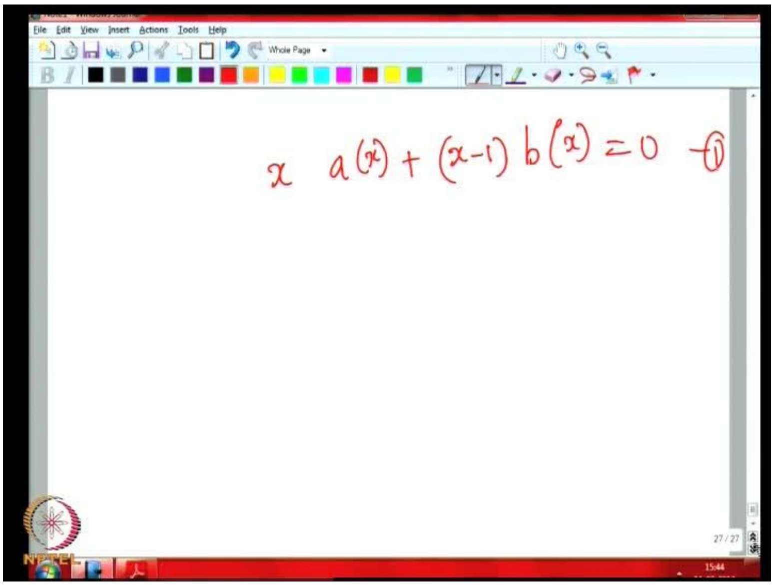

x into a of x plus x minus 1 into b of x equal to 0. This is what this will happen this.

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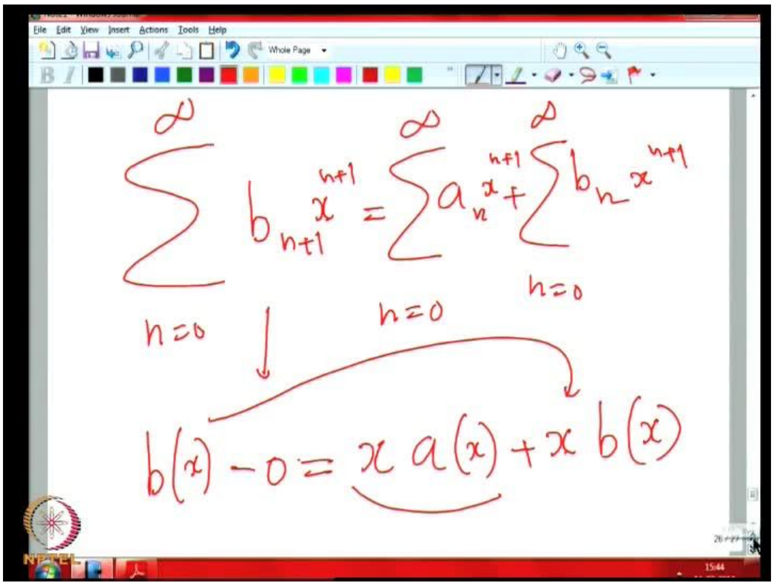

I just took x into a of x here, x and this b of x is brought here, x minus 1 into b of x is equal to 0.

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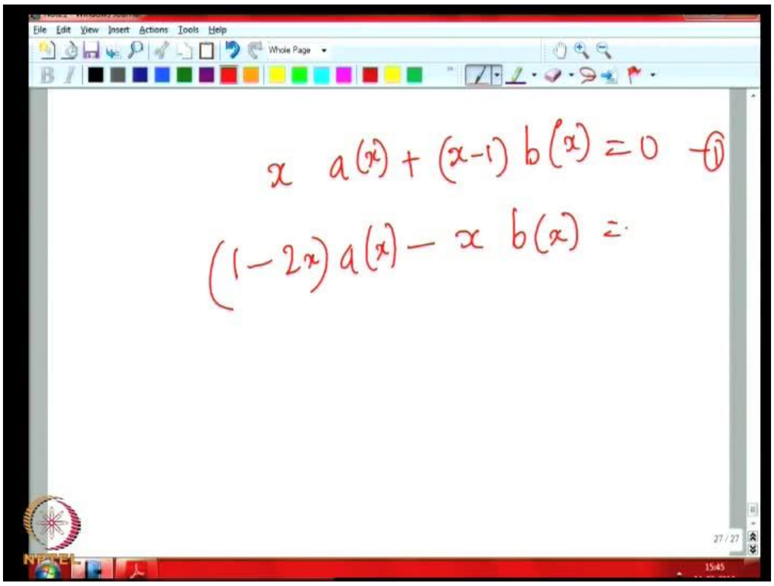

Here this one also I can rearrange, I can take x (( )) that is 1 minus 2 x into a x minus x b x. 1 minus 2 x into a x minus x b x, is equal to… So, I continue in the next class.