Combinatorics

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Module - 4
Lecture - 30
Solving Recurrence Relations Using Generating Functions -Part (1)

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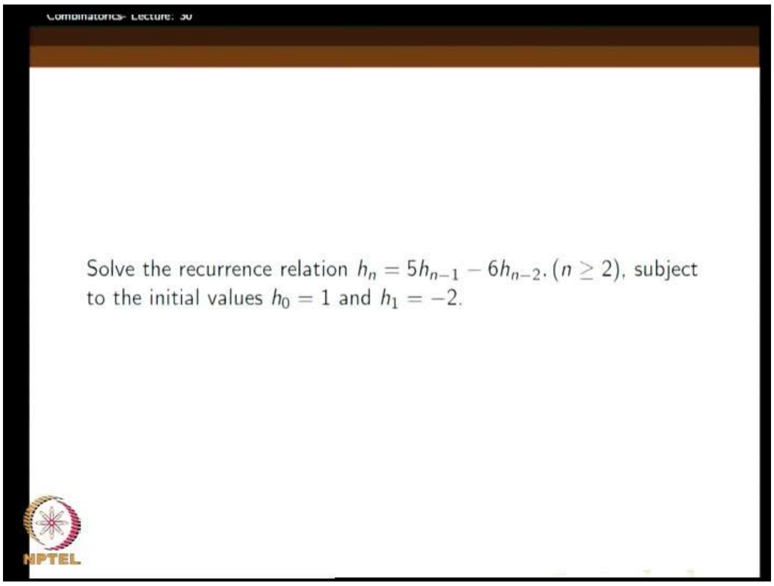

Welcome to 30th lecture of combinatorics. In the last class, we had started discussing the technique of solving recurrence relations using generating functions. So, we start with an example. So, this is the example we are considering. Solve the recurrence relation h n equal to five times h n minus 1 minus six times h n minus 2 n greater than equal to 2, right. So, this recurrence relation is valid only for n greater than equal to 2 because you need h n minus 1 and h n minus 2. So, that means h 1 and h 0 should be given. So, here the initial values h 0 equal to 1 and h 1 equal to minus 2 are given.

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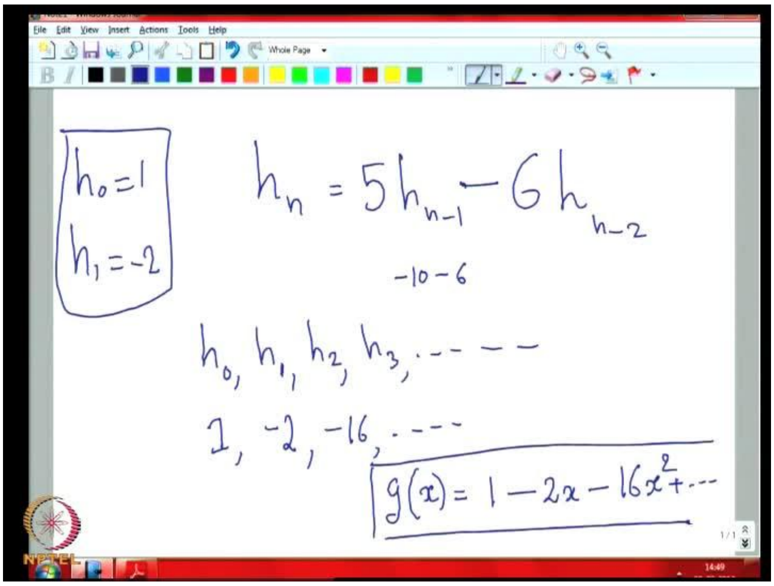

Now, how do we solve this thing? To solve this thing, this is what h n equal to 5, h n minus 1 minus 6 h n minus 2. So, I will write the initial values here, h 0 equal to 1 and h 1 equal to minus 2. So, not that the sequence will be like h 0, h 1, h 2, h 3 and so on. This is 1, this is minus 2. What is h 2? H 2 will be you substitute that will be 5 into minus 2, that is minus 10 minus 6 into 1, right minus 10 minus 6. That is equal to minus 16 and so on. It will go on, right. So, we know that our generating function, say if I write g of x as the generating function for this, this will be 1 minus 2 x minus 16 x square and so on, right.

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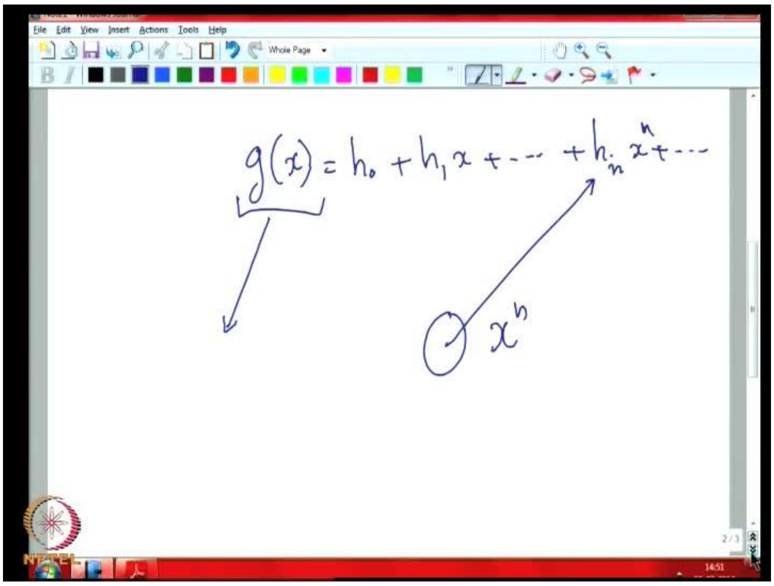

So, this is we want to find a close form for g x first. If you remember this is what we are trying to do initially and then once you get the close form, then we will try to get a general close form expression for the co-efficient of x raise to n in it. So, that will be the value of h n because the generating function g x is actually h 0 plus h 1 x plus like that, h n x raise to n plus.

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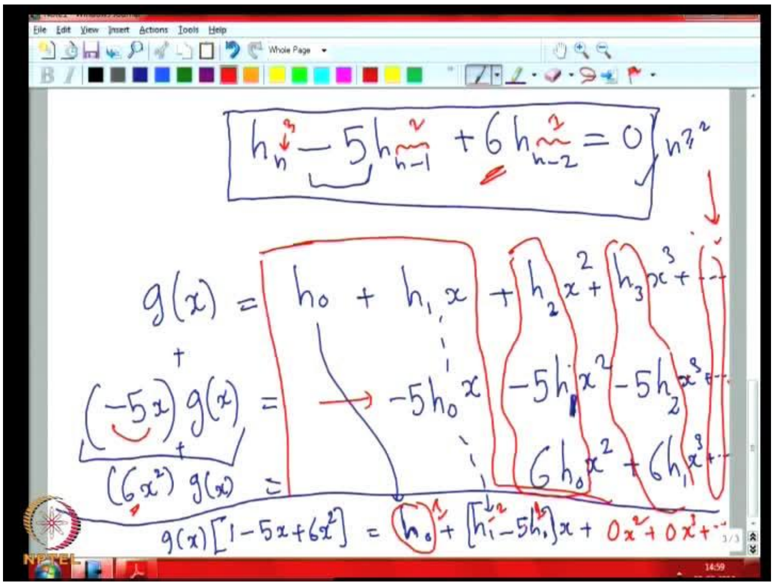

So, if you find the, sorry first find a close form for g x somehow and then from that infer somehow what would be the co-efficient of x raise to n. That would correspond, that should be x h n, right. That should be value of h n. This is our method, right. Now, this is what we want to do, right. So, how will we find out for this? Earlier, we had find some examples where we could infer the co-efficient of x to the power n, given some close form expression for g of x. Now, first our task is to get g of x, right. So, here we write like this. First, g of x. So, we start like this.

So, first we will write this in this formula, h n equal to 5, h n minus 6, h n minus 2 in a more convenient form. So, h n minus 5, h n minus 1 plus 6, h n minus 2 equal to 0. This is the way started off when we wanted to find the characteristic equation also, right. So, this is the line recurrence relation at this capture by this relation between h n minus 1 and h n minus 2.

Now, what I write is let g of x equal to h 0 plus h 1 x plus h 2 x square plus h 3 x cube plus so on. Now, multiply g of x by minus 5. This is a co-efficient of h minus 1 here into x minus 5 into x times g of x. What will be that? It will be h 0 into minus 5 x h 1 into minus 5 x into x. That means minus 5 h 1 into x square and so on, but when we write it, you know h 0 into minus 5 x. So, that is already got an x. So, the expansion of this, they would not be any constant term if we will start writing from here. So, this is minus 5 into h 0, right into x. This is the first one and then minus sorry here minus 5 into h 2, sorry h 1. So, this one, right h 1 into x square. So, I multiply h 1 x into minus 5 x minus 5 in h 1 into x square and then again minus 5 into h 2 into x cube and so on. This is what we will get, right.

So, next what we do is multiply g x by 6 x square. This g x, so you know that they would not be any constant term in the resulting expression. So, they would not be any term like something into x, no power of x will be there. So, it will start with x square onwards, right. So, therefore, we will write it here h 6 h 0 x square. That is the beginning plus 6 h 1 x cube and so on. We can write like this, right. Now, what I do is I sum of all these things together. So, that means I add this plus, this plus like this. So, what we get here? I get g x into 1 minus 5 x plus 6 x square. This is what I will get. This is equal to, so here this will come here h 0 plus h 1 minus 5 h 0 into x, and this I added here and put it here and then now h 2 x square plus minus 5 h 1 x square plus 6 h 0 x square.

So, if we take x square out, that is h 2 minus 5 h 1 plus 6 h 0 into x square, but you know from an equality two onwards, this for n greater than or equal to 2, this is valid. This relation is valid, this equation is valid. So, put n equal to 2. What does it say? H 2 minus 5 h n 1 a, sorry h 1 plus 6 h 0 equal to 0. So, that is exactly 1 h 2 here. If you read it out like this, h 2 minus 5 h 1 plus 6 h 0 into x square. That is actually 0. When I put 2 for this, this will become 1. This will become 0 that is h 2 minus 5 h 1 plus 6 h 0 equal to 0. So, this will become 0, right. So, there will be 0 x square. Only here plus here. What is happening when I take this thing into x cube? What I will get here? It is x cube x cube x cube here. So, h 3 minus 5 h 2 plus h 3. This is exactly for in sense if I put n equal to 3, this n minus 1 will become 2 and this will become 1, right.

So, that is h 3 minus 5 h 2 plus 6 h 1 equal to 0. So, exactly write h 3 minus 5 x 2 plus 6 h 1. That is equal to 0. Therefore, we have 0 x cube and we can consider from now onwards, everything will be like this. If I collect the coefficients of x raise to 4 from here, here, here and here and add them up, they will become 0 and 1 to the surprised. Why this is happening? Because that is a way we have arranged, right. So, for in sense we were multiplying by 5 x here, so that minus 5 x, so that we have pushed it one side for in sense for h n term. Here, h n x raise to n will be here. So, then what will be just below that? That will be h n minus 1 into this minus 5, right. So, that is exactly h n minus 1 into minus 5 here, right and what will be below because if multiply by 6 x square here, so it was pushed to steps towards the right. So, in the column of x raise to n, we will see h n minus 2 into the 6. The 6 came from here, right. So, 6 x minus 2, right into x raise to n.

So, if you collect all the coefficients of x raise to n from here, here, here and here, from each of this equation column wise, then what will you see, h n minus 5, h n minus 1 plus 6 h n minus 2, right in general. So, that will always be 0. So, from now onwards, they would not be thing. So, in this sum, only this portion will contribute. You can see write this portion will that is h 0 plus h 1 minus 5 h 0 times x and luckily, we know the values of h 0 and h 1. H 0 is what? That is 1 and this is what? This is minus 2. So, you have to put minus 2 for this and this is 1. Write h 0 is 1, so that is minus 7.

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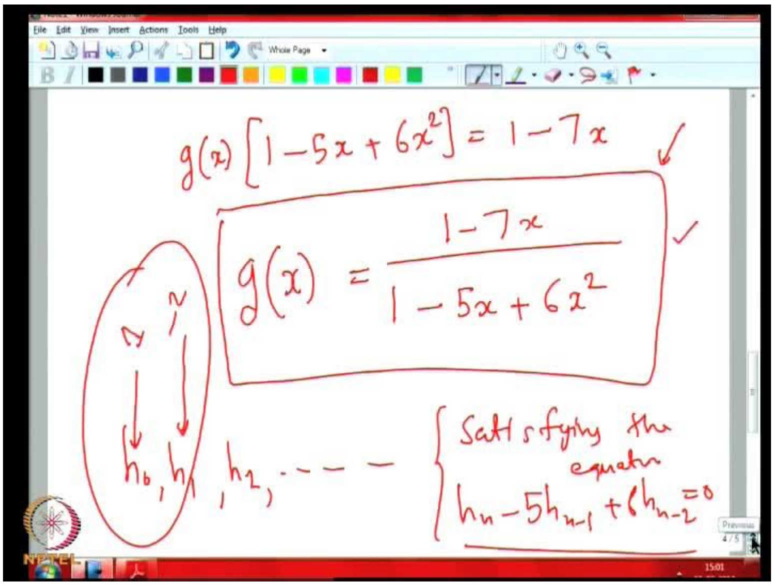

So, we get g x into 1 minus 5 x plus 6 x square is equal to, what we get is 1 minus 7x. This is what we get. So, that is g x equal to 1 minus 7 x by 1 minus 5 x plus 6 x square. So, this is the general expression for generating the function for that sequence given to us, h 0, h 1, h 2 satisfying the relation, the equation h n minus 5, h n minus 1 plus 6, h n minus 2 equal to 0. Not only that, with initial values 1 for h 0 and minus 2 for h 1, these are also incorporated here because after using that, only we derived this formula in the n. We substituted the values by 0 and h 1, right.

So, this is already considered the initial values, right and also this circulation, this is what? This is 1 0 2 l 0 does to cancel off most of the terms and the sum, right or the terms which comes after the term, right. Sorry after the second term. From the third term onwards, we have no contribution because of this relation only, right. So, I can see that is how we adjusted things and then we see that this is generating equation. Now, if this is the generating function, we can discuss how we can get the co-efficient of x raise to n. So, finally, giving close expression for h n, right.

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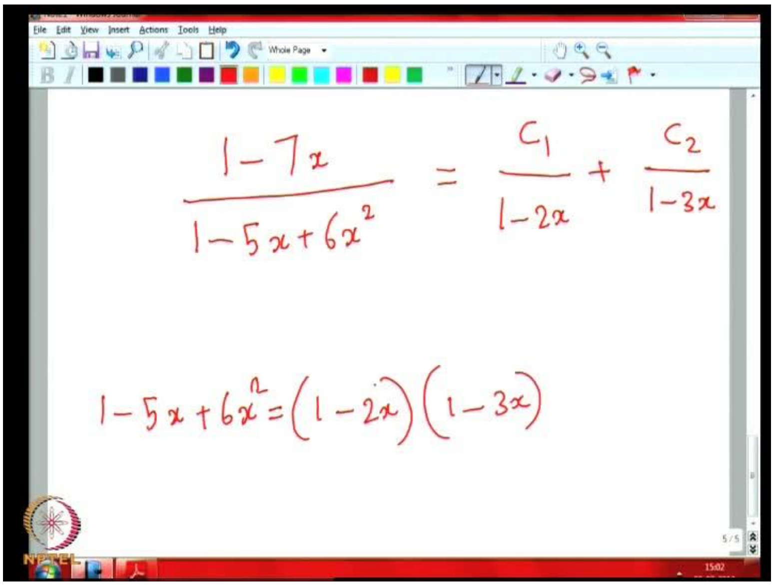

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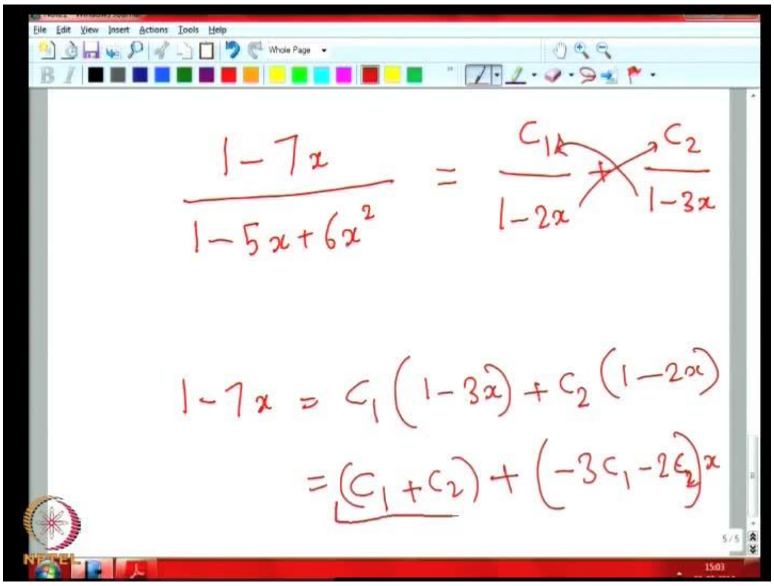

So, 1 minus 7 x by 1 minus 5 x plus 6 x square 1 minus 5 x plus 6 x square. So, what we do is to get the answer for that we have to simplify it. We use a method called the method of partial fractions. So, first, notice that this 1 minus 5 x plus 6 x square can be factorized to 1 minus 2 x into 1 minus 3 x. This is not ready to come up with, right. So, 1 minus 2 x into 1 minus 3 x is this. Now, we can see that. So, first some constants c 1 and c 2. So, we can say that c 1 by 1 minus 2 x plus c 2 by 1 minus 3 x. So, then what we do is we can compare the numerator. So, that means 1 minus 7 x is equal to, so this will multiply here and this will multiply here.

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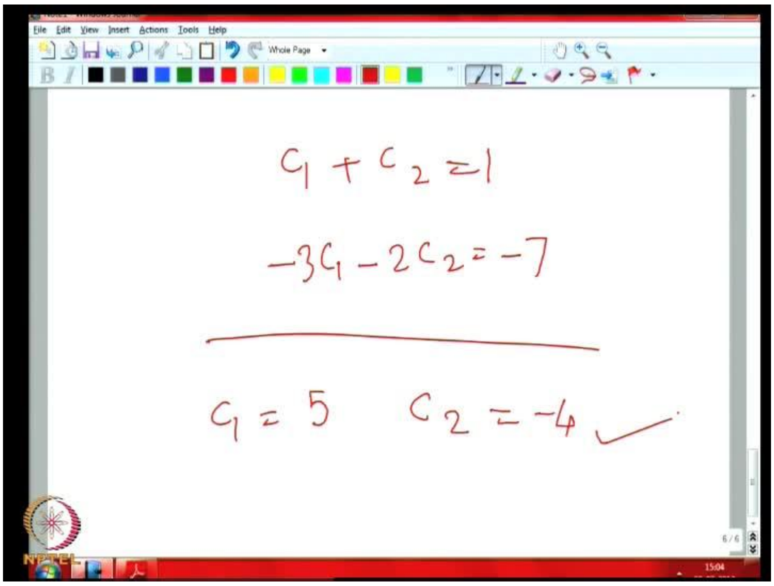

So, that is c 1 times 1 minus 3 x plus c 2 times 1 minus 2 x, right. So, this is c 1 plus c 2 plus. So, we have minus 3 c 1. Here, x minus 2 c 1, x minus 2 c 2, x minus 3 c 1 and minus 2 c 2 x. So, we can compare the constant terms here, that is c 1 plus c 2 has to be 1 c 1 plus c 2 has to be 1 and minus 3 c 1 minus 3 c minus 2 c 2 minus 3 c 1 minus 2 c 2 has to be equal to this one, right. 7 minus 7, sorry not this one. So, it should be minus 7 minus 7, right. Now, we can solve this simultaneous equation. So, this is you can do it easily. We will get c 1 equal to 5 and c 2 equal to minus 4, right. Now, you substitute in this equation. This is what c 1 is equal to 5 and c 2 will be equal to minus 4, right. So, we can put it as minus 4. This will be the end of the partial fraction.

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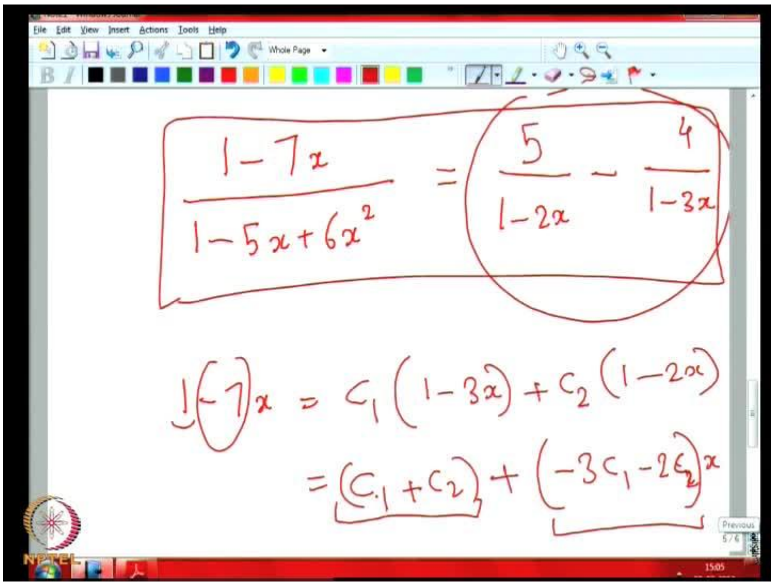

So, by using the method of partial fractions, we can say that this 1 minus 7 x by 1 minus 5 x plus 6 x square is equal to 5 by 1 minus 2 x minus 4 by 1 minus 3 x, right.

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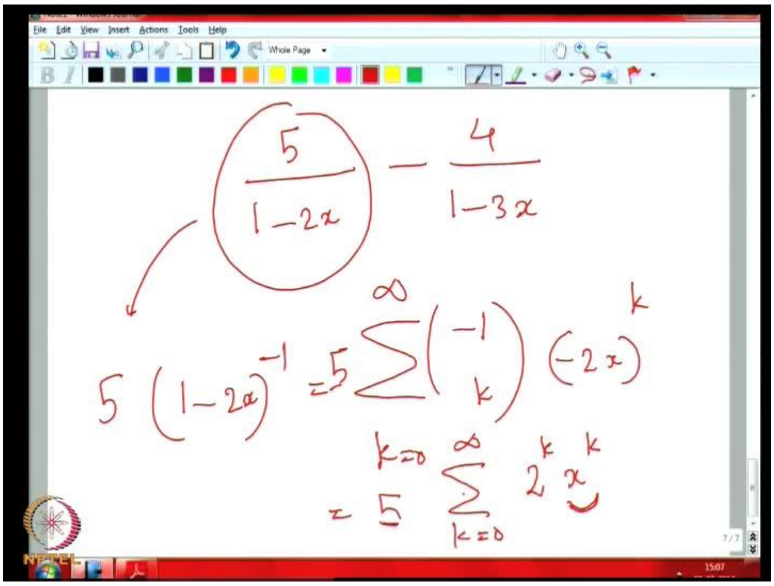

So, now what we have to do is, from this thing, we know this is score generating function 5 by 1 minus 2 x and minus 4 by 1 minus 3 x 5 by 1 minus 2 x minus 4 by 1 minus 3 x. So, this is easy to handle because this 5 by 2 x, you know 1 minus 2 x, it tends by 1 minus 2 x. What is this? This you know is sigma k equal to 0 to infinity minus 1 choose k. So, we can write it as 1 minus 2 x to the power minus 1, right.

So, minus 1 choose k into minus 2 x to the power k, this is what. Then if you want five times this, so there will be 5 here. Then there will be 5 here, right. Similarly, for the other thing. So, this we can simplify in our case. We know this is 5 into this minus 1 choose k. If you remember that is just 1 and m minus 1 raise to k. This will just become k equal to 0 to infinity, right. So, 0 to infinity 2 to the power k into x raise to k, right, 2 to the power k into x raise to k.

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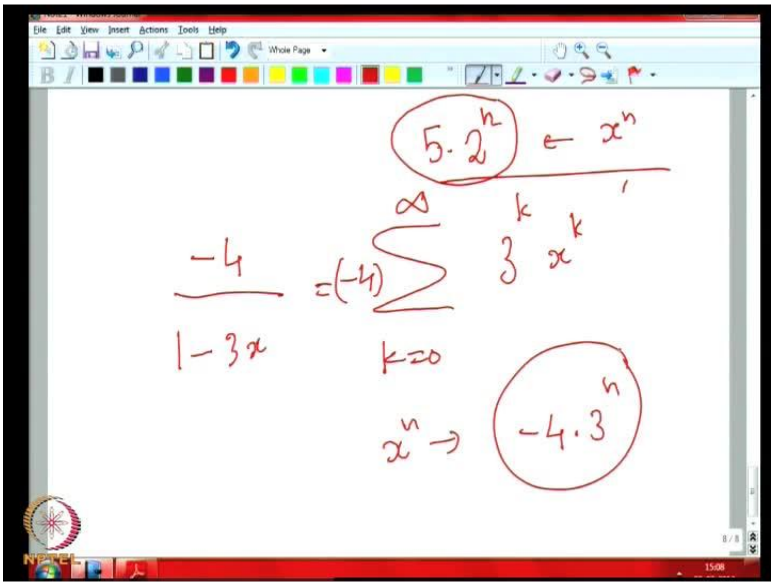

So, the co-efficient of, if I am seeking the co-efficient of x raise to n and this thing, that is 5 into 2, the power of n. So, co-efficient of x raise to n and in that will be 5 into 2 to the power n. Now, the other term. So, the minus 4 by 1 minus 3 x. That will also similar minus 4 by 1 minus 3 x will be sigma k equal to 0 to infinity. Same way, that will be minus 4 here, this minus 4 and will have 3 to the power k x to the power k. So, the co-efficient of x raise to n and if you look that will become the co-efficient of x raise to n. Here, it will be minus 4 into 3 to the power n.

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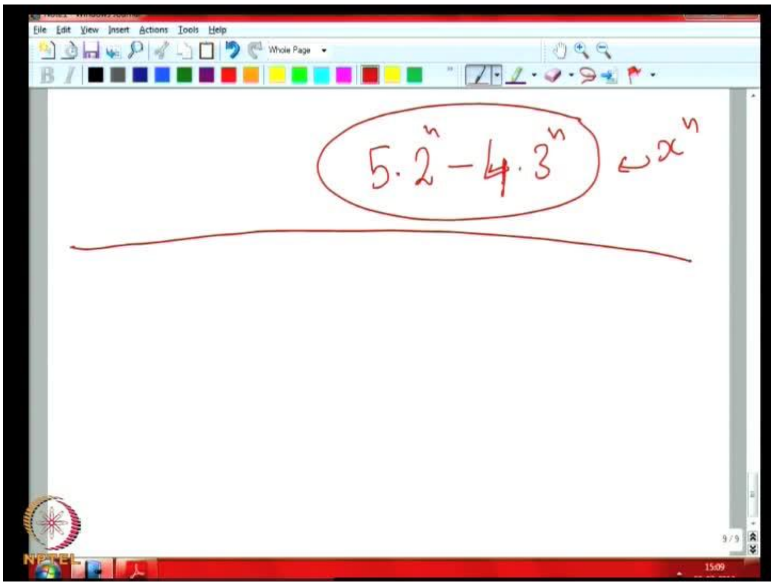

So, finally, when you sum up both the co-efficient of x raise to n, it will be 5 into 2 raise to n minus 4 into 3 raise to n. So, 5 into 2 raise to n minus 4 into 3 raise to n, this will be co-efficient of x raise to n. This will be the co-efficient of x raise to n. This is the way we can solve it completely. So, let me repeat. So, the first step was to somehow get that g of x and after getting the g of x, what we did is, we use the method of partial fractions to split it into more convenient terms and then each of the terms were familiar to us.

We know the expansion of that and then from that we collected the co-efficient of x raise to n from each term and then add up them together. This is what we did, right. Now, when we look at this method, we will say that this is quite general, right. We can do it for any homogeneous linear recurrence relation with a constant coefficient. Though, it is long as, sorry it is on order k suppose, so we can do it.

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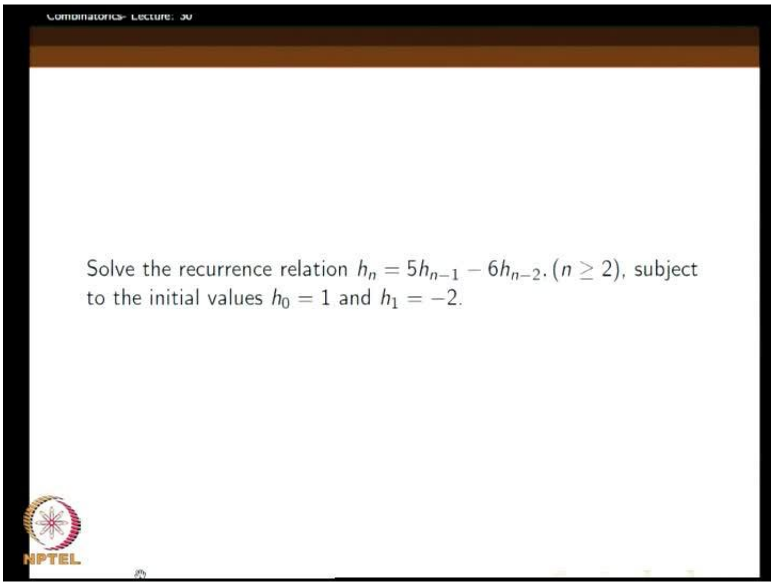

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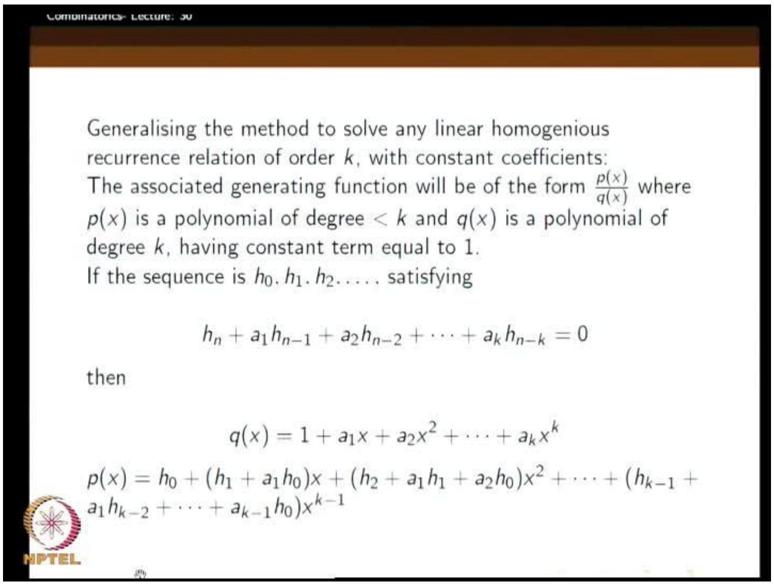

So, we will just catch out how we go about 2 in it, right. So, generalizing the method to solve any linear homogeneous recurrence relation of order k with constant coefficients, the associated generating function will be of the form p of x by q of x. There p of x is a polynomial of degree is less than k because this k is the order we are talking about, and then q of x is polynomial of degree k having constant term equal to 1. So, this is the way we do. If the sequence is h 0, h 1, h 2 etcetera, etcetera satisfying see h n plus a 1 times h n minus 1 plus a 2 times h n minus 2 plus, say a k times h n minus k is equal to 0, right. Then q x will be equal to 1 plus a 1 times x plus a 2 times x square plus a k times x k.

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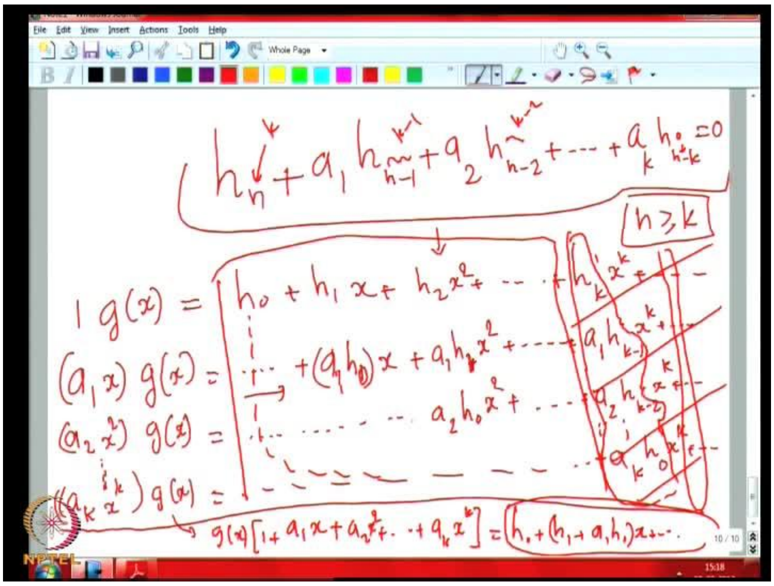

So, if this is easy, if you have carefully looked at it, what we did in the last thing. So, last time let us say this is the recurrence relation h n minus, sorry h n plus a 1 times h n minus 1 plus a 2 times h n minus 2 plus. So, a k times h n minus k equal to 0, this is our recurrence relation, right. Second zero, the 0 kth order recurrence relation. Now, this will be 2, 4 n greater than equal to k. May this will valid for n greater than or equal to k only because we are using h 0, h 1.

So, when n equal to k, we are already using all the 0 to h k minus 1, right. So, if you try to apply for h k minus 1, we will not be able to find h n h minus 1. This is not available. So, our initial values will be h 0. So, will be given h 0 equal to something, say h 1 equal to something. So, h k minus 1 equal to something. These many initial values have to be given because this will work only from n equal to k onwards, right. Now, if you remember the method was, so the method was that. So, we wrote g of x equal to h 0 plus h 1 x plus h 2 x square plus and so on. Here, we have h k x raise to k and so on.

Now, first we multiply by a 1 x into g of x, right. This is a 1 x will multiply by g of x and when we write, we will just skip this portion and will start a because every term now will have an x, right. So, we will write a 1 h 1 times x here and a 1 h 2, sorry this is h 0, this is a 1 h 1 times x square and so on. So, this here, what will be this? A 1 h k minus 1 x k, like this because 1 pushed one step here, right.

So, x raise to k will get h k minus 1, but a multiplier of a 1 is here, right. The next we will use a 2 times x square to multiply g of x, right. This will be pushed two steps. So, we will start from here onwards. Why? Because you know every term will have an x square now. So, we will start just below a square from here. So, this will be a 2 times h 0 x square plus. So, when we reach here, what we will get is a 2 times h k minus 2 x raise k will be here, right. So, like that we can write finally a k times x raise to k can multiply g of x here.

So, each of them after a to x, we will take a-th three times x cube and then will take a four times x, x a 4 x raise to 4 and so on. Finally, a k times x raise to k will multiply g of x. This will directly start here because x raise to k term here. So, that is a k times h 0 x raise to k. So, now when you sum up these things, you will get g of x into 1 plus a 1 x plus a 2 x square plus a 3 x cube plus. Finally, a k x cube. So, this is what you get here. Here, what you get is g x into 1 plus a 1 x plus. So, a 2 x square plus and a k x raise to k.

This is what you will get here. This will be equal to what? So, here when you sum only h 0 will come, right and then here when you sum h 1 plus a 1 times h 0 into x will come and so on up to here, up to here. Here, h k plus a 1 times h k minus 1 plus a 2 times h k minus 2 plus a k times h 0 up to here, right.

So, essentially when you put a n equal to k here, so this will become k minus 1, this will become k minus 2 and so on. This will become 0, right. So, before this cannot be 0. This will be 0 into x raise to k. From here put any equal to k plus 1. See, when you sum up, this will become 0 and x raise to k plus 1, 0 into x raise k plus 1 will be 0. So, this k thing will go away, right. Only we will have contribution, only this portion, right.

So, how will it look like? This will look like a 0 plus h 1 plus a 1 times h 0 times x h 1 plus a 1 into h 0 times x and x square co-efficient will be h 2 plus a 1 plus times h 1 plus a 2 times h 0 and so on, right. So, this will be finally just before the co-efficient of x raise to k minus 1, right, that is h k minus 1 plus a h a 1 times h k minus 2 plus a 2 times h k minus 3 and so on. So, this is the way to look like. Finally, a k minus 1 times h 0 will be there, right, so that only up to x raise to k minus 1 will come, right.

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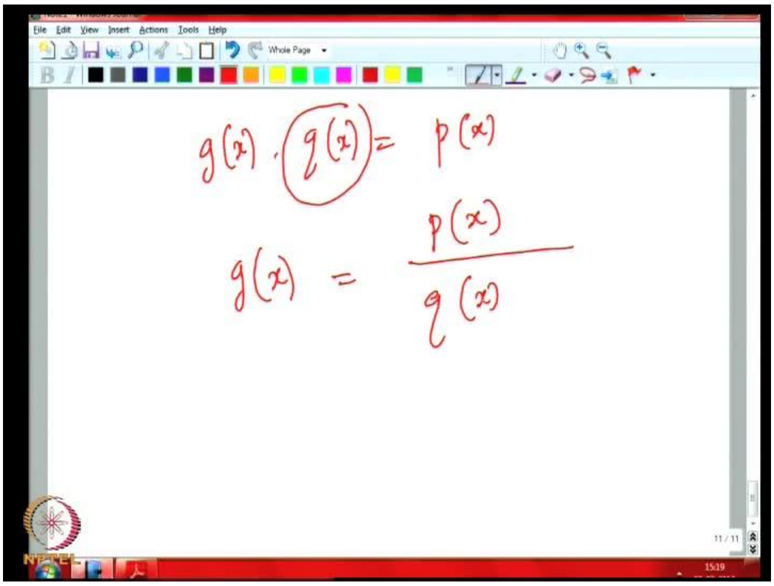

So, this portion will be our p of x and because g of x into something is equal to something, right, so this will look like g of x into some q of x is equal to p of x. The properties of this, p of x is that it has this constant term 1 because here we have always 1, right, this into g of x into 1.

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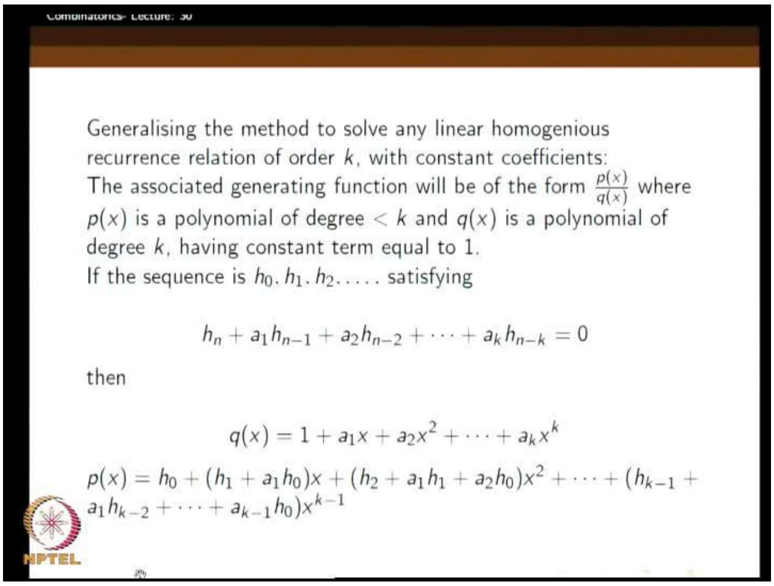

Then, this x raise to k that is it is k degree polynomial because k is always we know, this is non-zero, right. That is by definition of the order of the recurrence relation is k. This k has to be non-zero. So, this will be there. So, k raise to x raise to k will be there. Therefore, this is k 3 k polynomial and the first constant term is 1 here by that is property of this q of x, and this p of x is just i 0 plus this thing, but it is only up to x raise to k minus 1, right. Degree of p of x will be strictly less than k, right. We are not saying k minus 1; it will be strictly less than k. So, g of x will be equal to p of x by q of x p of x by q of x.

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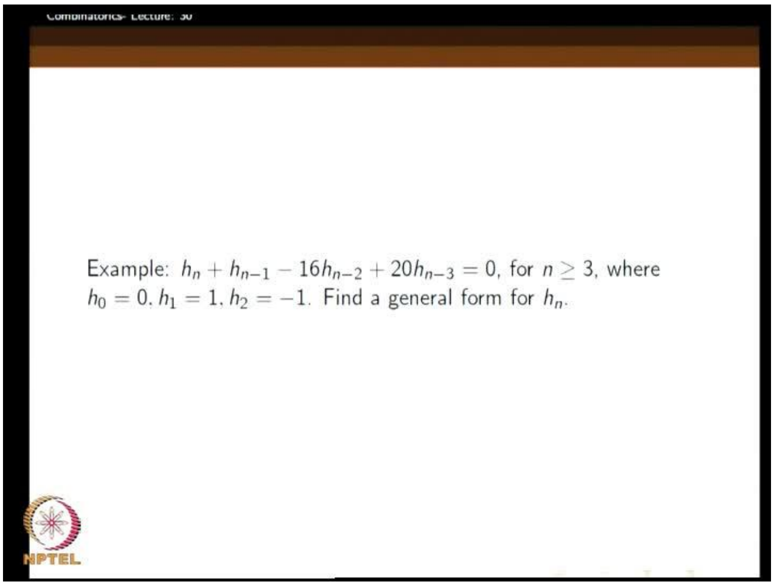

Now, for clarity I can show the q of x and p of x here. So, q of x look like this, q of x equal to 1 plus a 1 x. This a1 is directly coming from this and this a 2 x square that a 2 is directly coming from here and then k x square is coming from here. Similarly, p of x will be h 0 plus h 1 plus a 1 h 0 plus h 2 plus a 1 h 1 plus a 2 times h 0. Finally, h k minus 1 plus a 1 h k minus 2 plus a k minus 1 h 0 a k minus 1 h 0 into x raise to k minus 1. So, it is very easy. So, you do not have to rememorize. You just can work it out. Whenever you want, you can again read all these things and the procedure is very easy to remember once you understand it. Therefore, we can always do it when you want to write it on to keep these formulas in mind, right.

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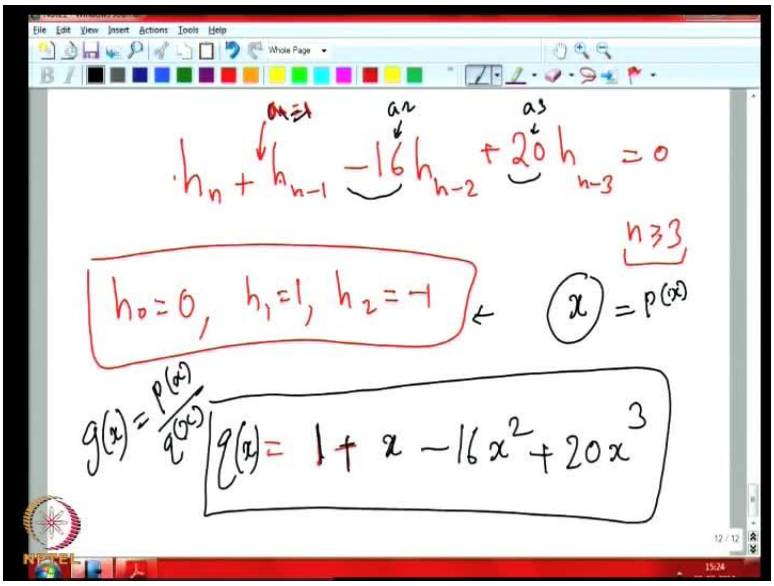

Now, let us look at one example, h n plus h n minus 1 minus 16, h n minus 2 plus 20, h n minus 3 equal to 0. So, h n plus h n minus 1 minus 16, h n minus 2 plus 20 h n minus 3 equal to 0. So, we need the initial values 4. So, this will work only from n greater than equal to 3, right. So, we want what is h 0, what is h 1 and what is h 2. So, what is h 0? Let us take it as 0 and h 1, let us take as 1 and let us take h 2 is minus 1, right. These are the initial values. Now, we can write down g of x straightaway. If you remember what we did last time, so what is that? That is here we have to start with 1, say 1 corresponds to this, right and here, this is a 1, right, a 1 equal to 1 here, right. Now, we can use a 1 equal to 1 here. Therefore, this you can take 1 into x. Now, here this is a 1. A 2 is minus 16, so that minus 16 x square, right and this is a 3 that is 20. So, plus 20 x cube.

So, this will be the 1 plus x plus minus 16 x square plus 20 x cube would be the generating function, right. Sorry, this is not the generated option. I am saying that g of x will be equal to some q of x by p of x, sorry p of x by q of x and this q of x will be this q of x. This is what will go to the denominator because now when you do that procedure g of x into this product will come, right and on the other side, what will be there first h 0, that is 0. So, let us try to write it from there, h 0 that is 0 plus then h 1 plus h 0 into a 1. So, anywhere other 0, that is just h 1 x that is x. Next one is h 2 which is minus 1 plus h 1 into a 1. This is 1, right and then h 0 into a 2. Anyway, h 0 will go away. This will be minus 1 plus 1 into x square, right and this is again 0, because minus 1 has become 0.

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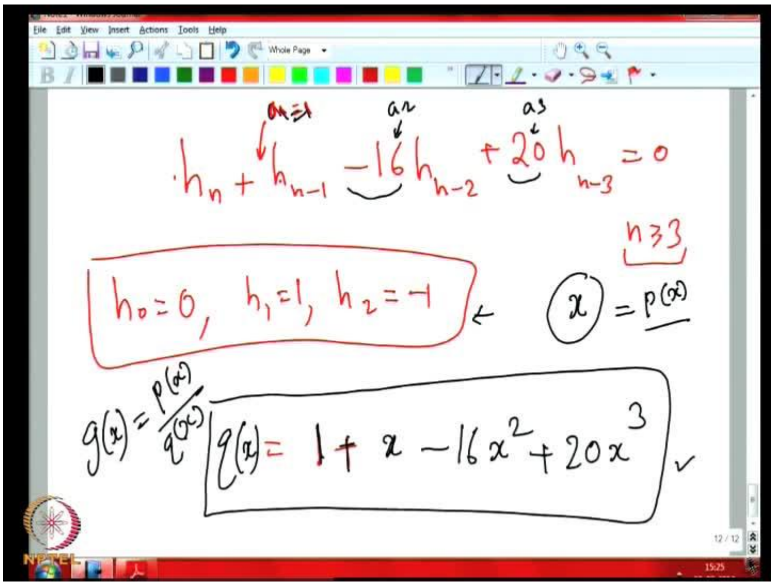

So, this is zero times x square, right and then there is nothing more because from a x cube onwards, third term onwards will be disappeared, right. So, this will be just x, right. This is third p of x p of x will be like this. So, if you are confused what I am doing is just substituted the values, where h 0 in this formula of p of x i. I just substituted the values for h 0, h 1, h 2 this one, right because k was equal to 3. I only have to go up to x square term, right x cube term one onwards will not be there.

So, that is I got x there, right. So, again if you remember how to do this thing, it is very easy, but even if you do not remember, you can go through the entire procedure and get p of x and q of x. So, p of x is x square, q of x is equal to this here. So, what we get is g of x, sorry g of x equal to p of x by q of x which is equal to x divided by, right. So, x divided by 1 plus x minus 16 x square plus 20 x cube.

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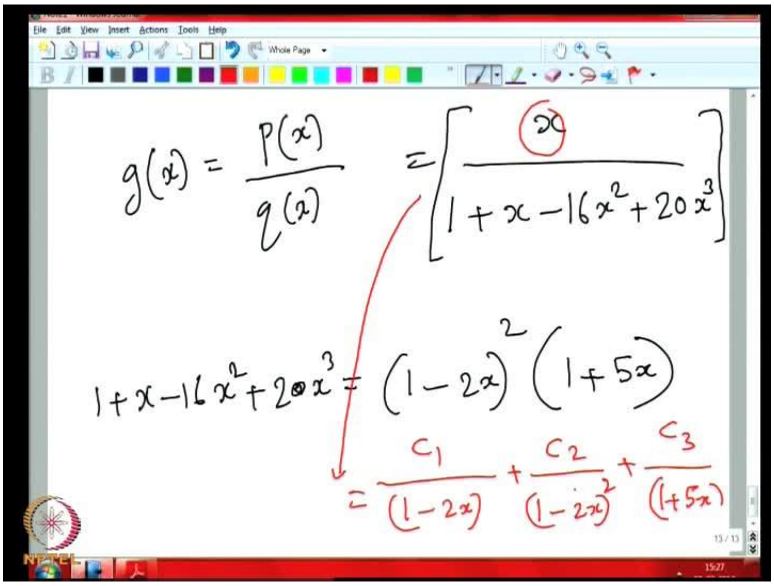

Now, if you want to get a close form, so if we want to get the value of h n, so you want to find the co-efficient of x raise to n and the expansion of this. So, we can again use the method of partial fractions. What we do is we write it as some c 1 times. First, we should know how this will be factorized, 1 plus x minus 16 x square plus 20 x cube. So, with some effort, we can show that this is 1 minus 2 x whole square into 1 plus 5 x. What we should do? Try it out and then once you know this thing, you can write this is equal to some c 1 times 1 minus 2 x plus c 2 times 1 minus 2 x whole square plus c 3 times 1 plus 5 x.

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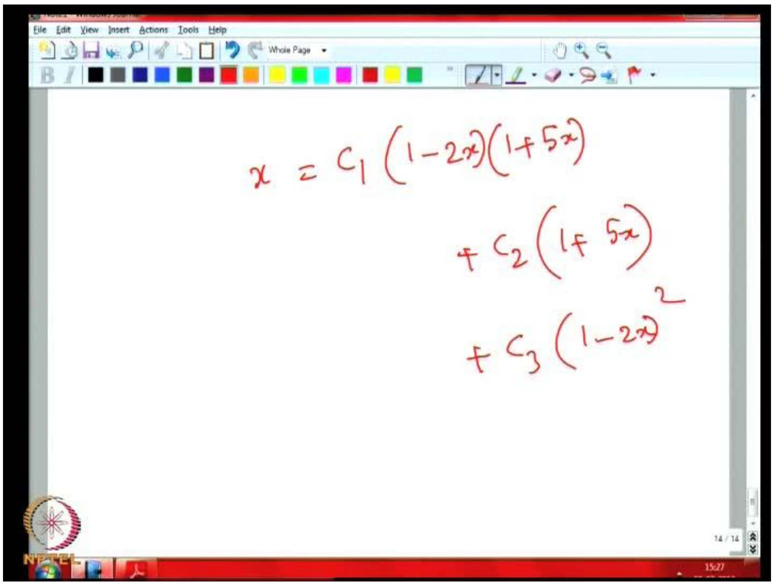

Now, the same thing we did last time, right. Equate the numerator is x here, right. Numerator is x here this c 1 times, yeah we will get x equal to c 1 times. This is routine of 1 minus 2 x into 1 plus 5 x plus c 2 times 1 plus 5 x plus c 3 times 1 minus 2 x whole square.

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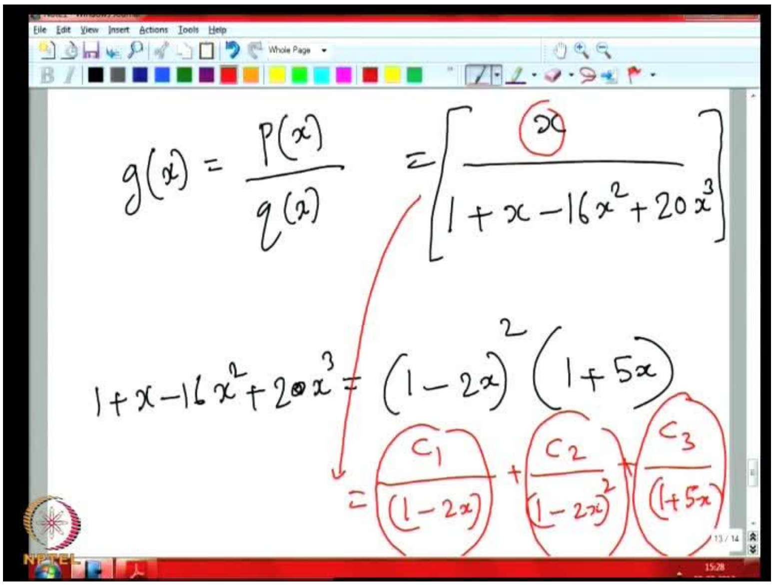

Now, we create the constant terms from both sides and then the co-efficient of x from both sides and co-efficient of x square from both sides. Then we get three simultaneous equations and solving it, we can get the value for c 1, c 2, c 3 and then we will be back to very familiar forms which we can use to infer the co-efficient of x raise to n. So, we have to get the co-efficient of x raise to n from this and then we have to get the co-efficient of x raise to n from this thing. We already know the trick by which we can infer the co-efficient of x.

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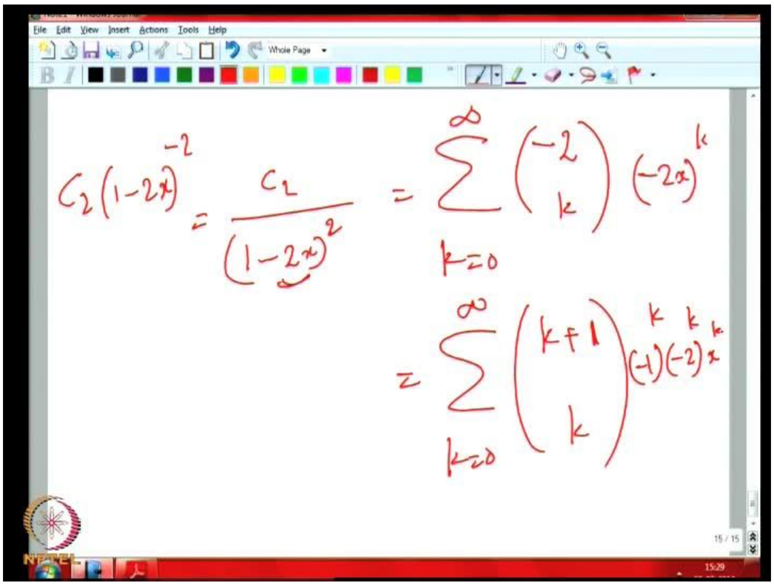

For instance, if you want to get the co-efficient of x raise to n in c 2 by 1 minus 2 x square, what we do is this will be equal to k equal to 0 to infinity because this is minus 2, right. So, remember this is c 2 into 1 minus 2 x whole power minus 2. So, this is just write, minus 2 choose k here, right and then this is x minus 2 x.

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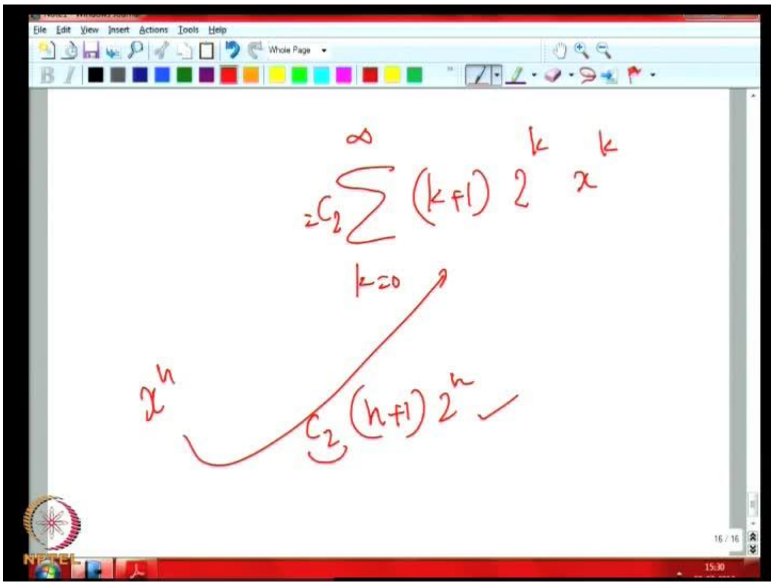

So, minus 2 x whole raise to k, right and this we know this is k equal to 0 to infinity. How did you do this k plus 2 minus 1 choose k. So, this is just k plus 1 choose k into choose minus 2 raise to k. Sorry, we also had a minus 1 raise to k coming from here and we have minus 2 raise to k here and that is x raise to k and this minus will go away. We will just have k plus 1 choose k. So, this will look like k equal to 0 to infinity. This will be k plus 1 choose, just k plus 1 into and here minus 1 raise to x is there minus 1 raise to k will come. Minus 1 raise to 2 k will disappear and then just 2 raise to k will be there into x raise to k 2 raise to k into x raise to k.

So, the co-efficient of x raise to n which is coming from this will be just n plus 1 into 2 raise to n, right. Similarly, we can find that those other terms are easier. For example, these terms are easier to handle. Yes, once we know c 1, c 2, c 3, we will easily be able to, right. So, we have this c 2 here, right, c 2 into, right. We will able to get the co-efficient of x raise to n. So, the thing is there again a something times 2 to the power n something times, right. Finally, we look like this, sorry you have to work it yourself because do not want to waste time on this.

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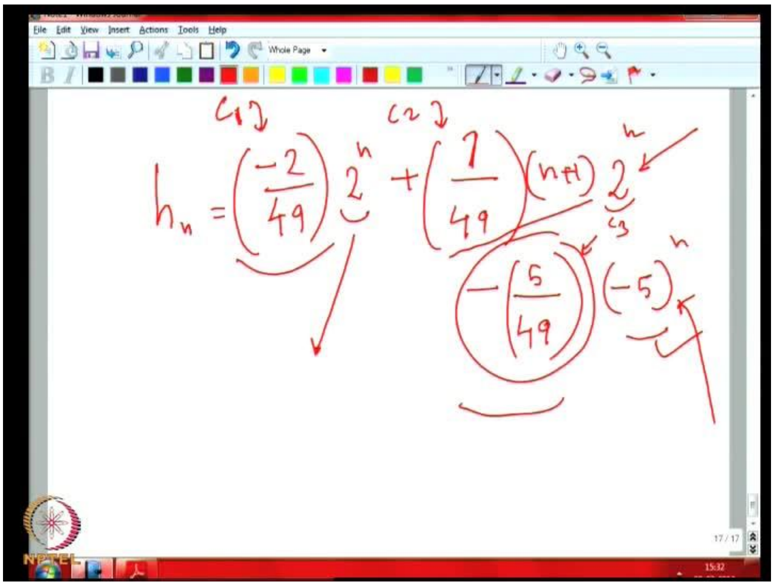

So, h n is equal to minus 2 by 49. So, this is the first co-efficient and this will correspond to c 1 into 2 raise to n plus 7 by 49 into, this will correspond to c 2 n plus 1 into 2 raise to n minus 4 by 49 into minus 5 raise to n. This will correspond to c 3, this one. So, this is what we will get, but interesting thing is that if you stare at it, this is when we try to solve the recurrence relation. This homogeneous recurrence relationship constant co-efficient kth order, we could get solutions of similar form, some constant to the power and some c n to the power n and c 2 to the power n c 3 to the power n, where they were the roots of the characteristics equation with something like this along with x, yes multiply s. This is what we got there also.

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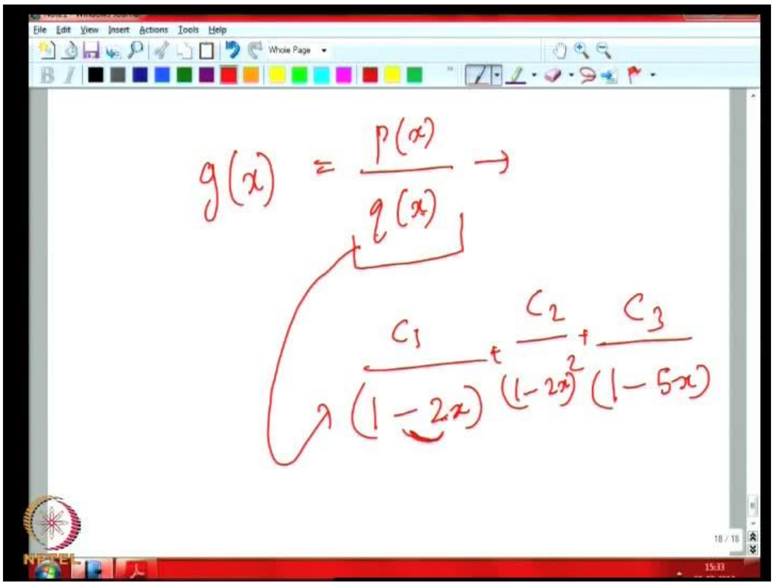

Now, we got some numbers here. So, we will tend to things that may be this r, the roots or the characteristic equation, but then we did not consider the characteristic equation here. I thought so is possible that they are again the roots of the characteristics equation. We just briefly discuss this issue, right. So, what is the relation between the characteristics equation and this q x? You know here, these numbers, this 2, 2, 2 to the power n and minus 5 to the power n, you see here, right they actually are the roots of q of x. Why? Because our g of x was written as p of x by q of x. Then when we consider the method of partial fractions, we factorized to q of x. So, the roots of q of x appeared. So, for a instant for c 1 by 1 minus 2 x 1 term, another term was c 3 by 2 minus 5 x, right.

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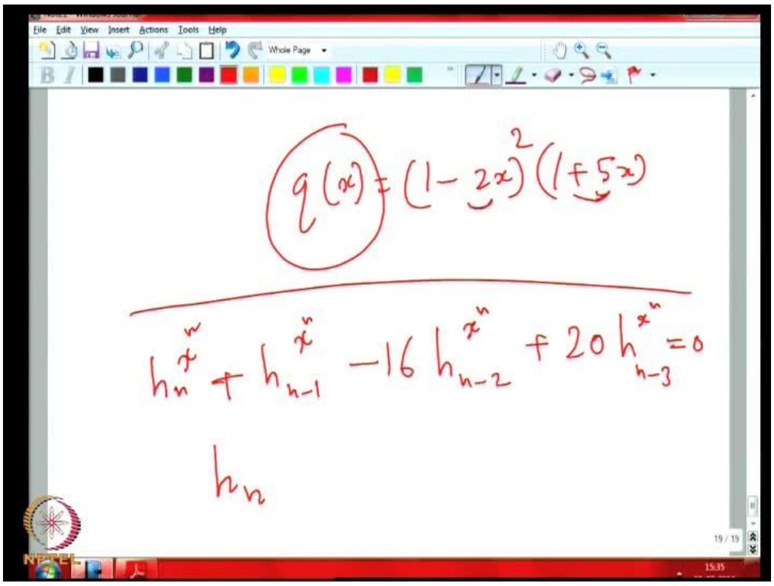

Another term was c 2 by 1 minus 2 x square, right. So, here this 2 has something to do with q of x, not with the roots, but it has something to do with the q of x because we could write q of x equal to 1 minus 2 x whole square into 1 plus 5 x. That is why those things came and that is the reason in the answer 2 to the power and 2 to the power n and 5 to the minus 5 to the power n plus. So, minus 5 to the power n appeared, right. So, what is the relation between this q x and the characteristics equation? What is the characteristics equation? See, here characteristics equation, we go back to the characteristic equation look like, for recurrence relation was h n plus h n minus 1 minus 16 h n minus 2 plus 20 h n minus 3 equal to 0, right.

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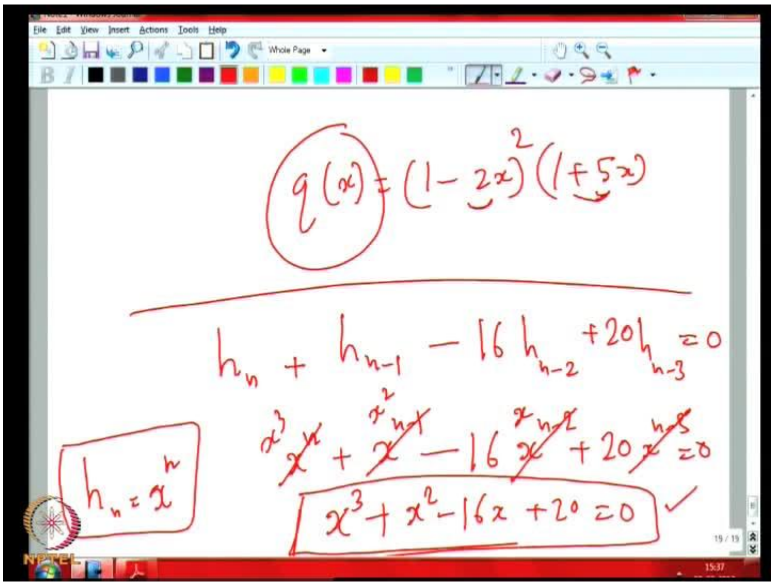

So, the characteristics equation automatically because they will be, if you multiply by x raise to n here, x raise to n, x raise to n, x raise to n and we cancel off x raise to n minus 3 from everywhere, so we will end up with this kind of relation, hn into, sorry so this is not the characteristic equation. So, our recurrence relation was h n plus h n minus 1 minus 16 h n minus 2 plus 20 h n minus 3 equal to 0.

So, how do you find the characteristics equation? So, we find the characteristic equation by thinking that suppose, h n is of the form some x to the power n, some constant to the power n we consider is valuable, some with 2. So, this will be x raise to n plus x raise to n minus 1 minus 16 into x raise to n minus 2 plus 20 x raise to n minus 3 equal to 0. This is what we get. Now, we cancel of x raise to n minus 3 from everywhere.

So, this will q n x. This will be q n x square and this will be given x cube, right. So, the characteristic equation is x cube plus x square minus 16 x plus 20 equal to 0. This is the characteristics equation and we know that if all the h n has a solution of the form, some constant to the power n, then that constant should satisfy this characteristic equation x cube plus it should be root of this characteristic equation, right.

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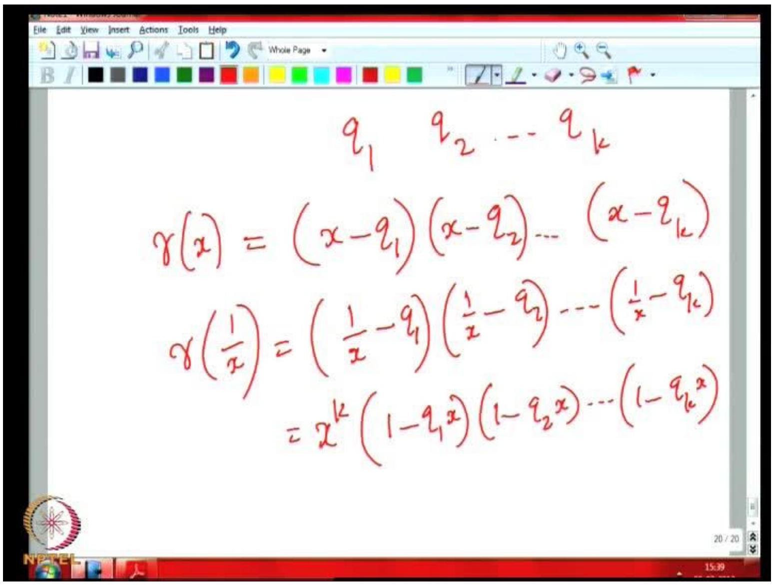

Now, let us say the characteristics equation has roots q 1, q 2, q k, right, then our characteristic equation r of x can be written as x minus q 1 into x minus q 2 into x minus q k. Isn't it? So, this is the way we can write the characteristic equation. Now, on the other hand, if we had taken r of 1 by x, so put substitute of x equal to 1 by x. So, you can substitute. Instead of x, we can put 1 by x. So, we will get 1 by x minus q 1 into 1 by x minus q 2 and so on, 1 by x minus q k. So, this will look like x raise to k into 1 minus q 1 x plus 1 minus q 2 x, right and 1 minus q k x, right but there is a relation between this handover q of x. So, what is the relation between this and the q of x? This q of x is actually we can see that q of x is actually the reverse of the corrections.

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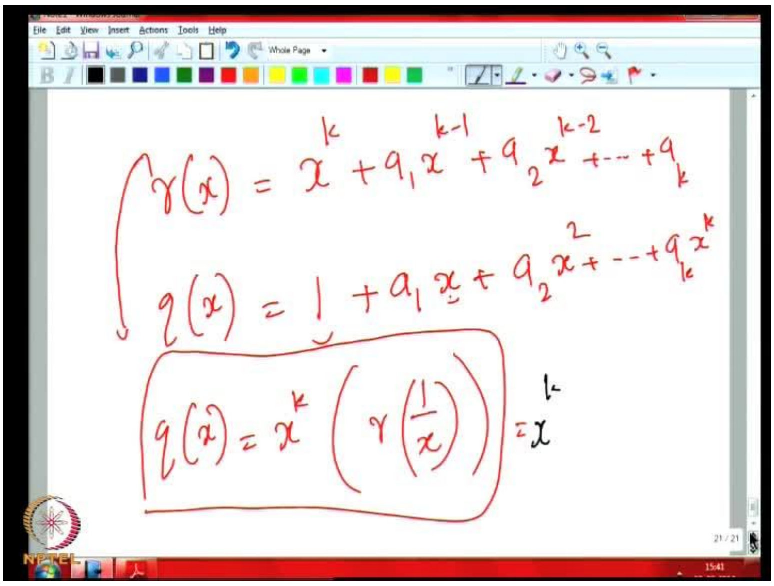

Let us write r of x and q of x together. R of x is x raise to k into a 1 x raise to k minus 1 plus a 2 times x raise to k minus 2 plus a k times. So, that is a k, right. This is the way we write the characteristic equation a k now. On the other hand, q k looks like q x, this 1 plus a 1 x plus a 2 x square plus. Finally, here a k times x raise to k. So, there should be a relation between this and this because coefficients are the same and just that this is x raise to k here, it is on there is x raise to k minus here is x. So, with some inspection, we can see that the relation is that q of x is equal to x raise to k into r of 1 by x. Isn't it?

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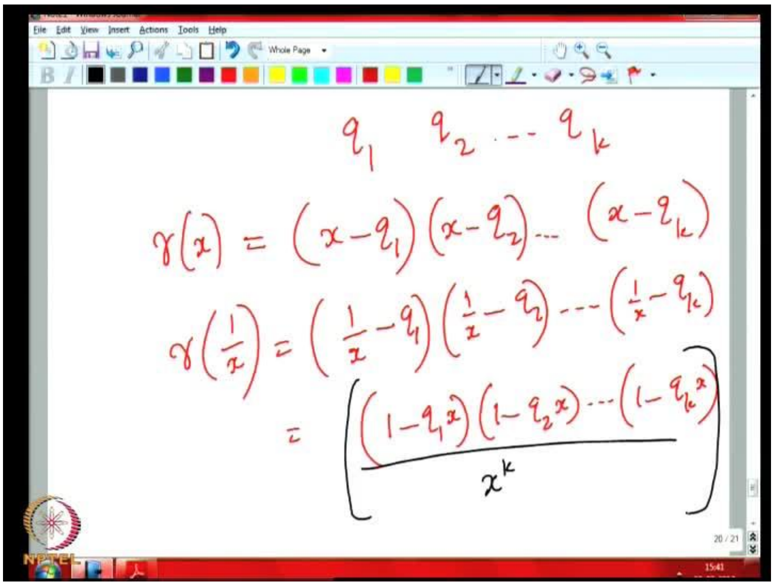

If you just put, we just have to substitute 1 by x n and this formula, so that is 1 by x raise to k. We multiply by x raise to k, this will become 1, right. Similarly, here 1 by x raise to k minus 1 multiply x raise to k will get an x here and a 1 like that. So, this is the relation between that, right. So, q x is actually x raise to k into r of 1 by x, but we know r of 1 by x is actually, so we can substitute for you.

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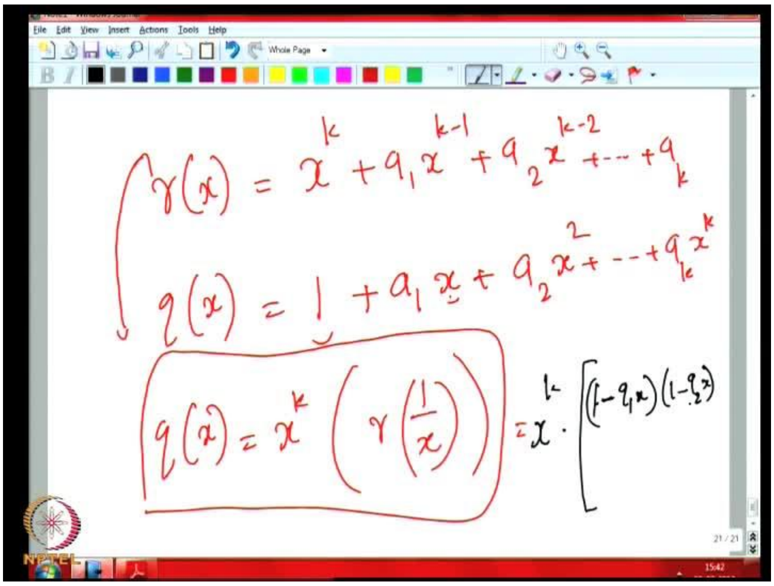

See, x raise k into r of 1 by x is already sketched out. Here, r of x is this, right. So, r of 1 by x is, so this what happens. It is here 1 by 1 minus q x. You have an x raise to k and the denominator, right. This is what r i raise. So, we can substitute there x raise to k into 1 minus q x 1 minus q 1 x into 1 minus q 2 x. Yes, we will do it in the next speech. So, I will repeat it. So, this relation we can see. This relation is easy to see once we write r of x and q of x.

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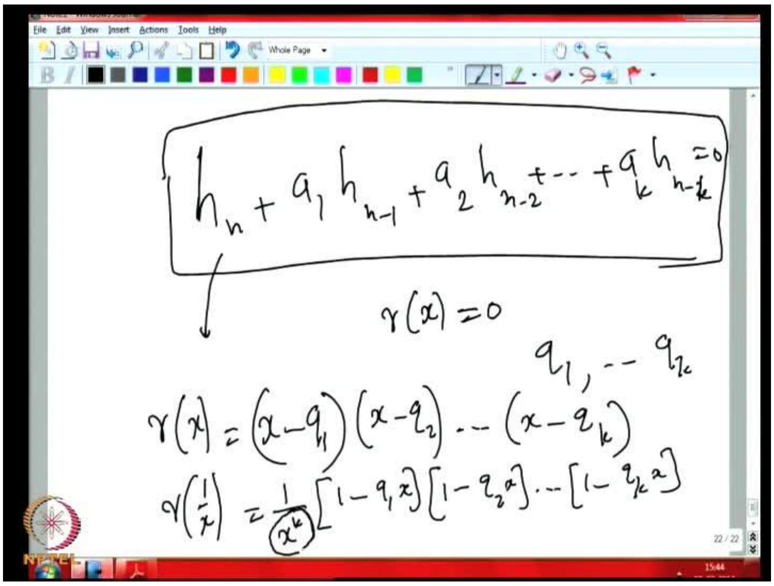

This is the characteristic equation for a recurrence relation of the form h n plus a 1 h n minus 1 plus a 2 h n minus 2 h n plus a 1 h n minus 1 plus a 2 h n minus 2 and so on, a k h n minus k equal to 0. So, this is the recurrence relation and now, the characteristics equation for this thing is written here, this one, right and q of x for that recurrence relation was written here, q of x in the sense, g of x is equal to p of x by q of x. That particular p of x will come like this and from this thing, we notice that q of x is actually x raise to k into r of 1 by x. Actually, the characteristics equation has roots.

Characteristic equation r of x equal to 0 has roots q 1 to q k. Then r of x equal to x minus q 1 into x minus q 2 into so on, x minus q k. So, then we can easily find out what r of 1 by x which is actually we have a 1 by x raise to k. Coming in the denominator, we will get 1 minus q 1 x into 1 minus q 2 x and so on, 1 minus q k x. So, we are just putting instead of x, we are putting 1 by x. So, 1 by x will, so taking of common denominator 1 minus q 1 x here by x, so here 1 minus q 2 x by x, here 1 minus q k x by x, so that all those x's in the denominator will combine to form a 1 by x raise to k here. So, that is what it is, right. Now, here we can substitute x raise to k into 1 by r of x.

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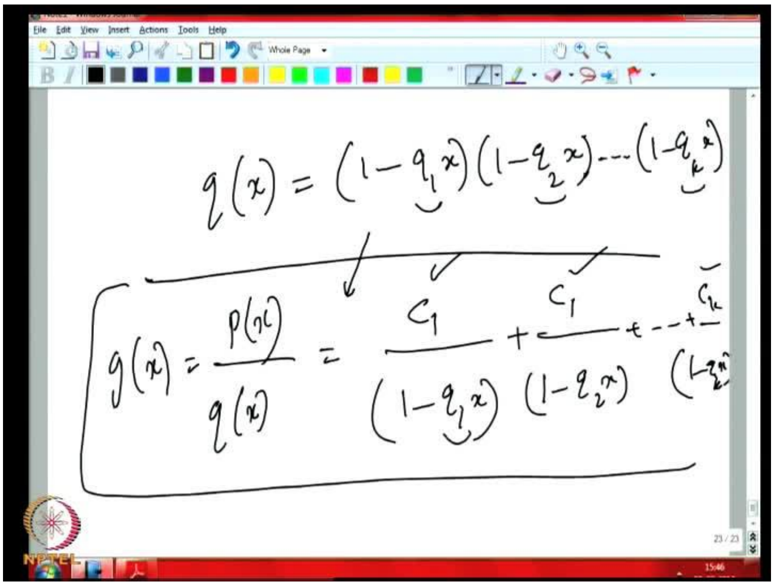

So, when we substitute here this formula, right that x raise to k and x raise to k will cancel and we just get q of x equal to 1 minus q 1 x into 1 minus q 2 x into 1 minus q k x. Now, it is not surprising at all that these are actually the roots of the characteristics equation and finally, when g of x is equal to p of x by q of x is converted into some form of this sort it will look like, right.

So, suppose in the case that q 1 q 2, all of them are different, we will write the simplified form due to by the method of partial fractions like this, 1 minus q 2 x and so on, c k by x. So, we will be finding the value of c 1, c 2, c k and it is not surprising that q 1 raise to n q 2 raise to n q k raise to n etcetera appear in the final answer because we know when we use the binomial, generalize binomial theorem to expand it, this q 1 raise to n q 2 raise to n etcetera will appear, right. That is why see the roots of the characteristics equation again. In the solution, when we use that method of generating function also, we will continue in the next class.