Combinatorics
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Lecture – 3
Pigeon hole principle – (Part 3)

So, this is third lecture of the combinatorics course. We were studying the pigeonhole principle, this is the third part of it.

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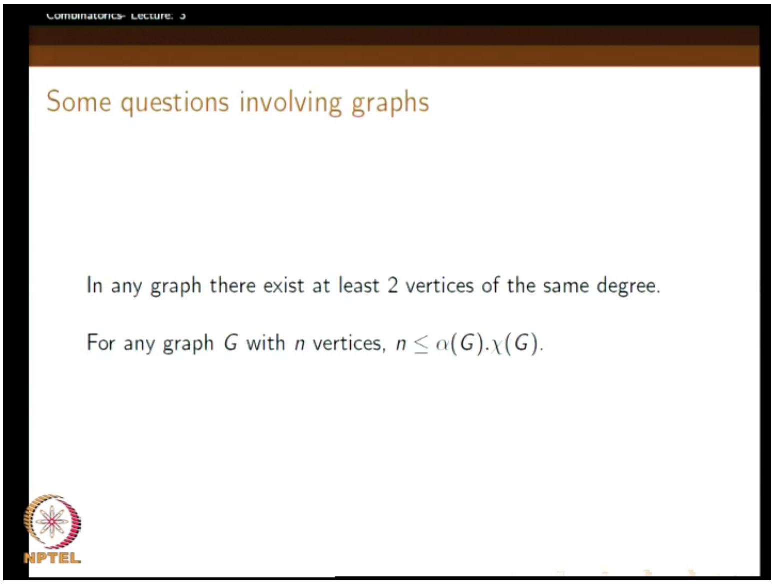

So, in the last class we considered some question involving graphs. So, of course somehow I assumed that the student is comfortable with a graph; comfortable means you know a little bit of graph theory. That I assumed about almost everything like, when I am talking about numbers I am assuming that student is to some extent familiar with the very basic notions like.

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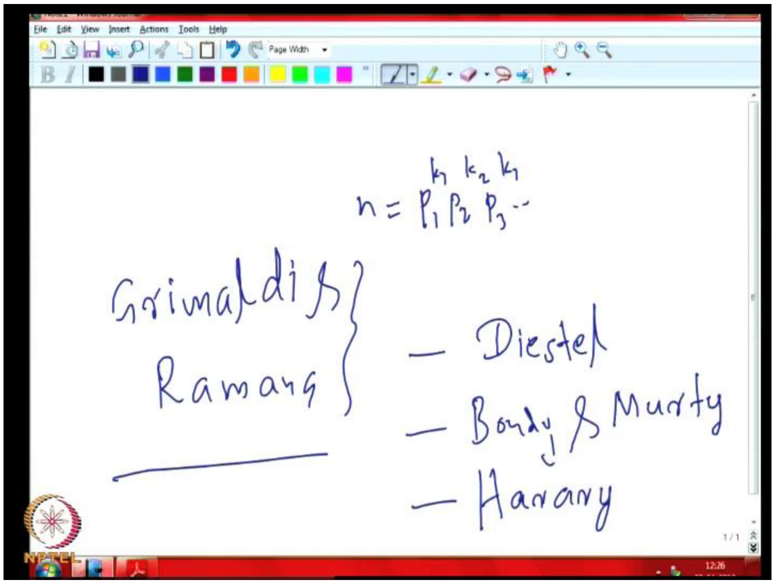

Any number can be decomposed into these prime factors factorization, right, like this; these kind of things and divisibility; that means what does it mean to say that some number divides another numbers, what is the reminder? Such things I am assuming, so which will be the twelve standard level knowledge. For graph theory I am assuming that the students are familiar with what is a graph? What are vertices? What are edges? And the very basic notation like what is the degree of a vertex, and also that student have seen it before at least once.

No in-depth knowledge is assumed definitely. So it is just some familiarity basic familiarity. So, in case you are not already familiar with those concepts what we can do is you can read the first chapter from any of the graph theory books available. So, it can be a book by Reinhard Diestel or it can be the book by Bondy and Murty, and it can be the book by Harary, but there are several several books.

You can just search in the net, say, go to the Wikipedia and read something about graph theory; even that would be enough, because we just need the basic things, and even those concepts like independent set, the coloring. I will remind, because see in that last class I introduced those notations though I just did it fast that is why I assumed that you have seen it before. So, I am just reminding it. Now in case you do not understand it, you can go back and see it once again in one of those books or wherever you have studied, or otherwise you can see the NPTEL lecture on graph theory, and the in this course graph theory though it is part of combinatorics, but it will not be thought in course.

So, if you want to study graph theory you have to view the separate course on graph theory, but still things about graphs, ideas about graphs will keep coming in this course again and again. Because here when we teach techniques to deal with combinatorial problems to give examples we will have to take problems from graph theory though. It will be just to illustrate certain points most of the time, but because there are several nice examples that which we can borrow from graph theory.

We will be doing it time and again; therefore, the student is supposed to know a little bit of it some basic familiarities always assumed and geometry also. When I talk of geometry I will assume that the student knows the twelfth standard level geometry and is familiar with most of the elementary concepts about Euclidean geometry. Even if he does not know he can always search somewhere and find out the corresponding notions.

But most of them whenever something reasonably sophisticated comes I will definitely explain, but of course I cannot keep on explaining what is a triangle, what is a rectangle and things like that, and also the area of triangle. Such notions I will definitely assume and also the other notions even from combinatorics. I will introduce the basic notations, but it will very quick. So, whatever you have studied till twelfth standard you are supposed to revise a little bit before coming to this course and of course what are functions, what are relations, such things I may mention in passing but not very carefully. So, incase you have difficulty what you can do is you can read one of these books which I mentioned initially this, say, Grimaldi and Ramana’s book because the initial chapters contains all these required material. So, you can revise from those books.

It is just a matter of when you do not understand something which I tell you can just go to the chapters and quickly see what it is, the definitions, if you just miss something, so then you can keep. But it is quite difficult for me to remember that what all like for when it comes to elementary notions I may forget that this is not introduced, because it is so common and I mean very elementary that I may miss to introduce that. In that case you have to just look at some of those books. Those books will anyway contain all these material, because our plan is to soon get into more nontrivial things. So, we cannot spend too much time on these elementary notations; this is the point. Now again so in the last class we dealt with two problems in graph theory. The first problem, remember it was about the degrees of the vertices in a graph. Again whenever I am talking about graph I am assuming that the graphs are finite, right.

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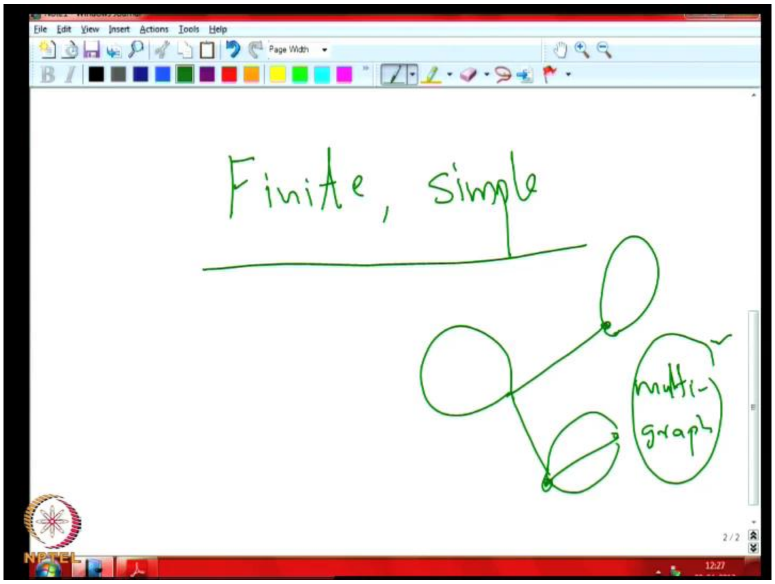

Finite though I may not always tell it finite and simple; when I say simple graph it means we do not have self loops. So, for instance not loops of this sought, right. So, these are not simple graphs because loops are there, and also I am assuming that there will not be things like this, right, multiple edges between two vertices. There is only one edge if at all, but in case if I am talking about multi graph, these are called multi graphs, then we allow loops self loops and multi edges. I will be mentioning that specifically, because it is more common to talk about finite and simple graphs than multi graphs. So, therefore rather than always assuming that graph is a multi graph and then always saying that it is a finite simple graph; better we assume that whenever I say graph I mean finite and simple graphs that will be our convention when we talk, and yeah in the last class the first problem was about the degrees. The second problem was about independent set and coloring.

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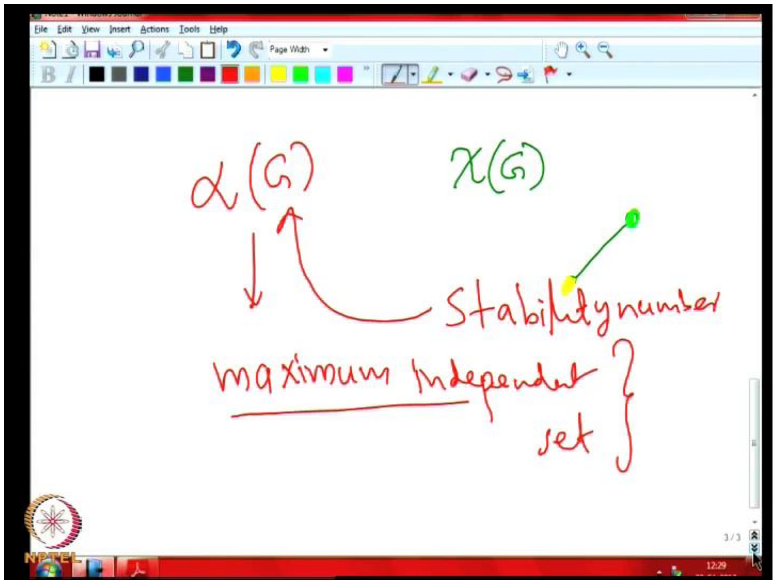

I introduce what is a vertex coloring and I mentioned that chromatic number of the graph refers to the smallest number of colors that are required to properly vertex color the graph. When I say properly vertex color I mean giving colors to the vertices in such a way that two nonadjacent vertices get different colors, sorry, two adjacent vertices gets different colors, right. Whenever they are nonadjacent they can get the same color, or they may get different color. It does not matter; we do not care. We can do it anyway, but whenever there is an edge between the two vertices they should get different colors, right, this is what we need. So, that minimum number of colors required to do this thing is called the chromatic number.

And now the other concept we introduced is alpha of g the cardinality or the maximum independence set. So, this independent set meant that it is a collection of subset of vertices it is a subset of vertices the graph such that between any two vertices in these collection there is no edge. That means they are pair wise nonadjacent, right, and we wanted the maximum such collection that was called maximum independent set; it is called the cardinality of the maximum independent set, and this has also got one name another name it is called as stability number; some of those use that name. So, this is the notations used for this chi of g for the chromatic number, right.

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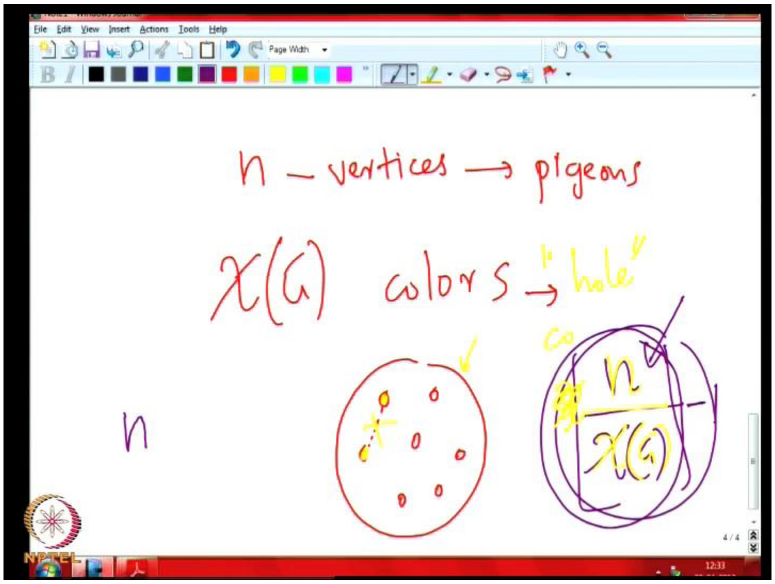

So, yesterday we told that when we consider the vertices as pigeons, there are n vertices, these are pigeons. Then there is chi of g colors, right, chi of g colors. So, then each color is a set of vertices each color the first two set of vertices namely the vertices which are colored by that particular color, and this is called the color class, right, color class, and the only observation we wanted was that if you consider one color class the vertices inside that color class form an independent set, why? Because you know if there was an edge between these two then it is not possible to give the same color to both the vertices; that would be wrong, right. Then you will have to give different colors here.

So, the fact that we have same color to all of them means that that forms an independent set. So, now each color forms a hole; that means colors forms the hole, right; the color classes are holes pigeon holes here, and the pigeons are the vertices. Now each vertex has to go to one of the pigeon holes, right, because in other words each vertex has to get one of the colors and say. So, now let us say number of colors is chi, right, and so we have to have because number of colors is chi, because there are n vertices. It means that we should have n by chi number of pigeons in one of the holes, right, n by chi number of pigeons in one of the holes, right, is it not, because this is just the averaging argument. So, you have n pigeons only chi classes here, right.

So, for instance, yeah this is we know that, right. So, therefore there should be at least, right; for instance if you take any number below this thing that number into chi will be smaller than n, n will be greater than chi into any number which is, say, if I put minus 1 here after that, right. So therefore, whichever integer is smaller than this things we can take the floor, right. If it is already an integer then take minus 1; if it is not an integer floor minus 1. So, n will be strictly greater than that; therefore, our generalized pigeonhole principle says that, right, they should have this many at least these many pigeons going into one of the holes; this is what we get.

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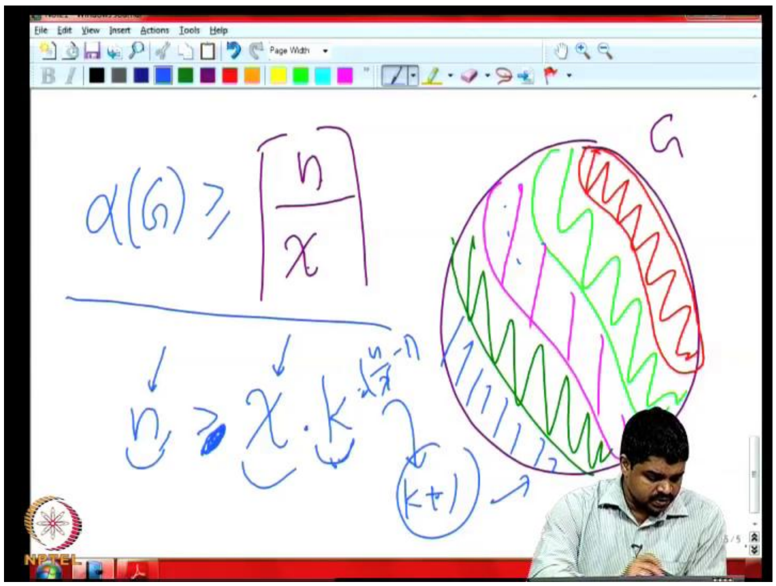

So, therefore, we see that there exists one color class in which at least n by chi vertices have come. The point here is this n by chi need not be an integer. What we mean is suppose it is an integer we have at least that much. If it is not an integer we have even, say, you can say seal of this thing, right, where you can take the next integer, because the number of vertices is always an integer, right. So, this many will be there; that is what, right. So, it should be clear pictorially this is what we mean. So, if this is G, so if I draw the color classes though it is after drawing giving the colors I just collect the vertices of those color glasses. So, let us see something like this, right, and so on, right, say these are the colors and now what we claimed was that so one of the color classes should get at least because there are chi of them these color classes at least n by chi vertices, right.

So, that is but then that each color class forms an independent set, right. So, therefore alpha of G the maximum independent set has to be atleast as much as this thing; that is what we get, right. So, this is coming from the generalized pigeonhole principle. So, if you really wanted to see is if n greater then or equal to chi into some k, right, chi being the number of color classes, these are number of pigeons, right, number of objects, chi is the number of classes into which we are dividing the object; if n is strictly greater than, suppose the number of object is strictly greater than chi into k, right. Then we know that they should be one of the classes where k plus 1 objects have come; that is what we told. Here there is a little difficulty because these are whether this k we have to find out as an integer.

So, what we do is so suppose we will see whether it is n by k, chi is an integer. If n by k chi is an integer we can take it as n by chi, k as n by chi minus 1, right. Then we will see that n by chi will come here. So, if n by chi was not an integer we just take the floor here as k; so that means we will get that because n is strictly greater than that. So, therefore yeah, we will get this from the generalized pigeonhole principle. This is what we did yesterday in the last class, and now the next question is again about graphs.

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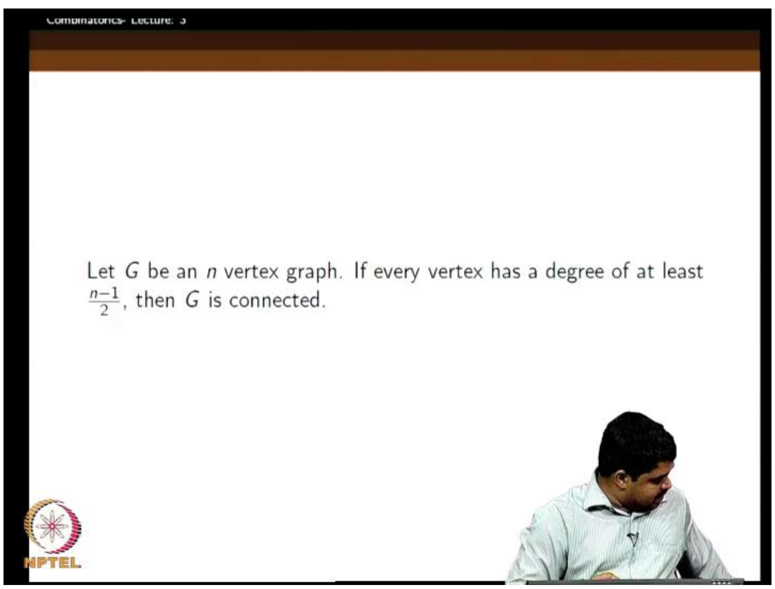

Let G be an n vertex graph. If every vertex has a degree of atleast n minus 1 by 2 then G is connected. So, here is again some notions which we have to explain.

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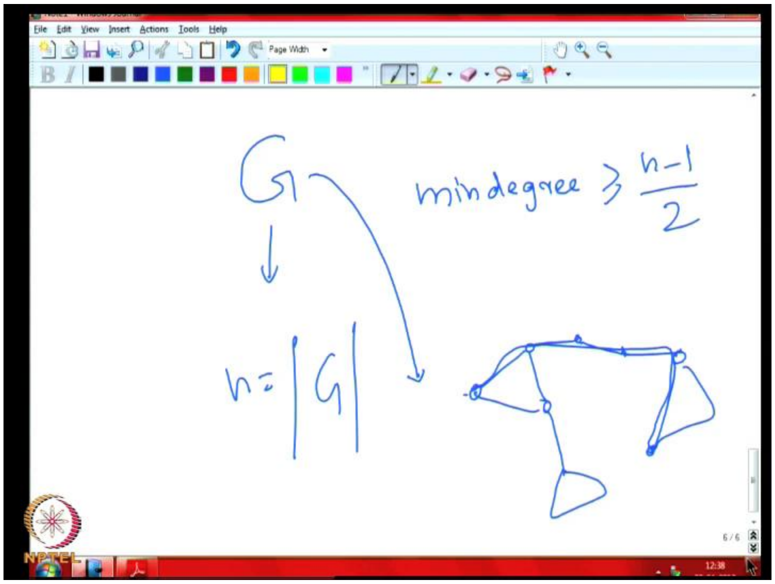

So, G and now every degree, so we will say minimum degree of the graph is at least n minus 1 by 2; n being the number of vertices, n is the number of vertices, we can also write like this; n is the number of vertices. Suppose the minimum degree is at least n minus 1 by 2, so what do you mean by minimum degree is at least n minus 1 by 2? That means every vertices degree at least n minus 1 by 2. This again need not be an integer; if it is a frication we will have the seal of this, right, because the minimum degree cannot be a fraction. Now we say that in this case G is connected; what do I mean by connected? Graph is connected if between any two vertices there is a path, right; there is a path. So, for instance this is a connected graph, right. This is a connected graph, because if you want to go from here to here you have a path here, right; you can reach from here without getting out of the graph from anywhere to anywhere, but on the other hand if I introduce, say, this graph.

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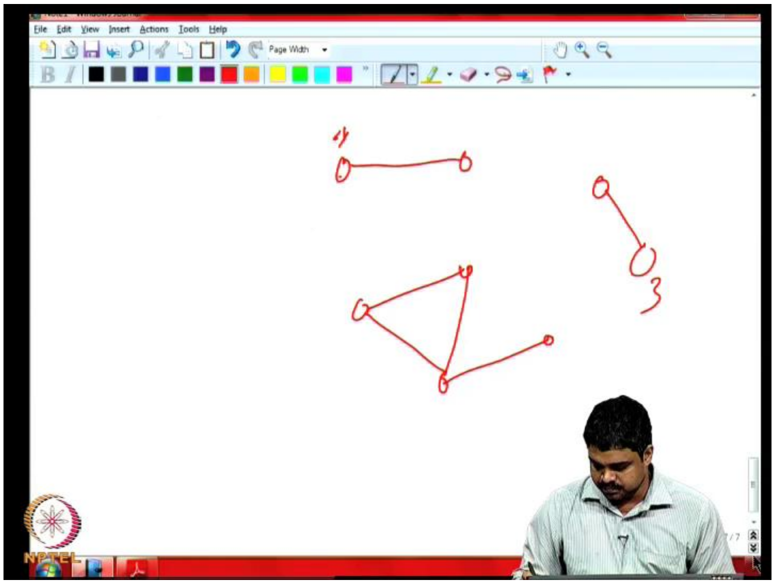

The next graph if I say consider this graph this is not connected. For instance if I want to go from this vertex number one to vertex number three, definitely I cannot, because I can go up to here, so I cannot reach, here right; it is not connected. This is intuitive notion of connectivity. So, this is a disconnected graph; so we need only this much for this problem. We are calming that suppose given a graph and if every vertex is big at least in minus 1 this problem will not come. So, there will be connectivity; that means there will a path from each vertex. So, each pair of vertices there will be a path, right; how will you show this thing? Now suppose four contradictions, right.

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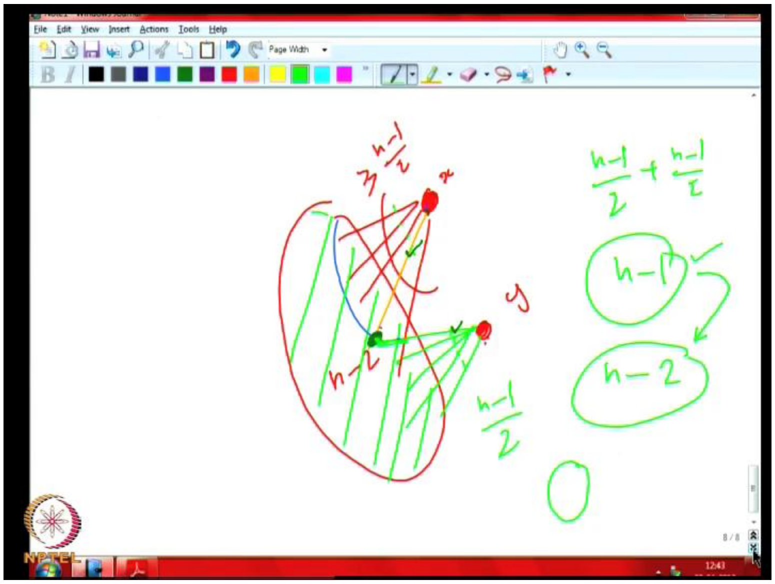

Let us assume that it is not connected; that means you can find some two vertices, say, x and y such that they are not connected in that. That means there is no path in the graph from x to y. So, that means there is no edge between them, right. If there is a direct edge between them then definitely they are connected, right. There is a path; there is a direct edge itself. So, it is like this. So, we do not have this, right; that is what we see, and now what about the remaining vertices? There are n minus 2 vertices here, because total n one two and n minus 2 left, and if I consider the degree of this thing, the neighbors of this thing, right. There are n minus 1 by 2 of them, right. I can only draw here because this vertex is not connected; this is not a neighbor of this thing.

So, this is atleast n minus 1 by 2 at least, and for this k also we have atleast n minus 1 by 2. Suppose if I have a common vertex here; that means this edge also is coming here, this edge also is coming here, and this edge also is coming here. Then definitely it is connected, because there is a path from here to here; not only that there is path, there is a path of length two, right. So, you can jump to here and jump to here; that means it should be such that this vertices I mean the vertices which are neighbors of this and the vertices which are neighbors of this thing, the vertices which are at the end of these red edges like the way we have drawn earlier, the vertices at the end of these red edges, and the vertices at the end of these green edges are to be disjoint.

But we claim that by pigeonhole principle it is not possible, why? Because there are which are the pigeons here; Here these edges will be the pigeons; these adjust will be the pigeons these edges, right. So, how many are there total? n minus 1 by two at least here and n minus 1 by 2 at least here, total n minus 1 pigeons are here. These edges are the pigeons, right, these total red edges and the green edges. Now the holes are these vertices. There are the vertices here, the vertices in this part that is only n minus 2. So, there are n minus 2 holes and n minus 1 pigeons. So, the n minus 1 pigeons are trying to occupy these holes which are one less than that; definitely the pigeonhole principle say's that it is not possible for each pigeon to go to a separate hole. You should have one hole; one hole means it should be one vertex here, say, let me draw this one vertex, right, such that two pigeons occupy that same hole.

That means two of these edges come on that, but then it is not possible for two red edges to fall on the same vertex. So, that is way we told that this is a simple graph; there is no multi edge, right. So, each of this edge starting from here has to go different different vertices. So, therefore, those two pigeons one has to come from here and one have to come from here; it is the only possible way. So, it should be something like this. So it is a contradiction to the fact that there is no connectivity in the graph. There is a path from this vertex to this vertex to hope path. So, we can jump from here to here, right, in two hops; that is what we see, right. So, this shows that if every vertex has degree at least n minus 1 by 2 then the graph has to be connected in a simple application of pigeonhole principle into graph theory.

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Now the next thing we will look is a more interesting, say, slightly more important theorem. It is one of the earliest theorems from extremal graph theory; it is called Mantel's theorem. If a graph G on 2n vertices contains n square plus 1 edges, then G contains a triangle.

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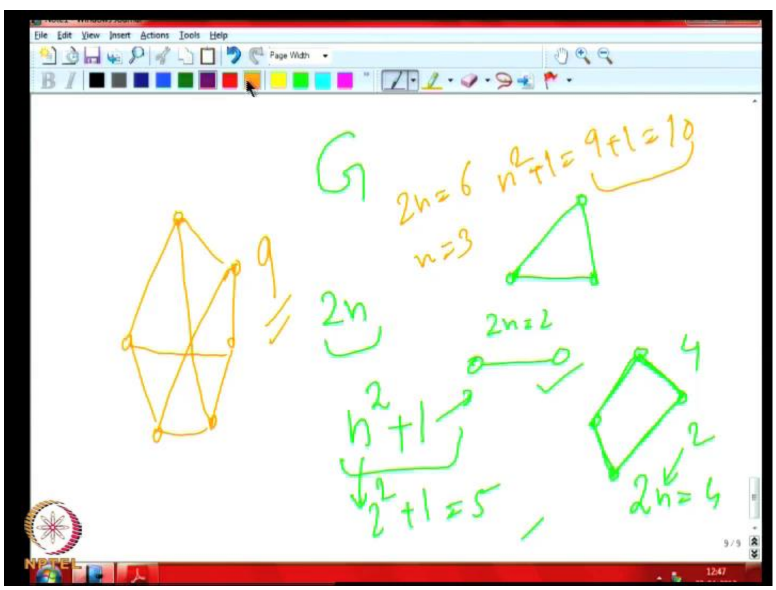

So, the question is like there is a graph G, right, it is 2n vertices. Now we want to add as many edges to these 2n vertices on these 2n vertices. We are talking about simple graphs only such that we do not want to make any triangle in the graph. Triangle means something like this, right; we do not want to create any triangle in this thing, right. How many edges, maximum how many edges can we put? Like for instance if there are two vertices you can just put one edge, fine; if there are three vertices you can only put this much so three vertices, I am saying 2n vertices. Four vertices we can put something like this, right, any extra edge will create a triangle, right; any extra edge will create a triangle.

So, there are four vertices; so four means here but what we are calming is if there are at least n square plus 1 edges then we will create a triangle is what we are seeing. Here this was 2n equal to 2, n was equal to 1 n square plus 1. So, this is not even possible to have n square plus 1 edges here, so that is correct here. So, now for this one 2n is equal to 4; that means n equal to 2 here, right, four vertices. So, n square plus 1 means 2 square plus 1, 5. So, you can put four edges without a triangle as you can see on four vertices, but if you put any more edge here it forms a triangle, whichever way you try and you can try it like this till the triangle. There is no other way you can try it.

So, what about six vertices? Six vertices also you can try. So, six vertices we say n equal to 2n equal to 6; that means n equal to 3. So, n square plus 1 equal to 9 plus 1 is equal to 10. If you put 10 edges you will definitely get a triangles is what it says, what this Mantel’s theorem says. So, up to nine edges you can put. See for instance you can see here so you have it is okay, no triangle. So, we can connect it to here. So, now if you connect it to here or here we are in trouble, right, because if you connect it to here we will get triangle connectivity here.

So, what about this? This you can connect to not this, this one you can connect, right, and then this one you can connect to here, yes, and this one you can connect to not this okay, already connected it, and we have three plus six, nine you have already put. Now we are clamming that if you put one more edge it is form a triangle, it look like, right, if you try to put any edge. So, for instance if you put any edge here it is forming a triangle, right. So, this need not be the only way; you can try it in various possible ways, see because I started with a cycle maybe I did wrong initially, but you can always see that if there are 2n edges.

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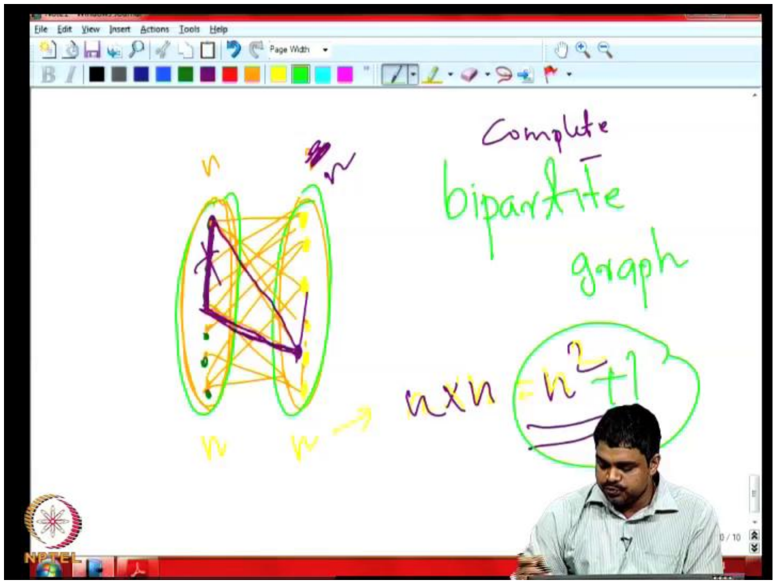

You can put n square edges without 2n vertices you can add n square edges without creating a triangle; this is the way to do it. So, there are n vertices here on these side n vertices, and you also put n vertices here on this side, right. There are total 2n vertices, and now add all the possible edges between this, right. This is a bipartite graph; bipartite in the sense that the vertices are partitioned into two classes here. This is one part, and this is one part, and the other is this part, and within a part there is no edge. The edges are always form one part to the other; the edges are always from one part to other, in that sense it is a bipartite graph. So, you can remember the vertex coloring idea we talked in the last problem. So, if we try to color bipartite graph we cam easily see that you need only two colors.

For instance you can give the same color to every vertex on this side, because there is no edge between them, and a different color here on the other side, right, just two colors are enough to color them. These kinds of graphs are called bipartite graphs. Now what I did is I connected every vertex on this side to the other side. So, n vertices here n vertices, how many edges are there by the so called product rule? n into n, right, 2n square edges have come here; n into n, n square edges have come here, right. So, n square edges, right. So, what is this product rule? Then I am counting the edges. I fix one of the n points here form this side and then you can select the other n point as any from here, sorry this is n, write any form here, right. There are n possibilities.

For the next one there are again n possibilities; for the next one there are again n possibilities, therefore, n square possibilities to make edges, right. So, this is so called product rule. So, therefore, there are n square edges here. Now the point is we have not created any triangle, why? Suppose there is a triangle here. So, suppose the triangle has to involve some vertex right side this is in that triangle, then follow the triangle. So, it goes to this side, so now follow the next edge, because the next edge has to go to the other side back, right, because anyway that is the only way we have edges, right. Now what will be the third edge? Third edge has to be here, but then we know that this edge is not present, because this is a complete bipartite graph; it is complete bipartite graph, complete because all the possible edges across we have put, right. So, this bipartiteness of this graph tells us that there is no triangle here.

So, we do not have any triangles in a bipartite graph and we can get. So, for instance n vertex here n vertex here if you put n square edges you can get. The point the Mantel's theorem makes is that if you have one more edge then we will end up with the triangle. Here it is very obvious, because any extra edge if you have to put you have to put in one of these classes, and you are forming a triangle there now, because already all the cross edges are there. So, therefore, that like this, right, the triangle will come. But you need not work with bipartite graph initially; you can try in anyway, but you cannot go beyond n square edge is what Mantel's theorem. We want to show this thing. So, the simple trick is to use induction in conjunction together with pigeonhole principle.

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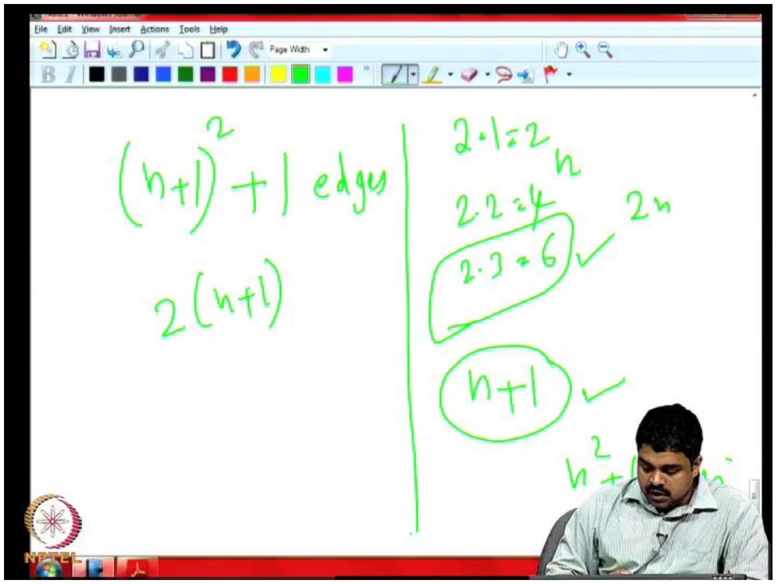

So, what induction? Induction on n, because we are talking about 2n, so we have seen that on 2 into 1, right, n equal into one case it is working; 2 into 2 n equal to 4 vertices it is working; 2 into 3, 6 vertices we have not verified, but we have seen some examples, but at least the first two cases we have n equal to 1 and two cases we have verified it. So, we can try induction. Let us assume that it works for up to n. So, we want to do it for n plus 1. Now by the way when I am talking about induction again I am not formally explaining what induction is; see I am not going to teach what induction is, because I think that it should be studied before twelve standard, right.

That is again I was mentioning initially that about numbers certain things I will assume, like induction is one of those techniques which is very basic and which you should have studied by this time and courses above that, though I call it elementary but not that elementary. So, induction n plus 1 we have to show. So, that means we will show that if there are n plus 1 square plus 1 edges in the graph on 2 into n plus 1 vertices then there is a triangle is what we want to show, up to now we know if there are n square plus 1 edges in a 2n vertex graph then we have a triangle in the graph.

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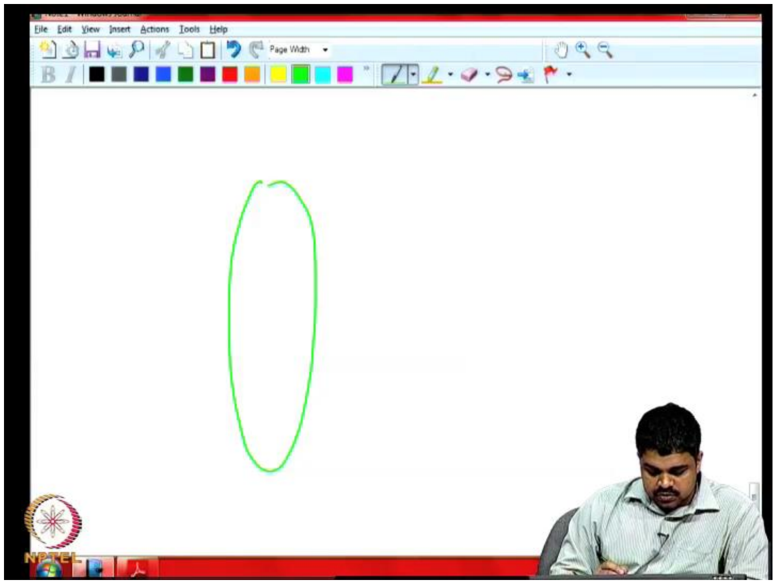

So now it is suppose this is not true for contradiction.

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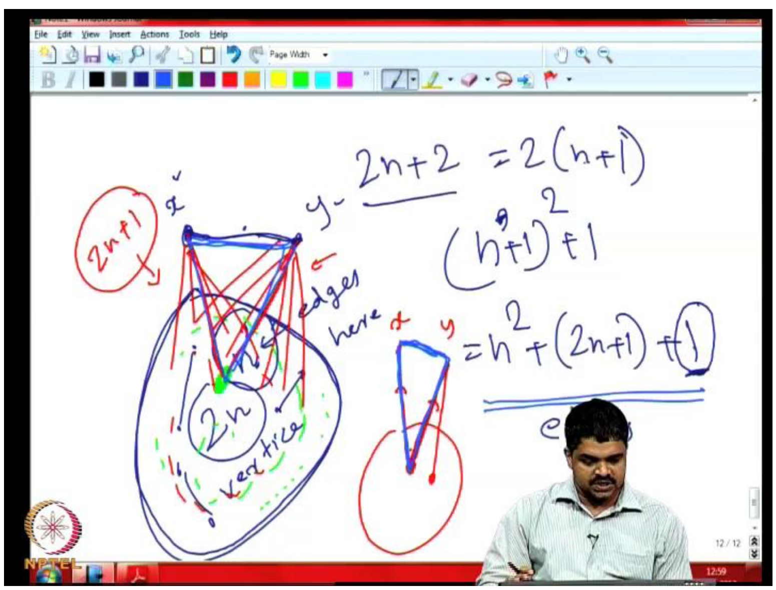

Let us say there are 2n plus 2 vertices in the graph that is 2 into n plus 1 vertex in the graph, and then we do have n plus 1 whole square plus 1; that is equal to n square plus 2n plus 1 plus 1 edges in the graph, and suppose there are no triangles, right. We will show that this is not correct; this cannot be true, right. How do we show? So, there are so many edges; we will pick up two vertices from the graph x and y which are connected to each other, right. See this is recent edge here, right, two vertices x and y will pick up. Now we will remove x and y form the graph and look at the remaining graph; this is the remaining, because total 2n plus 2 vertices were there. So, now 2 we have removed, so here we have 2n vertices, right, in the remaining graph. So, now in this 2n vertices see how many edges can be there? Clearly if in this 2n vertices in this induced sub graph on 2n vertices, when I say induced sub graph I mean that the graph which is just formed by these vertices alone.

We collect all the edges with both n points on these 2n vertices, we will discard. Suppose there are some edges going out this thing we will discard all of them, right. This is the induced sub graph, right. So, the induced sub graph on these 2n vertices can only have utmost n square edges, right, n square edges here, right, 2n vertices and n square edges, why? Because if we had n square plus 1 edges on these 2n vertices then by induction hypothesis we do have a triangle in that, and that triangle is present in the bigger graph also in these 2n plus 2 vertices also, but we know that it is not true. Before contradiction we assume that such a triangle does not access in this bigger graph, well and good.

So, we have hence utmost n square edges here. One edge I can see here, so that one edge is this, right, this one one edge, and yeah this one edge, and here n square edge. So, the remaining is this 2n plus 1, remaining is this 2n plus 1. So, this 2n plus 1 edge should be where? Because we have not thrown away any edge from within this thing; I mean if an edge at both n points inside this 2n like this, if an edge was like this, right, both n points inside this 2n vertices then we have not removed it. And this we have already accounted by this last one, so the remaining 2n plus 1 has to be across. So, I will use this red color, so it should be like this. So, either it starts from x and reaches here, or it starts from y and reaches here, right, reaches into this 2n, something like this, right.

Now these are 2n plus 1 of them at least 2n plus 1 of them. Now these are our pigeons; these are our pigeons. These red edges are our pigeons, and the holes are these 2n vertices. There are 2n vertices here in this part, right, within this 2n part there are within this we have 2n vertices and these are the pigeon holes. Now this 2n plus 1 pigeons are trying to occupy the holes 2n holes. So, there should be one hole which atleast two pigeons occupy; that means two of these red edges should end up on the same vertex, right. This is a hole, the same vertex. So, two of this edges should end up on the same vertex. So, this is the situation here. So, there is this one vertex and then this is x y, and then two edges are coming and sitting here.

So, why do I draw it one from x and one from y? Because both of them cannot be from y, right, because two different edges of y should end up in different vertices only. It is not possible for these two edges from y to occupy to come to the same vertex here. So, therefore, it should be one from x one from y; that is important, and then we remember here this edge was there. That means this edge was there, and this edge was there, right. So, this forms a triangle now, right. Here this is forming a triangle. So, we have proved Mantel's theorem. What we have done is we have done induction on n and we have used the pigeonhole principle very carefully here. So, this proves Mantel's theorem.

This is one of the earliest theorems in extermal graph theory like the kind of questions they ask is how big can a graph be if you want to make sure that it avoids a certain structure here, triangle has to be avoided. The number of the vertices is fixed. We are trying to see how big the number of edges can become at the same time keeping the property that there are no triangles in the graph, right, but anyway we are not going to get into the details of extermal graph theory. So, we just mentioned it impossible.

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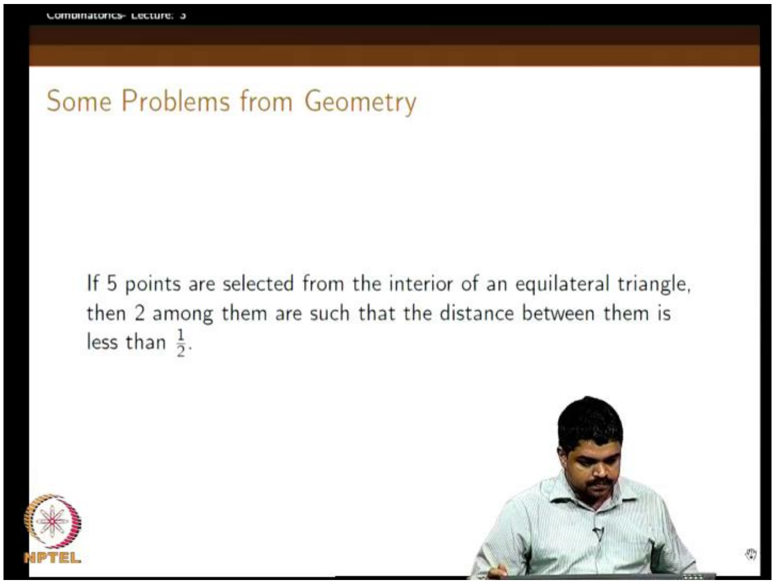

Now we have done a few examples from graph theory. So, now we will look at some examples from geometry some problems from geometry with simple examples. So, if five points are selected from the interior of an equilateral triangle, then two among them are such that the distance between them is less than half.

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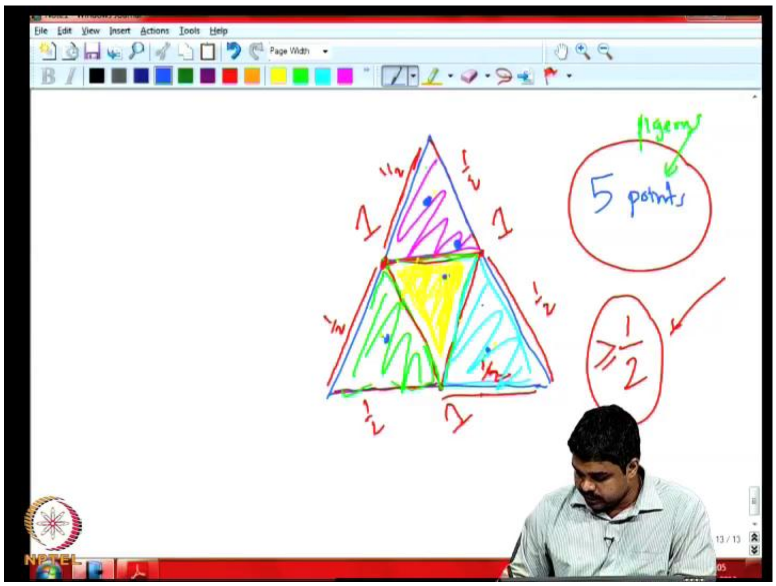

So, once again this says, so you consider an equilateral triangle; equilateral triangle means what? So, all the sides are equal, right, all the sides are equal. Now you ask your friend to select as simple as that. So, no restriction just that he has to select points, five points from the interior of the triangle. So, in this region, alright; in this region he has to select here. This interior of the triangle, not form the border, not form this edges of the triangle means inner part; five vertices he has to select, five points he has to select. Now you claim you ask him to select these five points such that they are all far apart; far apart means the distance between any pair of them is at least half means there are five points, any two you can select. So, between them if you measure the distance it should be greater than equal to half; this is what we are asking?

So, this is equilateral triangle is unit distance here. Now this is not possible is what we are trying to say. He can try as much as possible but as whatever he tries it will not be possible to get five points like that; that is what we want to prove. Why is it so? Of course if you had asked him to select five points from the equilateral triangle whose distance is more than one more than equal to one that would not be possible. He cannot even select two points like that because within the equilateral triangle the distances are only one, but now we are allowing him to keep the distance at least half. You can reduce the distance up to half but you should not go below half, but still you cannot get more then five points is what we are clamming, right. So let us see how it works, why it works, right.

So, here suppose he selects any five points you can draw five points two three five points, the point is suppose if you draw, say, take the midpoint here; that means this distance is half, and this distance is also half. And similarly the midpoint here that this is half, this is half, and this also half, right, and similarly the midpoint here; that means half distance here, and this is also half, and then connect these things; we can use a different color here. Here we see four triangles one two three four. Now the five points are selected; these five points are the pigeons for us now. So, now you can split these triangles into these points here, right, because interior points only; up to here we will consider it as one hole this one and this one this part we will consider as another hole. So, this is one hole, this is one hole including this maybe, and this is another hole and then this one can be taken as another hole, and the remaining thing here can be taken as the fourth hole.

There are four holes; I have marked it using different colors here, right. So, we have to put the borders into one of the, see this border will go into this thing, this border will go into this thing, this border will go into this thing; this will not have any borders, this is just the interior of inner triangle. Now if you take four points they should fall into these holes, right; that means these five pigeons are trying to occupy the four holes. There should be two of them which fall into the same hole say here.

Now it is very clear that it is again an equilateral triangle half, half, half, right, equilateral triangle. So, therefore, you can note the distances cannot be smaller than half; the distances cannot be bigger than half, because these are interior points only, they should be strictly smaller than half. That is what we have, right, because within an equilateral triangle. So, now this is a simple pictorial it is very pictorial; you can see the hole here as a four equilateral triangles, right, and the pigeons are the five points, and then they occupy it. Now another question another geometry question we want to consider is this.

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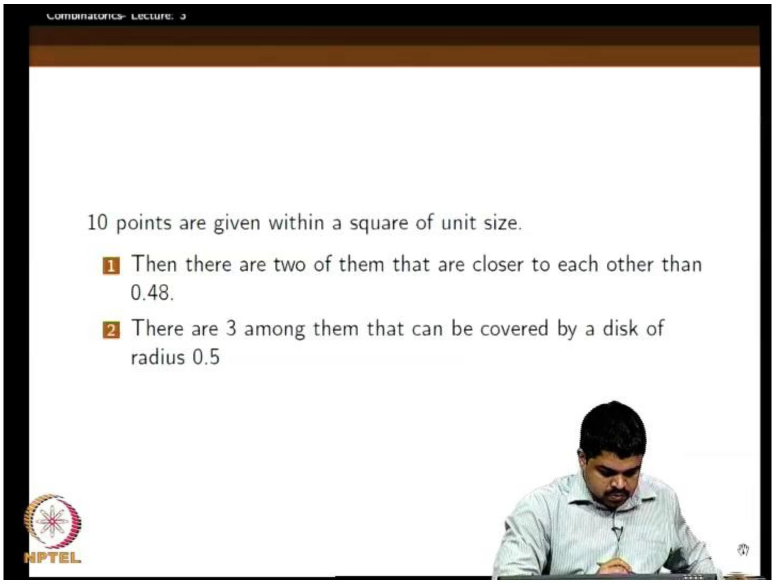

So, ten points are given within a square of unit size. Then we want to show that there are two of them that are closer to each other than 0.48 unit. Also there are three among them that can be covered by a disk of radius 0.5, how do we do this thing?

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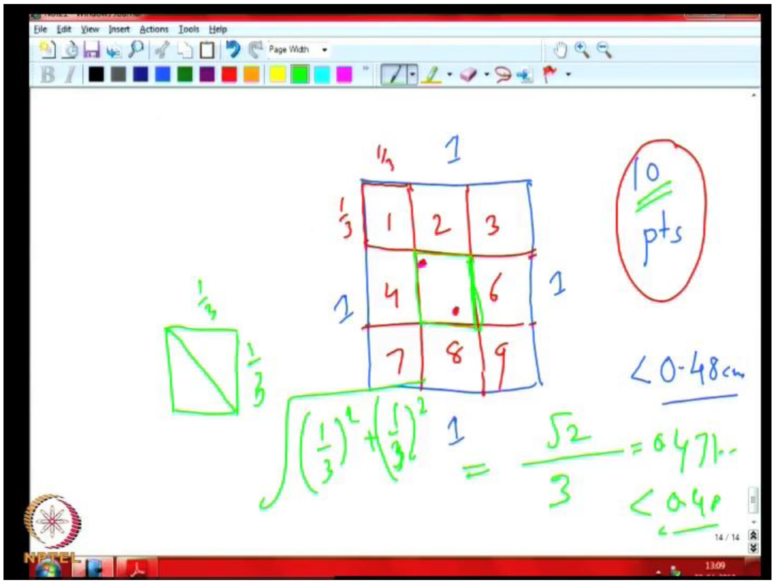

So, here we are talking about a square; I am not drawing it properly of course this is a square. So, all distances are one one one one, so unit square. Now I ask you to take ten points, you ask your friend to take ten points inside this square, inside interior of the square. Now we claim that however he tries he cannot avoid two points becoming too close; too close in the sense that they will become closer than 0.48 units. These is 1 unit, say, this is one centimeter, then this is 0.48 centimeter, alright; it will become strictly less than 0.48 centimeter, why is it so? The argument is like this. So, this time you cut this into three equal parts, so here three equal parts. Yeah, it will become like this.

So, we will get nine equal squares here, and you see this is 1 by 3 each, right; this will also be 1 by 3. This distance is 1 by 3; this distance is 1 by 3. Similarly for each square 1 by 3, 1 by three. Now we have ten points; these are the pigeons. We have nine holes you can see, hole number one two three four five six seven eight nine, and this ten pigeons occupy this nine holes. So, there should be on one hole where two other pigeons will say something like this, right. So, the question is this is a square of 1 by 3 1 by 3, so here 1 by 3 1 by 3. So, if suppose assuming that if this two pigeons fell into this, what will be their distance? Maximum distance is along the diagonal as you can see, right. So, I am not going for rigorous proofs here, because the diagonal is the maximum distance within a square.

So, this is 1 by 3, this is 1 by 3, and then what is this? Pythagoras theorem tells us so that is 1 by 3 square plus 1 by 3 square, root of this should be taken; that is root 2 divided by three which if you calculate you can see that something like 0.471 something like that will come which is strictly less than 0.48; that is all we meant, right, okay. So, therefore, that is where the theorem comes, first statement comes. If you take ten points we cannot avoid two points coming too close. See if the ten was selected because there are nine things which you can see here that is why we wanted the pigeons to be one more than the holes, right, for this argument to hold. Now the second statement says that if ten points are selected you can always find three points among them such that it can be covered by a disk of radius 0.5; that means if draw a circle of radius somewhere 0.5 half somewhere and you can capture three points from this thing.

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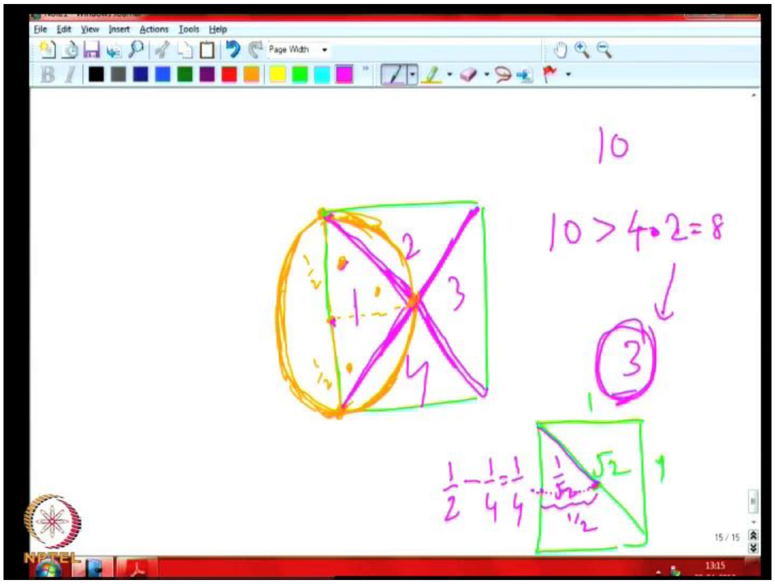

So, this is again like this again that square. We first draw these two diagonals here for the square. Now from elementary geometry you can see that, no I have not drawn it well, but these are all triangles here, right. These are all four triangles of equal size here. These diagonals are bisected by each other, yeah perpendicular this thing. So, diagonals are bisected by each other. Now we consider each of this four triangles one two three four as holes. Now you can use the generalized pigeonhole principle; ten pigeons are there. This ten is greater than, say, four into two, eight. So, therefore, there should be one hole which gets three pigeons more than two, one more than two, three pigeons, right.

So, let us say this hole gets that three pigeons; so this hole this hole gets that three pigeon, right. Now we claim that whatever is inside this triangle can be enclosed by a circle of radius most half; how will you do that? So, what we can do is you can find a circum circle of this triangle; that means a circle okay of course so one has to draw it well something like this. So, that means circum circle means each of these vertices for this is this corners of the triangle are on the circle, this whole circum circle, the triangle is inside this thing. Now you can see that if you consider the center of the circum circle comes over here, because here the distance is half, right, because this is also half.

Suppose I take the midpoint of this thing though the figure is not good at all, of course it should be something like this, okay. So, this is half, this is half, and then what would be this distance and that should be because this diagonal, say, if you consider as square of unit length, and if you consider the diagonal, this is one one. This should be root two, and if you consider the midpoint of it this distance has to be root two by two; that means one by root two, right. Now what should be this distance? If I take to the midpoint if I drop something from here to the mid point in case that is one by root two's; that means one by two is the square of one by root two and minus this is one by two that square is one by four, we get one by four. If you take the square root it will become one by two, right.

So, this point will be the centre of this circle; so therefore, that radius of the circle itself is a half, and then that means that this circle is actually capturing all of these three points, right. That circle is containing these three points; that means we have captured three of these points using a disk circle of radius half. That is what the other theorem stated again as far as the pigeonhole principle apart for the geometry which we use later which we can work out. Now the pigeonhole part was only this thing where we have ten points that those are the pigeons, the holes are this thing. So, these triangles; there are four of them. Therefore we use the generalized pigeonhole principle to argue that there are three pigeons in one of the holes, and then we used a circle which is the circum circle of this triangle, and we noted that the radius of that circle is only half.

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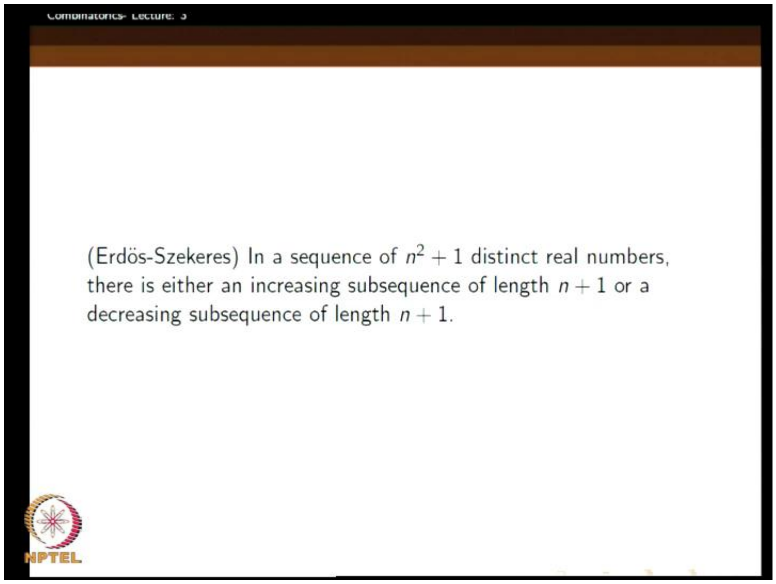

The next thing is a very famous thing. It is a statement about the sequences of distinct real numbers and what does it say?

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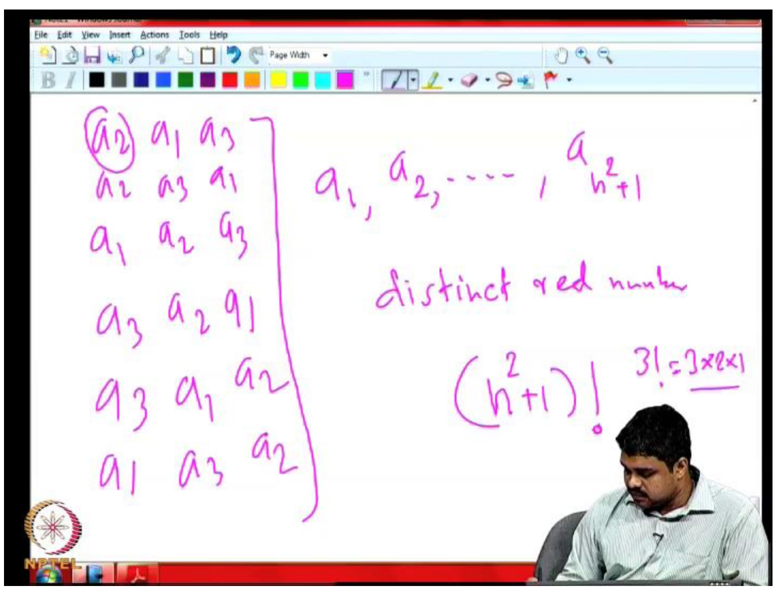

It tells that suppose n square plus 1 distinct real numbers are there n square plus 1 distinct real numbers; that means none of them are equal. Now you try ordering this n square plus 1 numbers in any possible way; you can ask your friend to do that. So, n square plus numbers one numbers means how many possible ways you can do this thing? So, if you know about permutations then there are n square plus 1 factorial ways of doing it, right. You can arrange in anyway n square plus 1 factorial ways. For instance you have just two numbers a 1 a 2 a 3, so you could have put it in a 3 a 2 a 1, or you could have put it as a 3 a 1 a 2, or you could have put it as a 1 a 3 a 2 and two more ways, right. We can start with a 2; a 2 a 1 a 3 and a 2 a 3 a 1, six ways of doing it, right; that is the 3 factorial, 3 factorial is 3 into 2 into 1, right, these many ways of doing it.

So, when you say factorial why it is factorial because you can fill the first position in n square plus one ways, because there are so many of them. The second one n square plus 1 minus 1 that is n square ways and so on, right, so by product or you can do it. So, now whichever ordering you take from this n square plus 1 factorial orderings, so the next thing what you are interested in is to get certain sequence sub sequence from this thing such that that is increasing or decreasing so you can visually say.

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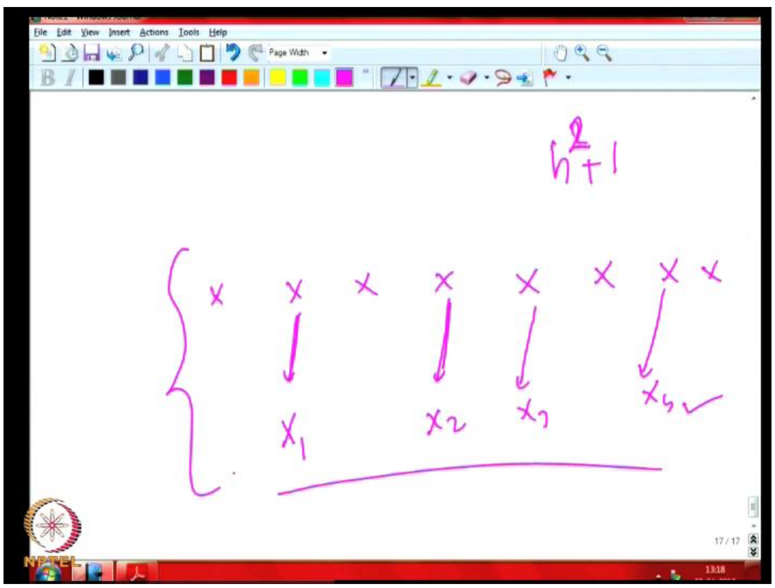

Suppose some n square plus 1 students are there in a class, and they are asked to go and stand on a queue. So, they stand on a queue like this, and assume that they all have these different, different heights. Now we can ask some people to step out, say, this one to step out, say this one to step out, this one to step out, step out. So, how many n of them to step out; this is say x 1, this is x 2, this is x 2. So, now in this order x 4 and the others will remain there. These people will just come to side of the queue, come out of the queue and stay here, and we can make sure that their heights are either in the increasing order or in the decreasing order. This is the statement, right. We will do the remaining in the next class.

Thank you.