Combinatorics

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Lecture - 29
Generating Functions - Part (2)

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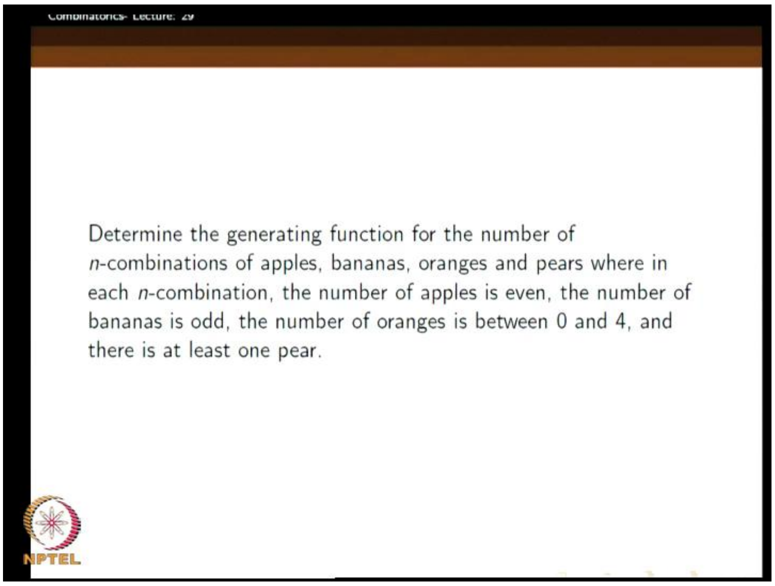

Welcome to the twenty ninth lecture of combinatorics. In the last class, we were looking at how to use generating function to solve certain counting problems. The last question we looked was… Finally, the issue is to extract the coefficient of say x to the power n from a generating function. When you expand the generating function, what would be the coefficient of x to the power n? This is what our… That will give you the final answer. So, the way we modeled certain problems as for certain part of the solution was getting as…

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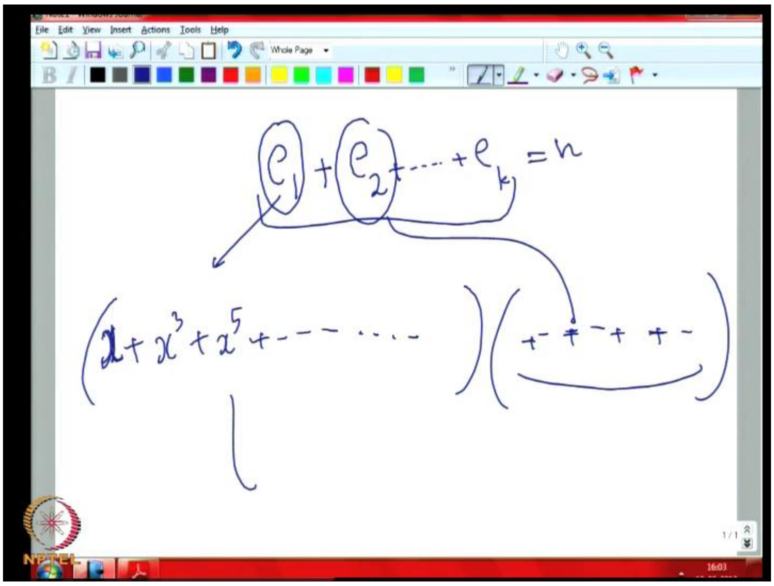

For instance, in the question as we considered up to now, we were counting the number of solutions for an equation like this – e 1 plus e 2 plus e k is equal to n with various conditions on e 1. The basic condition is that, e 1, e 2, e k, etcetera are non-negative integers. But we may have extra conditions like e 1 has to be an odd number or e 2 has to be an even number and such conditions are possible and also we can restrict the values that say an e i can take… We can… Several such conditions are possible.

And now corresponding to each e 1, we will have one term like this say 1 plus – say here x cube plus x raise to 5 plus like that we can put. When for instance… x plus x cube plus – this will kind of impose the condition that e 1 is always odd, because what corresponds to e 1 is the power of x, which will come from this first part. And e 2 will be coming from something like this. And we can cook up this stuff; construct this sum in such a way that the conditions on e 2 are satisfied. Like that we can form. And finally, we will get an expression for this product. And the coefficient of x raise n in that is what we are interested in.

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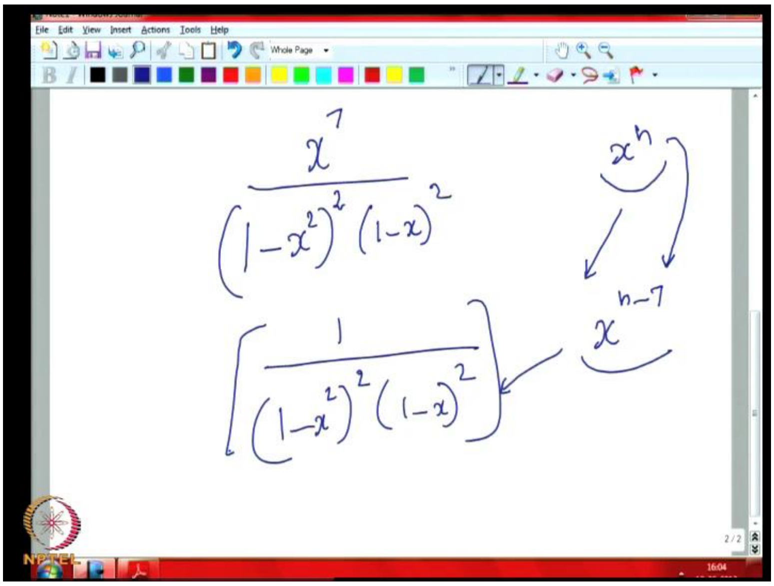

Yesterday, we came to some form like say x to the power 7 minus say 1 minus x square to the power 2 into 1 minus x whole square. So, what is the coefficient of x to the power of n unit? Such question. We first noted that, to find the coefficient of x raise to power n in this, we only have to look for the coefficient of x to the power n minus 7 in say 1 by 1 minus x square to the power 2 into 1 minus x whole square. We can drop this x raise to power 7, because that is already counted here, because every term is multiplied by x raise to 7. So, x raise to n is... It will come from x raise to n minus 7 of the expansion of this thing. So, now, we only have to concentrate on this thing.

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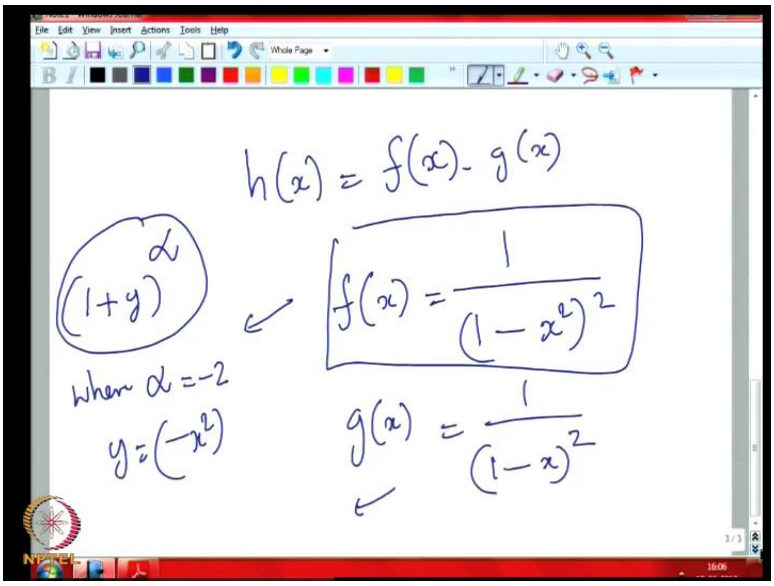

We identified that, this is of the form h x equal to f x into g x; where, f x can be written as 1 by 1 minus x square whole square and g x can be written as 1 by 1 minus x whole square. For both of this thing, we know how to get the expansion, because here we know this is 1 plus x raise to alpha form; where, alpha is equal to minus 2 and… Here instead of x, we may have to use… For instance, we can substitute y… Instead of x, we have to use minus x square; that is all.

Then we substitute or say use some substitution like y is equal to x square in it is y here. And then wherever y raise to i appears, substitute minus x square for y. That will give you the expansion of this thing. And similarly, g minus x – the expansion of this thing will be – put alpha equal to minus 2 and y equal to minus x we can take. And then you substitute using the formula for this; expand using the formula for this thing.

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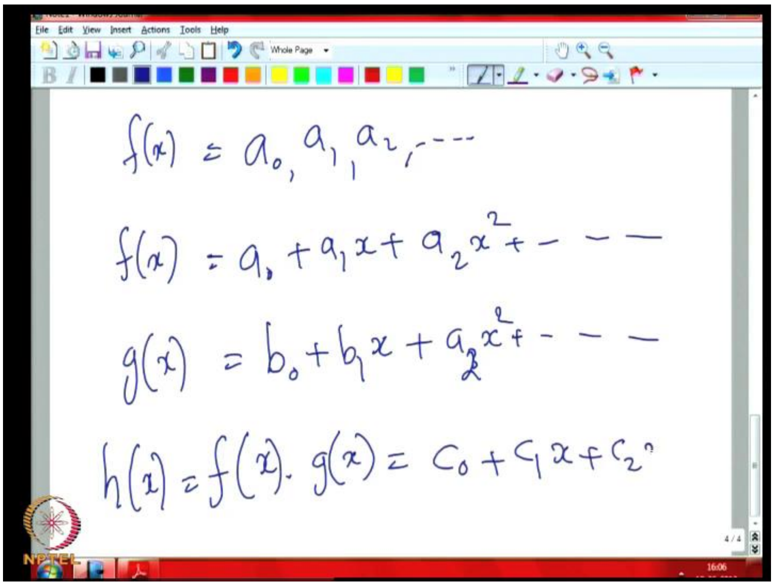

Now, for instance, if you know the sequence corresponding to say suppose a 0, a 1, a 2, etcetera correspond to f x; that means f x equal to a 0 plus a 1 x plus a 2 x square plus and so on; and g x equal to b 0 plus b 1 x plus b 2 x square plus and so on. Then let us say h x equal to f x into g x. Let us write this as c 0 plus c 1 x plus c 2 x square and so on.

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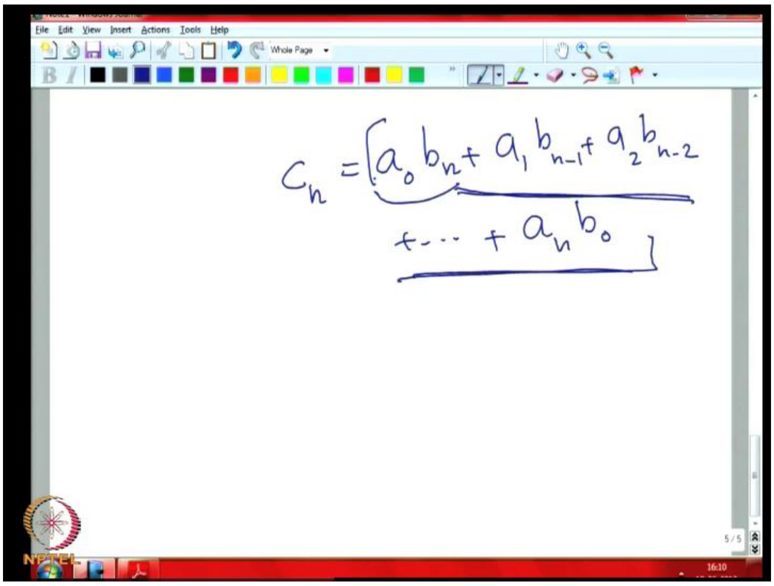

Then, we say that, c n… Then the coefficient of x raise to n will be given by a 0 into b n plus a 1 into b n minus 1 plus a 2 into b n minus 2 and so on all the way to a n into b 0. Why is it so? Because a 0 into b n – where is it coming from? Because this a 0 will multiply this – this is b 2 x square – say b n x raise to n. This is a 0 into b n x raise to n. So, that is why, that way we can form an x raise to n with coefficient a 0 into b n in f x into g x. Similarly, a 1 x into say b n minus 1 x raise to n minus 1 here; that x and x raise to n minus 1 will form an x raise to n and a 1 into b n minus 1; similarly, a 2 into b n minus 2. These all will form coefficient of x raise to n in the product and we have to sum up all those things. That is how we will get c n; that corresponds to c n.

Once we get the sequence for f x and g x by using this formula here; after that we can use this technique to get the formula for… Of case, we are not explicitly evaluating it, we are only giving it as a sum. I just wanted to introduce this concept of how to find the coefficient of h x, where h x is equal to f of x into g of x; where, the sequence corresponding to f of x is known to us and the sequence corresponding to g of x is also known to us. Then we can… How to cook up… How can we find out the coefficient of x raise to n in the expansion of h of x? That is what we discussed here. Of case, this kind of complicated formula – how it will be useful is not very clear, but at least we can compute it.

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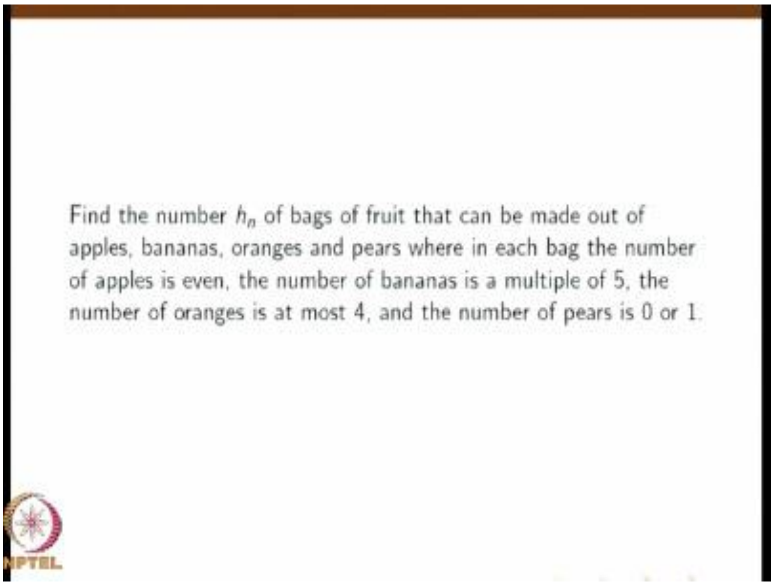

Now, we will look at an example, where things cancel out and finally, we will get a nice form for h of x. Just for like making you aware that, such a possibility is there in some cases at least. Find the number h n of bags of fruit that can be made out of apples, bananas, oranges and pears; where, in each bag, the number of apples is even – apples have to be even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1. These are the conditions.

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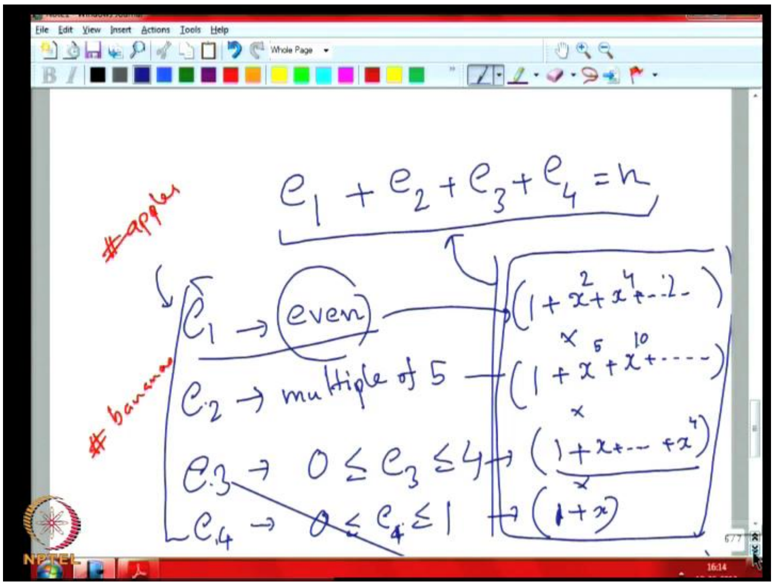

As usual, what we are looking for is the number of non-negative integer solutions for this equation – e 1 plus e 2 plus e 3 plus e 4 is equal to n with some extra conditions on e 1; namely, e 1 has to be even number and then e 2 has to be multiple of 5. Recall – this e 1 corresponds to the number of apples and e 2 corresponds to number of bananas; and the oranges namely, e 3 – the oranges – it is in between 0 and 4; that means e 3 has to be in between 0 and 4; and e 4 can take only 0 and 1; only two possible values are there for e 4. Therefore, we can cook up the corresponding terms for this.

How do we (( )) even terms. So, this is 1 plus x square plus x raise to 4 plus – this is the sequence; some infinite sum we will make for this thing, because e 1 has to get only an even value. So, the power of x's here has to be even number. And e 2 is a multiple of 5, that is, 1 plus x… See this 1 corresponds to x raise to 0. That is as good as saying that, x raise to 0, x raise to 0 here. Now, this is x raise to 5, because multiples of 5 only are allowed here; x raise to 10 and so on. This is easy. This is between 0 and 4; that is, x plus x raise to 4. And this is even easier. This is this, because 0 corresponds to this one and 1 corresponds to x.

Now, we multiply this out. The point here is that, the coefficient of x raise to n in the product of all these 4 terms will correspond to the number of solutions for this equation, where each e i satisfy the respective conditions given here. Why? That is clear, because e 1… So, if you get such a solution, definitely, one term will come corresponding to that from this thing, because there is an even number; I can take x raise to e 1 from this thing. And e 2 is a multiple of 5; you can take x raise to e 2 from this thing. And then e 3 is in between 0 and 4; then I can take from this thing x raise to that e 3. And e 4 is 0 or 1. Therefore, I can take from here either 1 or x corresponding to x raise to e 4.

Because they add up to n, I will get x raise to n when I multiply x raise to e 1 into x raise to e 2 into x raise to e 3 into x raise to e 4. On the other hand, consider any way you can make an x raise to n in this product. That will indeed corresponds to some e 1, e 2, e 3, e 4; where, the power of x, which came from here – corresponds to e 1; power of x, which came from the second term corresponds to e two; power of x, which came from here corresponds to e 3; and power of x, which came from here corresponds to e 4. And we know that, these things satisfy the conditions we want for e 1, e 2, e 3, e 4. That is why this is the correct solution.

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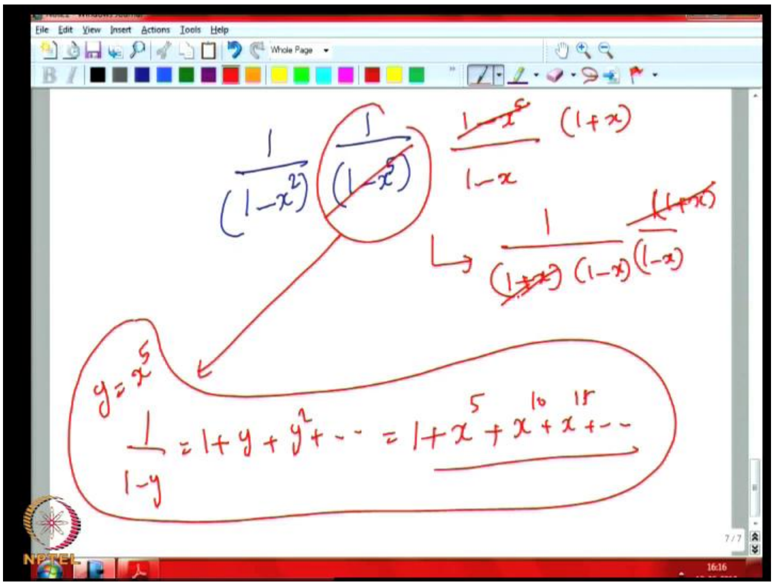

Now, how will you find the answer? Because the first term as we have seen in the last class, is 1 by 1 minus x square. The second term as we have seen in the last class… No, we have not seen it. But, we know that it is 1 minus x raise to 5, because… If you have any doubt on this, this one, you put y equal to x raise to 5 and consider 1 by 1 minus y. This is 1 plus y plus y square plus and so on. And when you substitute x raise to 5 in that, you will get 1 plus x raise to 5 plus x raise to 10 plus x raise to 15 plus… This is what.

Therefore, this should make it clear that, this 1 by 1 minus x raise to 5 corresponds to this sequence. And this third term is easy to find, because we already know that it is 1 minus x raise to 5 by 1 minus x. And this fourth term is just 1 plus x. There is no need to simplify it. Now, we can cancel this and this. And here we have 1 minus x square, which (( )) Now, what will happen to this? This 1 minus x square is 1 plus x into 1 minus x into… Here we have 1 by 1 minus x left. And upstairs, we have 1 plus x. So, in the numerator, we have 1 plus x. That we can (( )) this thing.

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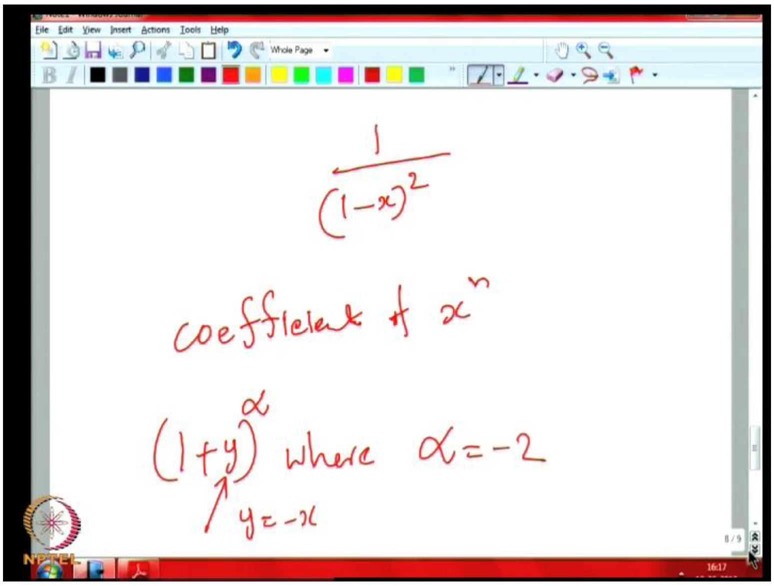

Final answer is 1 by 1 minus x square. Now, you know that, what we are interested in is the coefficient of x raise to n. What is the coefficient of x raise to n in this thing? That is… because this is of the form 1 plus x raise to alpha; where, alpha equal to minus 2. And we have a substitution y equal to minus x… Let us say 1 plus y raise to alpha and y equal to minus x here.

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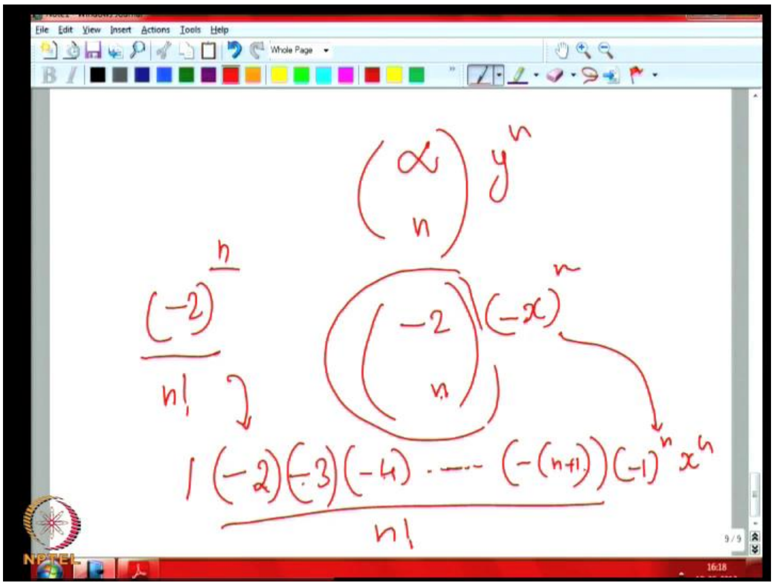

So, n-th term is going to be alpha choose n y raise to n; that is, minus 2 choose n minus x raise to n. This is what. This will be minus 2 minus 3 into minus 4 into – up to where we should go? We need n such sum. This is remember – what we need is minus 2 n falling – falling factorial divided by n factorial. That is what this is. So, up to minus of n plus 1 we will get here, we are starting with n. Below we have n factorial. So, even if we are starting with 2, we can assume that there is a 1 here. Actually this n factorial can sell out with 1 to up to n here. And what will happen to this? This will be minus 1 raise to n and x raise to n. There are how many minus signs here? There are minus 2, minus 3, minus 4 up to minus n plus 1.

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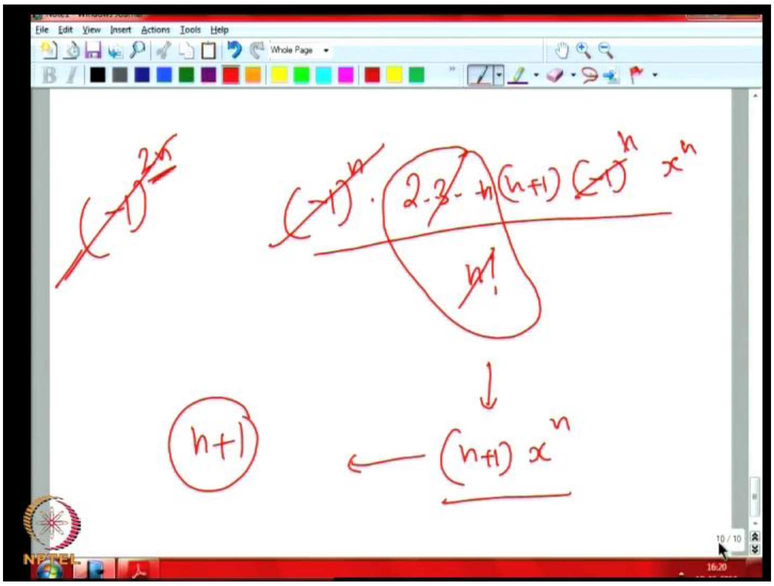

That means we have this as minus 1 raise to n into 2 into 3 into up to n plus 1 into we have minus 1 raise to n here and then x raise to n here divided by n factorial. So, this and this together will form minus 1 raise to 2n as usual; and 2n being an even number, that is just 1; that goes away. And this 2 into 3 into up to n if you consider, that is already n factorial; that goes away. We just get n plus 1 into x raise to n.

The coefficient of x raise to n in this thing is just n plus 1. So, the number of integer solutions for the equation we wrote – this equation satisfying all these conditions, is indeed n plus 1. Interesting thing is we did not even think what we are doing; we were just manipulating equations and we came up with this answer. It would be nice to try it out combinatorial whether we can get the same answer using counting arguments without using these generating functions. Of case, this was an example to show that, sometimes the terms involved may nicely cancel off and we may get a very neat answer in the end.

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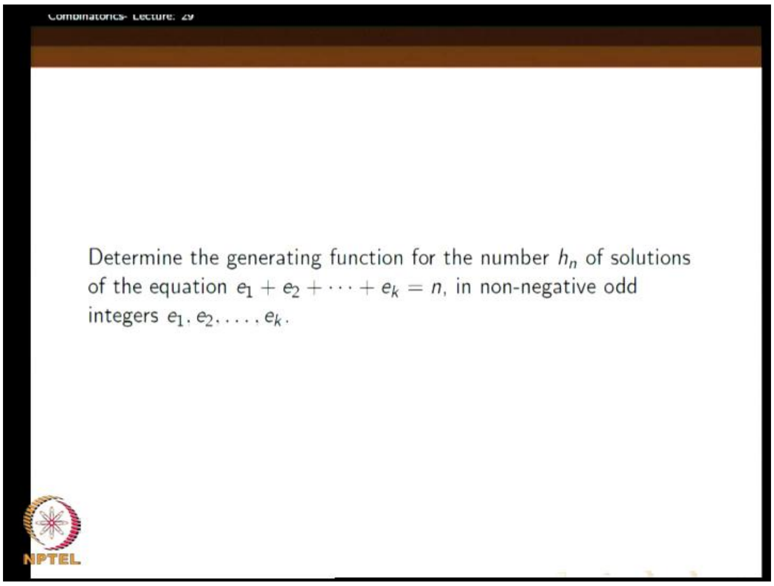

Next question – determine the generating function for the number of h n of solutions of the equation e 1 plus e 2 plus e k equal to n in non-negative odd integers e 1, e 2, e k.

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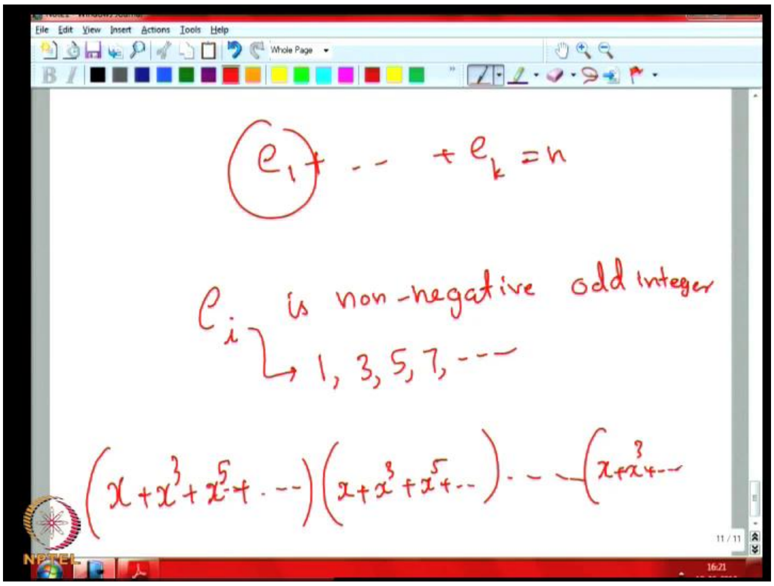

How is that? We want e 1 plus up to e k to be n. So, each e i is non-negative odd integers; that means it can be 1, 3, 5, 7 and so on. e i can take these values. So, for corresponding to e 1, we can put one term namely, x plus x cube plus x raise to 5 plus so on. Similarly, we had term corresponding to e 2 – x plus x cube plus x raise to 5 plus so on and like that; e k – we will get x plus x cube plus and so on.

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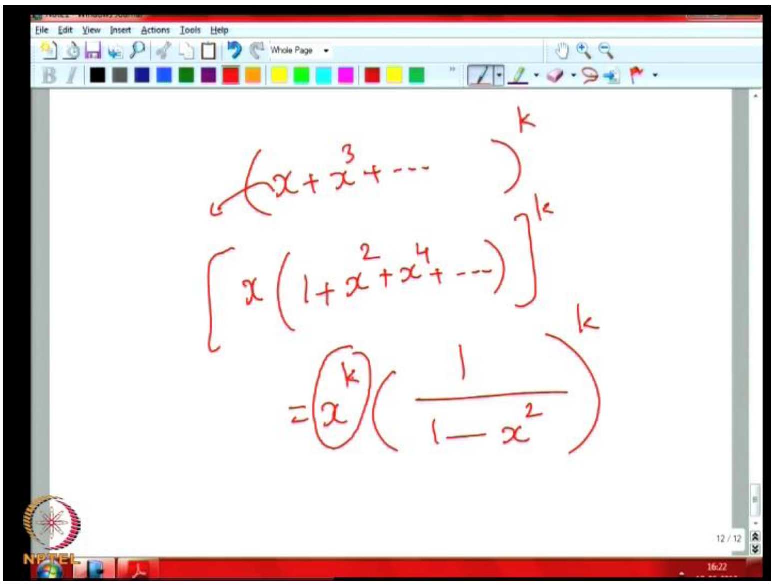

So, this is x plus x cube plus to the power k, because there are k terms here. And this is what? x raise to k into… I am taking x out from this thing. So, what I do is I will take x out here; that means x into 1 plus x square plus x fourth and so on whole power k, which is x power k into 1 by 1 minus x square whole power k. This is what. If you want the answer, we look for the coefficient of x raise to n in it. As we have discussed, the coefficient of x to the power n will correspond to the coefficient of… because every term is being multiplied by x raise to k.

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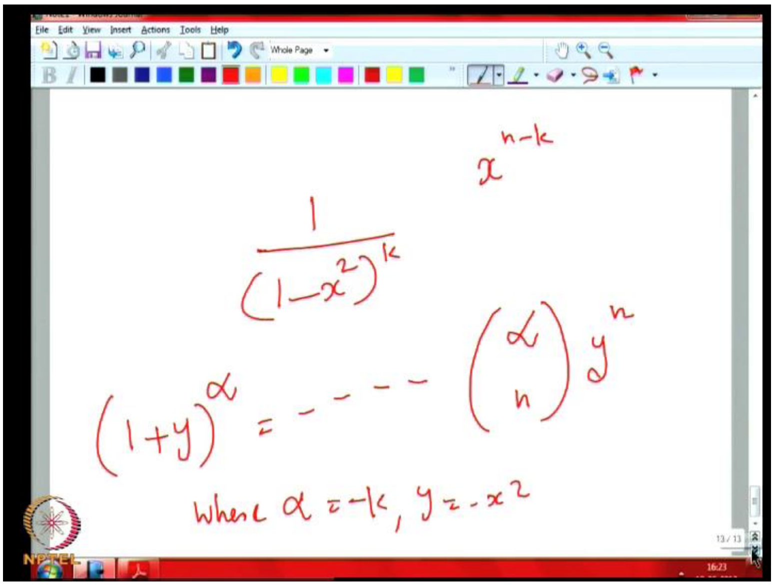

So, the coefficient of x raise to n will correspond to the coefficient of x raise to n minus k in the expansion of 1 by 1 minus x square whole power k. And this is familiar. What do we do of case? We have this form – 1 plus y to the power alpha; where, alpha is equal to minus k here and y is equal to minus x square. Now, the n-th term of this expansion is alpha choose n into y raise to n.

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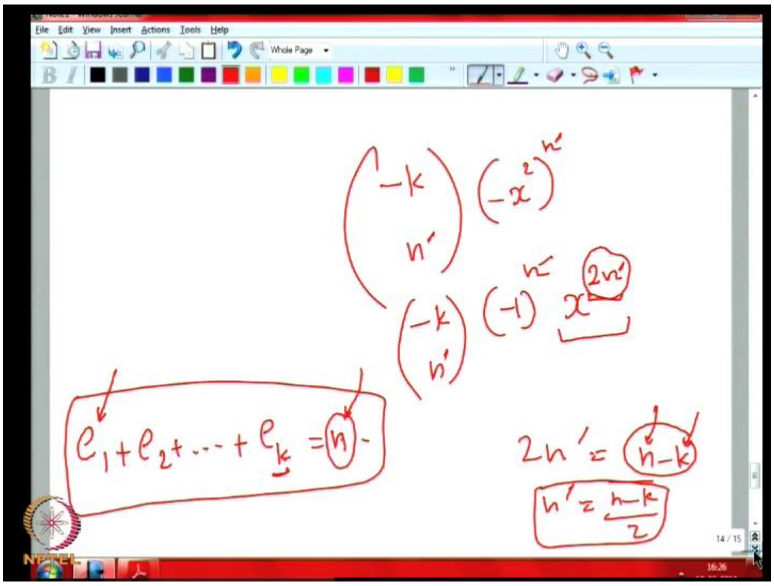

Now, what is that? Alpha being minus k; that will be minus k choose n into – substitute y equal to minus x square – minus x square whole power n. That is the n-th term. So, here we are getting x raise to 2n into minus 1 raise to n and minus k choose n. But, what we are interested in is the coefficient of x raise to n here. So, what could you do? We were looking for the coefficient of x raise to n minus k, because that is what we told, because in the earlier, we had x raise to k into 1 minus x square whole power k. So, we are looking for the coefficient of x raise to n minus k. This is a general thing. The coefficient for x raise to 2n is going to be this. Of case, this is always going to be an even power of x here.

Now, you know… See we were looking for the solution of e 1 plus e 2 plus e k is equal to n; where, each e i is odd. Suppose k was an even number; this n is going to be also an even number, and n minus k is also going to be an even number. And if you are looking for n minus k here, we can actually take… For instance, let us put this as n dash; n dash is the general term here. So, n dash. Now, we can put n dash equal to – 2n dash is equal to n minus k; and then n dash equal to n minus k by 2.

So, this we can easily find out. Put n minus k by 2, because n minus k by 2 is going to be an integer; where, we can substitute here. But, on the other hand, if k was an odd number, then this n is going to be odd, because the odd number of odd numbers add up to an odd number. So, n minus k is going to be again even, because it is an odd number and this is an odd number. So, n minus k by 2 is going to be again an integer n dash. So, we can find the coefficient for that. Anyway that was just to give like… Usually, we have some techniques to find the coefficients like this. To some extent we can proceed. Of case, I hope this gives some idea about how to go about doing it.

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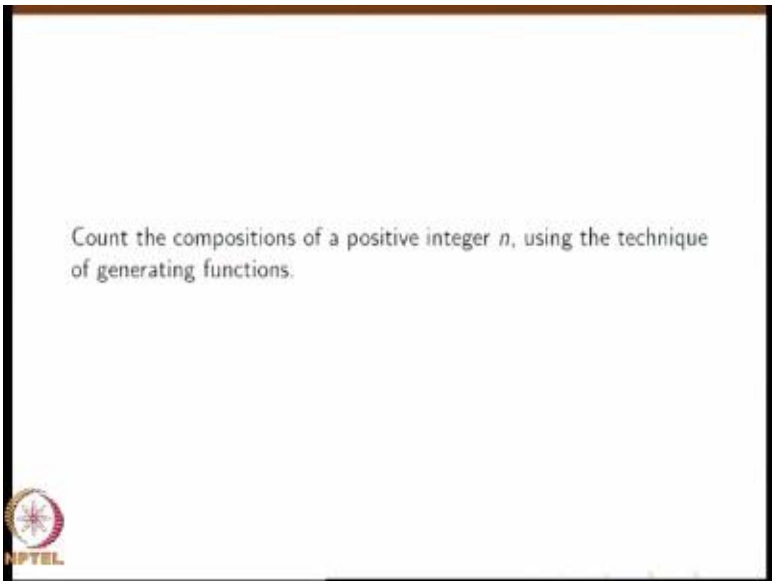

Now, we will look at the next question; this we have already discussed. Count the compositions of a positive integer n using the technique of generating functions. This is something, which we have earlier looked at.

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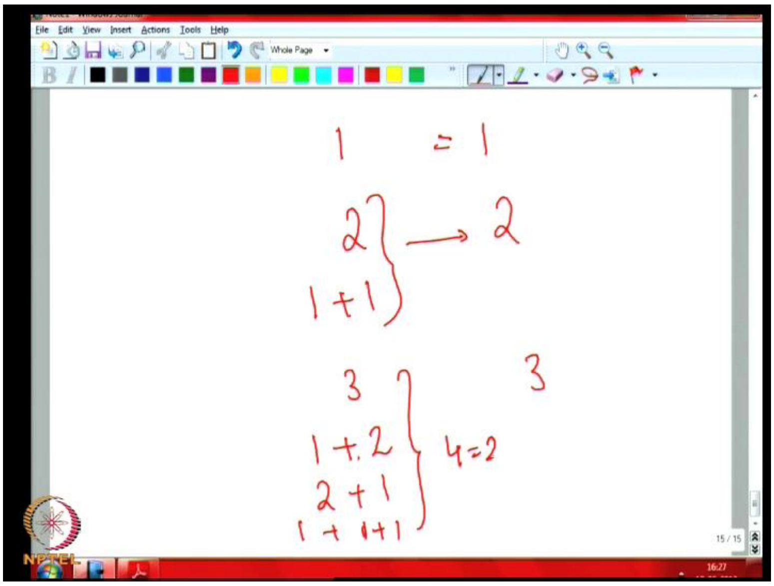

What was the composition, for instance, 1? 1 can be written as 1 here. Only one composition here; we had seen this before. For instance, 2 – we can write it as either just 2 or 1 plus 1. There are two compositions for 2. When I take 3, we can either write it as just 3 or 2 compositions namely, 1 plus 2 or 2 plus 1. These are two possible ways. Or, we can write it as 1 plus 1 plus 1. There are four compositions. 4 equal to 2 raise to 2; compositions for 3.

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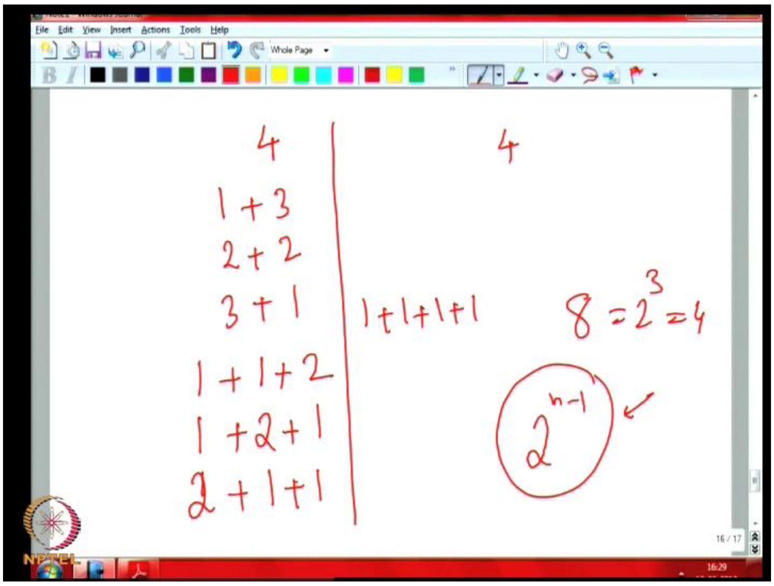

And for 4, we have say 4. So, 1 plus 3, 2 plus 2, 3 plus 1. And now, 3. For instance, I can start with 1 plus 1 plus 2, 1 plus 2 plus 1; and 1 plus… That is it. We can start with 2 here – 2 plus 1 plus 1. That is all. And we have the composition 1 plus 1 plus 1 plus 1. This is also (( )). So, there are 8 compositions; that is, 2 to the power 3. We have seen that, in general, for n, 2 to the power n minus 1 compositions are there.

So, what we are doing is we are splitting it into the sum of positive integers. But, we are again importance to the order here. In what order we are doing and how many of them are there is the question. What we do now to find out this is this. We will use generating functions to get the same answer 2 raise to n minus 1. We already know the answer, but we want to demonstrate the use of generating functions to get this thing.

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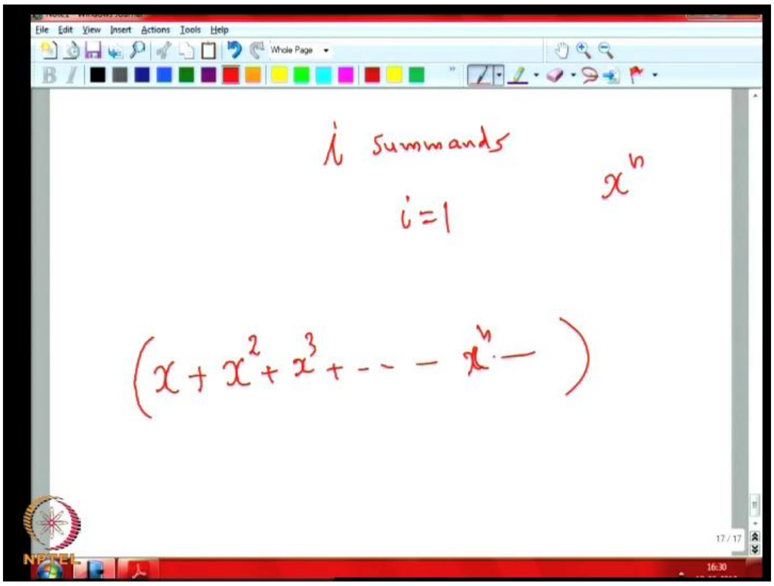

We first look for the compositions with exactly i summands. This is what we are going to do. How will I do that? For instance, if it was i equal to 1; then I can create something like this – sum like this – infinite sum like this – x plus x square plus x cube… And now, see… Look for the coefficient of x raise to n in this. Definitely, there is only one way to get one summand composition for n; that is, x raise to n coefficient is just 1 here.

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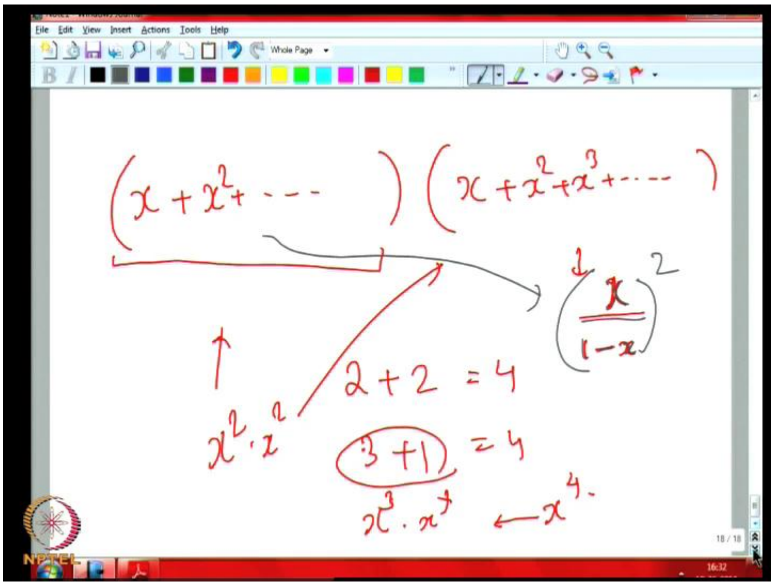

Now, let us see if… I am looking for the 2 summand compositions of n. What I can do is I create x plus x square plus – this term; and then similarly, one more term – x plus x square plus x cube plus… See not this corresponds to the first term in the summand, the sum. For instance, if I write 4 equal to 2 plus 2 or 4 equal to 3 plus 1; I want to… I will look for the coefficient of x raise to 4. One possible way to get x raise to 4 from the product is to get x raise to 3 from here and x raise to 1 from here. That is why 3 plus 1 is formed. This will be formed. Another way is x square from here and another x square from here. So, 2 plus 2 is forming.

So, that way, we can get all possible summands. We are not allowing 0. That is why x raise to 0 is not here at all, because in that compositions, we are not allowing zeros. And the order is important here, because either 3 comes from here or 1 will come here; or, 1 comes from here and 3 comes from here. It is separately counted, because if the coefficient of x raise to 4 counts for both such ways for forming x raise to n. Both of them adds one each more to the coefficient of x raise to 4. Now, what is this? This is just x by 1 minus x because this is what… This we know, because we can take x out from here – x into 1 plus x plus x square plus; that is it; that is, x by 1 minus x. And this is also the same; that is square.

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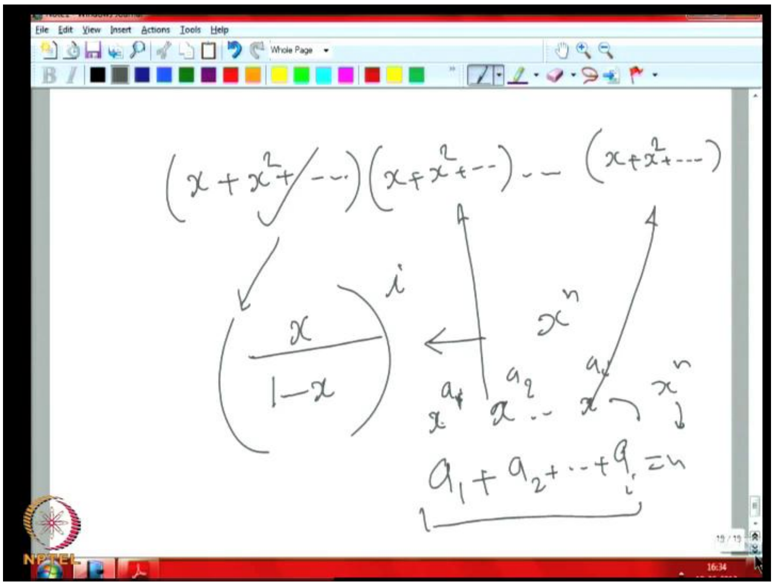

The i summand thing is just this – x plus x square plus – first term; second term is x plus x square plus… Similarly, i such terms are to be put – x plus x square plus so on. What does it give? This will be giving x by 1 minus x whole power i. And look for the coefficient of x raise to n in the expansion of this. It is clear enough, because anything of this form – a 2 plus say a i is equal to n. So, corresponding to this, we can create an x raise to a i from this and x raise to a 1 from this, a 2 from this, and finally, x raise to a i from this thing. So, we will get an x raise to n corresponding to that. That adds 1 to the coefficient of x raise to n. On the other hand, any way in which you can form an x raise to n by taking some x raise to a 1 from this thing, some x raise to a 2 from this thing – that a 1, a 2, a i will add up to n and that is one of the i summand composition of n. Therefore, we are actually counting what we are looking for. Now, of case, we want the total number of… We could have gone and figured out the coefficient of x by 1 minus x raise to n in x by 1 minus x whole power i. But, that is not what we want; we want the total number.

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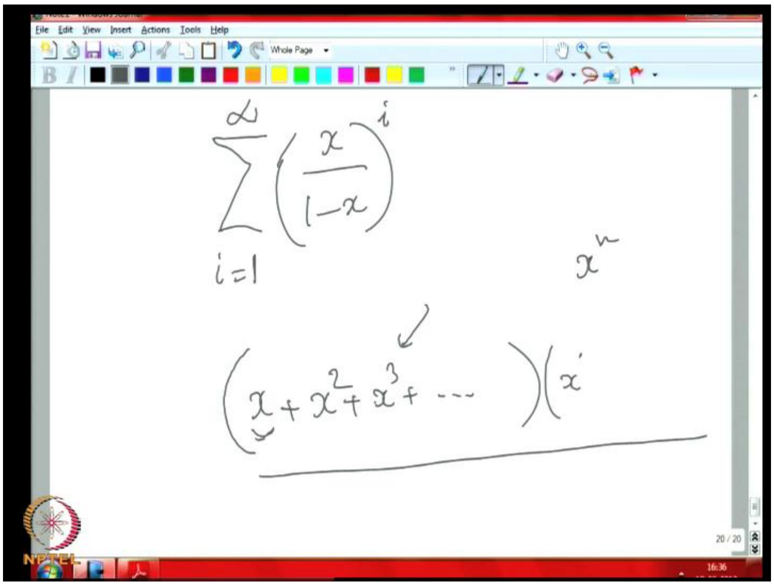

Namely, we want to find the coefficient of x raise to n in this – x by 1 minus x whole power i and this i is… because 0 is not necessary, because we are starting with 1 summand only. There is no 0 summand composition of n. So, i equal to 1 to n; not more than n, because there is no n plus 1 summand composition of n. It is not possible; one of them has to be 0; we are only allowing positive integers here. 0 is not being allowed.

So, this only is required. But, because you are looking for the coefficient of x raise to n in the product of all these terms, we can as well put this till infinity, because… Why? Because that is not going to change. The reason is because we are taking the terms of the form x plus x square plus x cube. So, the first term has to contribute at least one x. It is not that there is an x raise to 0 possible. So, x raise to 1. The second has to contribute at least one x. So, by the time you cover n terms, the product already accumulated x raise to n.

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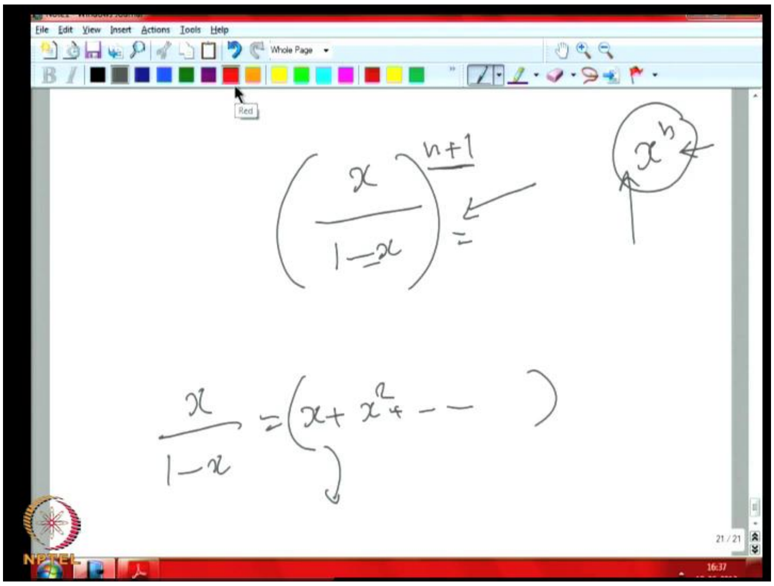

Therefore, these terms, that means, x by 1 minus x to the power n plus 1 kind of terms will not have x raise to n with nonzero coefficients. Why? Because there are x by 1 minus x pairing n plus 1 times here. Each of this x by 1 minus x is of the form x plus x square plus this thing and each are contributing at least one x. So, if there are n plus 1 such things, the smallest power of x in the product has to be n plus 1. It is not possible to have x raise to n there.

So, the coefficient of x raise to n in the expansion of this thing is going to be 0. So, that is not going to trouble us. Here because this is sum; the first n terms only will contribute. We can add the remaining things – x by 1 minus x to the power n plus 1 onwards freely. That is not going to disturb us. Why am I doing the summation till infinity? To make things easier; that is all, because the calculations are easier when you sum all the way to infinity, rather than the partial sums. Now, what will happen to this? We need to do some manipulation.

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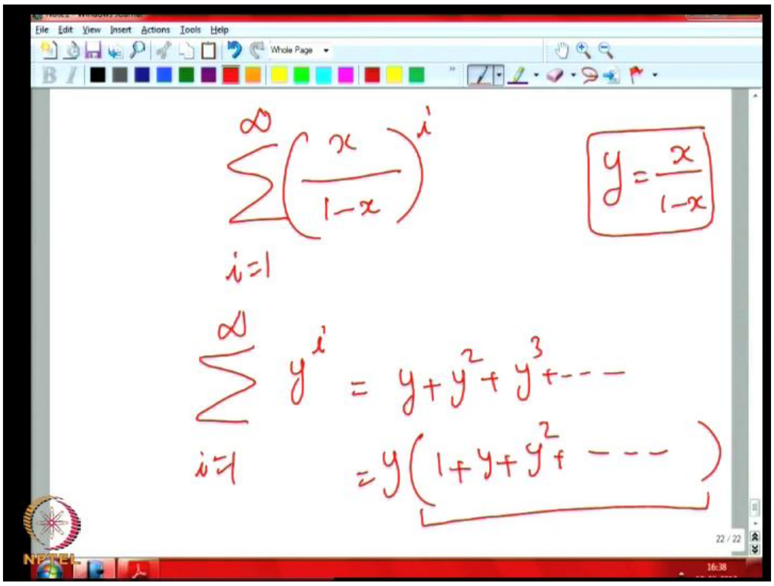

First, what we do is we look at again the sum i equal to 1 to infinity x by 1 minus x whole power i. Put y equal to x by 1 minus x; some change of substitution. Then this will read i equal to 1 to infinity y raise to i. How will it look like? This will be y plus y square plus y cube plus and so on. You can pull out one y from it. So, this is y into 1 plus y plus y square plus so on. Why did we do this? Because this is easier to sum. We know the formula for that.

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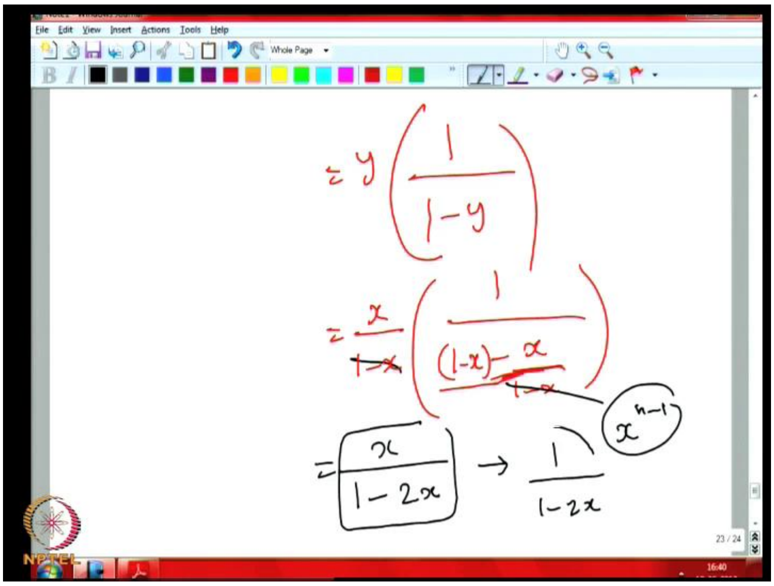

Therefore, we have y into 1 by 1 minus y. So, that is 1 by 1 minus y. This portion is 1 by 1 minus y. So, we have y into 1 by 1 minus y. Now, we can substitute this one – y equal to x by 1 minus x in this thing. So, x by 1 minus x is substituted here and here we have 1 by 1 minus x by 1 minus x. Here we can do 1 minus x minus. So, this will become same. And now, this can cancel off with this. So, this will become x by 1 minus 2x. This is the answer. Now, answer is namely, the number of compositions of n, will be equal to the coefficient of x raise to n in this. And we already know that, the coefficient of x raise to n in this will correspond to the coefficient of x raise to n minus 1 in 1 by 1 minus 2x.

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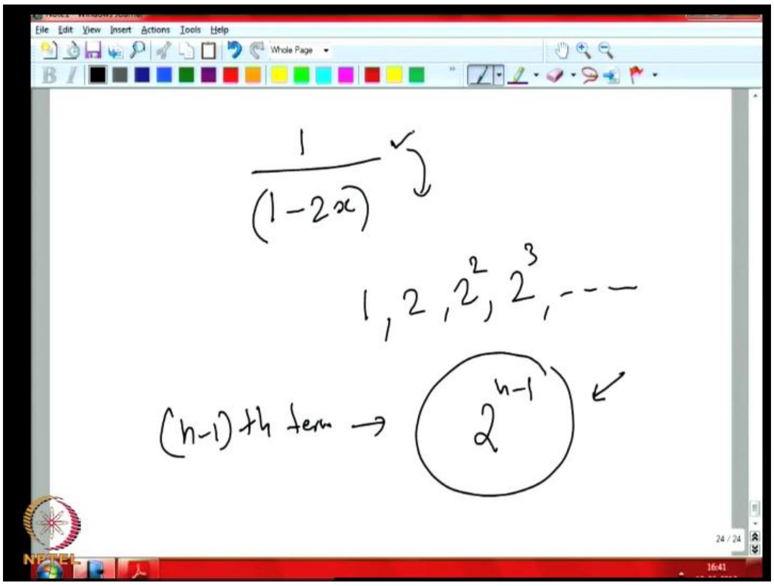

So, we only have to look for the coefficient of x raise to n minus 1 in 1 minus (( )) this one. And this is familiar. We know this corresponds to the sequence of 1, 2, 2 square, 2 cube and so on. And we are looking for the n minus one-th term here. What is that? That is 2 to the power n minus 1. This is what we got earlier also. So, we manage to get the same answer using the method of generating functions. So, this is a nice example, because here things were like… We were getting familiar generating functions – close forms till the end. Somehow it was cancelled off and came to this nice final generating function. So, that way it was nice.

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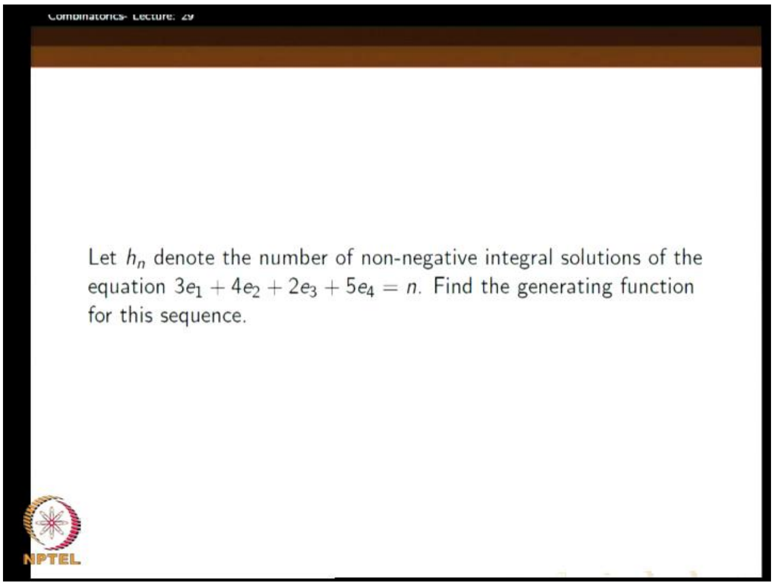

And now, let us look at another example. What about compositions of this form? h n denote the number of non-negative integer solutions of the equation – 3 times e 1 plus 4 times e 2 plus 2 times e 3 plus 5 times e 4 is equal to n. Find the generating function for this sequence.

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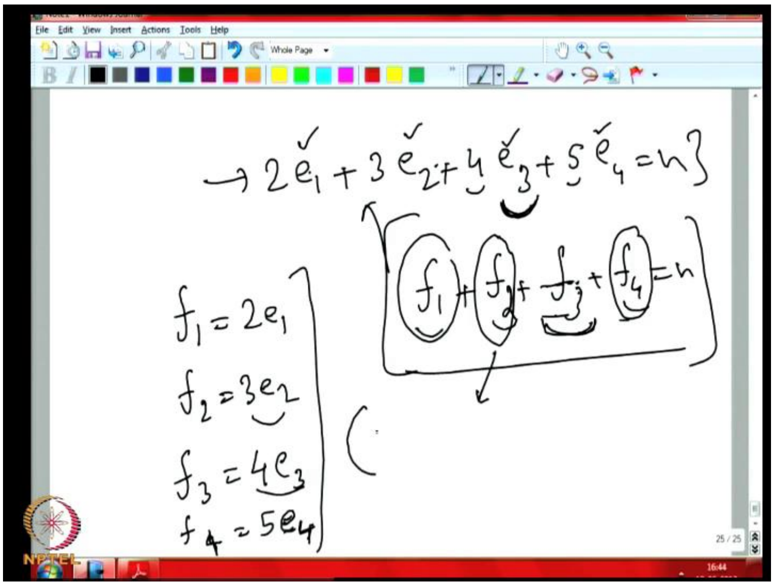

What we do is here we do a kind of conversion. So, 2 times e 1 plus 3 times e 2 plus 4 times e 3 plus 5 times e 4 is equal to n. What we do is we make some substitutions first. Let us say f 1 is equal to 2 e 1; f 2 is equal to 3 e 2; and f 3 is equal to 4 e 3; and f 4 is equal to 5 e 4. For e 1, e 2, e 3, e 4, we have just the condition that, they are non-negative integers. But, now, we are looking for the solution of f 1 plus f 2 plus f 4 equal to n; where, f 1, f 2, f 4 has some extra conditions. Namely, f 1 has to be a multiple of 2; that has to be an even number; f 2 has to be a multiple of 4; and f 4 has to be a multiple of 5. How many such solutions are there?

We can easily verify that. That count will give the count for this thing, because if we get a solution for e 1, e 2, e 3, e 4 here, that corresponds to a solution of f 1 plus f 2 plus f 3 plus f 4 equal to n with a specified condition namely, because f 1 is going to be 2 times e 1; that is going to be an even number. f 2 is going to be 3 times e 2; it is going to be a multiple of 3. f 3 being 4 times e 3; it is going to be a multiple of 4. And f 4 being 5 times e 4, it is going to be multiple of 5.

On the other hand, if we get a solution here; you can find solutions for e 1, e 2, e 3 and e 4 by dividing by the corresponding factor. For instance, f 3 value – we have to divide by 4; we will get an e 3. Why? Because f 3 is indeed… Here the solution restricts f 3 to be a multiple of 4. You can divide by 4 and we will get an integer – non-negative integer. Therefore, that will be e 1, e 2, e 3, e 4, which is obtained that way, will be a solution for the first equation. So, we just have to work with this stuff. Now, what is the generating function for this thing?

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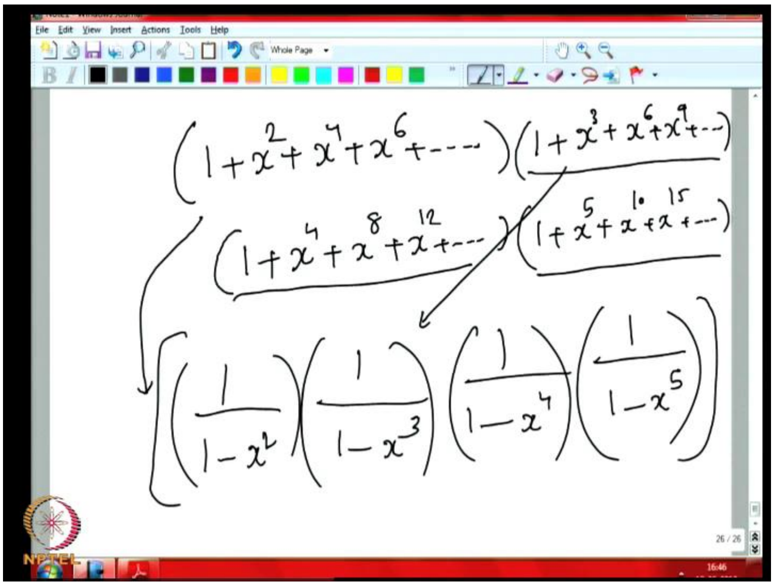

This is easy, because f 1 – we will have a term like this; for f 1, we will have a term like 1 plus x square plus x raise to 4 plus x raise to 6 plus and so on; that means only the e 1 powers of x are taken here. For f 2, it has to be multiple of 3. So, we will have 1 plus x 3 plus x raise to 6 plus x raise to 9 plus; only multiples of three will come as powers of x. And the third form – for f 3, we will have 1 plus x raise to 4 plus x raise to 8 plus x raise to 12 and so on. And finally, we have 1 plus x raise to 5 plus x raise to 10 plus x raise to 15, etcetera for f 4, because it has to be multiple of 5. That is the way we have set up these things.

The power of x here is always multiples of 5; the power of x here is always multiple of 4; and power of x here is always multiple of 3. Now, we can simplify this thing. This is going to be 1 by 1 minus x square; this one is 1 by 1 minus x raise to x cube; and this is 1 by 1 minus x raise to 4; and this is 1 by 1 minus x raise to 5. Multiply this out. And we have to find the coefficient of x raise to n in the expansion of this thing, if we want to get the answer. So, you can get it as a complicated sum probably, but we would not try it out here.

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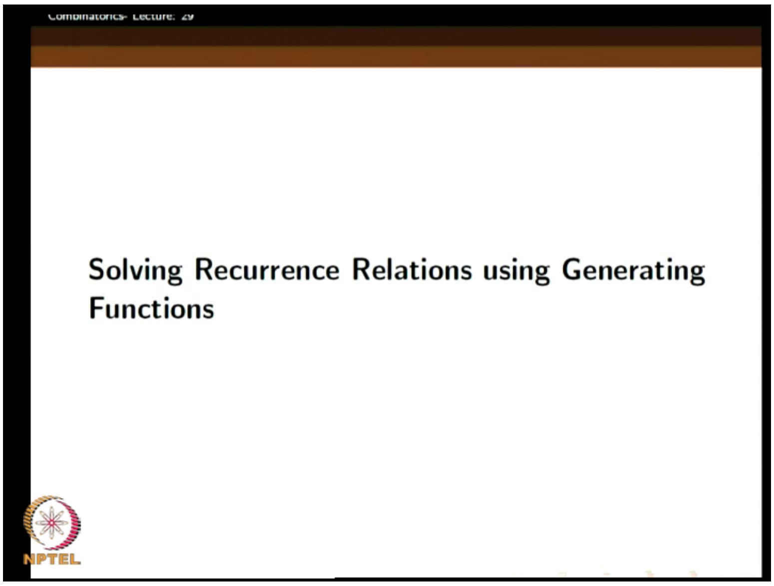

Now, the next theme we have. For instance, there is a very interesting use for this generating functions in the context of recurrence relations. In the last… before few classes, we were discussing recurrence relations in detail and we have given several methods to deal with linear recurrence relations with constant coefficients – homogenous or non-homogenous. We were giving several methods. And here we will see how generating functions can be used to solve recurrence relations.

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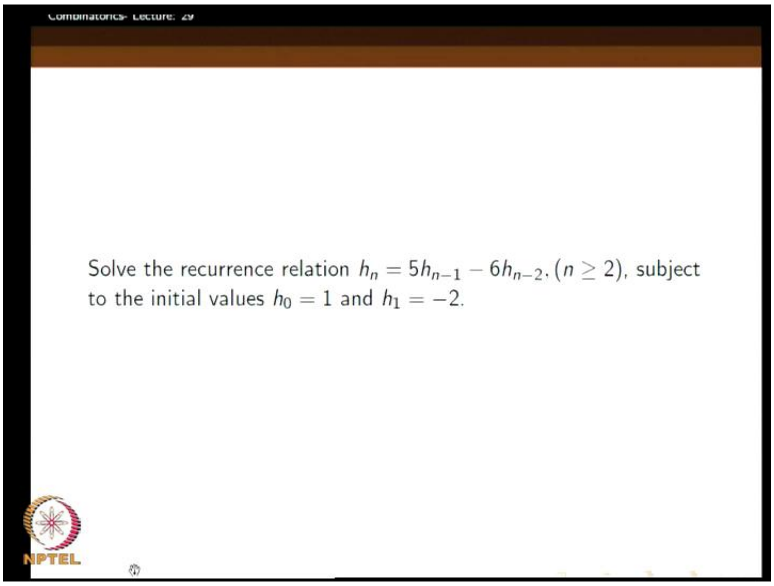

In the case of linear homogenous recurrence relations with constant coefficients of say order k, we will show that, this method will always work. So, this is like a recipe, which you can simply use. You do not have to think much here; just use it. Of case, there is some difficulty later, because once you get the generating function, we have to find a way to extract the coefficients. That is the real difficult part. But, there also, we will see some methods are helping.

So, that is what we are going to discuss. Now, we will take some example and do. The first example is this one – h n equal to 5 h n minus 1 minus 6 h n minus 2 and greater than equal to 2. Of case, this will work only for n greater than equal to 2, because you are depending on two previous terms: 0, 1, then 2. Initial value – 0 for h 0 and h 1 has to be given. h 0 is equal to 1 and h 1 equal to minus 2 is given here. Now, how will we solve it?

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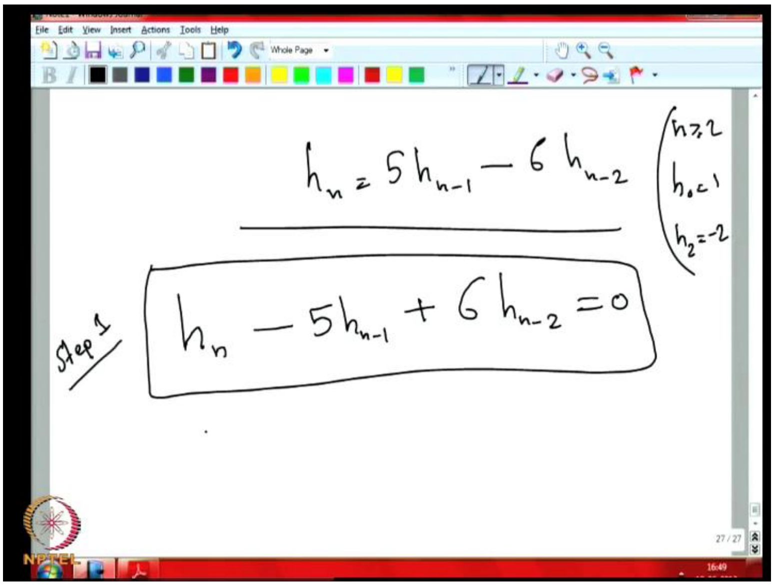

5 h n minus 1 minus 6 h n minus 2. Now, this is only true for n greater than equal to 2. And here h 0 is equal to 1; h 1 equal to minus 2; h 0 is equal to 1 and this is minus 2. So, what we do is we write this in this form – h n minus 5 h n minus 1 plus 6 h n minus 2 equal to 0. This is the first step. This is step 1. In the second step, we write like this – the generating function say g of x or we can write say h of x – say h 0.

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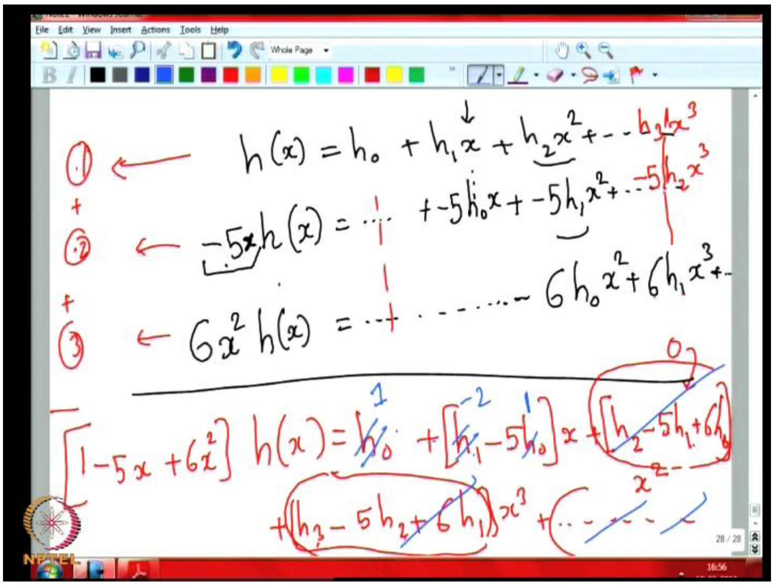

h of x equal to h 0 plus h 1 times x plus h 2 times x square plus and so on. Now, what we do is we take this minus 5 and multiply this h of x here. So, minus 5 into h of x we find.

So, what we get? minus 5 times h 0 minus 5 times h 1 times x minus 5 times h 2 times x square. So, what we do is that we will align this x terms. What I do is minus 5 into x – minus 5x I use to multiply this h of x. So, what happens is minus 5 x times h 0 will come; minus 5 x times h 1 x will come; (( )) x square and so on. So, to align x terms and that power of… For instance, x raise to i term should come just below this; that way we will write. Therefore, we will put a gap here and we will start here. So, that means this is minus 5 into h 0 into x plus here minus 5 into h 1 into x square and so on; minus 5 into h 2 into x cube and so on.

Now, in the next step, what we do? We will go back and see. Here it is plus 6. So, we take that plus 6 and we will use 6 x square to multiply h x. Now, what we get? h 0 into 6 x square plus h 1 into 6 x square into x; that means 6 x cube and so on. To align the power of x raise to i same column; for instance, below here I need the power of x square. So, here the first term itself will contain x square. Therefore, I will put a gap here all the way up to here. And here I start 6 into h 0 into x square plus – next term will be 6 into h 1 into x cube and so on. Like that it will keep on going.

Now, what we do is we sum this up. We add all the… So, this is say equation 1; this is 2; and this is 3. We add 1 plus 2 plus 3. So, here we have h of x into… This is 1 minus 5x plus 6x square into h x. This is what I get here. Here what do I get? I get h 0 first, because there is nothing below here. Here I get an h 1 minus 5 h 0 into x. Next term will be an h 2 minus 5 h 1 plus 6 h 0 into x square plus – next term will be h 3 minus 5 h 2 plus 6 h 1 – this term. So, here we have h 2; h 3 here – h 3 x cube; here we have minus 5 h 2 x cube; and here we have 6 h 1 x cube.

So, h 3 minus 5 h 2 plus 6 h 1 x cube. And from now, it will be the same thing – h 4 minus 5 h 3 plus 6 h 2 into x 4 and so on. So, what is good about this is that, this one – h 2 minus 5 h 1 plus 6 h 0. This is familiar. Why, because h n minus 5 h n minus 1 plus 6 h n minus 2 is equal to 0. We put n equal to 2 here. So, this will become 1; this will become 0. So, that will become h 2 minus 5 h 1 plus 6 h 0 is exactly what is written here. So, this coefficient is getting 0 by the recurrence relation given.

Similarly, this coefficient will also become 0 if you put n equal to 3; say h 3 minus 5 h n minus 1; that is, h 2 plus 6 h n minus 2; that means h 1. This is also going to be 0. And the next term is also going to be 0, because that is h 4 minus 5 h 3 plus 6 h 2. So, all of them will become 0 and they all will vanish. This will vanish; this will vanish. It will go away. So, we only have this h 0 plus h 1 minus 5 h 0 times x in this side. This side we have this. That is interesting. But, this h 1 and h 0 we already know. What is this here? h 0 equal to 1; h 2 is equal to minus 2. We can substitute that here. Say here I use 1 and this is minus 2; this is 1; that is, 1 minus 7x.

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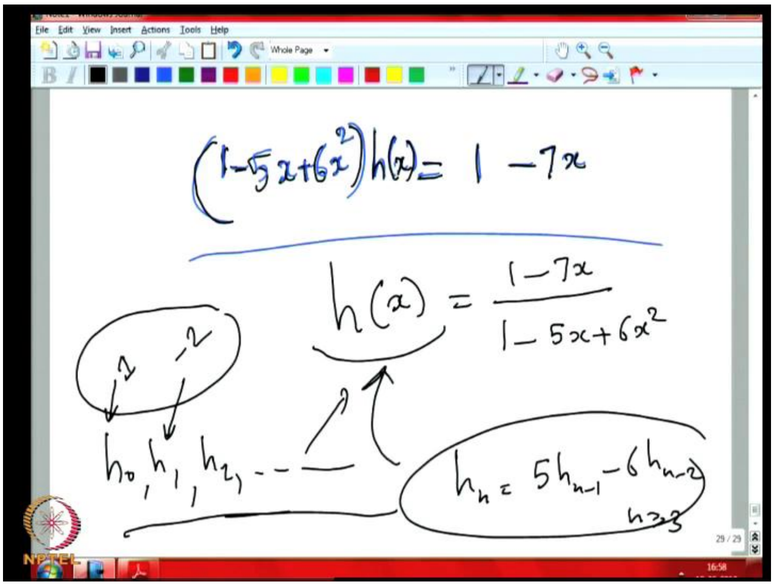

1 minus 7x – this is equal to 1 minus 5x plus 6x square into h of x. This is what we got. 1 minus 5x plus 6x square into h of x is equal to 1 minus 7x. This is what we got. So, h of x is what? h of x is equal to 1 minus 7x by 1 minus 5x plus 6x square. So, we got the generating function for h of x. See the sequence of h 0, h 1, h 2, etcetera and h 0 value was 1; h 1 was given to be 2.

And for the remaining things, we just had that recurrence relation namely, h n equal to 5 h n minus 1 minus 6 h n minus 2. This is what we (( )) From this thing, we could derive the close form for the generating function h of x. Generating function h 0 plus h 1 x plus h 2 x square plus h 3 x cube – its close form we got; that is, 1 minus 7x by 1 minus 5x plus 6 x square. Note that this is already captured the initial conditions also; we do not have to struggle to fit the initial conditions. It has also captured the recurrence relation. Now, we just have to get the coefficient of x raise to n from this thing.

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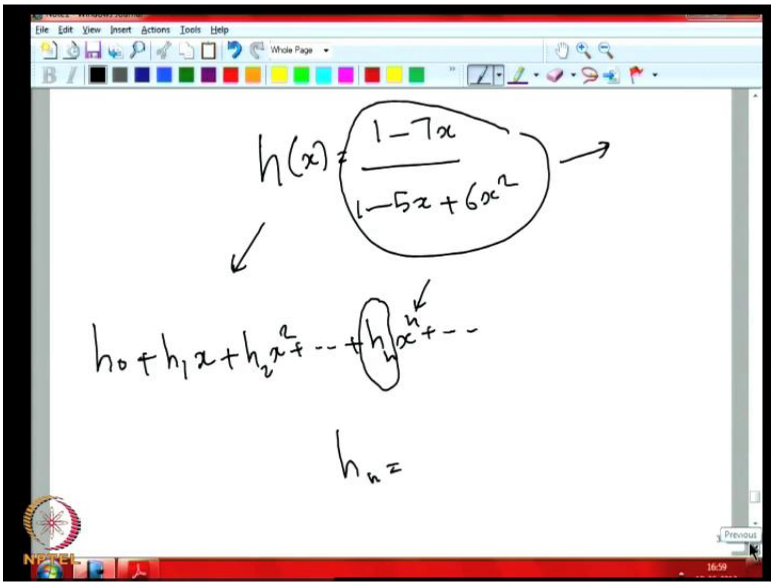

That will be h n, because this generating function h of x, which we have written as 1 minus 7x by 1 minus 5x plus 6x square can also be written as h 0 plus h 1 x plus h 2 x square plus – that is, h n x raise to n and so on. So, the coefficient of x raise to n is indeed h n. So, if you can expand this and get the coefficient of x raise to n in some different way, (( )) close form; then we got an expression for h n. So, how will you do that? Here we have to use the method of partial fractions; not necessarily, but that is a very helpful (( )).

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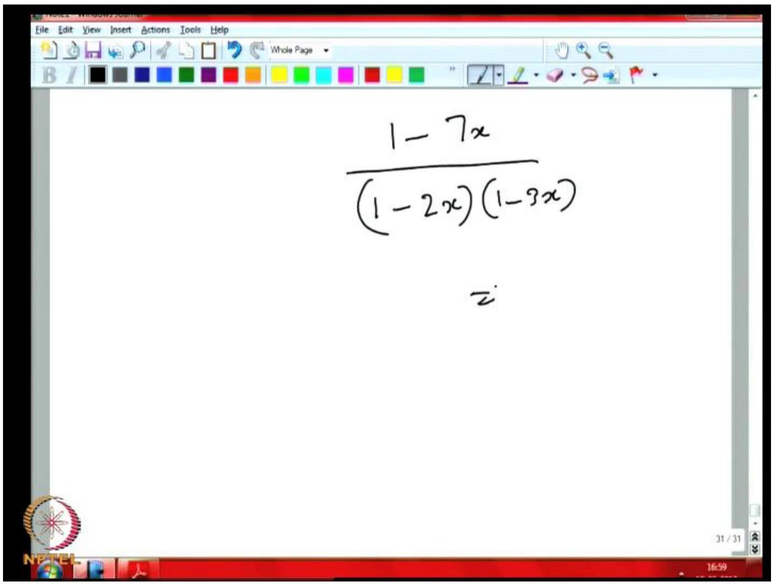

1 minus 7x… First you factorize this thing – 1 minus 5x plus 6 x square is in fact 1 minus 2x into 1 minus 3x. I will do it in the next class. I will complete this in the next class.