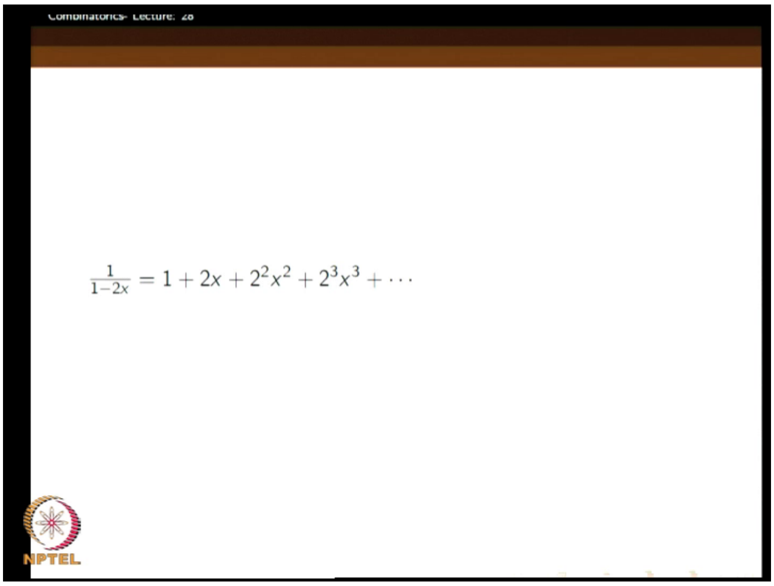
Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 28
Generating Functions – Part (1)

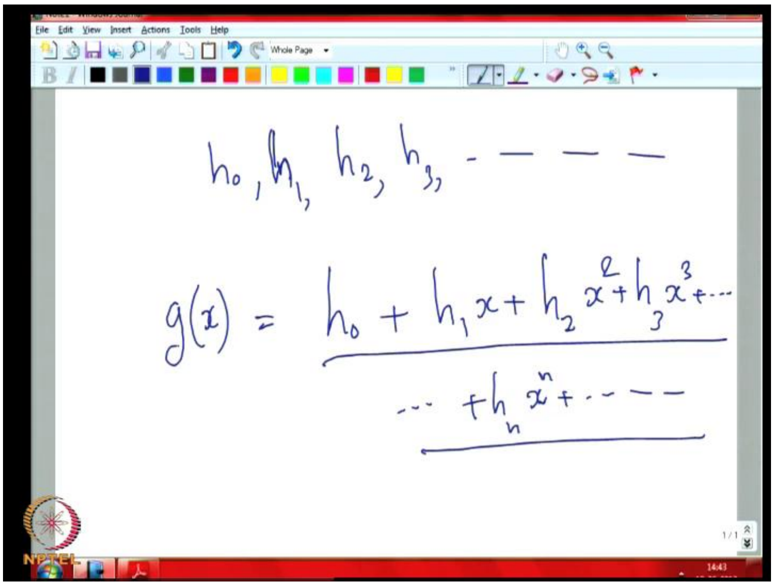
Welcome to the twenty eighth lecture of combinatorics.

(Refer Slide Time: 00:20)



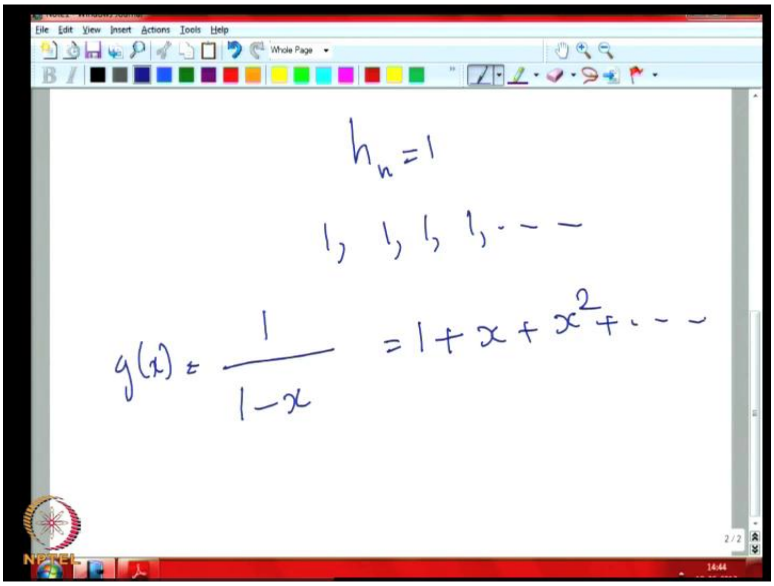
We were discussing generating functions and we saw several examples, how a sequence can be represented by a generating function; to remind you.

(Refer Slide Time: 00:40)



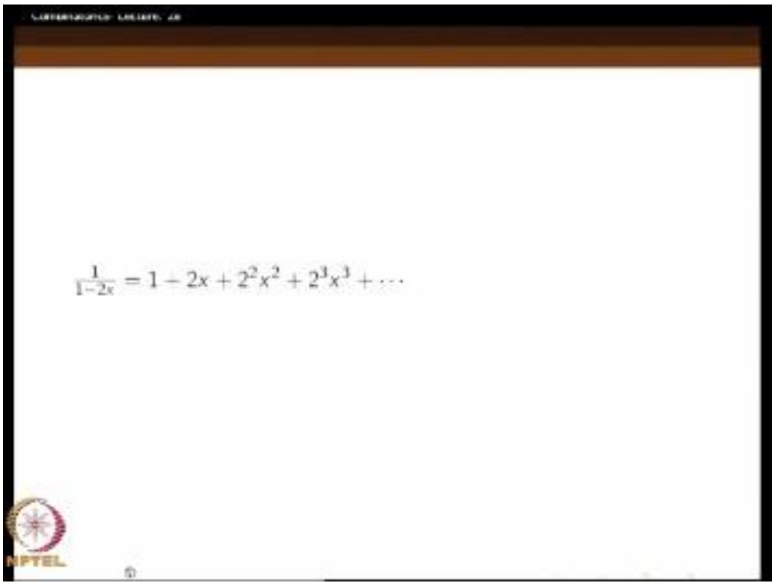
Suppose the sequence is h 0, h 1, h 2, h 3 and so on. Then, the generating function g of x is h 0 plus h 1 x plus h 2 x square plus h 3 x cube plus and so on plus h n x raise to n plus… We have seen the several situations, where the sequence is such that its generating function has a very nice form.

(Refer Slide Time: 01:30)



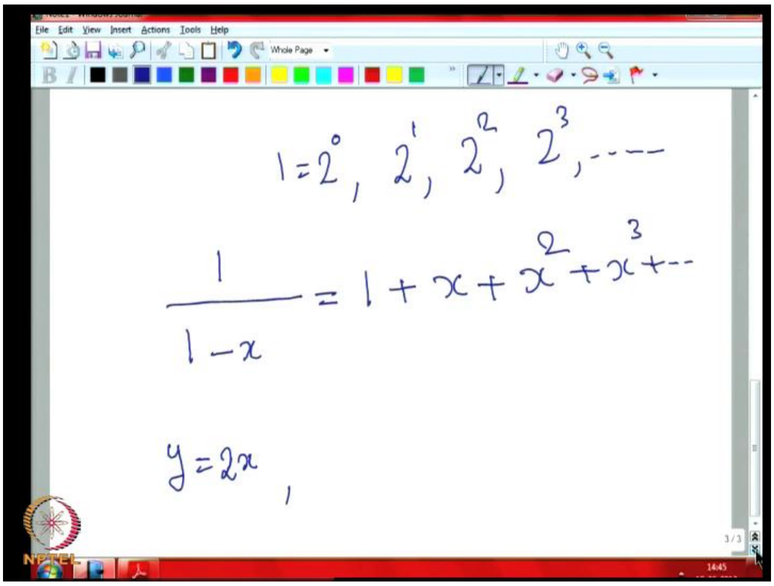
For instance, when h n is just equal to 1; that means the sequence 1, 1, 1, 1, 1, 1 corresponds to 1 by 1 minus x; g of x will be equal to 1 by 1 minus x, because this expands to 1 plus x plus x square plus and so on.

(Refer Slide Time: 01:53)



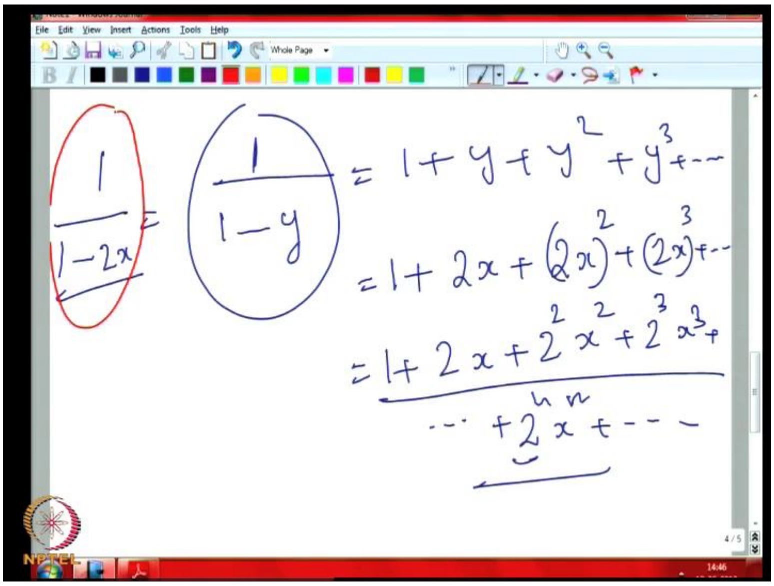
Now we will continue with the discussion. We will consider few more examples.

(Refer Slide Time: 02:22)



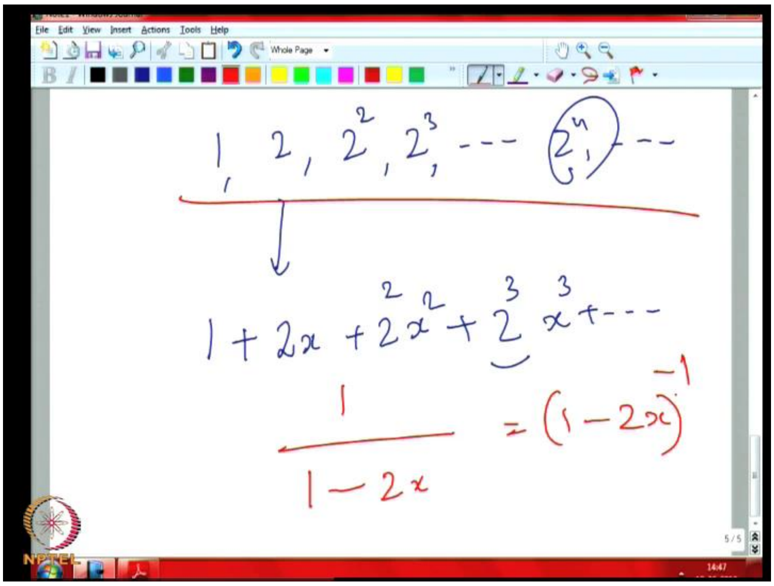
From that, we can get the… From that 1 by 1 minus x is equal to 1 plus x plus x square plus x cube plus… we can get the generating function of another interesting sequence namely, 1; which is equal to 2 raise to 0; 2, that is, 2 raise to 1; then, 2 raise to 2, 2 raise to 3 – this sequence. How will I get it? I know that, 1 by 1 minus x equal to 1 plus x plus x square plus x cube plus so on.

(Refer Slide Time: 03:01)



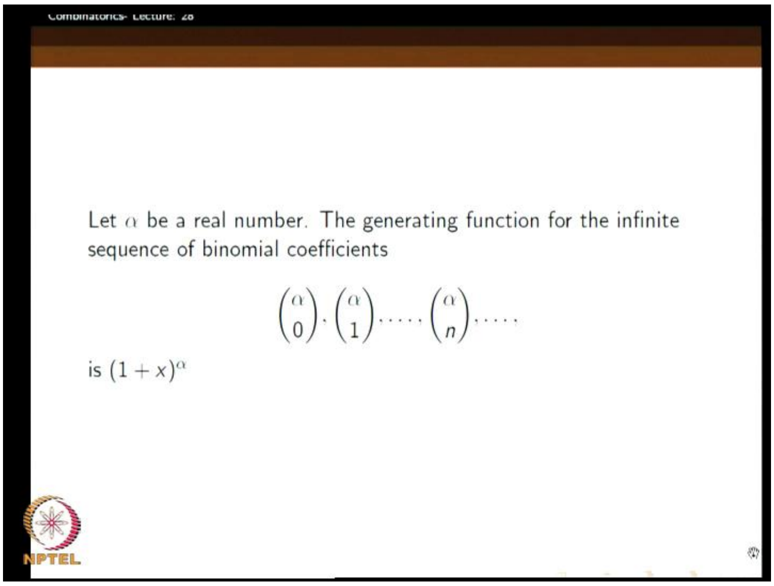
Now, what if I write y equal to 2x and then evaluate 1 by 1 minus y? It will be definitely 1 plus y plus y square plus y cube plus and so on. But, this y being 2x is equal to 1 plus 2x plus 2x whole square plus 2x whole cube plus so on. This is 1 plus 2x plus 2square into x square plus 2 cube into x cube plus general term – will be 2 raise to n x raise to n. So, the coefficient of x raise to n is 2 to the power n.

(Refer Slide Time: 03:48)



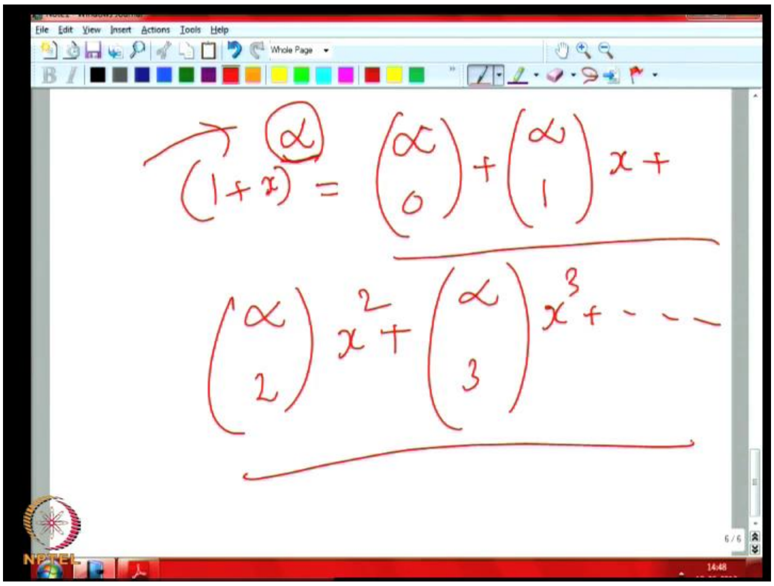
If you had looked at the sequence 1, 2, 2 square, 2 cube, 2 raise to n and so on; the corresponding generating function would have written as 1 plus 2x plus 2 square x square plus 2 cube x cube plus and so on. The coefficient of x raise to n being 2 to the power n. So, that is exactly what we are getting here. And, this is 1 by 1 minus y; that is, 1 by 1 minus 2x. So, this is what is the generating function. So, we get the generating function for this sequence is 1 by 1 minus 2x; or, we can also write it as 1 minus 2x power minus 1.

(Refer Slide Time: 04:47)



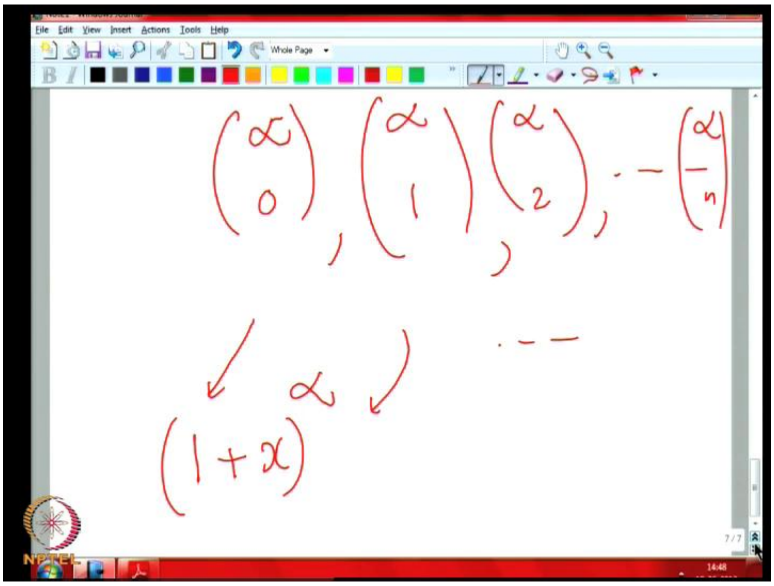
Next, we will… We have seen some special cases; we will see somewhat more general generating function.

(Refer Slide Time: 05:02)



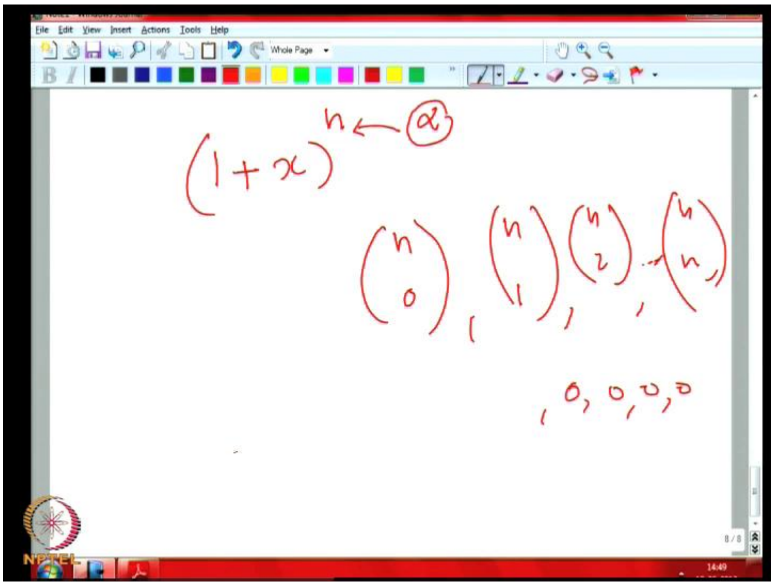
What we told, 1 plus x raise to alpha. We have already seen this in some earlier class, previous class that, this is alpha choose 0 plus alpha choose 1 x plus alpha choose 2 x square plus alpha choose 3 x cube plus so on. Here alpha is any real number or even complex number. That is what we told. So, we can define it. So, we have seen it when we studied the generalized binomial theorem. This is what it is. What does it say?

(Refer Slide Time: 05:51)



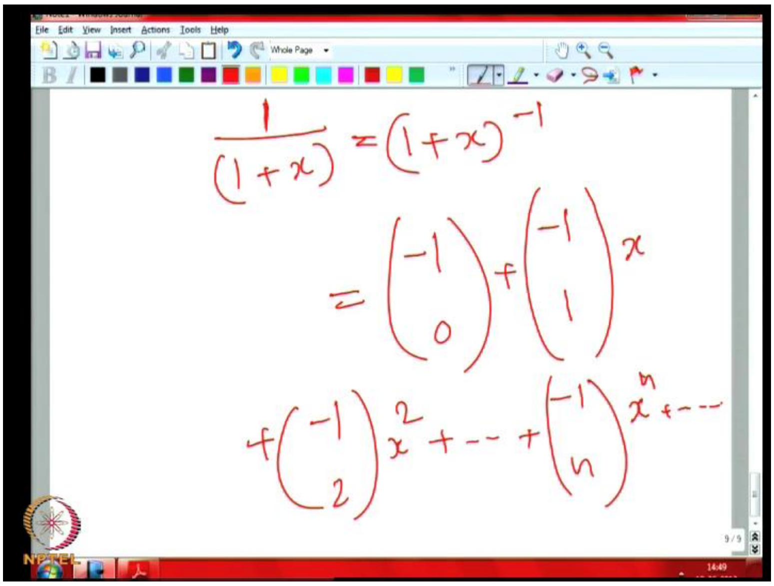
It says that, this sequence for a given real number alpha – say alpha choose 0, alpha choose 1, alpha choose 2 – this sequence to alpha choose n being the general term. And, this goes on; has the corresponding generating function 1 plus x raise to alpha. So, I told that, this is a more general generating function than what we have earlier seen, because…

(Refer Slide Time: 06:23)



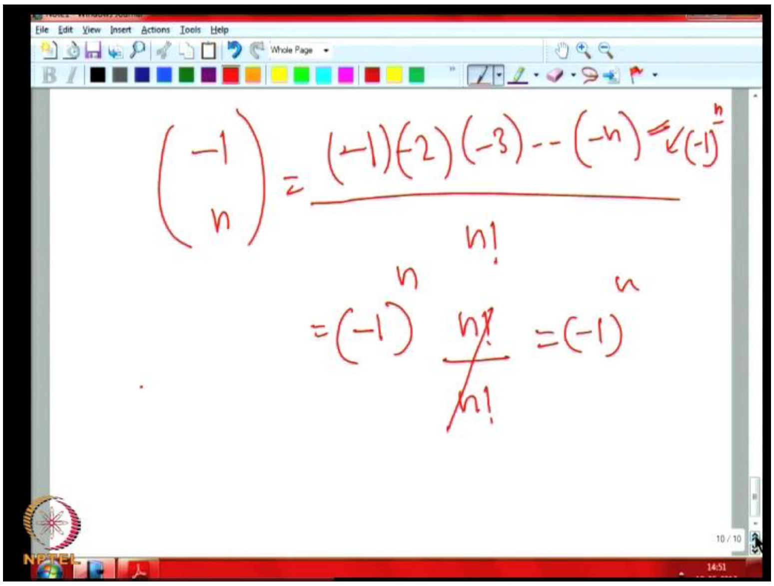
For instance, if you… One generating function we have earlier seen is the special case when alpha is taken as some positive integer (( )) This corresponded… We saw that, this – the coefficient – n choose zero, n choose 1, n choose 2 up to n choose n. And then, zeros in that case. This is the usual binomial theorem.

(Refer Slide Time: 06:50)



And, this 1 by 1 minus x we saw; that is, 1 plus x to the power of minus 1. This is actually… When I take alpha equal to minus 1, this will become minus 1 choose 0 plus minus 1 choose 1 x plus minus 1 choose 2 x square plus… So, in general, the general term will be minus 1 choose n to the power x raise to n and so on.

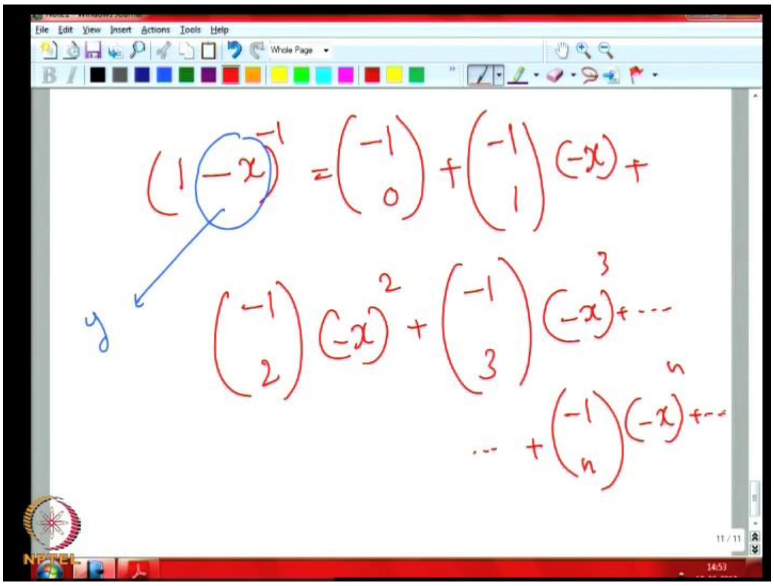
(Refer Slide Time: 07:31)



How does it look like? For instance, this minus 1 choose n will be minus 1 into – the next is minus 1 minus 1, that is, minus 2; then, minus 3 up to it will go to minus n… n terms we have write starting from here – 1 to n terms we will write. And then, this is decreasing. For instance, you see this is the following factorial. What we have written here is minus 1 n following. Below it is n factorial. This is minus 1; like that only this will go – minus 1, minus 2. That will make it minus 1 raise to…

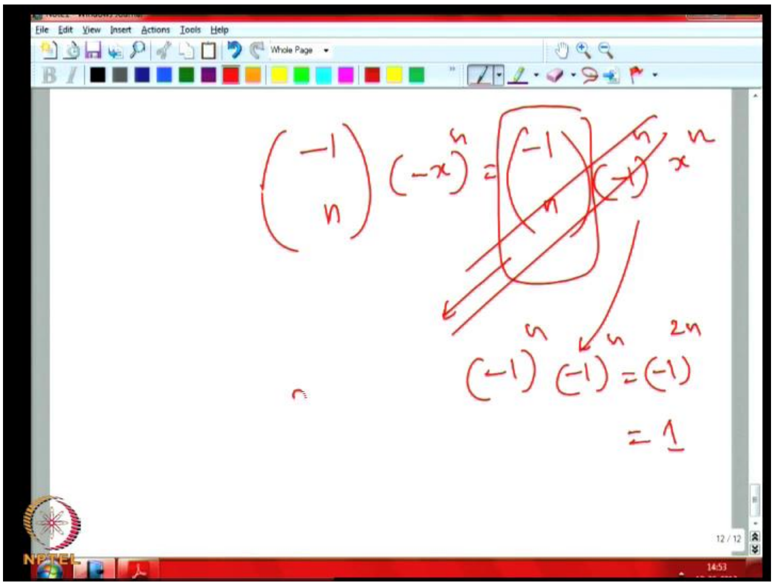
Here we collect all the minus 1's out. So, minus 1 to the power n. And, we get an n factorial here. And, here we get an n factorial below. So, this cancels off. We get minus 1 power n. But… This is 1 plus x. Suppose if I wanted to consider… See earlier we had considered… We wanted to say that this is more general form compared to the earlier things we considered. See 1 minus x is what we considered here. So, let us say we will consider 1 minus x there; like this; instead of this, we will put minus here. Then, what will happen? Then, minus x here; minus x whole square here; minus x cube; and, the general term will be …

(Refer Slide Time; 09:55)



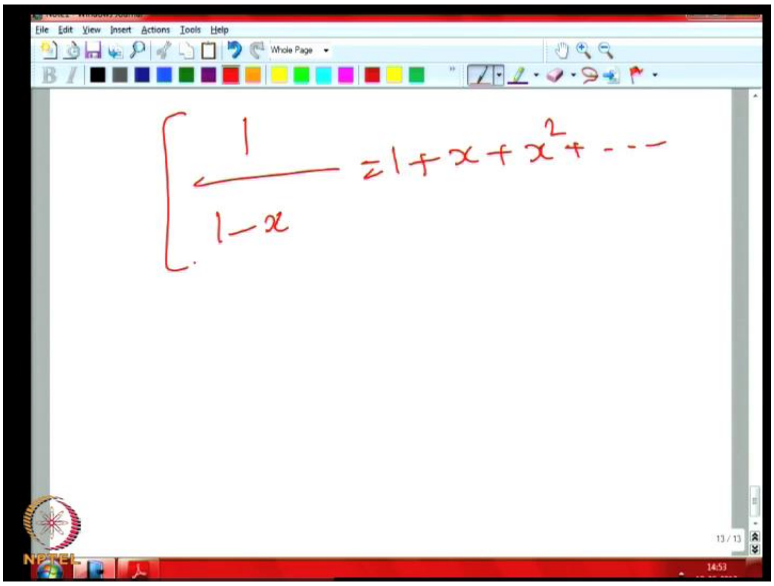
Here once again, if I consider 1 minus x to the power minus 1, what I get is minus 1 choose 0 plus minus 1 choose 1 to the power minus x. Here if we want to replace this with y, you can get the earlier expression and substitute minus x in that. Rather than going round about, we directly write it minus 1 choose 2 minus x square plus minus 1 choose 3 minus x cube and so on. The general term is minus 1 choose n minus x to the power n and so on.

(Refer Slide Time: 10:44)



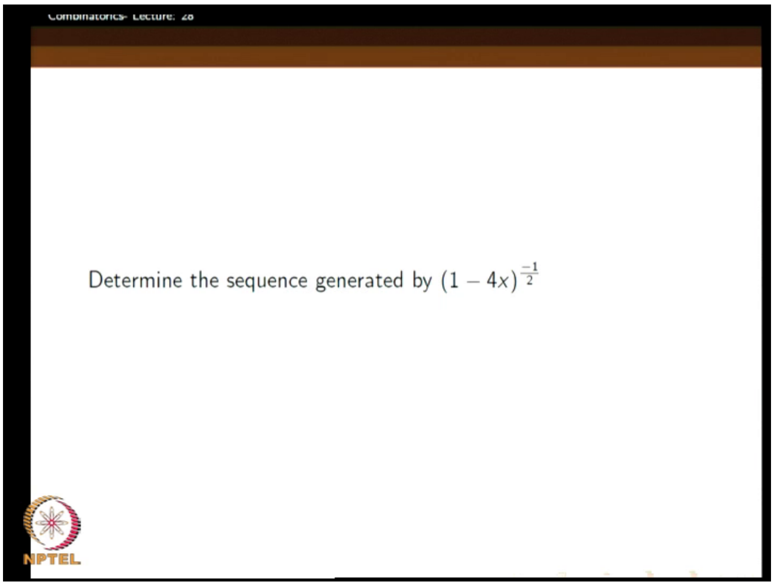
So, that general term… This being minus 1 to choose n minus x to the power n, which is equal to minus 1 choose n minus 1 to the power n into x to the power n. This happens what? This as we have seen, is minus 1 to the power n; and, this is another minus 1 to the power n. This will become minus 1 to the power 2 n, which is always 1. So, there would not be any coefficient now. So, we will just get x raise to n. Going back to this formula; all these things will go away.

(Refer Slide Time: 11:21)



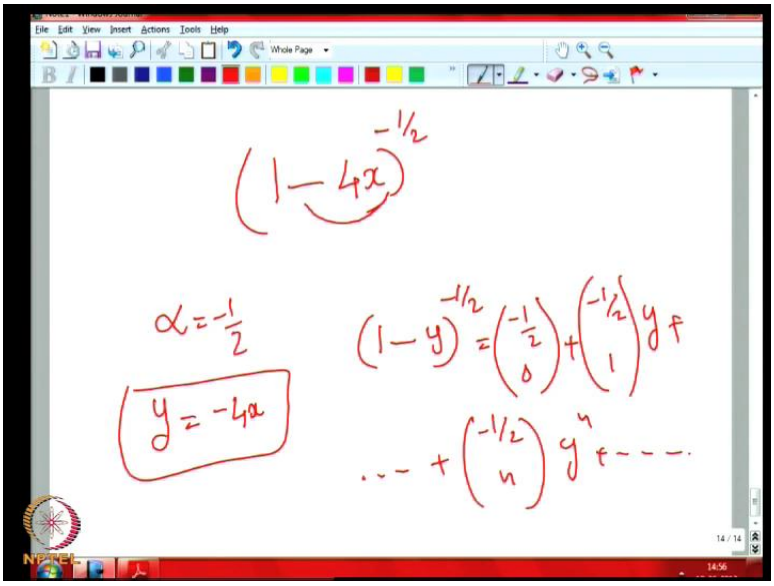
It will just be 1 by 1 minus x will just be 1 plus x plus x square plus like this. So, that way if you remember this formula… Which formula? This general formula we have written here; we changed it here – 1 plus x. This formula if you remember; then, all those previous cases can be derived by putting appropriate value for alpha and sometimes taking x equal to minus 1 and so on.

(Refer Slide Time: 12:08)



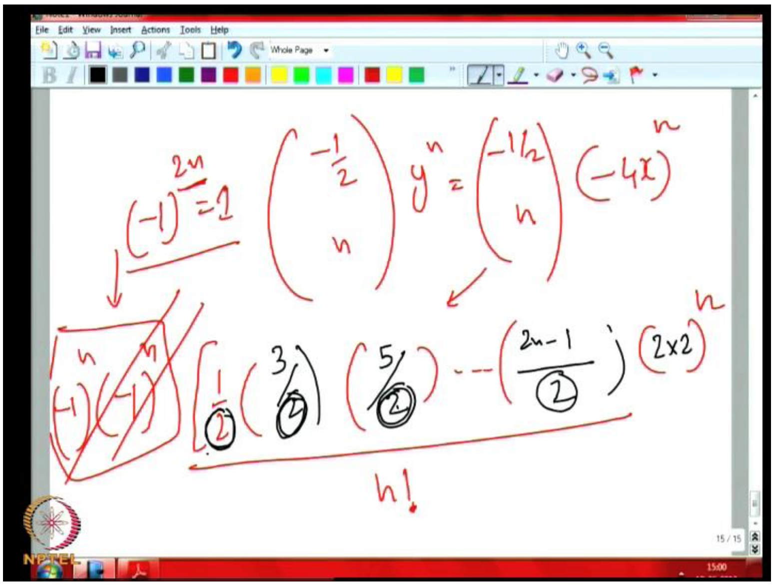
In that context, we should mention that… x need not be… x can be… We can take x as some a times x. Then, substitute a x equal to y and then use the formula; get the expression in terms of y. Then, y can be substituted by a x. So, wherever y raise to n is coming, we will get a to the power n into y to the power n, so that a to the power n will go to the coefficient, merged with the coefficient. It will become part of the coefficient of x to the power n. Now, let us take… To illustrate this point, we will take one example. Determine the sequence generated by 1 minus 4x whole power minus half.

(Refer Slide Time: 13:06)



1 minus 4x to the power minus half. This is what we want (( )) Here alpha equal to minus half. And, we can substitute say y equal to minus 4x. This is what I was trying to tell. And then, say 1 minus y to the power minus half. How will it look like? This will be minus half to choose 0 plus minus half choose 1 y plus… So, minus… General term will be minus half choose n y to the power n and so on.

(Refer Slide Time: 13:52)



That general term – minus half choose n y to the power n will become minus half choose n; y being minus 4 x to the power n. This will become… We can expand this. This will be minus half into minus half minus 1 into minus half into minus 2 and so on minus half minus n plus 1. We will get minus 4 to the power n here and we will have n factorial in the denominator.

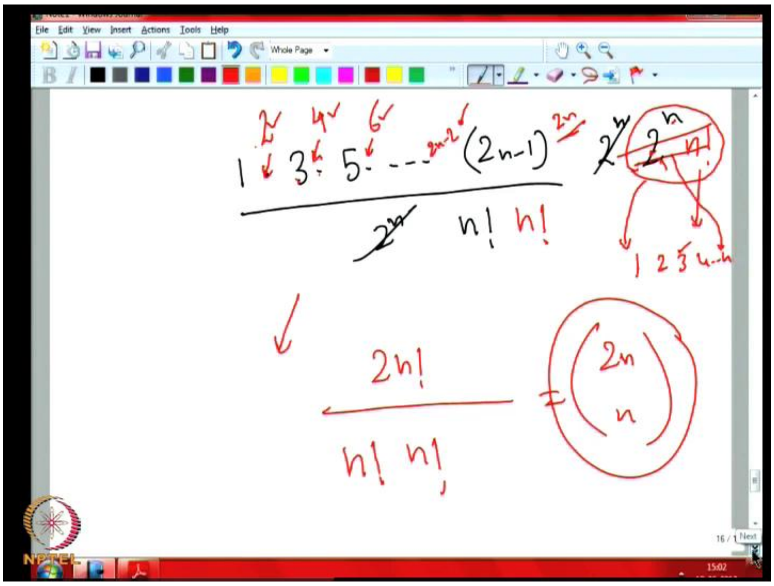
Now, see we can take a term here. Say here this term. This is minus 1 by 2; this is minus half minus one and so on. So, what I do is I first take this minuses out; from here I take minus out. What will happen? This will become half plus 1. Here I can take half plus 2 and so on. This will become plus minus… What I have done is I have removed, taken out minus 1 from each of the terms in the numerator up to here.

So, I got minus 1 to the power n from here, because here n terms here – half half plus minus half minus 1. So, minus half became half now. Minus half minus 1 became half plus 1 now; minus half minus 2 became half plus 2 now and so on. Then, this minus n can be merged with… Here we have minus 4 raise to n. Here if we take this minus; this is n. So, here 1 minus 1 raise to n can be taken from that. And, this will become minus 1 raise to 2n, which is equal to 1 always, because 2n is any one number. So, this will go away.

Now, we have only this part. Here is a 4n. Now, what we can do is to… We can… Because this is half plus 1 and so on, what I do is; this 1 I can write as 2 by 2. So, that will become 3 by 2. Similarly, here I can write it as 4 by 2; this will become 5 by 2. So, similarly, in the end, this 2n plus n minus 1 – that can be written as 2 into n minus 1 by 2. How does it help me?

Because I have 2 here from this term, I have half plus 2 by 2, which I can write it as 3 by 2. And, there is a 2 here. And then, this is half plus 4 by 2, which I can write as 5 by 2, which is here. And, this is… This can be written as 2n minus 2… See 2n minus 2 plus 1; that is, 2n minus 1 by 2; like this. So, we have 2 from each term; half in each term; 1 by 2 here; 1 by 2 here; 1 by 2 here. This I can… Because I got n such 2's; so, 2 to the power, which I can cancel with 2 to the… This 4 is actually 2 into 2. That one of the 2 to the power n's will go away.

(Refer Slide Time: 17:48)

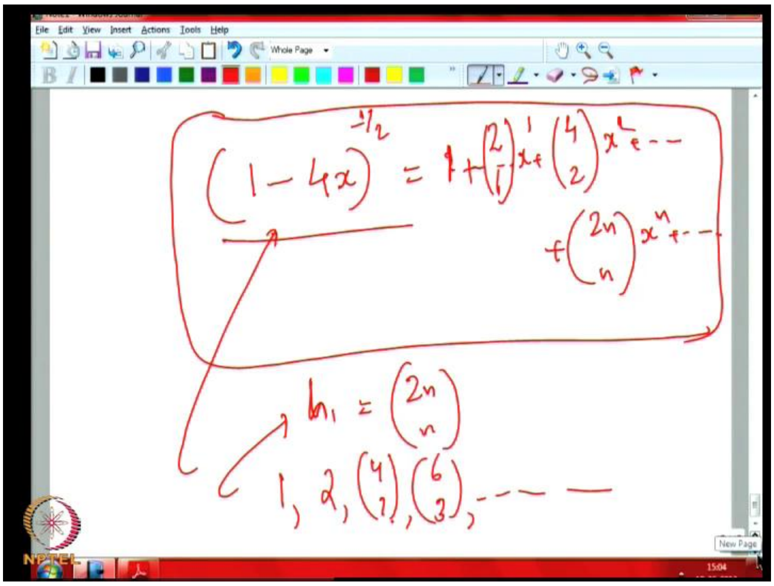


In the end, what I get is 1 into 3 into 5 into up to 2 n minus 1 above. The 2 to the power n, which was here got cancelled off with one of the 2 to the power n's. And, we still have 2 to the power n below and there is an n factorial below. This is what is happening – 1, 3, 5, because every time we were adding an even number. Next even number is being added. Therefore, we get this sequence of odd numbers – 1, 3, 5 up to 2n minus 1 in the denominator. And, 2 to the power n here – that gets cancelled with one of the 2 to the power n's. That 2 to the power of n remains and we still have n factorial below.

Now, what we will do is we introduce… Here I introduce one more n factorial. And, see to balance it, I have to add it above also, because this thing if you look, this is 1 into 3 into 5 into 2n minus 1. I would like to introduce 2, 4, 6 – the missing numbers, the even numbers in between. Till here I will introduced 2n minus 2 and here I want to introduce 2n, because 2 to the power n I have. The 2's which I need… n 2's I have here. And, this n factorial will provide the remaining figures; n factorial is 1, 2, 3, 4 up to n. One of the 2 I take from here and multiply it; I get this 2. And, another 2 I take and multiply it; I get this 4; another 2 from here I take and multiply this 3; I get 6 and so on. So, the last two will multiply by this n and we will get 2n. So, this entire this thing can be distributed in these gaps here. So, this will become…

Above what will you get then? 1, 2, 3, 4, 5, 6 up to 2n; that is, 2n factorial. Below we have n factorial into n factorial – this is 2n choose n. So, what do we get? The term is 2n choose n apart from the x to the power n we have. So, the term has become… See here still we have x to the power n, because we were just manipulating the coefficient alone. So, the final term will be this into x to the power n. So, we see that, this minus half choose n into y to the power n; which is equal to minus half choose n. Minus 4 x to the power n will become 2n choose n into x to the power n.

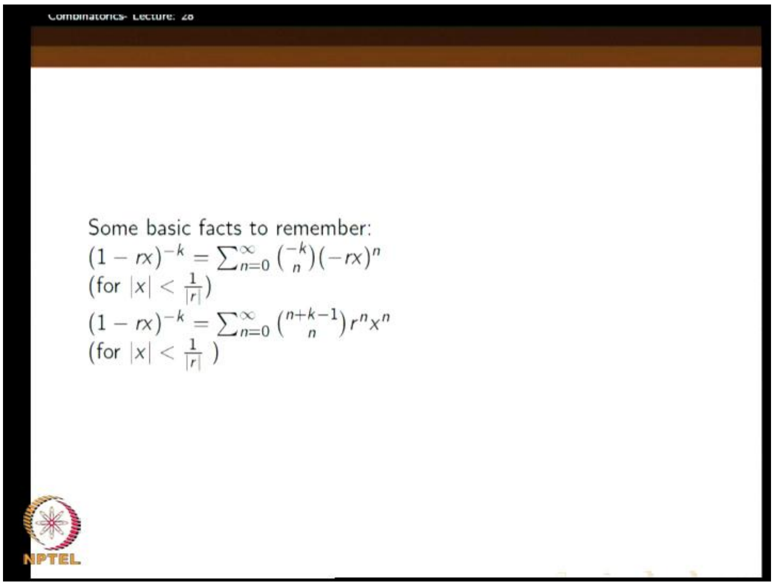
(Refer Slide Time: 20:34)



Therefore, this thing – 1 minus 4x to the power minus half expands to 2n choose n x to the power n. You can give values for n. And then, for instance, when it is 0, that will be this 0 choose 0; that is, x to the power 0. This is 1 only. When this is 1, it will be 2 choose 1 x 1. And then, when it is 2, that will be become 4 choose 2 x square and so on. In general, it is 2n choose 1 x to the power n. This is the formula for 1 minus x n.

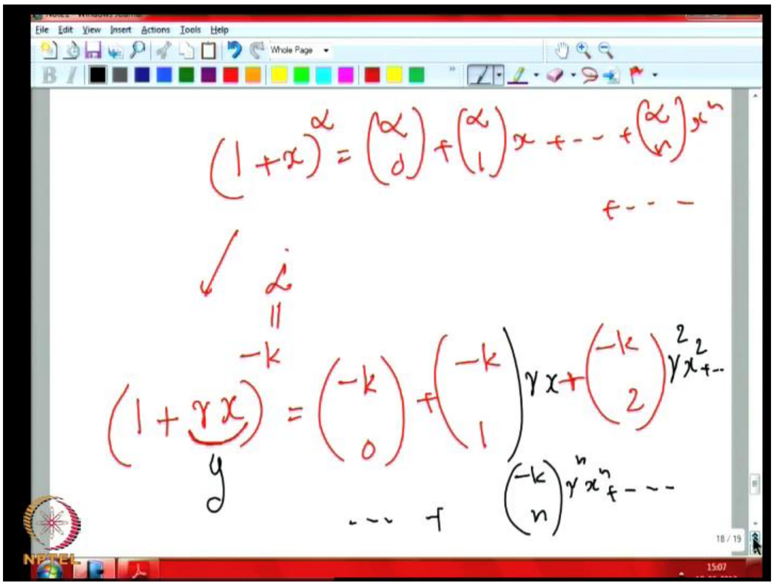
So, this is actually 1 minus 4x to the power minus half turns out to be the generating function for this sequence 2n choose n; h n equal to 2n choose n; that means 0 choose 0, which is 1; 2 choose 1, which is 2; and, 4 choose 2 and then 6 choose 3 and so on – this sequence. This happens to be the generating function for that sequence – 2n choose n sequence.

(Refer Slide Time: 22:12)



Now, let us look at some basics. Now that we have mentioned the general form, these are some special cases – 1 minus r x to the power of minus k. So, how will… So, it is good to…

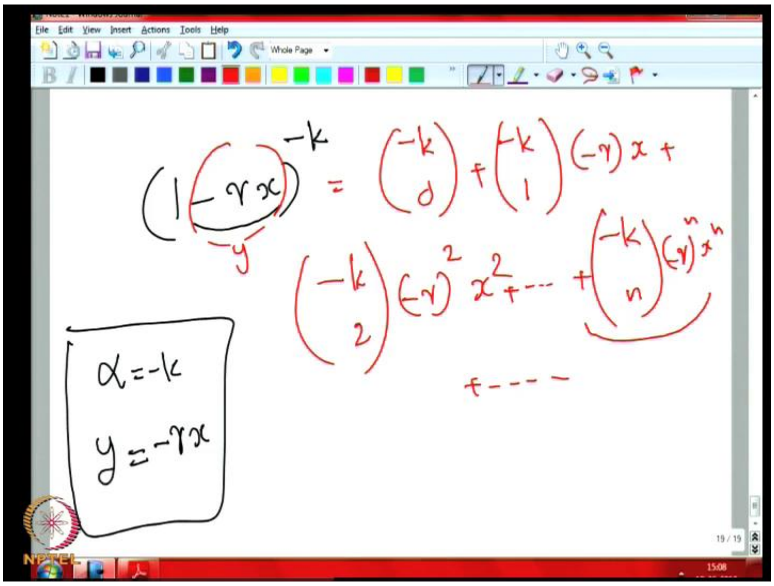
(Refer Slide Time: 22:33)



We can always derive it from the general formula namely, 1 plus x to the power alpha equal to alpha choose 0 plus alpha choose 1 x plus alpha choose n x raise to n form. So, we can always derive all these things. But, some of these forms – they will occur more frequently. Therefore, we just have a look at them. 1 plus rx to the power n – what will happen? 1 plus rx to the power minus k; minus k; what will happen? Minus k is taking the role of alpha. So, if I put it here, that will become k choose 0 plus minus k choose 1 x plus minus k choose 2 x square. See as we have mentioned, this we should took it as y or something; and then, this will become y; and then, y square and so on.

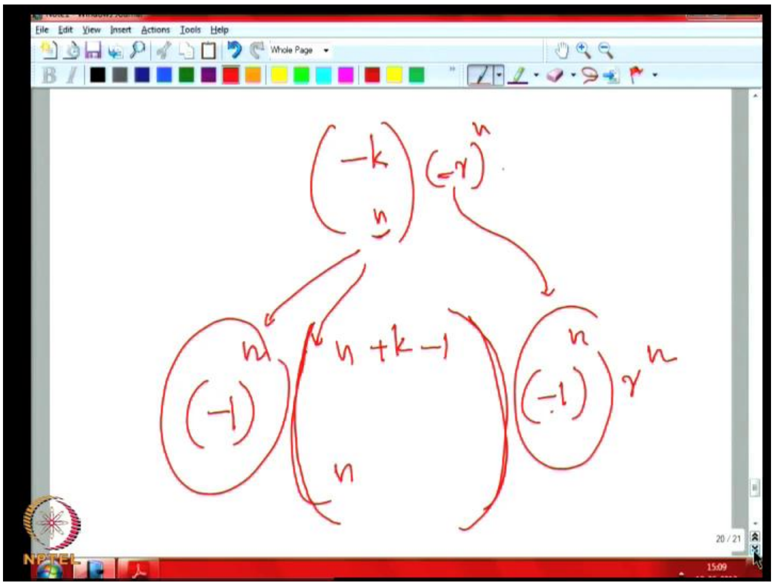
The general term is what? minus k choose n y to the power n. Now, y can be substituted as rx. So, this will become r times x; this will become r square times x square; this will become r raise to n times x raise to n. So, the general term is minus k… We wanted minus… I copied it wrongly; I wanted minus rx here; but, this is correct. So, let us… rx but… Off case, this can be done like this. But, the form we wanted was slightly different.

(Refer Slide Time: 24:41)



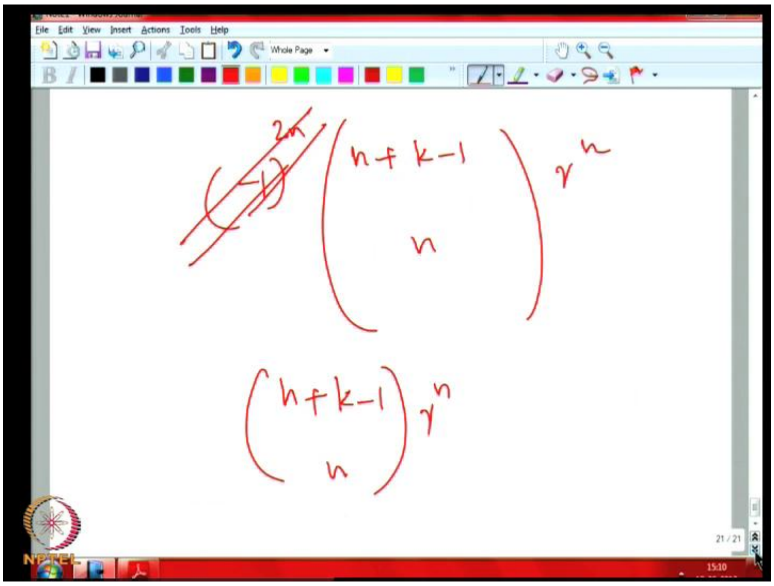
We wanted 1 minus rx to the power minus k, is what we wanted. Here alpha is equal to minus k and y is equal to minus rx, not rx. This is what we wanted (( )) Here this is y; this is y. Minus rx is y. Now, this will become minus k choose 0 plus minus k choose 1 into minus r raise into x plus – next is minus k choose 2 into minus r square into x square and so on. And, general term is minus k choose n into minus r raise to n into x raise to n. This is what the general term and so on. And, what is this? We can simplify this general term a little bit if you remove… This is already simple – minus k choose n minus r raise to n.

(Refer Slide Time: 25:53)



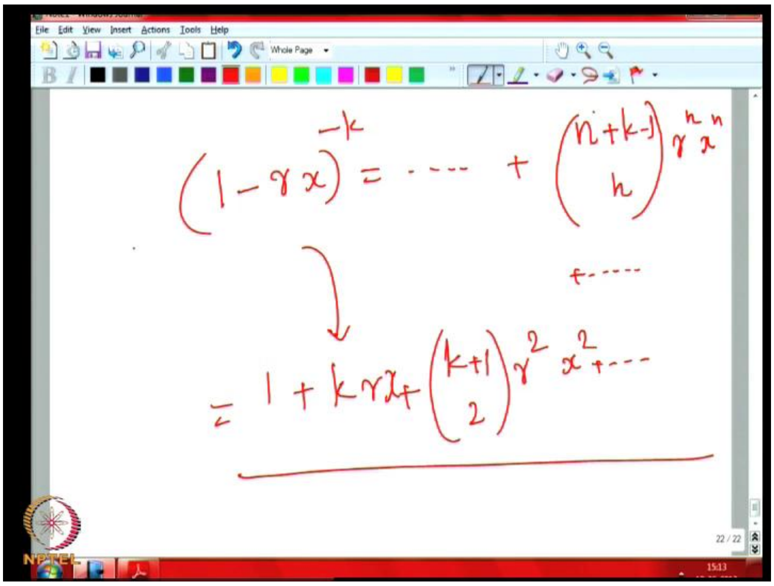
But, what we can do is to convert it to a more familiar form, because minus k choose n into minus r raise to n is what? This we know; can be converted to n n… We have learnt this trick of change of sign. This was equal to this. This we studied when we studied the binomial coefficients especially when we studied the generalized binomial coefficients. So, this is same as this. And, this became minus 1 raise to n into r raise to n. This also needs this thing – minus 1 raise to n; whatever they say. Therefore, this minus 1 raise to n and this minus 1 raise to n – this minus 1 raise to n is coming from here. This is a negative r. So, we are getting minus 1 raise to n into r raise to n here.

(Refer Slide Time: 27:00)



This minus 1 raise to n into minus 1 raise to n will combine to form minus 1 raise to 2n into n plus k minus 1 choose n into r raise to n. This will go away, because this is just 1. So, we will get n plus k minus 1 choose n r raise to n. This is the general term. General term means this term. This corresponds to the same thing. There is n plus k minus 1 choose n into r raise to n will become the general term here.

(Refer Slide Time: 27:43)



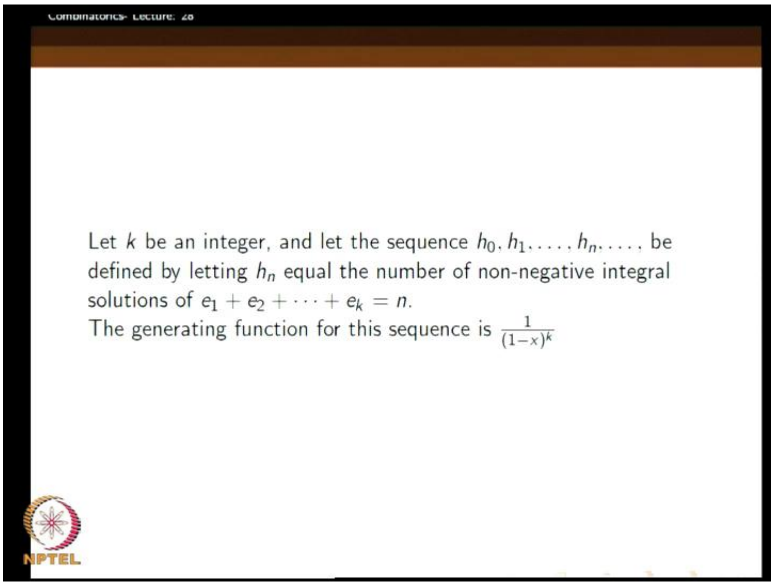
Therefore, what we see is 1 minus r x to the power minus k is same as… This is the general term – minus… – n plus k minus 1 choose n into r raise to n into x raise to n. This is the general term. We can substitute for the value of n. For instance, if we put n equal to 0 here; this will become k minus 1 choose 0 into r raise to 0 into x raise to 0; that means k minus 1 choose 0; which is essentially 1.

Then, put n equal 1. That will become k minus 1 plus 1; that is, k choose 1 into r raise to 1 – k choose 1; that is, first term is this one only. The second term will be k choose 1; that is, k into r – k r. When I put n equal to 2, what will happen? This will become k plus 1 choose 2 into r square x square and so on. This is the sequence, which we… Here we have an x. So, this is the expansion we will get. So, we can…

Off case, we can derive it. If you recall the original formula namely, 1 plus x whole power alpha; we substituted y equal to minus rx and alpha equal to minus k in that formula and simplified; that is all. But, it has a nice form. That is why we did it. And, recall that, this formula will only… This is equal to 0 to m. Now, this is valid when the mod x is less than 1 by r, because generalized binomial theorem if you remember, had this condition that, mod x has to be less than 1 by r; that means the mod of this rx has to be less than 1. That is why this condition is coming.

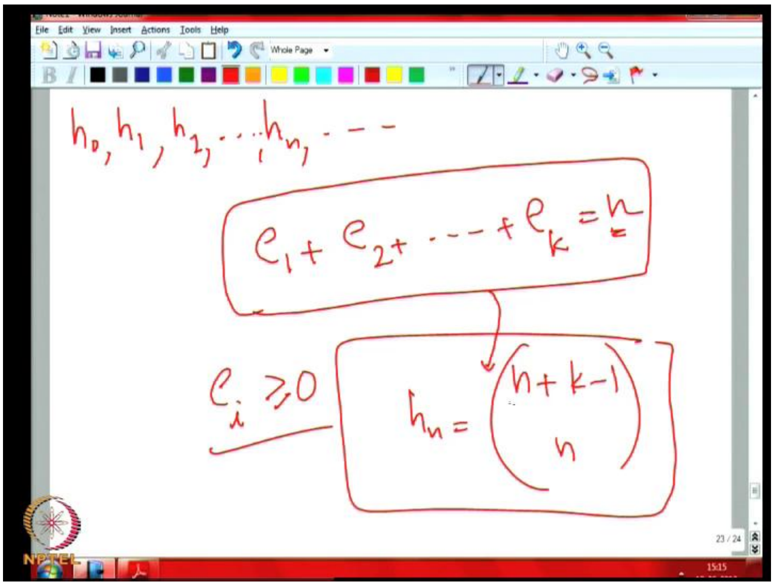
Off case, when we used generating function, we do not have to really worry about this issue of whether it converges or not, because we will be picking up the coefficient and comparing. As long as it converges in the set and range, that will work. So, we do not have to worry about whether it really converges for all as long as we are not going to submit up. Most of the times as you will see, we will just pick up the coefficient of x raise to n and compare. That is what we are going to do.

(Refer Slide Time: 30:44)



Now, we consider an example, which is again familiar; k be an integer. And, let this sequence h 0, h 1, h n be defined by letting h n equal the number of non-negative integral solutions of e 1 plus e 2 plus e k is equal to n.

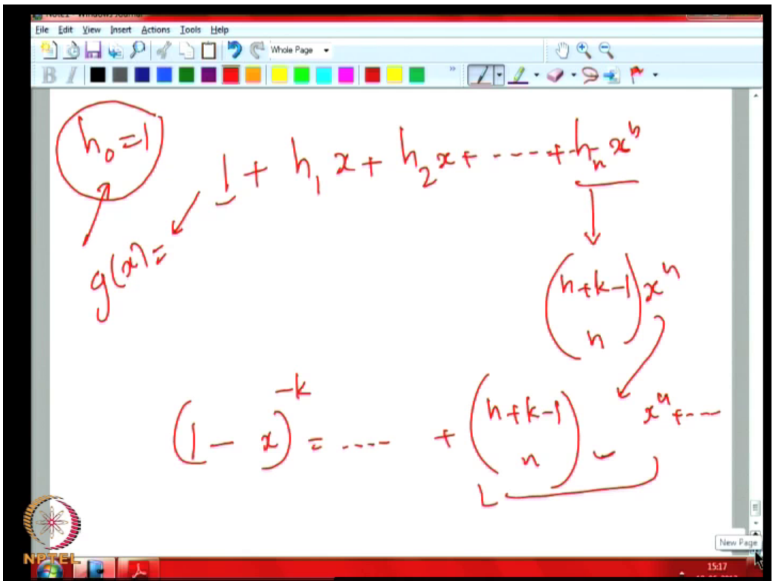
(Refer Slide Time: 31:10)



What about the generating function for this sequence? See this is a very familiar question; we have already solved this thing; that means we have this equation e 1 plus e 2 plus up to e k; there are k variables; this is equal to n. What we are interested in is non-negative integer solution for this thing. So, this e 1, e 2, e k should get values, which are integers, but non-negative integers; that means e i has to be greater than equal to 0. We have discussed this problem earlier. We know that the answer – the number of such solutions, which we can get – different possible assignments of non-negative integers e 1, e 2 up to e k, so that they add up to n is n plus k minus 1 choose n. This is the number. But, we see that…

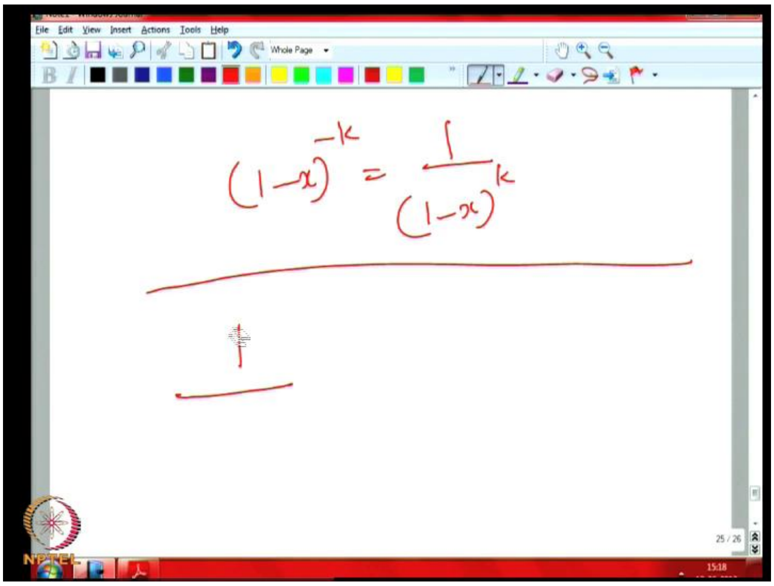
Now, suppose this number of integer solutions is written as – for this n, we write h n. So, h 0 will be the number of solutions when n equal to 0; h 1 corresponds to the number of solutions when n equal to 1; and, h 2 corresponds to the number of solutions when n equal to 2 and so on; h n for the general when this is n and so on. So, there is a sequence like this. And, you know that, actually h n value is this. This is from our previous knowledge; we have not solved it here; we have done it before. But, now, we can write the generating function for this thing easily. Why? Because we see that, the generating function, whose coefficient, which is such that its x raise to n has coefficient h n equal to n plus k minus 1 choose n is this one (( )) n plus k minus 1. This is 1 minus… Here we have to put r equal to 1, because we are only interested in this thing. So, we want r equal to 1 here. So, 1 minus x to the power minus k will be the answer for this thing. Write this here…

(Refer Slide Time: 33:19)



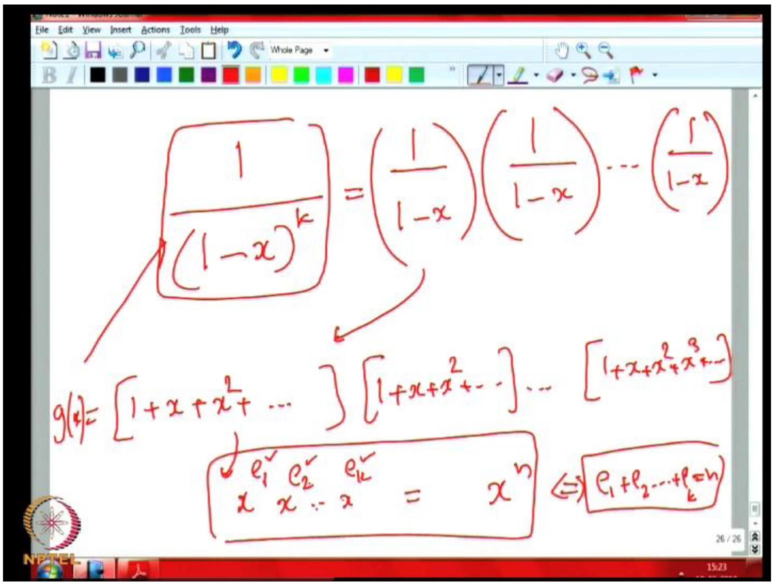
We want a generating function like this – 1 plus h 1 x plus h 2 x plus and so on. I am taking h 0 is equal to 1. The number of integer solutions 0; that we will fix it as 0. For n equal to 0, in how many ways we can do? That is 0, 0, 0, 0. It is definitely equal to 1. This generating function, where h n x raise to n. This is what I want to evaluate. What is this g x? Let this is g x. But, we know that, this h n equal to n plus k minus 1 choose n into x raise to n. Now, you know if we take r equal to 1. This earlier formula, previous formula we have derived; that means 1 minus r x whole power minus k. The n-th term is coming as n plus k minus 1 choose n into r raise to n x raise to n is the general term. Now, we do not want r here. Put r equal to 1. So, that means the general term will be as we want – h n x raise to n – this will become the general term. Now, r has to be 1. So, 1 minus x raise to minus k will give the generating function we are looking for.

(Refer Slide Time: 34:46)



1 minus x to the power minus k – 1 by 1 minus x whole power k. This is the generating function for the sequence we are looking for. How did we derive it? We knew the answer before; we knew that, the answer for the question namely, the number of non-negative integers solutions for this equation is indeed this n plus k minus 1 choose n. And, in the previous discussion, we have seen a generating function, where the coefficient of x raise to n is this. Just I have to take r is equal to 1. So, that is how we solved it. Now, we can also solve it directly without referring to the earlier calculations namely… So, let us say 1 minus…

(Refer Slide Time: 35:40)



Let us look at this one. Suppose if we can solve it in the sense (( )) At least we can establish that, this is indeed the generating function for the number of non-negative integer – the sequence of numbers, where n-th number h n is the number of non-negative integer solutions for that equation e 1 plus e 2 plus plus e k equal to n. This is what we are going to do. How will you do this? You write this as 1 by 1 minus x into 1 by 1 minus x into 1 by 1 minus x. There are k terms here. But we know this one expands to 1 plus x plus x square plus – this sequence. And, this also expands to 1 plus x plus x square plus and so on. Like that the k-th one also expands to 1 plus x plus x square plus x cube plus and so on.

Now, in this expansion… For instance, if i multiply it out; suppose this is g of x; this is… So, if I have an expansion for this thing, it will come from this, if you expand it out. What would be the coefficient of x raise to n in that? x raise to n can be formed in several different ways from this product. For instance, I can take an x raise to e 1 from this and I can take another x raise to e 2. So, a term x raise to e 2 from here; and then, finally, x raise to e 3 from the next; and then, x raise to e k from the last. So, this can become equal to x raise to n if… When will this become?

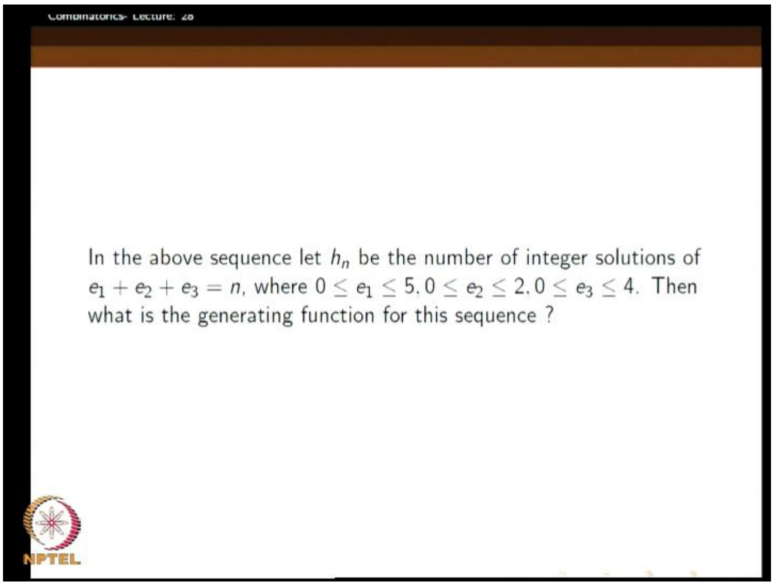
When e 1 plus e 2 plus e k plus equal to n; that means this selected e 1 into e k form a solution for that equation; it is a non-negative integer solution when assignments… So, e 1, e 2, e k such that they add up to n. So, this is what we can do. But, for every possible e 1, e 2, e k such that they add up to n, that will correspond to… That will give another x raise to n. To get the coefficient of x raise to n, we have to add up all of them.

Suppose some t is the coefficient of x raise to n in the expansion of this thing; that means how many ways we can form n as a sum of e 1 plus e 2 plus e k, where each e i gets non-negative integer values, because in that case, you can see that, that is the way you can get it. More clearly; suppose you take a non negative integer solution for e 1, e 2 up to e k to the equation e 1 plus e 2 plus e k equal to n. Then, what we can do is for that corresponding solution, we collect x to the power e 1 from this first term, x to the power e 2 from this term and x to the power e k from the last term. Then, that will indeed form x raise to n by multiplying out. On the other hand, if we consider a particular way in which we can form x raise to n; that indeed corresponds to a solution of that equation. Therefore, that indeed counts; the coefficient of x raise to n in this product counts the number of ways we can assign non-negative integer values to e 1, e 2 up to e k such that they add up to n.

We get directly from this argument that 1 by 1 minus x raise to k is the generating function for the sequence h 0, h 1, h 2, h 3 up to h n and so on, where h n is the number of integer solutions for e 1 up to e k, where e 1 plus e 2 plus e k has to be equal to n for the integer solution for that equation. This is one thing. But, this argument is very useful, because we get how this x raise to n – the coefficient of x raise to n – how is it getting formed; in how many different ways x raise to n can be formed from the product.

The first term gives something; the second terms gives something; and, third term gives something. And, in various ways, they form this x raise to n. Everytime an x raise to n is formed, they add up together and then they get it. So, that is why it is interesting. Why is it interesting? Because we can put some restrictions on the way we can take e 1. So, we have considered such kind of questions when we discussed intrusion-exclusion principle.

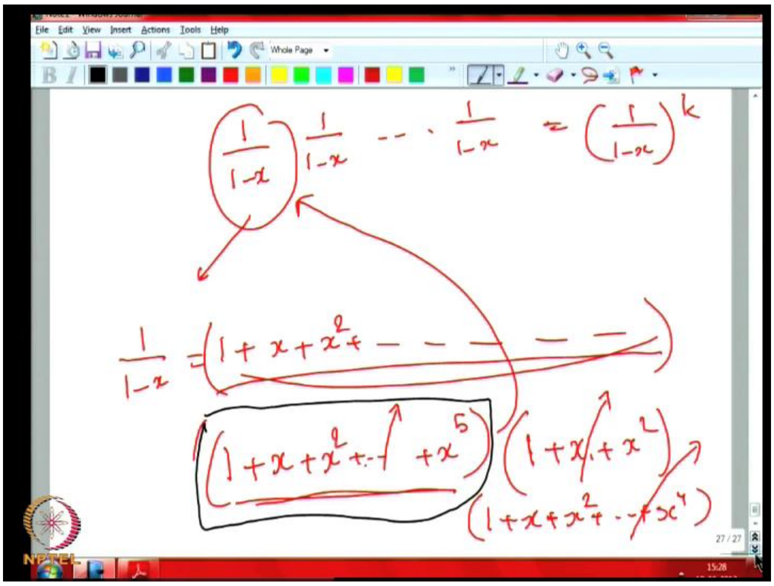
(Refer Slide Time: 41:20)



But, we release such things in a different way now. For instance, you can look at this question. In the above sequence, let h n be the number of integer solutions of e 1 plus e 2 plus e 3 is equal to n. I am taking k equal to 3 now for simplicity. Let k equal to 3. So, e 1 plus e 2 plus e 3 equal to n. And now, e 1 cannot take any non-negative integer value. It is restricted to be between 0 and 5; that means e 1 can only take either 0 or 1 or 2 or 3 or 4 or 5. Only 6 possible values e 1 has. Similarly, e 2 can take only values – 0, 1 or 2;

that is, e 2 is between 0 and 2. Similarly, e 3 can take only values 0, 1, 2, 3 or 4; it is between 0 and 4. Then, what is the generating function for the sequence?

(Refer Slide Time: 42:19)



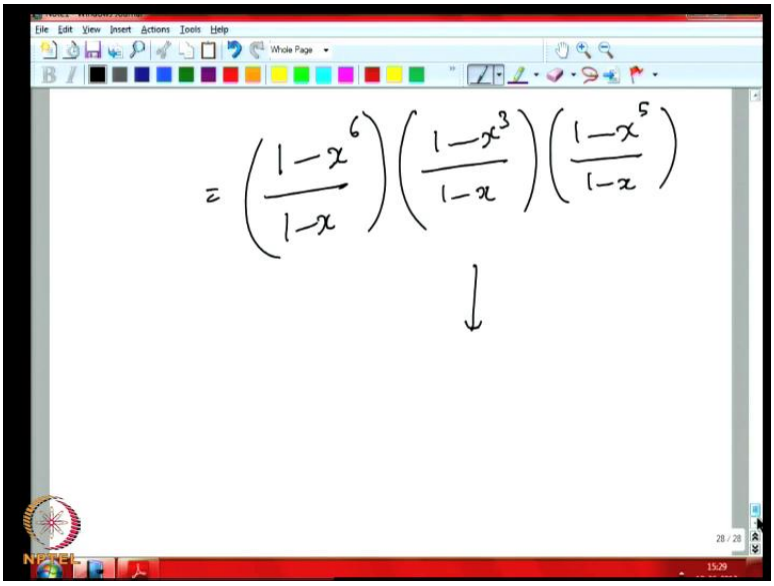
You can see that, earlier, we had considered the first term as 1 by 1 minus x; the second term as 1 by 1 minus x; the third term as also 1 by 1 minus x. And, we had taken k such terms – 1 by 1 minus x. That is why it is 1 by 1 minus x whole power k. That is because this 1 by 1 minus x was equal to 1 plus x plus x square plus – all the possible powers of x are there here; that means we are allowing any power of x to be taken (( )) including x raise to 0, x raise to 1, x raise to 2 to any power of x.

But, now, I do not want it to be like that; I want only either x raise to 0 or x raise to 1 or x raise to 2 or up to x raise to 5 is to be allowed for e 1. So, this first term – I will rather substitute with this; rather than this infinite sequence, infinite sum, I will have this 1 plus x plus x square plus up to x raise to 5. I will use this here. Similarly, for the second term, instead of allowing all possible non-negative integer values for e 2, we are now allowing only x raise to 0 or x raise to 1 or x raise to 2; that is what we told. So it has to be between 0 and 2.

The third one has to be between 0 and 4. So, up to here only we allow. And, the third term – we only allow 1 plus x plus x square plus x raise to 4. So, this will indeed give the solution for this thing, because if now you look at the coefficient of x raise to n in this product, what will happen? Because this x raise to n can be formed by giving some x raise to e taking some x raise to e 1 from here, some x raise to e 2 from this term and x raise to e 3 from this third term. But, this is making sure that, x raise to that e 3 is in between 0 and 5. So, similarly, e 2 is in between 0 and 2. Here we are making sure that, e 3 is in between 0 and 4. And, if they add up to n; then, we will get one more to the coefficient of x raise to n. If they do not add up to n, then it is not contributing to the coefficent of x raise to n. And, any thing – any e 1, e 2, e 3, which can add up to n and such that they are satisfying the conditions, will come from this also.

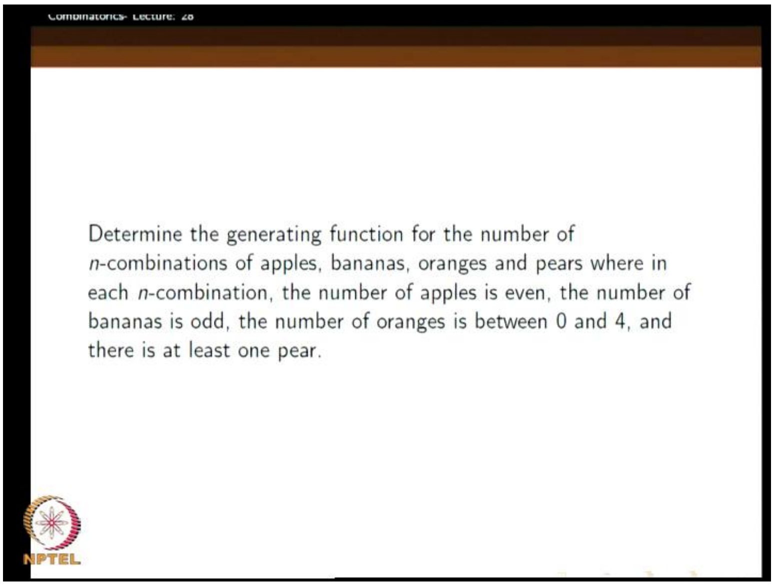
First – e 1 can be taken from this thing; second – e 2 can be taken from this thing; and, third – e 3 can be taken; third that term – third variable e 3, whichever value it was given, it can be taken from this also. Therefore, you see that, it corresponds to the kind of solutions we look for. But, now, if you see, this will be a product like this. We can indeed simplify this, because this 1 plus x plus x square plus x raise to 5 – we know how to submit.

(Refer Slide Time: 45:35)



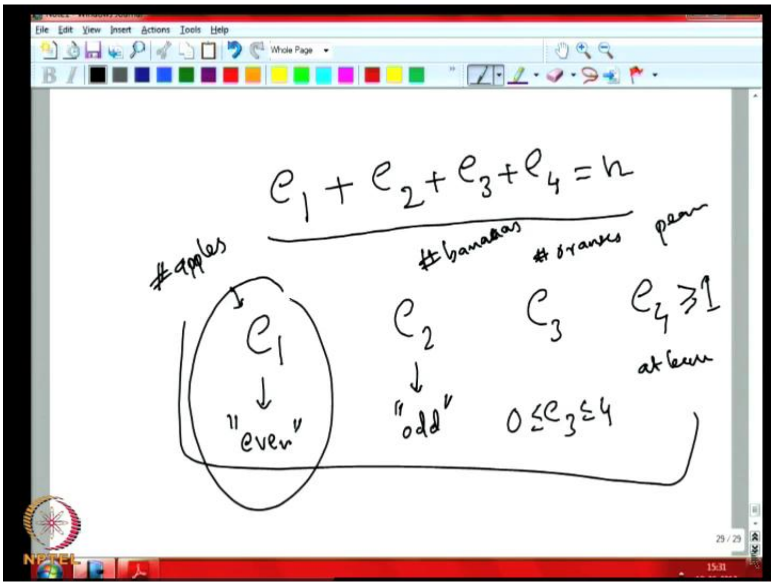
How much is that? That is 1 minus x raise to 6 by 1 minus x. Similarly, next term – this term – 1 plus x plus x square is 1 minus x cube by 1 minus x. And now, this one – 1 plus x plus x square plus x raise to 4; that is, 1 minus x raise to 5 by 1 minus x. We can indeed simplify this thing. So, the coefficient of x raise to n in this product will give the number of integer solutions for the problem with restrictions for e 1, e 2 and e 3. Now, let us look at the next question.

(Refer Slide Time: 46:30)



We will give some examples for the same problem, but with some more concrete examples. Determine the generating function for the number of n-combinations of apples, bananas, oranges and pears; where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

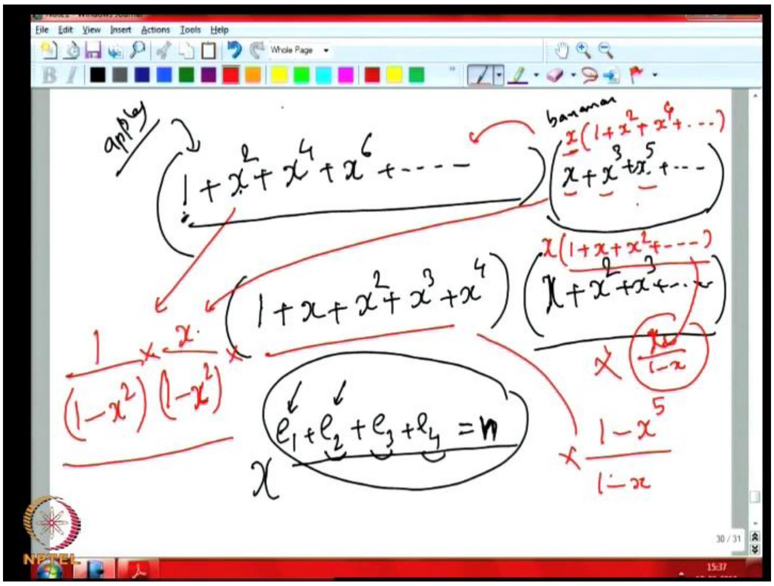
(Refer Slide Time: 46:57)



See here it is very clear that, we are asking for the number of integer solutions for e 1 plus e 2 plus e 3 plus e 4 equal to n with conditions that e 1 has to be the number of…

We say that the number of apples is even, e 1 corresponds to the number of apples; that has to even. And similarly, the number of bananas is odd. So, e 2 corresponds to the number of bananas. So, this corresponds to the number of apples. So, this e 2 will correspond to number of bananas. So, this e 1 is to be even; e 2 has to be odd. These are the conditions. And, then one more condition here – the number of oranges is between 0 and 4 and there is at least one pear; that means e 3 – this is oranges; this corresponds to the oranges. We have e 3 is between 0 and 4. And, e 4 – that is pears. It is at least 1; that is, the condition – that is, e 4 is greater than equal to 1. So, these are the conditions. With this… Respecting these conditions, how many non-negative integers solutions can be given to this is what we are asking. So, how will we find it out?

(Refer Slide Time: 48:42)



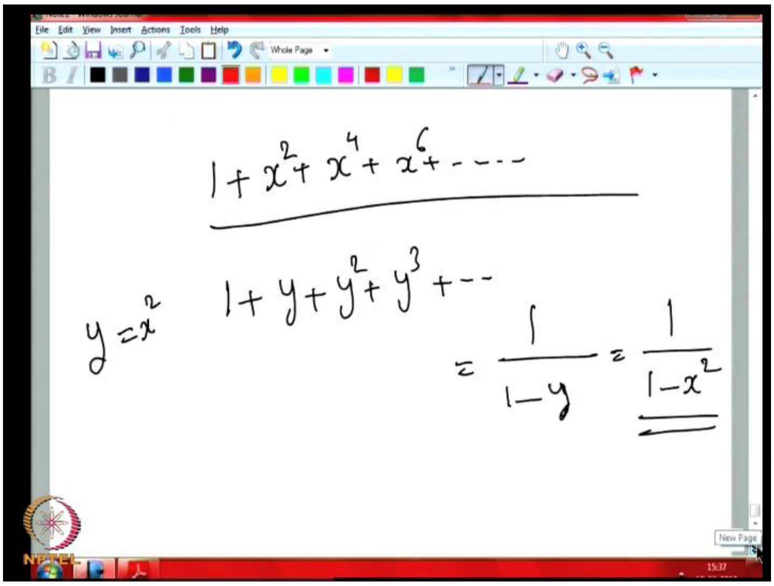
Now, corresponding to this apples that, namely, this e 1, we will have one term. So, we will add like this – 1 plus x square plus x raise to 4 plus x raise to 6 plus and so on. Why? Because this is x raise to 0; this is x square, x 4, because I am only interested in getting an even number of apples. So, I do not allow x raise to 1, x raise to 3, etcetera, because they are all powers of x. So, I do not want… When I create x raise to n, I do not want the first term in the product to contribute.

And, x to the power t; where, t is an odd number. So, I make sure that, all the powers of x here in the first term are indeed even; that is the trick. And, second term, which… This corresponds to apples. And, the second term will correspond to bananas. They have to be odd. This will correspond to bananas. So, that means I cannot put even (( )) 1; I can only put odd powers of x – x plus x cube plus x raise to 5 plus; like that. And, the third one is easier, because we have already seen that. Third one says you need oranges and that should be either in between 0 and 4. So, that third term will be 1 plus x plus x square plus x cube plus x raise to 4. So, the powers of x are all between 0 and 4 here. We are not taking anything more than 4 – x raise to 4. x raise to 5 onwards we are not considering.

And, the last one says we should have at least one pear; that means we cannot take 1, we have to start with x – x plus x square plus x cube plus so on. This term will not contribute one; that means this last term should contribute at least 1 to the power of x when I form x to the power n. Now, when you take the product of this thing, we see that, some x to the power e 1 plus e 2 plus e 3 plus e 4 will be formed, where this is equal to n. This e 1 plus e 2 plus e 3 plus e 4 is equal to n.

But, this e 1 is coming from here – the first term. That will definitely is going to be an even number. And, this e 2 is coming from this second term; that is going to be an odd number as we require. And, e 3 is in between 0 and 4. And, e 4 is going to be at least 1 as we want. (( )) trick. How many ways you can form it will be easily obtained if we consider the coefficient of n, because in the product, we will get the coefficient of x raise to n when we consider as the number of solutions of this equation, which satisfies the required conditions. And, to evaluate the coefficient of x raise to n, we can actually consider the close forms and multiply and see whether we can get something. What will happen? This one…

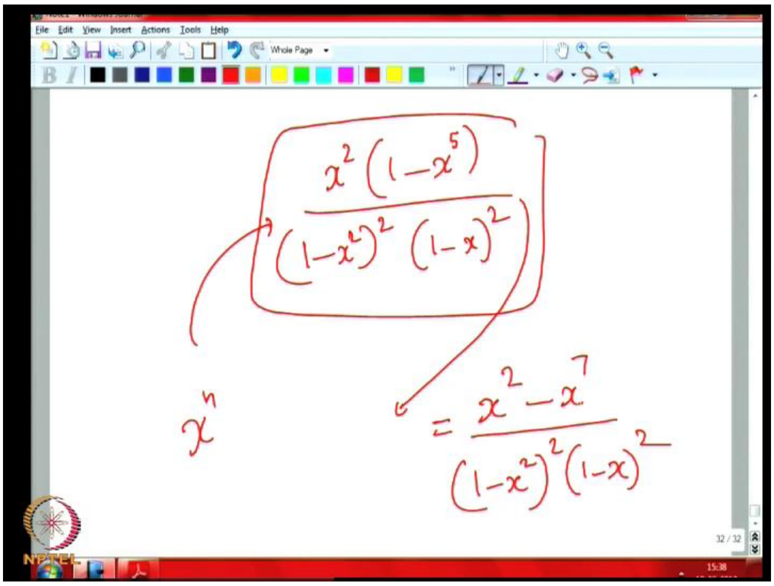
(Refer Slide Time: 51:53)



Now, off case, one will ask, how will I get a formula for 1 plus x square plus x raise to 4 plus x raise to 6 and so on. Now, what I suggest is put y equal to x square. Then, what will happen to this sum? This will become 1 plus y plus y square plus y cube and so on. We know this is actually 1 by 1 minus y. And, substitute for y now. So, that is this one – 1 by 1 minus x square. So, the first term is actually… This will be actually 1 by 1 minus x square. And similarly, this term is what? This term – what I can do is I can take x out from this thing and this corresponds to the same (( )) So, this is actually x into 1 plus x square plus x 4 plus and so on. You see this x into… Other than x, the remaining is the same as this term; that is, the second term is actually x by 1 minus x square. Instead of 1, we have x, because there is a multiplier x here. And, this is indeed 1 by 1 minus x square.

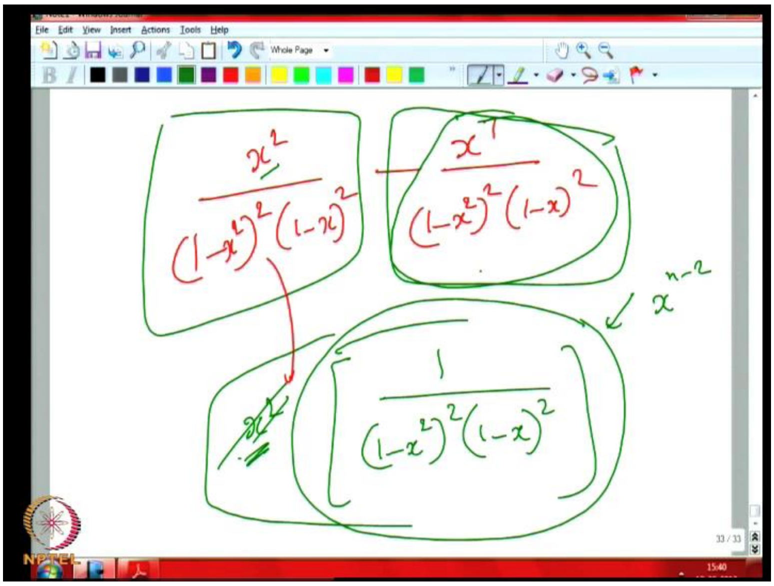
And, this is what… This we already know. This is 1 minus x raise to 5 by 1 minus x. And, this last term – the last term is easy to say; what I do is I take x out from here. So, I write it as 1 plus x plus x square plus and so on, because this is anyway… This second term is actually… After x whatever you see is 1 by 1 minus x. So, you put x also here. Thus, that will give you x by 1 minus x. So, we have this total product as 1 by 1 minus x square into x by 1 minus x square here. And, here x by 1 minus x and 1 minus x square by 1 minus x. When you multiply all the four terms together, what do you get?

(Refer Slide Time: 54:28)



We get x square into 1 minus x raise 5 in the numerator. And, in the denominator, we have 1 minus x square whole square into 1 minus x whole square. Now, if you want to find the coefficent of x raise to n in this thing, how will I go about doing this? Because this is the… Usually, if you want to get the final answer, we have to really find out the coefficient of this kind of an expression – coefficent of x raise to n and this kind of expression. For instance, I can try like this. I will rewrite this as x raise to 2 minus x raise to 7 divided by 1 minus x square whole square into 1 minus x whole square.

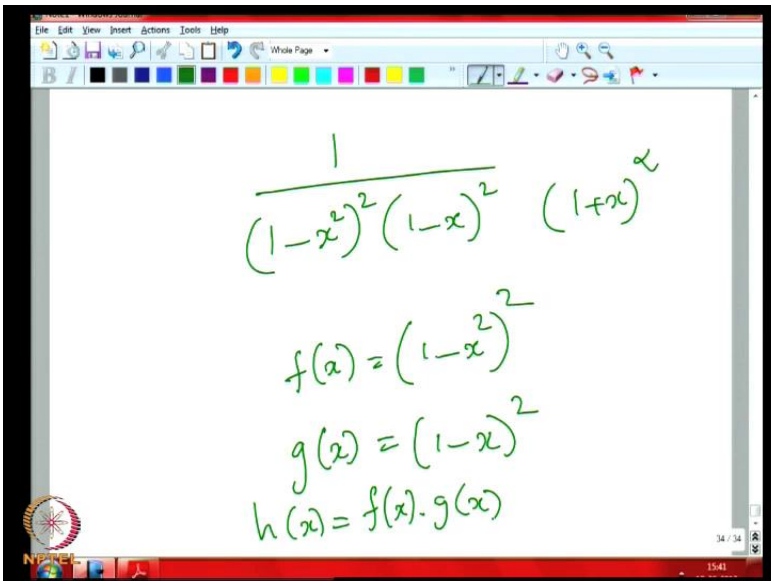
(Refer Slide Time: 55:37)



Now, this will be two different sequences: one is x square by 1 minus x square whole square into 1 minus x whole square and minus x raise to 7 by same thing – 1 minus x square whole square into 1 minus x whole square. Now, I can take… When I am interested in the coefficent of x raise to n, I can get the coefficent of x raise to n in the expansion of this and I can get the coefficent of x raise to n in the expansion of this portion; and then, take the difference – this coefficent minus this coefficent. So, I can concentrate on say one of them say, because if I can do this thing, I can do this also. So, here you see this is x square into something – x square into 1 by 1 minus x square whole square into 1 minus x whole square.

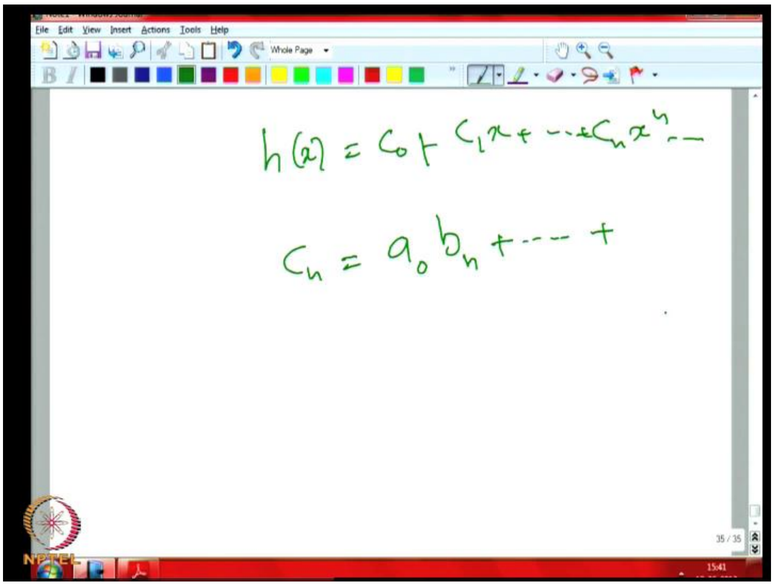
Suppose I find the coefficent of x raise to n minus 2 in the expansion of this. And, that will correspond to the coefficent of x raise to n in the expansion of this thing, because x square anyways is multiplying each term there. So, if you are looking for the power of x raise to n, we should rather look for the power of x raise to the n minus 2 in the remaining. I discard this and look for x raise to n minus 2 – the coefficent of x raise to n minus 2 in this thing. And, in this case, what will you do? You should look for the power of x raise to n minus 7, because x raise to 7 is multiplying every term. So, if we are looking for x raise to n, we should actually look for the coefficent of x raise to n minus 7 in the coefficent of this term. Discarding x raise to 7, the remaining is 1 by 1 minus x square whole square into 1 minus x whole square.

(Refer Slide Time: 57:30)



We will be interested in the power of some x raise to n dash. Either it is x raise to n minus 2 or x raise to n minus 7 or whatever; 1 minus x square this thing. So, here if this is… Suppose if I take f of x is equal to 1 minus x square whole square and g of x is equal to 1 minus x whole square; both of this thing, I know how to expand using the formula 1 plus x raise to alpha. Now, I am interested in the coefficent of x raise to n in h of x, which is f of x into g of x.

(Refer Slide Time: 58:36)



Now, we see that, if h of x is equal to f of x into g of x; and, say if h of x is equal to c 0 plus c 1 x plus say c n x raise to n; then, c n will be equal to a 0 into b n plus… So, I will discuss this… Because the time is over, I will discuss in the next class.