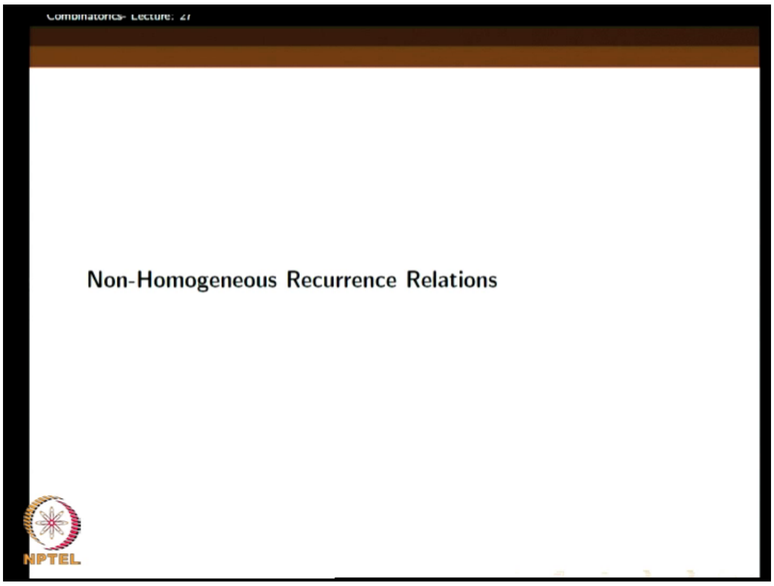
Combinatorics

Prof. Dr. L Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 27
Recurrence Relations – Part (5)

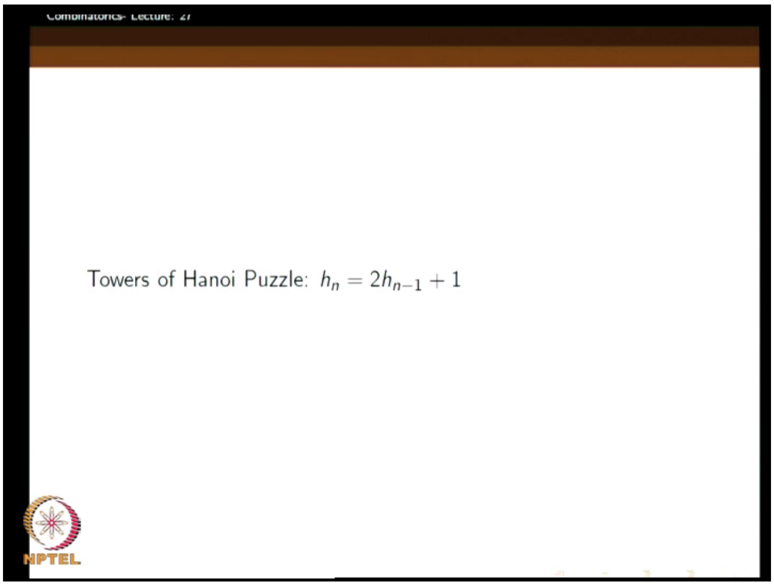
Welcome to the twenty seventh lecture of combinatronics.

(Refer Slide Time: 00:20)



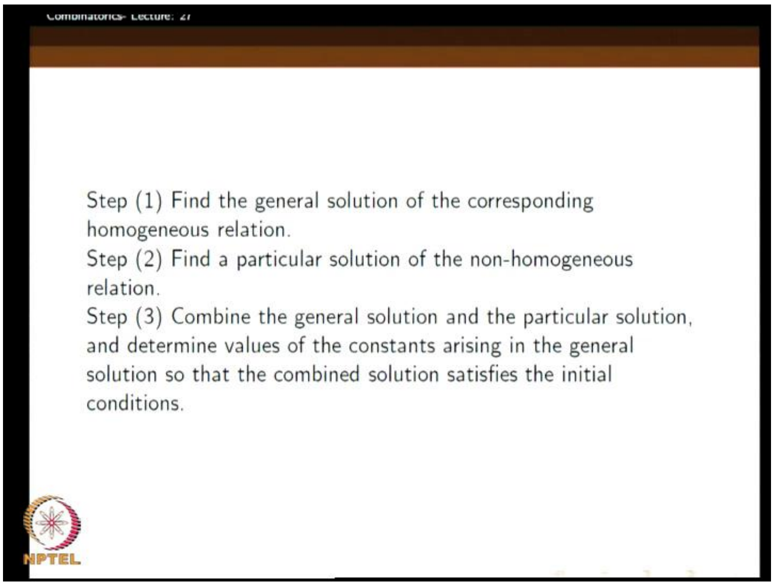
In this class, we will look at non-homogenous recurrence relations and propose some methods to solve it. It is not that these methods will work always, but this is the way one can try to solve it.

(Refer Slide Time: 00:42)



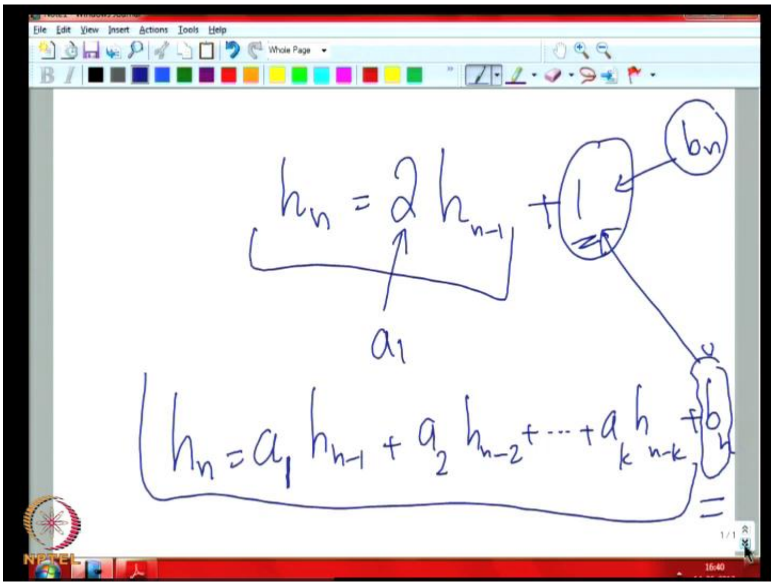
In the last class, we had discussed this tower of hanoi puzzle. And, we show that, we can formulate. So, the puzzle will give rise to a recurrence relation – h n equal to 2 into h n minus 1 plus 1. So, the question was how to solve this recurrence relation. We show how to unroll this recurrence relation; that means h n equal to 2 into h n minus 1; that is equal to two times – 2 of h n minus 2 plus 1. Like that we got a sequence of numbers, which we added in the end. And, it was like 1 plus 2 plus 2 square plus 2 cube plus like that. And we got in the end, the answer 2 to the power n minus 1. So, now, the intension here is to look at a different procedure, which probably can be called a little more general; that we can try to apply whenever we have a situation like this.

(Refer Slide Time: 01:59)



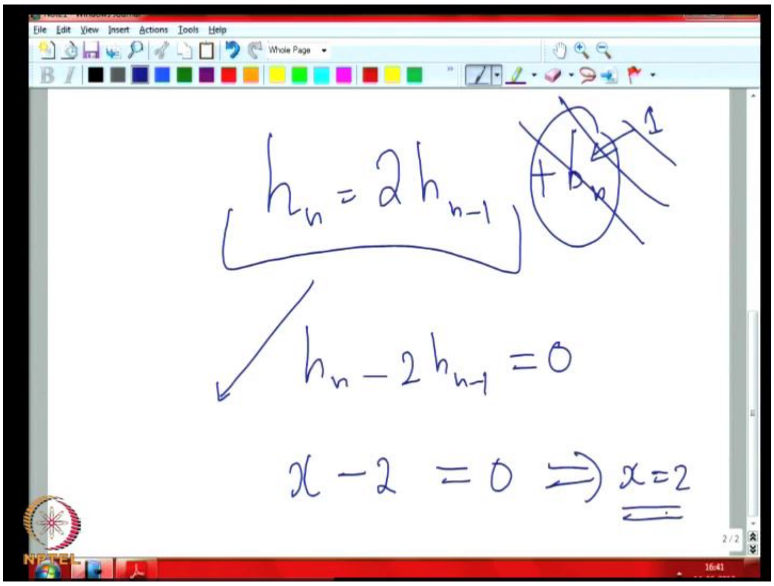
There are three steps. First step – we find the general solution of the corresponding homogenous relation. What do we mean by that?

(Refer Slide Time: 02:14)



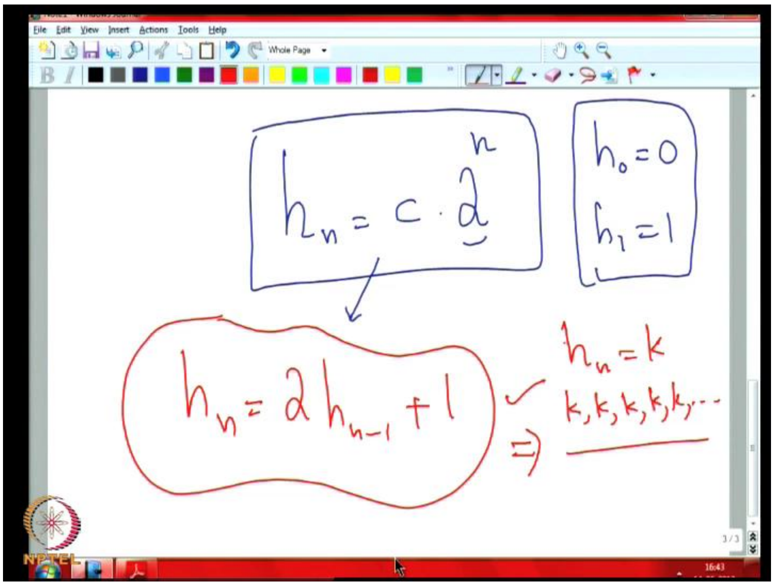
For instance, in this tower of hanoi case, we have h n equal 2 into h n minus 1 plus 1. So, what is the homogenous part here? We know that, the homogenous part is this, because this is that b n, remember – b n when we formulated the k-th order recurrence relation as h n equal to a 1 into h n minus 1 plus a 2 into h n minus 2 plus finally a k into h n minus k plus b n. This is what we wrote. And we told this when this b n is 0; then it is a homogenous recurrence relation of order k and linear off case – linear homogenous recurrence relation of order k; that is what it is. But then, suppose b n is not equal to 0; then, we say that it is non-homogenous. Here this is a non-homogenous recurrence relation, because here h n equal to 2 into h n minus 1; here a 1 is equal to 2 and h n minus 1. And, there is nothing more, because k is equal to 1 here. This is the first order recurrence relation; and then, this plus 1; plus 1 corresponds to this b n here.

(Refer Slide Time: 03:53)



Now, there are two parts to it. There is a homogenous part to it. The homogenous part is h n equal to 2 into h n minus 1; which means we just discarded the other part; this plus b n part plus b n; in which case, here it is 1; this – we just remove; and, this is called the homogenous part of the recurrence relation. This we can easily solve, because this is h n minus 2 h n minus 1 is equal to 0. And, the characteristic equation is x minus 2 is equal to 0. And, that implies x equal to 2, is the solution of the characteristic equation.

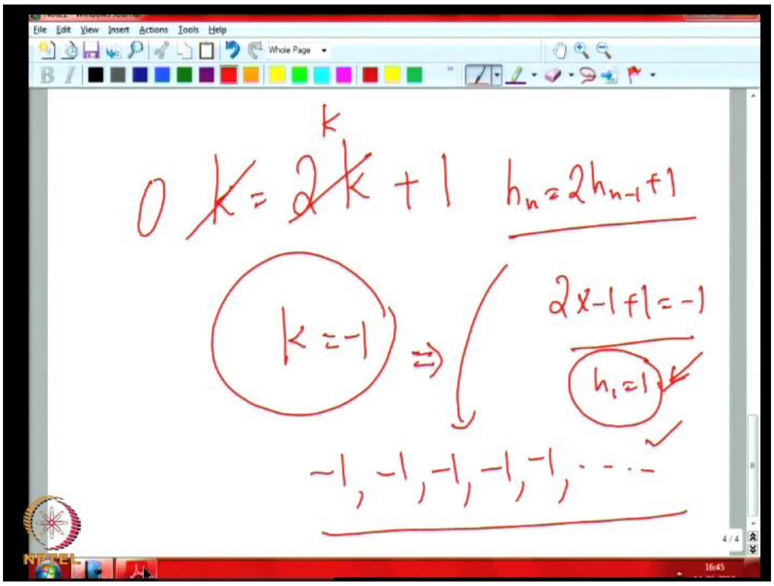
(Refer Slide Time: 04:36)



And then, the general solution for the homogenous part is h n equal to some constant c times 2 to the power n. Remember – 2 was the solution of the characteristic equation. This we remember. Now, what we do is remember – we have the initial conditions h 0 equal to 0 and h 1 equal to 1 for the tower of hanoi. There are zero d's; there is zero moves required. There is just one disc on the first peg; we just need one move. These are the initial conditions. We have to fit this thing. And, not only that; this only fits the homogenous part; the actual recurrence relation is not h n equal to 2 into h n minus 1, but rather h n equal to 2 h n minus 1 plus 1. So what we do is; we find the particular solution of this thing. We will go back to the steps (Refer Slide Time: 05:33) – the first part was find the general solution of the corresponding homogenous relation.

Then, we find a particular solution of the non-homogenous relation; non-homogenous relation means this full relation. What is this particular solution? What we do is; we will find the solution, which will work for some initial condition – some particular value of h 0. For instance, what I do is I will substitute. So, I will see whether if I take some constant, it will work or not. For instance, take a number say d or we can say a number k, let us say; k being a constant. So, some solution like h n equal to k is what I am saying; that means for some number k, can this sequence k, k, k, k, k – the sequence satisfies h n equal to 2 times h n minus 1 plus 1? This is the question.

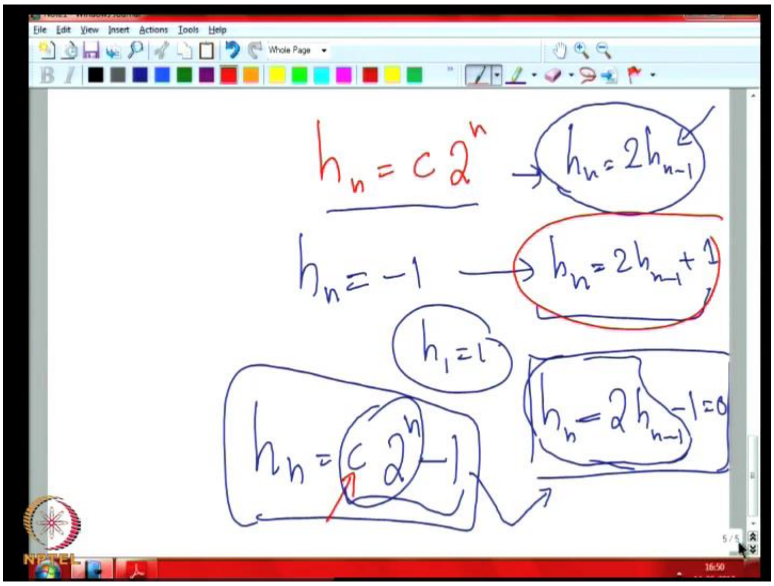
(Refer Slide Time: 06:51)



Let us try it. h n has to be equal to k; that is, 2 into h n minus 1. That is again has to be k, because h n minus 1 and h n – both are k. All of these h i's are k's – plus 1. So, is it possible to get some solution for this thing? This off case if we cancel k here, we get zero here. And, this becomes 1 k; that means k is equal to minus 1. So, if we take k is equal to minus 1, it will indeed be true. And, this sequence – consider a sequence minus 1, minus 1, minus 1, minus 1, minus 1, so on. So, it is true that, if you add two times minus 1 plus 1, that is indeed another minus 1. 2 into minus 1 plus 1 is minus 1. So, it is satisfying the recurrence relation h n equal to 2 times h n minus 1 plus 1. So, this is a particular solution. We could find this particular solution.

The good thing now is what, why did we find a particular solution like this? See this particular solution will not be a valid solution for us, why? Because this indeed satisfies the recurrence relation, but it does not satisfy the initial condition namely, h 0 equal to 0, h 1 equal to 1. These things are not satisfied. So, we need only one initial condition. So, h 0 is equal to… We can start with say h 1 is equal to 1 for instance. That is more clearer that, this is what we are talking about, because when one disc is there, we need one move, because these are all minus 1s. Then, h 1 is not equal to 1 here. So, this is not a correct solution for us. But, then we can use this particular solution (Refer Slide Time: 08:43) and the general solution to create a lot more solutions. How? Combine the general solution and the particular solution and (( )) and the values of the constants arising in the general solution, so that the combined solution satisfies the initial conditions. This is what we should do. This is the third step.

(Refer Slide Time: 09:02)

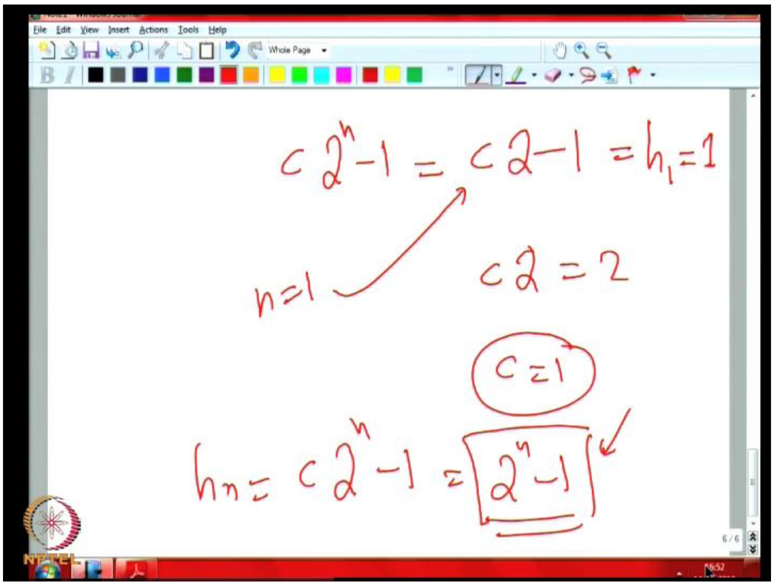


Again, I am saying, if h n equal to c times 2 raised to n will satisfy what? This will satisfy this relation – h n equal to 2 times h n minus 1. Now, this sequence – minus 1 – h n equal to minus 1 will satisfy this entire relation h n equal to 2 into h n minus 1 plus 1. So, what is not satisfied is the initial condition; that means h 1 equal to 1. This is what is not satisfied. So, to satisfy this also, what we should do is; we should create another solution; namely, we will add up these two. For instance, h n equal to c times 2 raise to n plus 1 – will this be a valid solution for the recurrence solution? It is indeed, because if you substitute see h n equal to c into 2 raise to n minus 1 – this h n minus 1. So, what I did is; I added these two things together. Then, what happens is; if we consider this part h n minus 2 h n minus 1 minus 1 equal to 0; this is the way we write the recurrence relation.

And, if we substitute this to here; c 2 raise to n minus 2 times c 2 raise to n minus 1 – that will become 0, because this is homogenous part of the recurrence relation; that means h n equal to 2 h n minus 1 or h n minus 2 times h n minus 1 is (( )) equal to 0 is satisfied by c times 2 raise to n for any c. So, it does not have any effect; that is what. This part does not have any effect on it; it simply disappears from this itself. But, the rest is minus 1. If you put minus 1 for h n and here also minus 1, we will get minus 1, minus

2, another minus 1. So, what happens is, this minus 1 and minus 1; that is, it will become minus 2; here minus 2 times minus 1 will become plus 2. So, that is, indeed becomes x 0. That is clear, because when I put h n equal to minus 1, this entire thing is anyways satisfied. The extra c times 2 raise to n has no effect on this, because h n minus 2 h n minus 1 itself will nullify the effect of them, because if we substitute c 2 raise to n here and 2 into c 2 raise to n minus 1 here; that becomes 0. That is what the meaning of this; h n equal to c times 2 raise to n is the solution of h n equal to 2 into h n minus 1. So, we understand that, since h n equal to c times 2 raise to n is a solution of the homogenous part – that h n equal to 2 into h n minus 1; and, h n equal minus 1 is a particular solution for this one – h n equal to 2 times h n minus 1 plus 1. So, indeed h n equal to c times 2 raise to n minus 1 is a solution for this recurrence relation – this part.

(Refer Slide Time: 12:57)

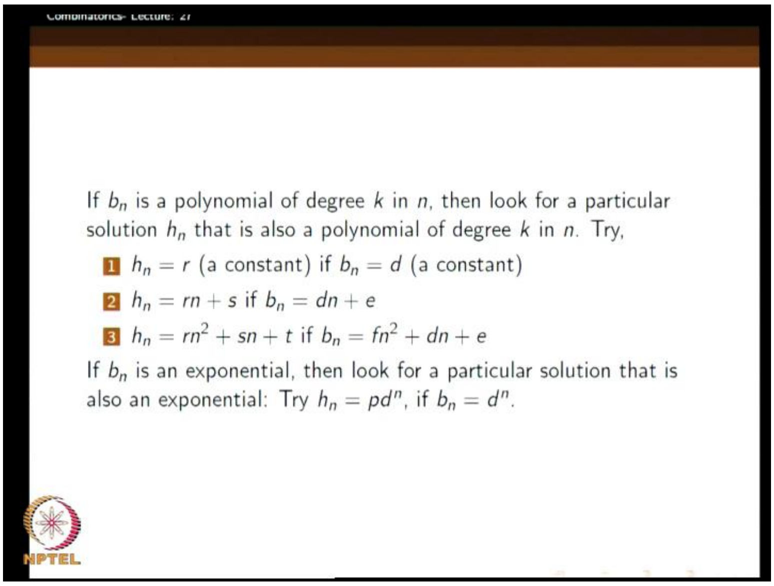


Now, we have a c there; that is a constant here; which we can decide the way we want; that means c times 2 raise to n minus 1. Suppose if you put n equal to 1; that is… Then, what will happen? This will become c times 2 minus 1. But, when n equal to 1, we need the answer to be h 1 equal to 1. So, what is this? c times 2 equal to 2; that means c equal to 1. c equal to 1 will satisfy our initial condition; which means c times 2 raise to n minus 1 is a solution is what we told. Put c equal to 1; that means 2 raise to n minus 1 is indeed the solution for the recurrence relation, which satisfy the initial condition. So, this is exactly what we got in the other (( ))

We elaborately described it to explain the methodology. There are three steps in it. First, we consider only the homogenous part and solve the homogenous part first by the techniques we have already learnt for solving the homogenous linear recurrence relation with the constant coefficients. Then, we get a particular solution for the entire non-homogenous recurrence relation. It will just work. It will work for some initial values, but not for every possible initial values. In particular, the initial values given to the problem may not be satisfied by this particular solution.

Now, what we do is; we combine the general solution for the homogenous part plus with a particular solution for the non-homogenous recurrence relation we have. And, that is indeed a solution for the non-homogenous recurrence solution given to us. But, then the good thing is there is a constant available there, which we can manipulate; which we can fine tune to meet our requirement. What is the requirement? We have to meet the initial condition. We substitute n equal to whatever; maybe 0; whatever. And then, we get an equation involving c and we solve for it; that is what. We will take yet another example and see what happens.

(Refer Slide Time: 15:24)

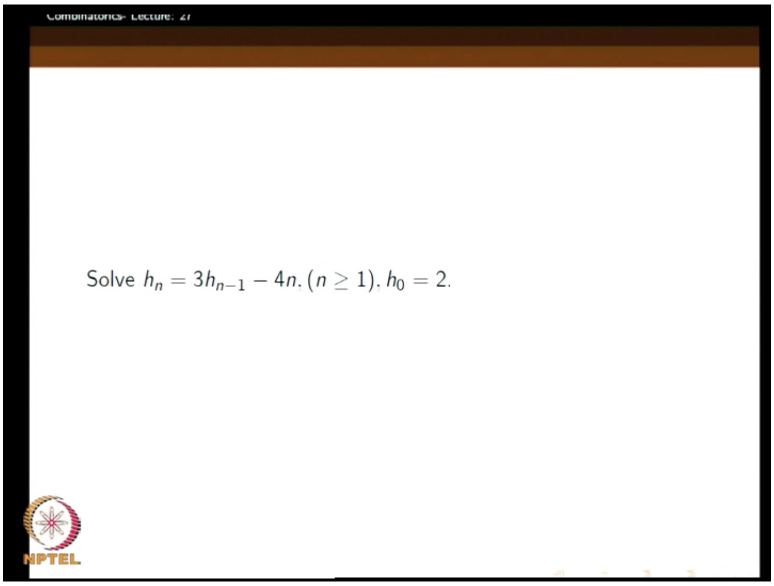


If b n is a… Before getting further, we will just mention that, here off case, there are two parts. One is first find the general solution for the homogenous part. There you have to solve the characteristic equation off case. We can easily find the characteristic equation. The difficult part is to get the solutions for the characteristic equations; which may not be so easy, because we do not have techniques to solve any polynomial equations. For instance, if it is a higher degree polynomial, we may not be able to do it so easily. So, that is the difficulty in it in using this method. Somehow we can find the roots of the polynomial. So, we can get by the earlier methods some particular solution for this thing. We have given general solution for this – general solutions in the case of all distinct roots and also when the roots are repeating, we have given solutions for this thing.

But, the next part is to find the particular solution. Particular solution – how will you go about finding the particular solution? In the tower of hanoi case, what we did is; we just substituted; we saw whether a constant would work or not; will a sequence like k, k, k, k – I mean unchanging values – will it fit the given recurrence. Definitely, it need not fit in all cases. So, here we need… Sometimes we may need some different kind of substitutions also. Here there is a list – what kind of particular solutions we may try. For instance, if the b n is indeed a constant; like now, previous example, it was 1 there. Then, we can try a constant like this if it may work.

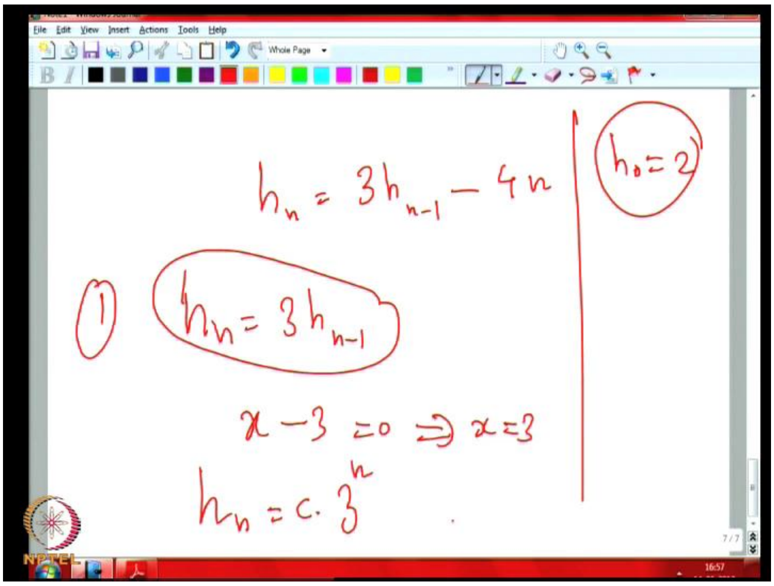
If b n is of the form dn is plus e – some linear in n; that d times n plus e; then, we can use h n equal to r n plus s. See not that always if we substitute this thing, this will always work; but, this is the thing to try. Then, similarly, if b n is equal to f times n square plus d n plus e; then, we can try h n equal to r times n square plus s n plus t; where r, s, t, etcetera are some constants; f, d, e – all of them are constants here. If b is an exponential; that means like b equal to d raise to n for some constant d; then, what we do is try h n equal to p times d raise to n for some other constant p. So, these are some guidelines for finding the particular solution. Now, you see once you get the particular solution, then see what happens is; you can sum up the general solution for the homogenous part and the particular solution and then solve for the constant, so that the initial conditions are met; that is what we should do.

(Refer Slide Time: 18:53)



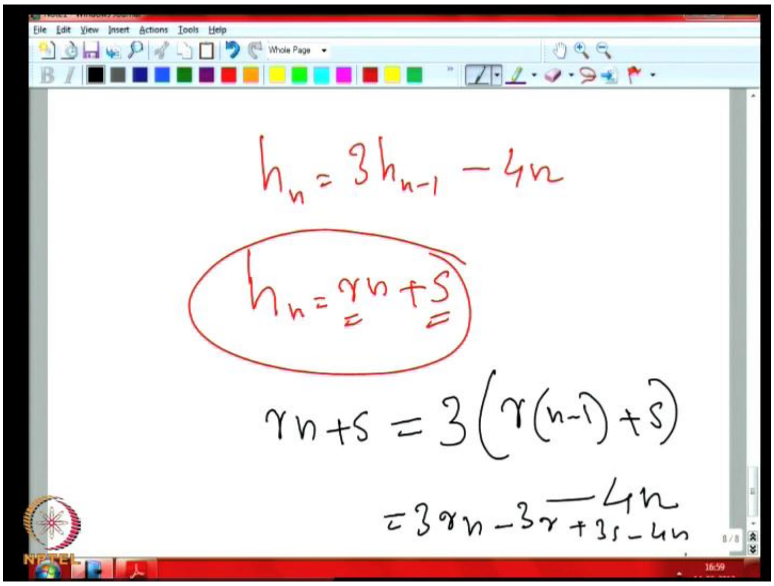
Now, we will get some experience by looking at some problems. For instance, look at this problem – h n equal to 3 times h n minus 1 minus 4n for n greater than equal to 1; and, for h 0, you have 2. Initial condition is h 0 is equal to 2. h n equal to 3 times h n minus 1 minus 4n. Here you can see that it is a non-homogenous recurrence relation, which is linear and of order 1. And, non-homogenous, because b n is not 0 here; b n equal to minus 4 and here.

(Refer Slide Time: 19:31)



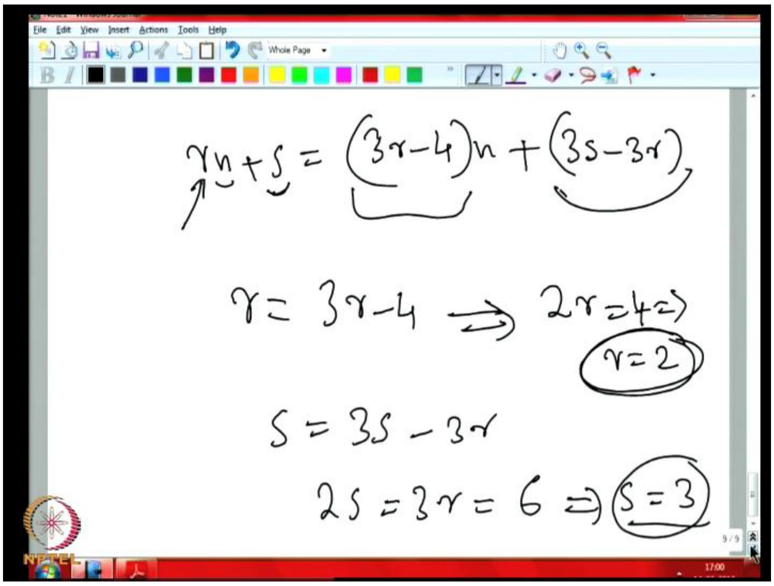
Now, this is the recurrence solution – h n is equal to 3 times h n minus 1 minus 4 n. We have h 0 equal to 2 for our initial condition. The first step is find the general solution of the homogenous part. Homogenous part is h n equal to 3 times h n minus 1. Clearly, the characteristic equation here is x minus 3 equal to 0; which means x equal to 3. So, general solution for this one is h n equal to some constant c times 3 to the power n. For this thing, you can substitute c times 3 n; whichever is the constant – c times 3 n; it will work; you can see that – c times 3 n minus you put here; c times 3 n is coming. And, that is exactly the value for (( ))

(Refer Slide Time: 20:31)



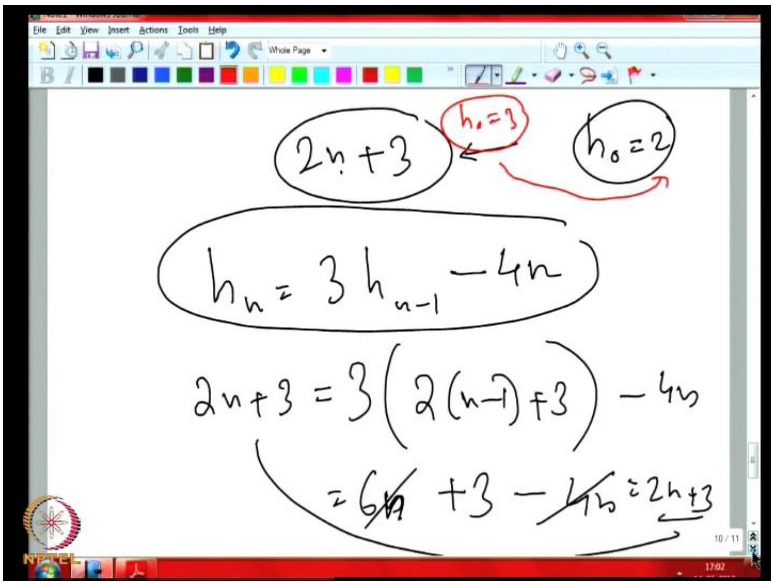
The next step is to find a particular solution for this one – 3 into h n minus 1 minus 4 n. Now, we are saying that, this is of the second type. Second type means we will go for the chart here (Refer Slide Time: 20:48) d n plus e. Then, rn plus s is what we should try. If this is minus 4n plus 0; this is of this form – second category. So, we will try h n equal to rn plus s for the further particular solution. This is what we will try – h n equal to rn plus s. Here this r and s – both are constants. See whether will work or not; that means you substitute for h n. So, what we get is rn plus s equal to 3 times r into – here n minus 1 – plus s minus 4 into n. Now, we have to simplify. So, this is 3 rn or we get here 3 times rn minus 3r plus 3s minus 4n here.

(Refer Slide Time: 22:07)



Now, we can collect the terms involving n together; that is, 3r minus 4 – this is equal to rn plus s is equal to 3r minus 4 into n plus 3s minus 3r. This is what we say. Now, we can compare the coefficients of n. If we have the coefficient of n as r here; r has to equal to the coefficient of n here namely, 3r minus 4; which would imply that, 2r is equal to 4 is equal to… which will imply that r equal to 2. Now, we can compare the constant terms. Here it is s; s equal to 3s minus 3r is the constant term here. So, that would be 2s equal to 3r; that is, 3 into 2 – 6. What does it mean? s equal to 3. We get r equal to 2 and s equal to 3.

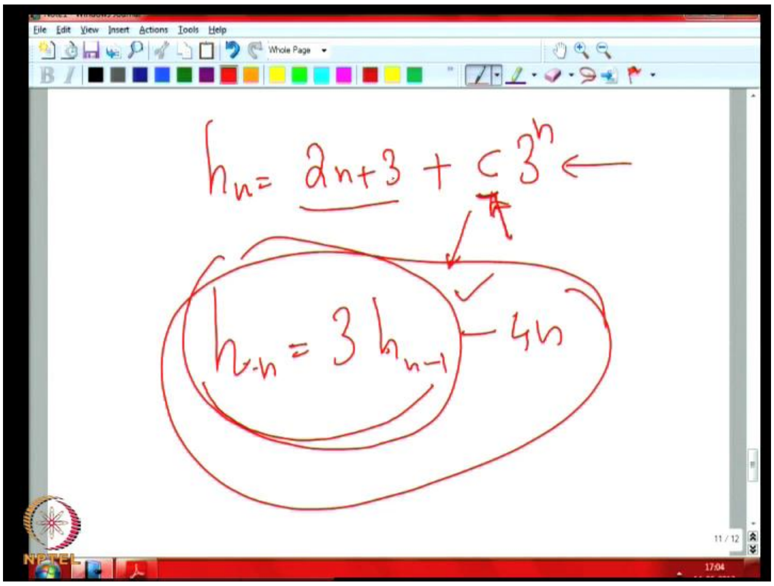
(Refer Slide Time: 23:26)



So, our particular solution is 2n plus 3. So, we can try it out. See if we substitute 2n plus 3 in the recurrence relation h n equal to 3 times h n minus 1 minus 4n. It should indeed work, is what it says. It will work. If we want to try it, we can just check it – 2n plus 2 equal to – it will become 3 into 2 into n minus 1 plus 3 minus 4n. So, here we have 6n; here we have minus 6 plus 9; that is, plus 3 minus 4n; which will make it 2n plus 3s we want. This and this are same. I just verified this particular solution is indeed one.

But, only one problem here – if 2n plus 1 is the solution, this will not meet the initial condition. For instance, we wanted h 0 to 2. This is what we mentioned before – h 0 equal to 2. h 0 equal to 2 is what we wanted. But, in this case, if we put n equal to 0, what will happen is 2 into 0 plus 3; that will be 3; h 0 will 3 here. But, we want h 0 equal to 2. What we will do? This particular solution is indeed valid for the recurrence relation, but it is not satisfying the initial condition. And, we do not have any thing to tune, because there is no constant, which we can select according to our requirement.

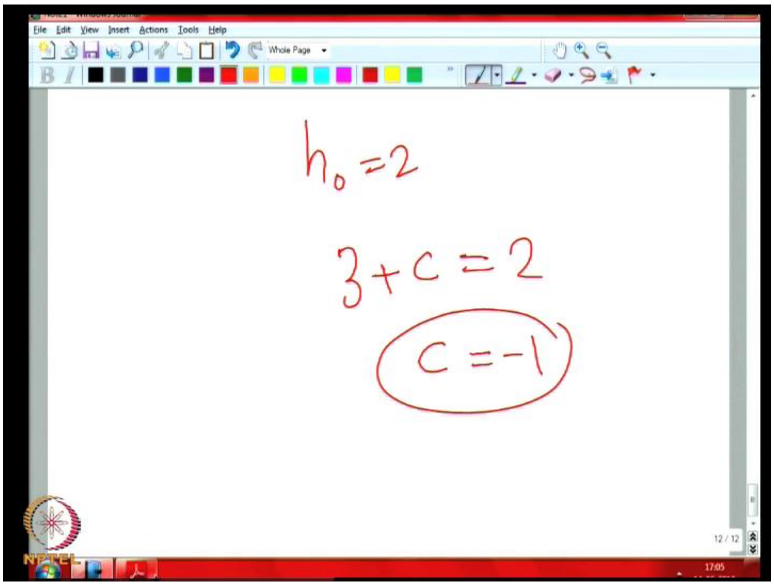
(Refer Slide Time: 25:33)



So, what we do is, we will combine the particular solution – 2n plus 3 with a general solution of the homogenous part; which was c times 3n. This is indeed going to be a solution for your… If I take h n is equal to this; this will indeed satisfy h n equal to 3 into h n minus 1 minus 4n. Why? Because you know this part is a solution for this thing. And, when I substitute this, the contribution due to this will be cancelled here. So, here and here the same contribution. And now, we only have to worry about, whether this will balance both sides when I give this thing here and this thing is for instance, 2n plus 3 part is given to n equal to n minus 1 here. Substitute; will it satisfy it or not? That is all we have to worry. And, we have already seen that, this solution h n equal to 2n plus 3 will satisfy this thing. Contribution from this thing is anyway balanced due to the fact that, this was the solution for this h n equal to 3 into h n minus 1 – the homogenous part.

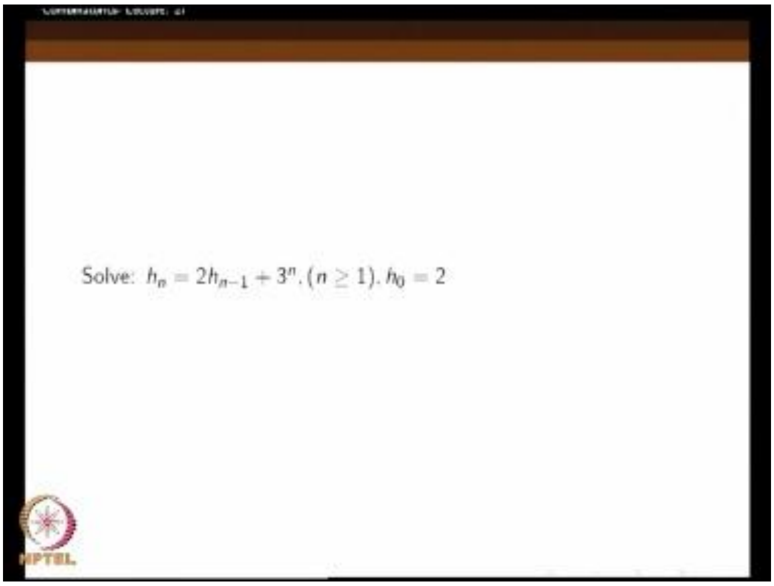
Therefore, this is indeed a solution for this thing. What is good about this new solution? The good thing about this new solution is that, there is a c here. And, this c we can choose; it is irrespective of whichever c we put. This relation will be satisfied, because that does not depend on this c. But, on the other hand that, tuning this c, fixing the c at a correct value will help us to get our initial conditions satisfied.

(Refer Slide Time: 27:18)



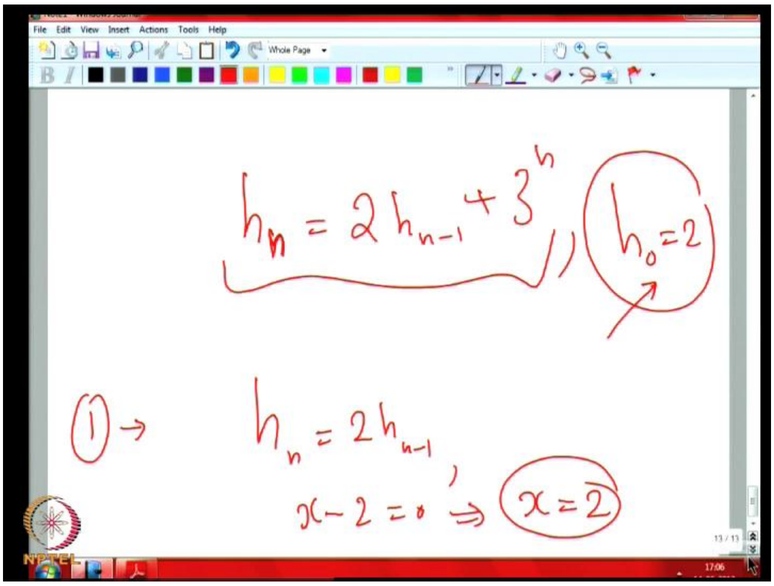
What is the initial condition? h 0 equal to 2. Now, you try putting (Refer Slide Time: 27:22) n equal to 0; we get… This will go away. We get 3 plus c; 3 plus c equal to… that means 3 plus c has to be 2 is what we say. So, we can decide c to be minus 1. If you decide c to be minus 1, it will work; that means here we have to put minus 2n plus 3 minus 3n; that is all. So, we could find the value of c, so that this not only satisfy the recurrence relation; but, it also satisfy the initial condition. h 0 equal to 2 here. So, that is is the idea. So, we can recall that, how we have gone through three steps and then how we got the solution (Refer Slide Time: 28:18) in the end by solving for the constants.

(Refer Slide Time: 28:23)



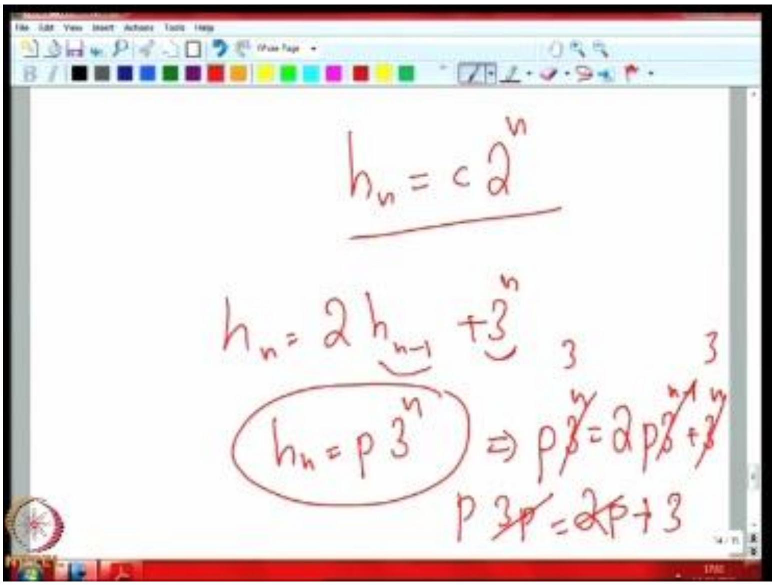
Now, we will try one more example that, h n equal to 2 into h n minus 1 plus 3 raise to n. This is a slightly different example to show that, this may not always easily work.

(Refer Slide Time: 28:40)



h n equal to 2 into h n minus 1 plus – what is that? 3 raise to n. Let us see what this is. n (Refer Slide Time: 28:56) greater than equal to 1 (( )) h 0 equal to 2. We have h 0 equal to 2. This is the initial condition, because we can only… This is the first order relation. So, we have this reccurence relation working starting from n equal to 1 onwards; n equal to 0 – we have to give the initial value right. Now, to solve this thing, the first step – as usual, separate out the homogenous part – h n equal to 2 into h n minus 1; and, its characteristic equation – that is, x minus 2 equal to 0. The solution of the characteristic equation is x equal to 2.

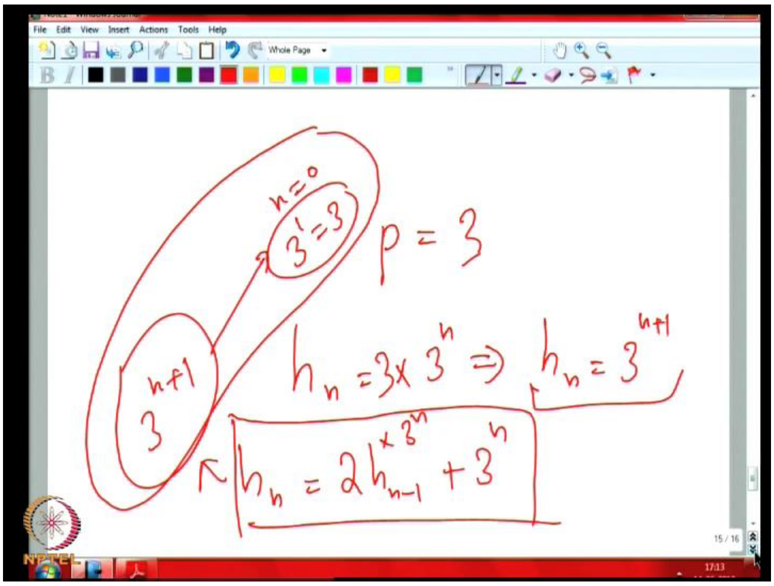
(Refer Slide Time: 29:55)



Now, if the solution of the characteristic equation is x equal to 2; then, we know that the general solution is h n equal to c into 2 raise to n. This is for the homogenous part. Now, we have to find a particular solution for the entire non-homogenous equation namely, h n equal to 2 into h n minus 1 plus 3 raise to n. How will you find the particular solution and for this thing? We look at the chart; when we look at the chart, we see (Refer Slide Time: 30:28) that, if b n is an exponental – like b n is equal to d to the power n form; then, we can try h n equal to something like p into d raise to n. Here 3 raise to n.

So, we will try h n equal to some p times 3 raise to n. This is what we will try; that means we subsitute it – p into 3 raise to n equal to 2 into p into 3 raise to n minus 1 plus 3 raise to n. Now, we can cancel off this 3 raise to n minus 1 from all here; we get 3 here, 3 here; that means 3 p equal to 6 p plus 1. This is what we get. So, we put h n equal to p into 3 raise to n. So, 2 into p into 3 raise to n minus 1 we put, because h n minus 1 only we consider; and, this is 3 raise to n. Off case, we can cancel off 3 raise to n from everywhere. But, (( )) wheter I am (Refer Slide Time: 31:58) (( )) this 2 only. So, we get…

(Refer Slide Time: 32:07)



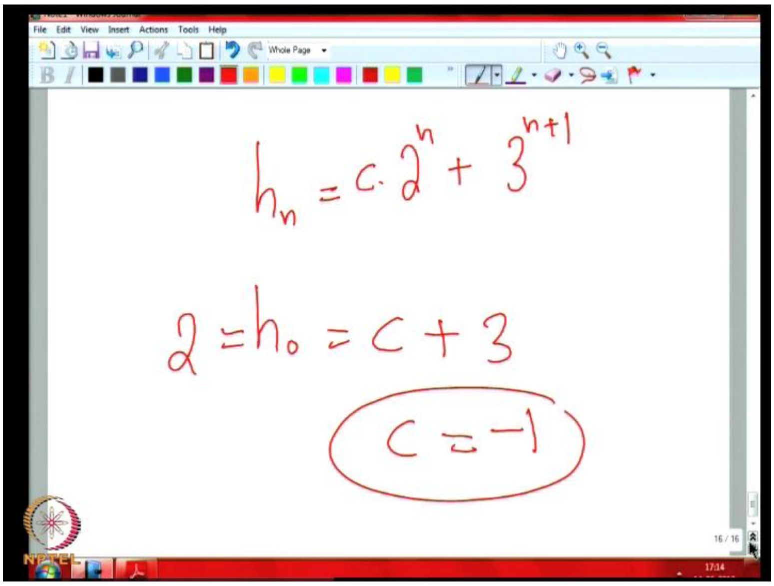
From this thing, what do we get? 3 p plus 1 equal to 0 is what we are getting. So, we get p equal to minus 1 by 3. Now, off case this (Refer Slide Time: 32:19) will… minus 1 by 3; what will be h n then? h n equal to p times 3 raise to n; that is, minus 1 by 3 into 3 raise to n; that is, minus 3 raise to n minus 1. This will be the particular solution. If you want to check whether it is working; it is minus 3 raise to n minus 2 here. And then, minus 3 raise to n minus 1. This is 2 times 3 raise to n minus 1. This is 3 times 3 raise to n minus 1; this is, 1 times 3 raise to n minus 1 minus will stay here. And then (( )) We wanted to find a particular solution; I put h n equal to 2 times h n minus 1 plus 3 raise to n.

So, we get p times 3 raise to n equal to 2 times p into 3 raise to n minus 1 plus 3 raise to n. Now… This is not 1; this is 3 – plus 3 (Refer Slide Time: 34:06) because here I cancel 3 raise to n minus 1; here 3 is remaining; here 2 p is remaining; this is 2 p – 2 p is remaining. And then, here it is 3 p. So, we can see p equal to 3. So, this was wrong. p equal to 3 is what is coming. So, p equal to 3. So, this will be h n equal to 3 raise to n plus 1.

Now, we will check whether it is working out on this thing that, h n equal to 2 times h n minus 1 plus 3n, because this 3 raise to n plus 1. When I substitiute h n equal to 3 raise to n plus 1, h n minus 1 will be 3 raise to n – 2 into 3 raise to n plus 3 raise to n is 3 into 3 raise to n; that is, 3 raise to n plus 1. So, its working out. So, that is indeed a particular solution for the given non-homogenous recurrence relation – this one. Now, we will combine this particular solution. See again not that this particular solution – it is working fine with the recurrence relation, but the initial condition will not be met, because the initial condition is again h 0 is equal to 2.

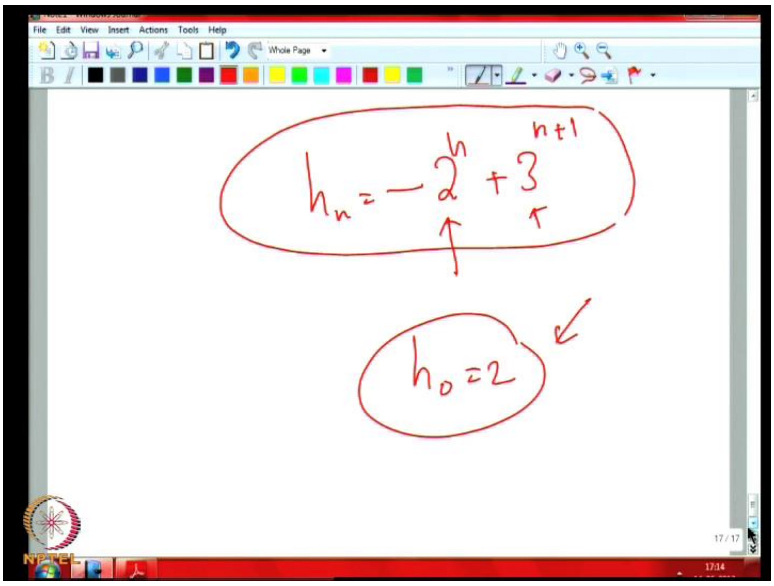
So, that is not met here, because we see 3 raise to n minus 1. If we put 0; that 3 raise to n plus 1 is… Put n equal to 0. That will only give 3 raise to 1 equal to 3; that is not 2. For n equal to 0, we want h n to be 2; h 0 is equal 2. We are given that. But, if we take this particular solution alone, it is not true; it is actually 3. So, how will we find a solution, which will satisfy our initial condition also?

(Refer Slide Time: 36:20)



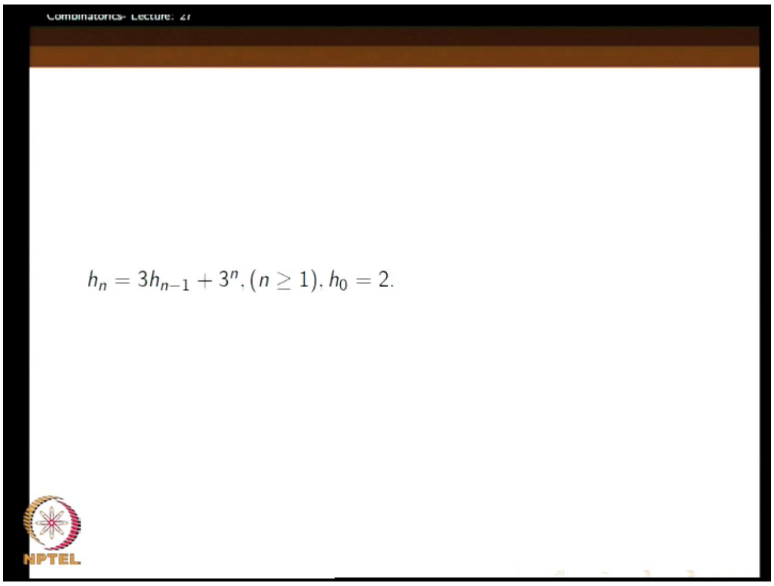
What we do is we combine the general solution for the homogenous part namely, c times 3 raise to n – c (Refer Slide Time: 36:31) times 2 raise to n. This is the homogenous part of solution. c times 2 raise to n and our particular solution – 3 raise to n plus 1. And, try putting n equal to 0. You get h 0 equal to 2. So, putting n equal to 0, we get c plus 3 equal to 2; that is, c equal to minus 1.

(Refer Slide Time: 37:04)



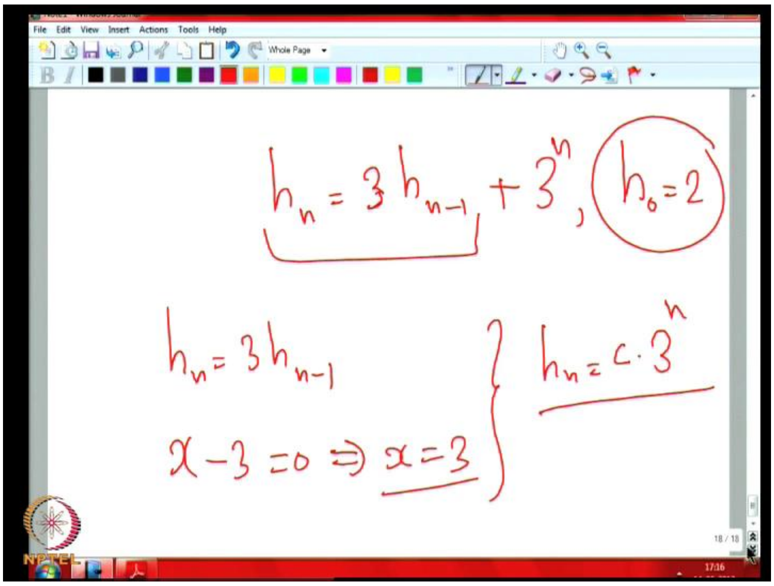
That means our solution will become h n equal to minus 2 raise to n plus 3 raise to n plus 1. So, indeed this will satify the recurrence relation. Now, if you put n equal to 0, what you get is here 3. This is 3 and this is minus 1 (( )) 3 minus 1 is equal to 2. As we want, h 0 is indeed 2 here. So, it is satisfying the initial condition also. This was our last…

(Refer Slide Time: 37:43)



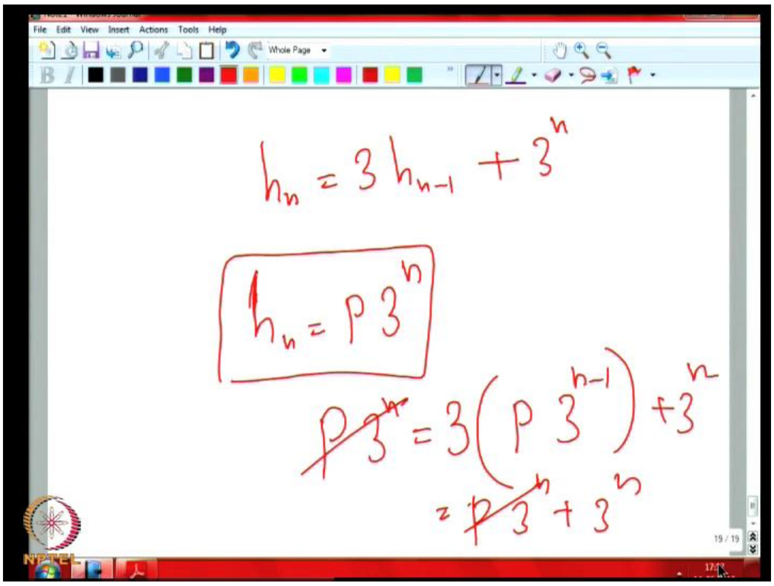
Now, we will look at the last example namely… So, it is a slight modification – h n equal to 2 into h n minus 1 plus 3 raise to n is what we looked now.

(Refer Slide Time; 38:01)



We will look now h n equal to 3 times h n minus 1 plus 3 raise to n. h n equal to 3 times h n minus 1 plus 3 raise to n. So, this is only valid for n greater than and equal to 1. So, we need to give the initial condition h 0. That is again 2. Now, particular solution… First, we take the homogenous part; that means h n equal to 3 into h n minus 1. We write the characteristic equation x minus 3 equal to 0. Solve it; that means x equal to 3 here. Now, we infer that, the general solution for this thing is h n equal to c times 3 raise to n for any constant c. This is (( )) solution.

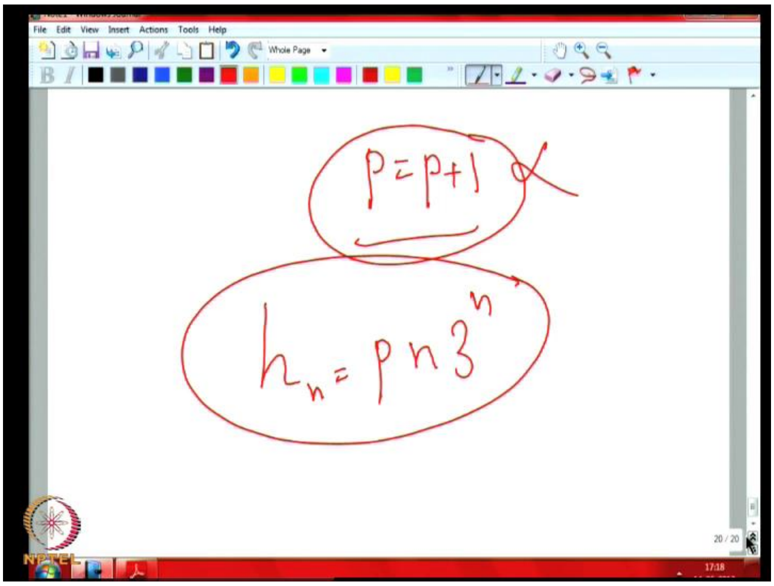
(Refer Slide Time: 38:52)



Now, we look for a particular solution for this h n equal to 3 into h n minus 1 plus 3 raise to n. What we can do, we go back to the chart. If you recall the chart; chart would say because its (Refer Slide Time: 39:07) an exponential b n equal to d raise to n. Then, try p times d raise to n. That is what it will say. So, h n is equal to p times 3 raise to n. That is what it proposes to try.

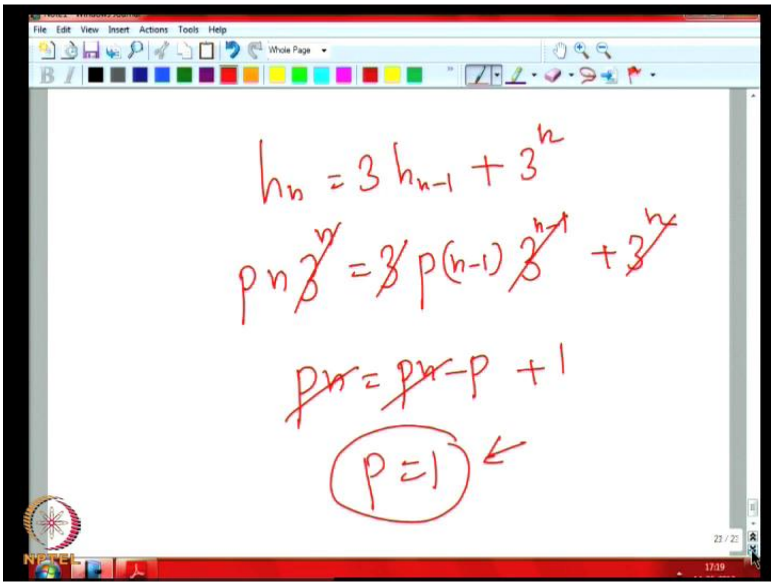
But, we have tried it earlier for h n equal to 2 into h n minus 1 plus 3 raise to n twice. Let us try it again. So, this is p times 3 raise to n is equal to 3 times p into 3 raise to n minus 1 plus 3 raise to n. Now, this is what. This is again… This is p into 3 raise to n plus 3 raise to n. Cancelling this, we get 3 raise to n equal to 0. This is what; which is not true. When you cancel three raise to n everywhere, p equal to p plus 1 is what we are getting.

(Refer Slide Time: 40:15)



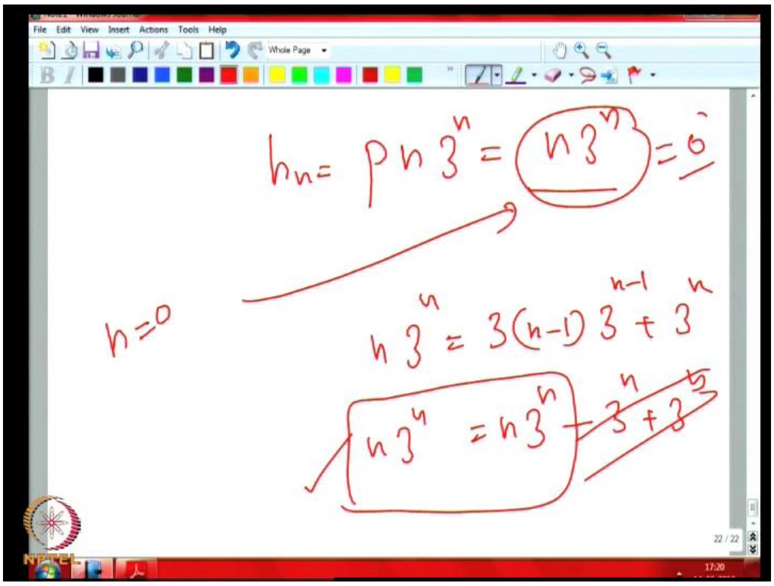
p equal to p plus 1 is what we are getting; which is absurd. So, we cannot get such a value of p; that means (Refer Slide Time: 40:24) h n equal to p times 3 raise to n. If you take h n equal to p times 3 raise to n, there is not going to be a particular solution for this non-homogenous recurrence relation of that form. So, what we do which… We then change the pattern of this particular solution a little bit. We will try h n equal to p times n into 3 raise to n. This is a slight change we give and then we again try that.

(Refer Slide Time: 40:57)



Now, h n equal to 3 times h n minus 1 plus 3 raise to n will become p into n into 3 raise to n equal to 3 into p into n minus 1 into 3 raise to n minus 1 plus 3 raise to n. Now, we can cancel off all the three raise to n's here; that means p n equal to p n minus p plus 1. This is what we get. So, p n also goes away. So, we get p equal to 1. So, we get a solution for p now.

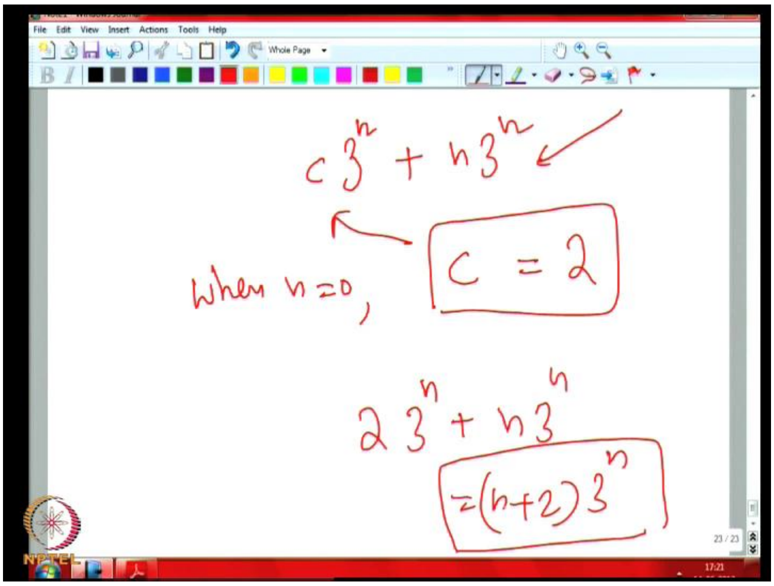
(Refer Slide Time: 41:54)



We get a particular solution h n equal to p times n into 3 raise to n; that is, p being 1 as n into 3 raise to n. This is the solution. We can try it out here (Refer Slide Time: 42:07).

We will see that it is correct, because this is n into 3 raise to n. This is 3 into n minus 1 into 3 raise to n minus 1; n into 3 raise to n equal to 3 into n minus 1 into 3 raise to n minus 1 plus 3 raise to n. So, this is n into 3 raise to n here minus 3 raise to n plus 3 raise to n. So, here n equal to 3 raise to n. This is true. This is indeed correct. So, n into 3 raise to n indeed a particular soultion. Again as we were always seeing, if you try whether the initial condition is met; it is not met, because if you put n equal to 0, this will become 0. So, n into 3 raise to n is indeed 0. But, we want h 0 equal to 2. So, how will it happen now?

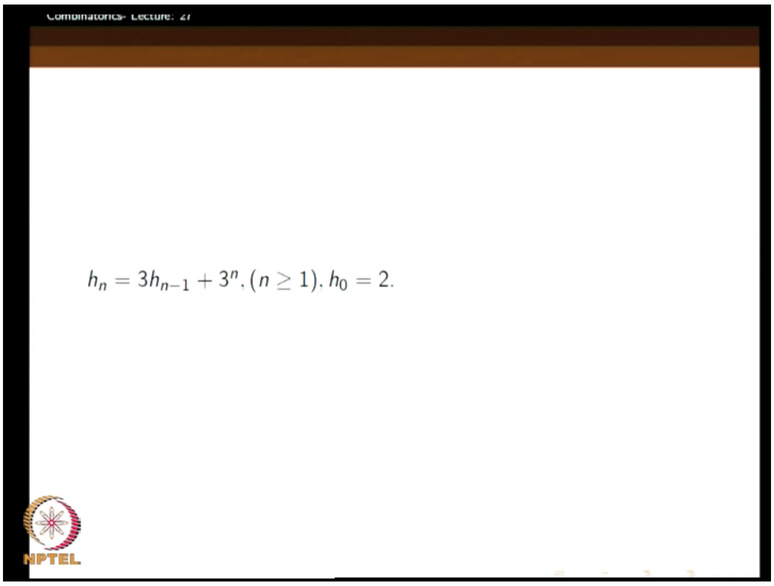
(Refer Slide Time: 43:21)



What we do is we combine the general solution equation c into 3 raise to n. This general solution was for the homogenous part alone – h n equal to 3 times h n minus 1. This is combined with the particular solution that we obtained for the non-homogenous recurrence relation – the full recurrence relation namely, n into 3 raise to n. Now, we want… When… We will check when n equal to 0, what happens. When n equal to 0, this will be c; this will be simply be c, because this will go away; this is c. And, this has to be equal to 2.

Therefore, we select c equal to 2 and substitute here. So, 2 into 3 raise to n plus n into 3 raise to n; that is, n plus 2 into 3 raise to n is indeed a solution for the given recurrence relation, which satisfies the initial condition also. That is what we get, because if we put n equal to 0, we will get 2 here; indeed by our construction of the solution we know that satisfies the recurrence relation.

(Refer Slide Time: 44:45)

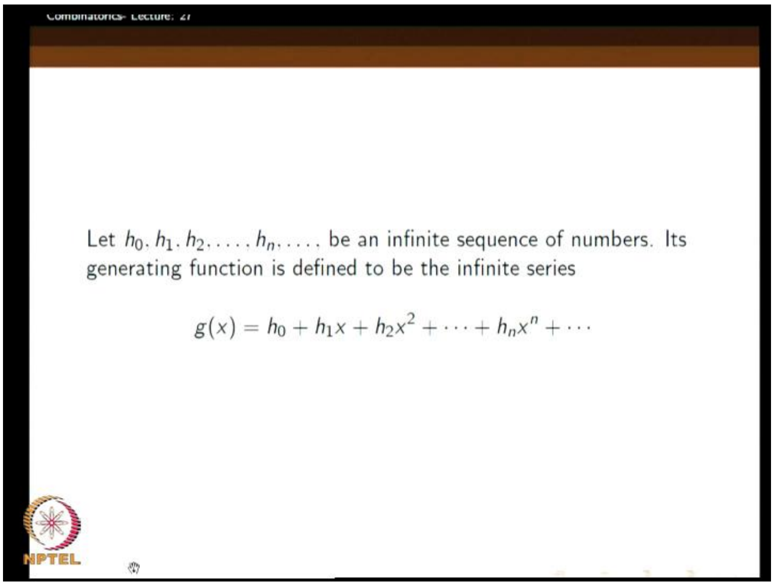


Now, we are going back to the… This is our last example. Off case, this whatever we have done are some examples. But, it illustrates say general strategy to attack this kind of kind of problems. But, success is not guaranteed always. Sometimes it may feel; in which case, you may have to do some slight adjustments and go forward or sometimes it may not workout. So, I have taken some examples to illustrate all those points.

(Refer Slide Time: 45:22)

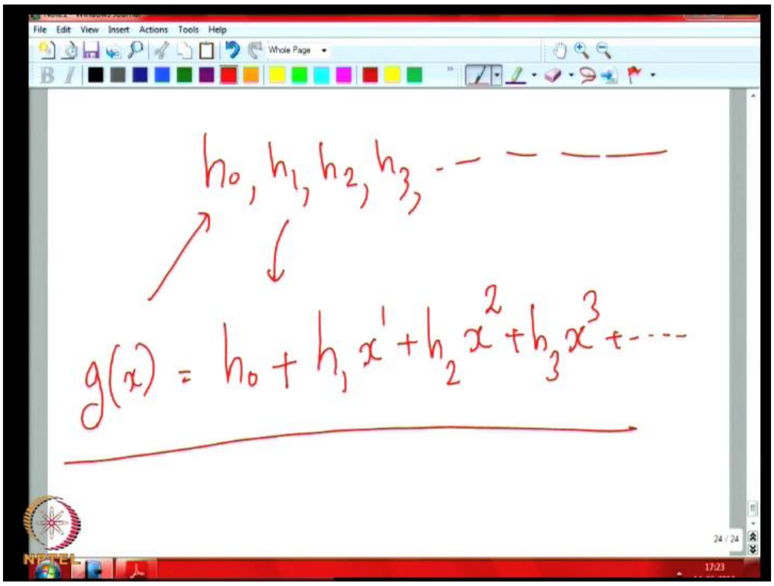
Now, what we do is we will look at the next topic namely, generating functions.

(Refer Slide Time: 45:29)



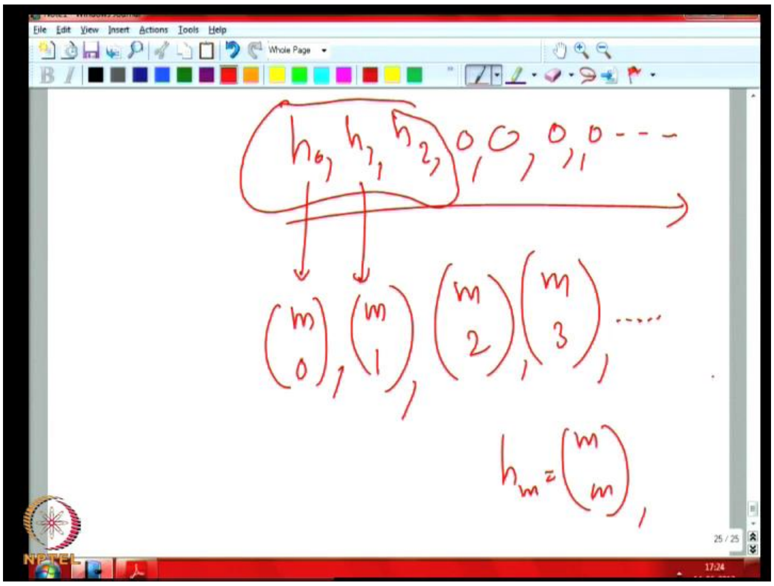
What are these generating functions?

(Refer Slide Time: 45:34)



Consider a sequence h 0, h 1, h 2, h 3 like this. Now, the generating function corresponding to this sequence is given by this power series h 0 plus h 1 raise to x 1 plus h 2 into x raise to 2 plus h 3 into x cube plus so on. This is the power series. Let us say you write it as g of x. This g of x is the generating function of this sequence h n.

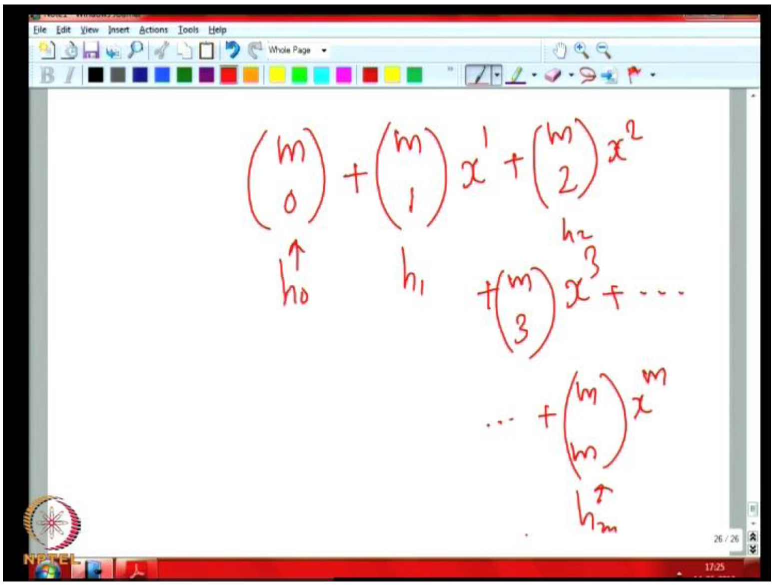
(Refer Slide Time: 46:31)



Now, it is not always necessary that we should have all these h i's to be an infinite sequence. Suppose in some cases it may be just h 0, h 1, h 2. And, after sometime, all of them may become 0. So, this is indeed an infinite sequence in that sense; otherwise, only the relavant nonzero terms come only in the first m terms say finite number of terms. That is also possible.

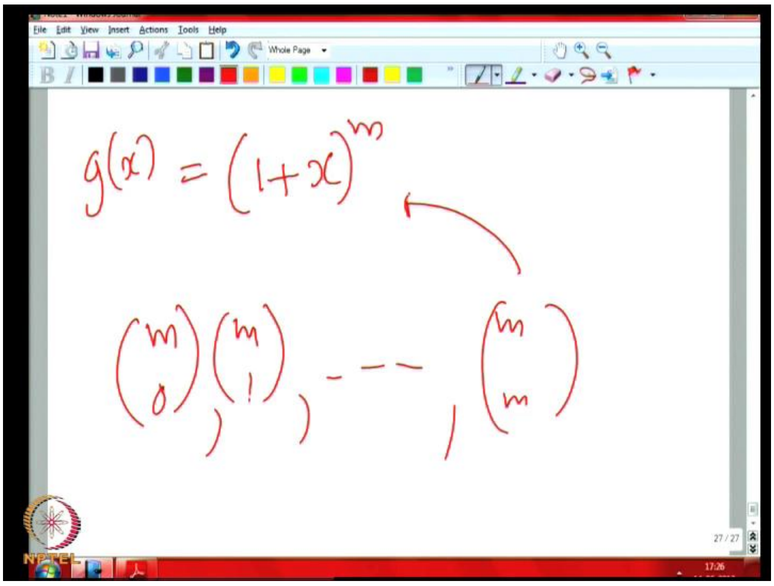
We will look at some examples. We will start with the most fimiliar example namely, let h 0 equal to m choose 0; let h 1 equal to m choose 1 – second number; third number be m choose 2; and, fourth number be m choose 3 and so on. So, h m equal to m choose m; and, h m plus 1 onwards – it is all 0. Now, h m plus 1 onwards – that is 0, 0, 0, 0. So, this is a sequence. What is the generating function of it?

(Refer Slide Time: 47:38)



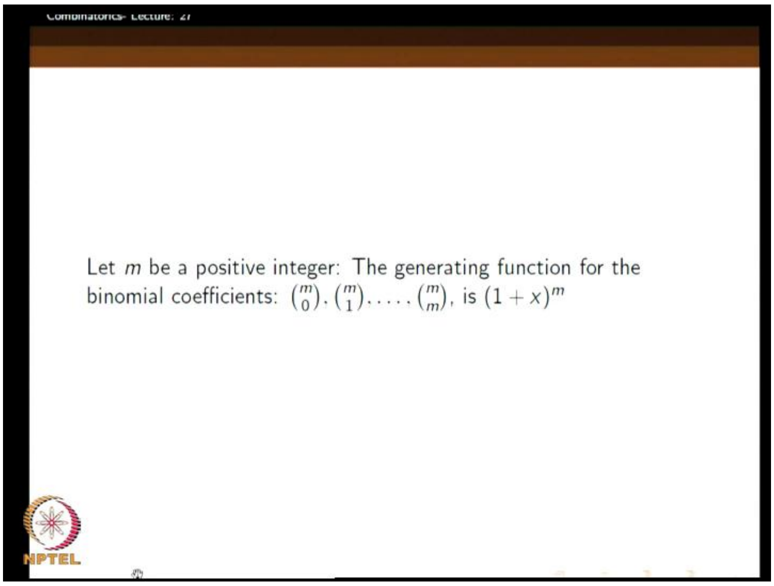
The generating function of this is given by m choose 0 plus – because we are writing this as h 0 off case – m choose 1 into x raise to 1 plus m choose 2 into x square plus m choose 3 into x cube plus like that until you reach m choose m into x raise to m. So, the rest of the terms are all zeros. So, we can discard them. So, this corresponds to h 0; this is h 1; this is h 2 and until… This is h m.

(Refer Slide Time: 48:24)



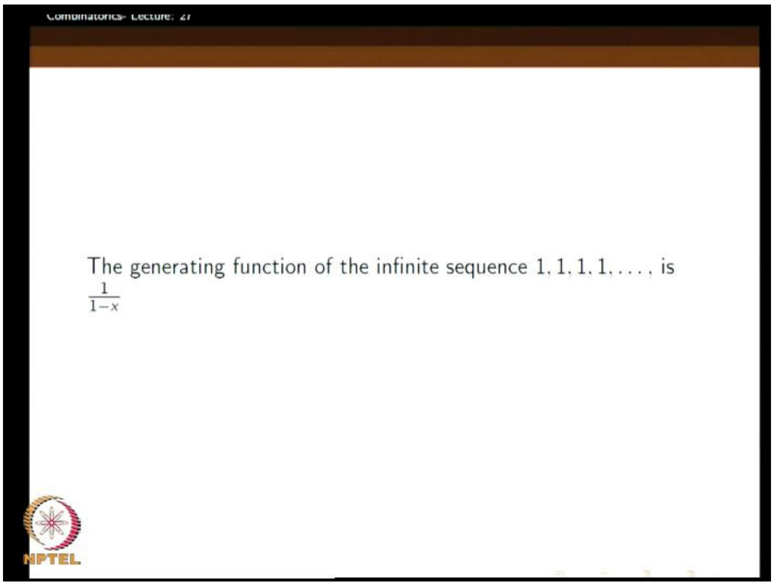
Now, what is this? This we know already; that is.. This is 1 plus x whole power m. So, the generating function g of x for the given sequence m choose 0, m choose 1 to m choose m is 1 plus x raise to m.

(Refer Slide Time: 48:46)



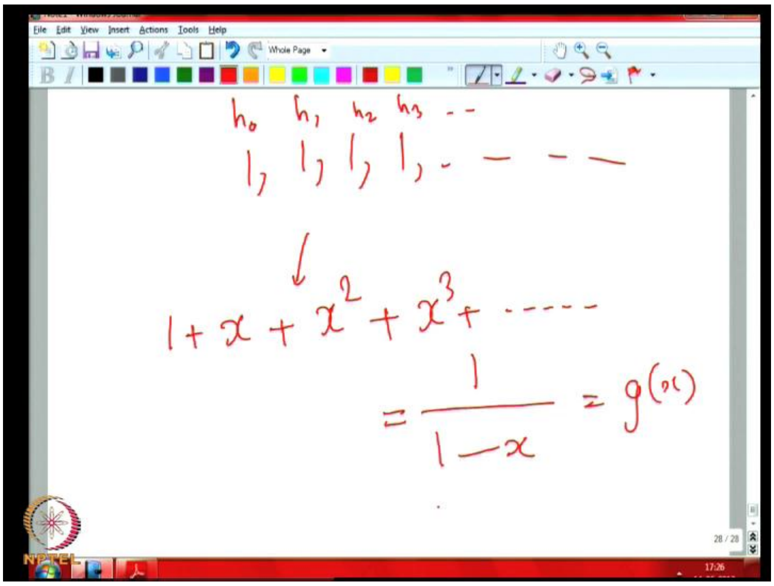
Now, that is probably the most familiar generating function for us.

(Refer Slide Time: 48:48)



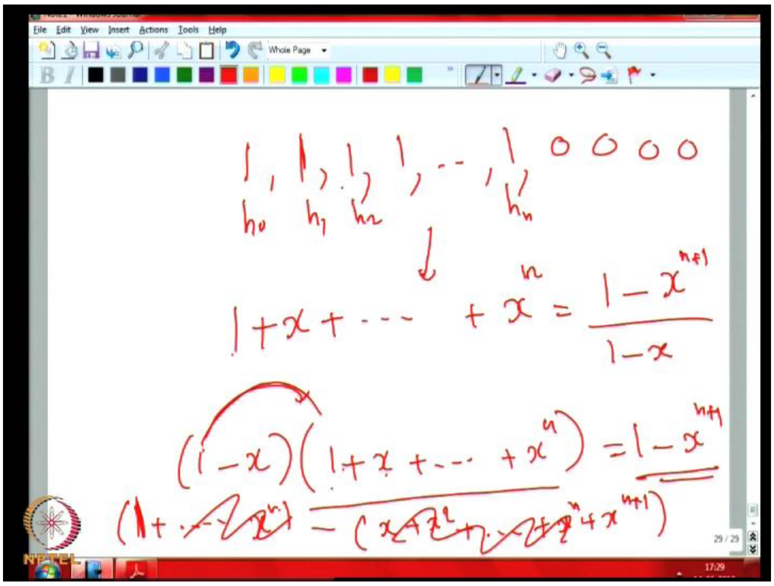
Now, we will look at some infinite series.

(Refer Slide Time: 48:56)



This is 1 plus 1 plus… So, this is infinite series. This is h 0, h 1, h 2, h 3 like that. So, 1, 1, 1, 1, 1, 1. So, the genrating function is 1 plus x plus – see 1 into x – 1 into x square plus 1 into x cube plus this infinite series. So, this is equal to 1 by 1 minus x. So, this is the generating function for it.

(Refer Slide Time: 49:39)

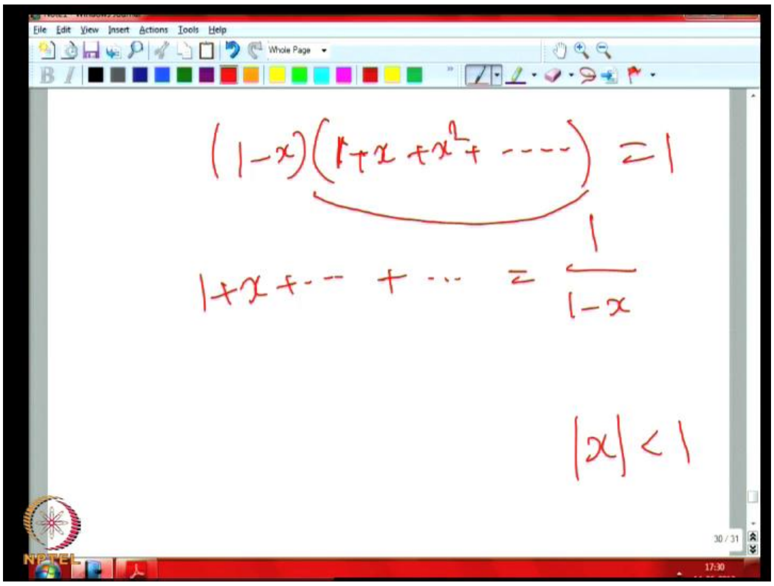


If you want to convince yourself… You can recall that… Suppose if 1, 1, 1, 1 – suppose there are n terms – n 1's and later all are zeros. Then, what will be the corresponding generating function? It will be like this – 1 plus x plus x raise to n. n trems means including 0 upto n plus 1-th term; that means this corresponds to h 0; this is h 1; this is h 2… So, we are writing upto h n. So, we have 1 plus x plus up to h n. So, this we know is 1 minus x raise to n plus 1 by 1 minus x. For instance, you can try multiplying 1 minus x into 1 plus x plus x raise to n. This will be indeed 1 minus x raise to n plus 1, because when you multiply by 1, you will get this. And, when you multiply by minus x, you will get minus x here, minus x square from here. And finally, minus x raise to n plus 1.

So, all the terms except the first one and the last series, n will be… For instance, if you want to see this, it is like x raise to… When I multiply by 1, here this (( )) will come – x raise to n minus… When I multiply by x, we will get x plus x square plus x raise to n plus x raise to n plus 1 will come. So, these terms will go away. So, we get 1 from here; minus x raise to n plus 1 here. This is what we get.

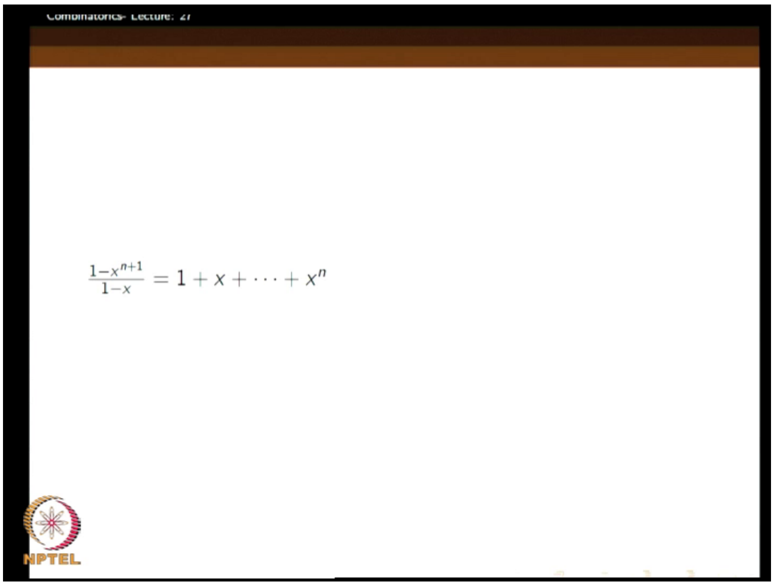
So, that is why this sequence has n rating function – 1 minus x raise to n plus 1 by 1 minus x. Now, we can see that, the same technique will tell us… For instance, if this was infinite sequence, it is not ending at x raise to n; it is 1 plus x plus x square plus x cube plus… The same argument would tell us that, that is equal to 1, because there is no… In the end, what happens is this x raise to n plus 1 – that is tending to 0.

(Refer Slide Time: 52:06)



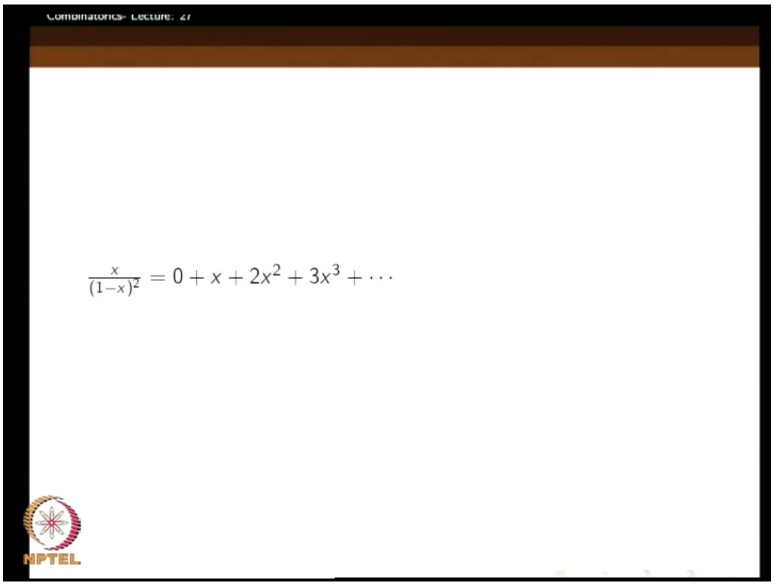
So, we can say that, 1 minus x into 1 plus x plus x square plus is equal to 1; that means here 1 plus x plus – this sequence is equal to 1 by 1 minus x. Only thing we have to worry about is… When we argue this thing, we have to worry about whether this converges or not. See it will converge when x is less than 1. In all of our discussions, we will not worry too much about this part – convergence, because we will be worried more about the coefficients of the powers of x, which is coming here and we will not be discussing the issue of convergence all the time. Therefore… And, we will be dealing with familiar forms, where… Therefore, some ranges – it indeed some converges; which range converges will not be relavant to our discussion. We will not be trying to sum up the series anyway.

(Refer Slide Time: 53:38)



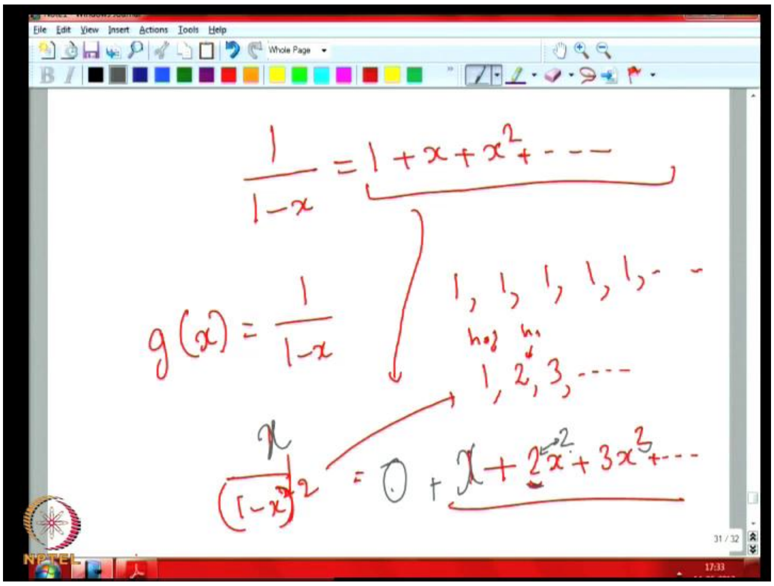
Now, the next one. So, we will… Now that we have seen this one;

(Refer Slide Time: 53:41)



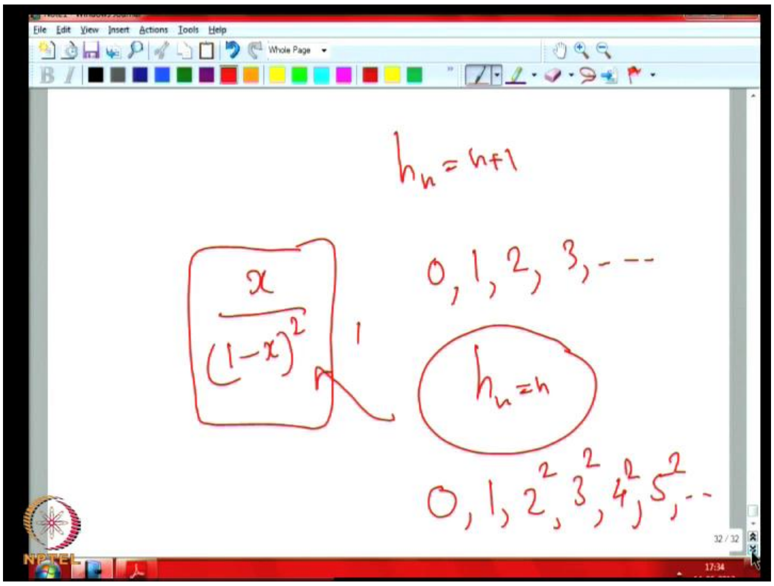
So, we cannot also cook up some other sequences.

(Refer Slide Time: 53:46)



See from this thing – 1 by 1 minus x – this is 1 plus x plus x square plus this thing; that is, this g of x equal to 1 by 1 minus x is the generating function of the sequence 1, 1, 1, 1, 1, etcetera. Now, what we can see is that, this part – if you differentiate, we will get 0 plus 1 plus 2x plus 3x square plus and so on. So, if you differentiate this portion, we will get minus 1 by x whole square, because this is minus 1 by x square and there is a minus x here. Therefore, it is plus 1 by x square. But, then here we have 0. That we can anyway remove – 1 plus 2x plus 3x square. The only thing is here 2 is the coefficient of x; that either we can say this is the generating function of the sequence 1, 2, 3 and so on. So, h 0 is equal to 1; h 1 equal to 2; and, h 2 is equal to 3 and so on.

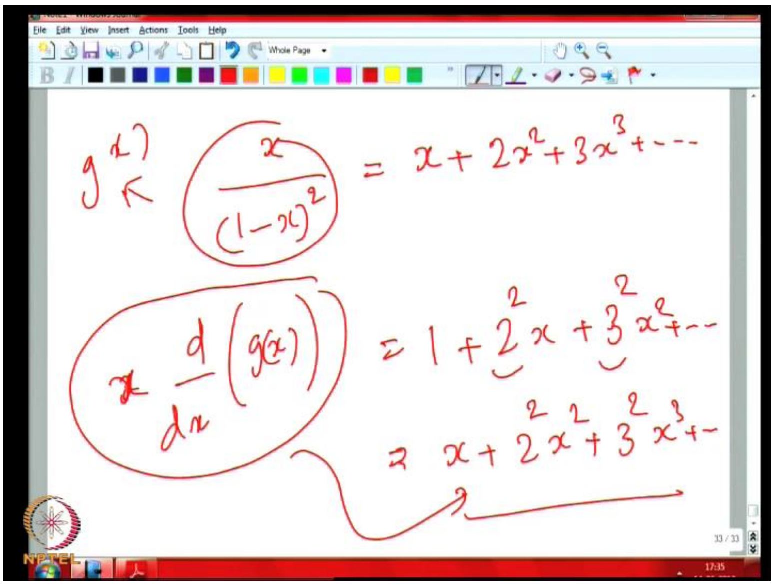
(Refer Slide Time: 55:17)



That means the sequence h n equal to n plus 1. That is what it is. So, this is the generating function 1 by 1 minus x whole square, is such a sequence. Or, what we can do is, we can multiply here this side by an x; this side by an x. So, x into 1 minus x whole square. That will introduce an x everywhere here like this; that means this exponent, which is coming x to the power what?

That will be the same as the coefficient in that term. Therefore… Just that we are starting with x; not with a constant term. Therefore, we can put the constant term as 0. The first term is 0 – 0 plus x plus 2 x square plus 3 x cube and so on. So, we see that, x by 1 minus x whole square is the generating function of the sequence 0, 1, 2, 3 and so on; that is, h n equal to n. Before getting into the one, what we can mention is… If you want to generate a… So, it is a good exercise to try to find the generating function for 0, 1, 2 square, 3 square – this sequence – 4 square, 5 square and so on.

(Refer Slide Time: 57:10)



How do we do that? Because we know that x by 1 minus x whole square is the generating function for the sequence 0 plus 0, 1, 2 and so on; that is x plus 2x square plus 3x cube plus… This is the sequence that is representing – This one. So, you can see that, if you diffentiate it further – d by d x of this one – this g x if I take; this being g x. Then, what will happen here is – this will give 1; this will be 2 square x plus 3 square x square and so on.

So, we see the terms, which we want here. But, just that 2 square is the coefficient of x, what we can do is we can mutliply by what? x here. So, this will be x… So, if you multiply by x here; then, we will get x plus 2 square x square plus 3 square x cube and so on here. Now, this function will be the generating function of this; that means this is the generating function corresponding to 0, 1, 2 square, 3 square etcetera. And, if you consider this function; that will correspond to that thing. We will discuss in the next class.