Combinatorics
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Lecture - 26
Recurrence Relation –Part (4)

Welcome to the twenty sixth lecture of combinatorics. In the last class, we were discussing the linear homogeneous recurresnce relations with constant coefficients and order k. We saw a general method by which we can solve such recurrence relations, provided the roots of the characteristic equation are all distinct. So, we saw that irrespective of the initial conditions, we can get a solution for that using that method. The only difficulty is to solve the characteristic equation, if you want to get all the rules of the characteristics equation. Yeah, it can be difficult, but of case, if we have a reasonably small recurrence relation, I mean a recurrence relation of small order, most of the time we can do that. But yeah in the case of k being large, it is questionable whether how we will be get it. So, otherwise the rest of the things are working out, right.

But yeah that is all correct and also see once you see the technique to apply the technique is quite routine. There is nothing clever about it and one does not have to, you just have to remember the technique and you can apply it without much effort. Of case, so these recurrence relations themselves are coming from problems. So, the more difficult thing most of the time is to come up with the recurrence relation. So, we should not, when during this discussion, we should not forget that, that is where the counting is actually involved.

This is after the first processing and we get this recurrence relation then we have this technique to solve. Of case I mean, what we have to learn is this thing, otherwise other part is usually depends on the problem. Still let us take some example, we have seen some examples, where the Fibonacci recurrence relations are coming earlier. So, now we will take some examples, where some other recurrence relation results. So, for instance, we can look at this problem.

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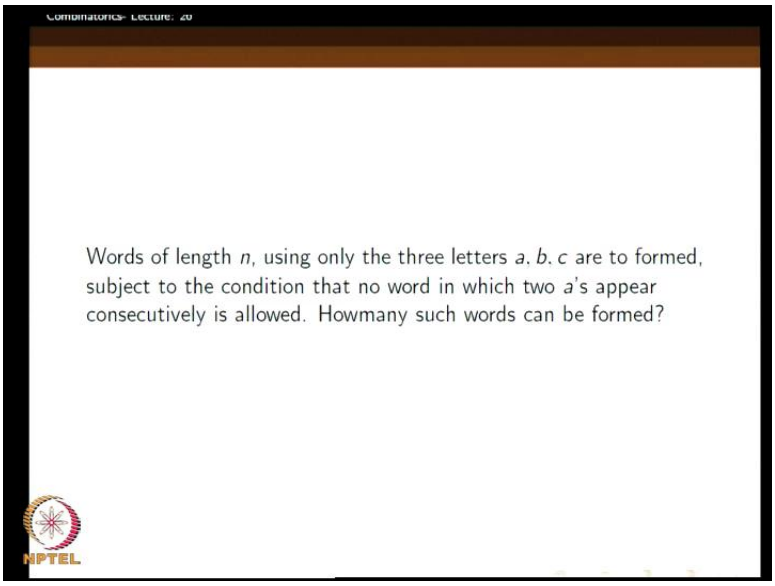

Words of length n, using only the three letters a, b, c are to be formed, subject to the condition that, no word in which 2 a's appear consecutively is allowed. How many such words can be formed?

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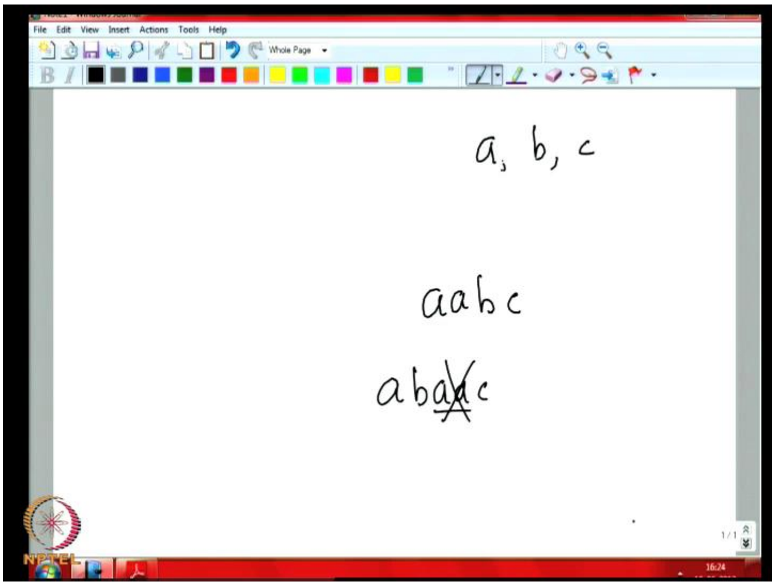

I hope you understand question. So, we have 3 letters a, b, c. We have to form words of length n. The condition is that, there should not be an a a like this in the word. For instance, this is not allowed or this is not allowed; these kinds of words, consecutive a's are not allowed. Now, how many words can be formed is the question. So, of case, obviously whether you have other methods or not, you are not interested. As of now you want to write a recurrence relation and then solve it to get the answer for this thing, right.

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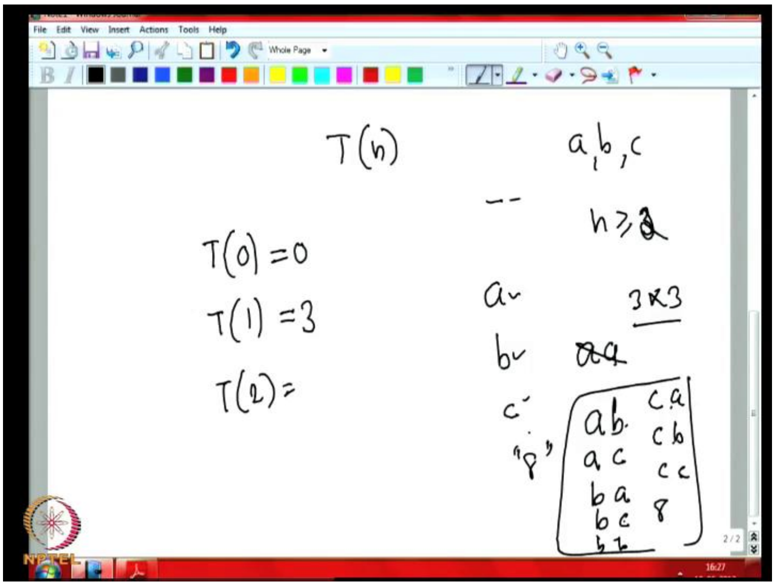

So, what kind of recurrence relation can be written here? So, let the number of words be t of n. The number of words satisfying these constraints of length n, which can be formed using the three letters a b c be this much, right. So then consider an n length word. So, let us assume n is greater than that equal to 3, right or n is greater than equal to 2 would be enough, let us say. So, we will see 0 length word is equal to 0, 1 length word is just 3, right. This much is correct.

So, why it is so t n? Because, if we are not, yeah, there is no 0 length word at all and 1 length word can either have a b or c. Two length words if you ask, that should be how much? So, a a is not allowed. Any other two combinations are allowed, right. So, that is you can have a b, a c and b c. So, b c, b a, b c, b b, that is also allowed and c a, c b, c c. These are the two length words, right. A lot of them, of case, 1 2 3 4 5 6 7 8 of them are there. Out of the nine, we have 3 into 3, 2 letter words and out of that, a a should be removed and that is why 8 is coming. But anyway, so we do not need t of 2 here. We will get it using the recurrence relation. So, we will just use this two, because this two are very easy.

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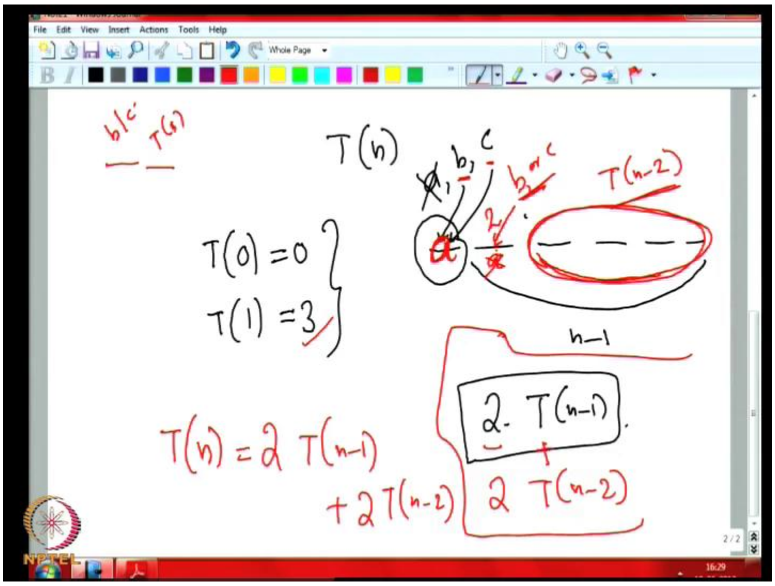

Now, how will you, suppose there is, we are thinking of n length words. So, the first letter can be a, b or c. Let say either b or c. Suppose this not a, then we do not have any restriction on this part. This is the n minus 1 length part, right. So, if it is either b or c, then we can just form all the t of n minus 1 possible words here. That is 2 into t of n minus 1.

On the other hand, suppose it is a here. Suppose, a is sitting here. This is 2 comes because we have either b or c for this case. Suppose a is sitting here, then there is a constraint, that here we cannot have an a, right. So here, we cannot have an a. That means, here we have b or c. Two choices here and then this portion, we can fill in any possible, that is t of n minus 2 is, right. So, that will be 2 into t of n minus 2 total. So, once you fix a here, here we have two choices only. Three choices are not there; b or c. For each choice, we have t of n minus 1, t of n minus 2 is to fill this portion, right. So, that is, this plus this. So, we get t of n equal to t of 2 times t of n minus 1 plus 2 times t of n minus 2. This is the recurrence relation which is coming from here. So, not that even when n equal to 2, this argument works, because if n was equal to 2, our argument told if it was b or c here, this is t of 1 ways you can fill it. That is 3, in 3 ways you can fill it.

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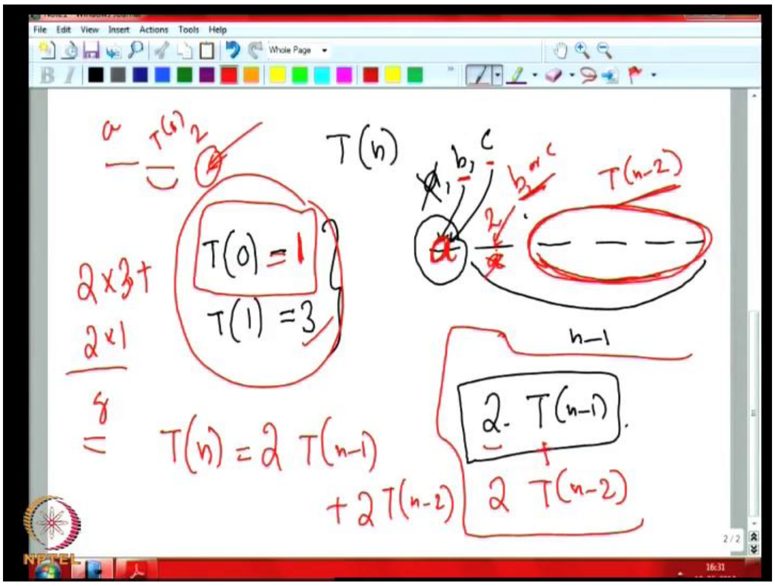

If it was a, then here we have only 2 choices, 2 into t of 0, right. That means, we have to set it as 1 here. Correct? Because, we have how many 0 length towards are there? We should put it as 1. So, earlier it was 2 into 3 plus, now it is 2 into 1. So, total 8. Correct? So, t of 0 should be set as 1 for this purpose. See, of case, how many words of length 0 are there? That is, of case, see if you ask like that, so depends on our requirement. Here, our requirement tells us that it has to be 1. So, or otherwise we should have started with giving t of 1 and t of 2 and then trying to set up the recurrence relation. But, this is ok. We just defined t of 0 is equal to 1, right.

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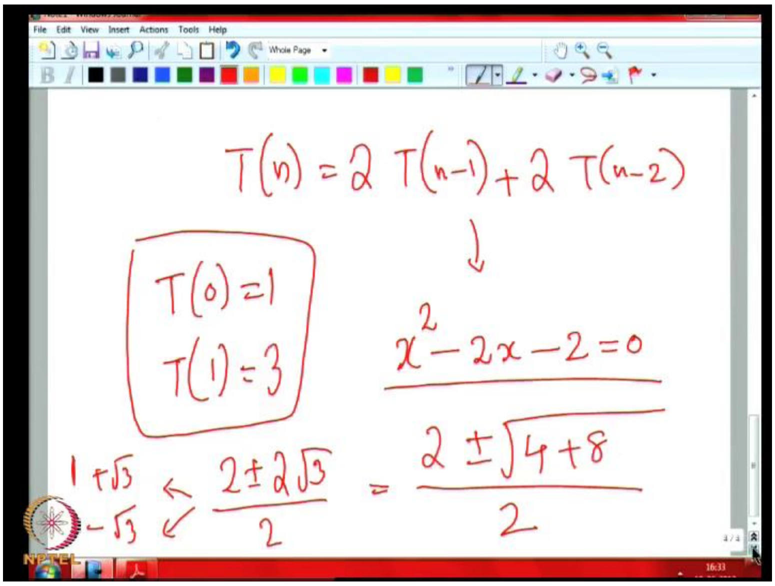

Now what we do is, we try to solve this t of n equal to 2 times t of n minus 1 plus 2 times t of n minus 2 with t of 0 equal to 1 with t of 1 equal to 3. These are the initial condition. We get the characteristic equation first. So, without much effort, we can see that it is x square minus 2 x minus 2 is equal to 0, right. x square minus 2 x minus 2 is equal to 0. We can solve it using the method to solve the catalytic equation minus b plus or minus b square, so that is 2 plus or minus 4 plus 8 divided by 2. So, this is 2 plus or minus 2 into root 3 by 2, right. So, we end up getting two solutions here. One is 2 plus root 3, sorry, 1 plus root 3 and other is 1 minus root 3. These two solutions will come. Correct? So, these are the possible values that we can give for q.

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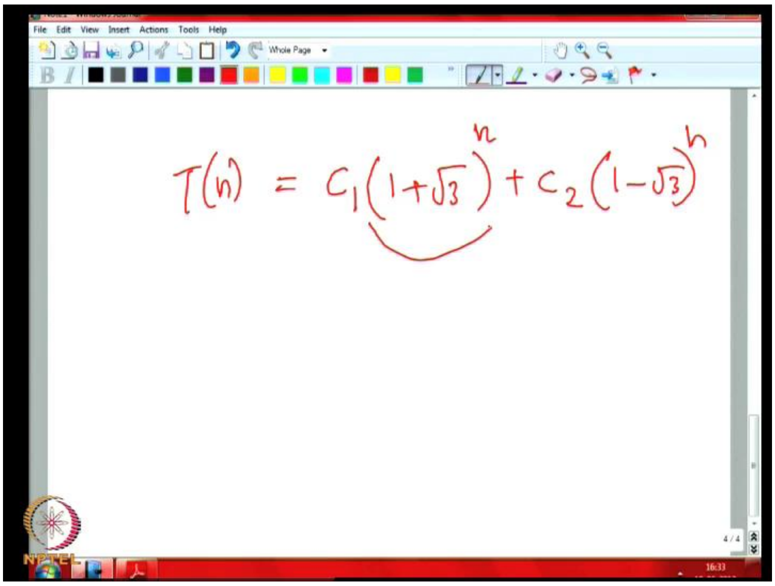

You remember from the last class, so you were calling q as the possible solutions, so therefore, we get that c, some c 1 and c 2 we can select, such that, c 1 into 1 plus root 3e raise to n plus c 2 into 1 minus root 3 raise to n is a solution for t of n. What you should remember is, if you just substitute 1 plus root 3 raise to n for t of n, this recurrence relation will be valid because 1 plus root 3 is a solution for this thing for the characteristic equation. Similarly, if I substitute 1 minus root 3 raise to n for t of n, then this recurrence relation will be valid, because 1 minus root 3 is also a solution for this recurrence relation and for any constant c 1 times 1 plus root 3 raise to n is a solution for t of n.

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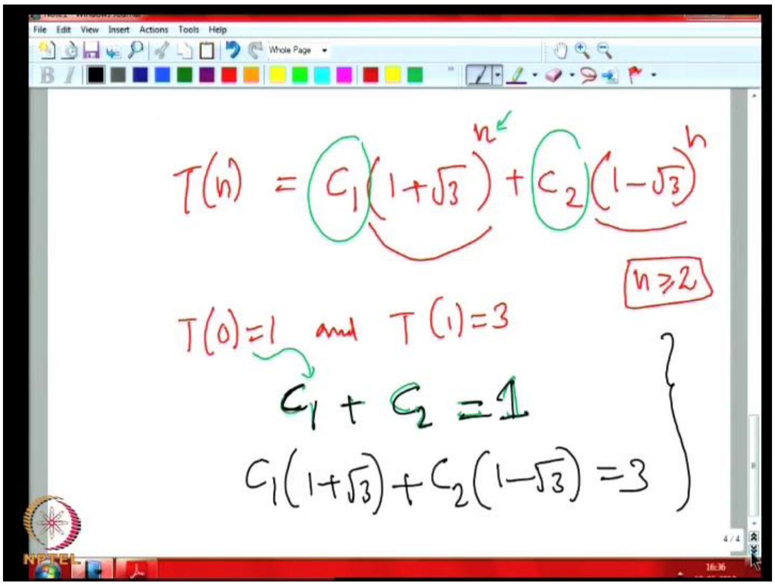

Any constant c 2, 1 minus root 3 raise to n is a solution for t of n, right and so on. But, this will be valid for n greater than equal to 2, right. But, this only satisfies the recurrence relation. If we also have to satisfy the initial condition, namely t of 0 equal to 1 and t of 1 equal to 3, if this is to be satisfied, we have to carefully select the c 1 and c 2, right. How will we select it? We just put n equals to 0 and write down what will happen.

So, c 1 plus c 2 because putting n equal to 0, that is how this will become 1 and this will become 1 and c 1 plus c 2 is equal to 1 is what this says. t of 0 equal to 1 implies this and similarly, c 1 plus c 2 equal to 1. Now, when I put n equal to 1, I get c 1 into 1 plus root 3 plus c 2 into 1 minus root 3 equal to 3. Now, we have to solve for this. As we have elaborately explained in the last class, this will always have a solution, right, because we have distinct root here; 1 plus root 3 and 1 minus root 3. So, we just have to solve this thing. We know how to solve it.

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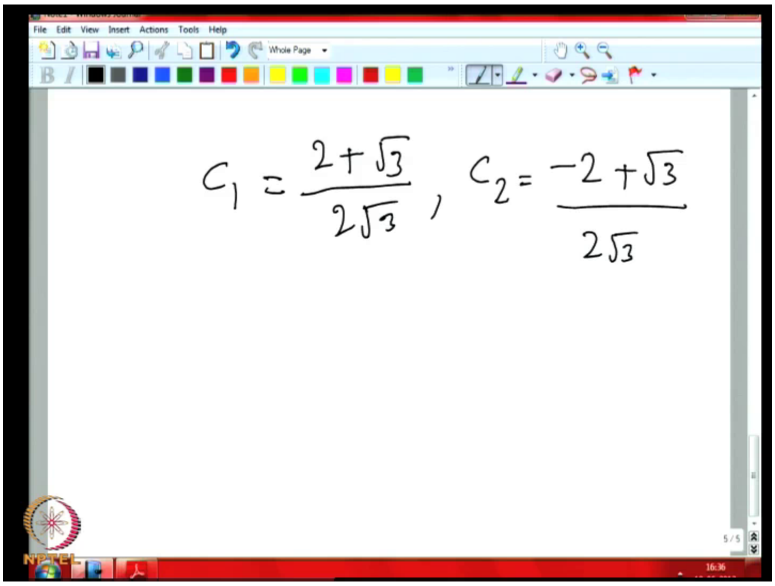

So, the answers will be c 1 equal to, I am just trying to copy, c 1 equal to 2 plus root 3 by 2 root 3 and c 2 equal to minus 2 plus root 3 by 2 root 3. These are the solutions we will get for this thing. Now, we can substitute for c 1 and this will work for n greater than equal to 0, right. So, this will work for n greater than equal to 0.

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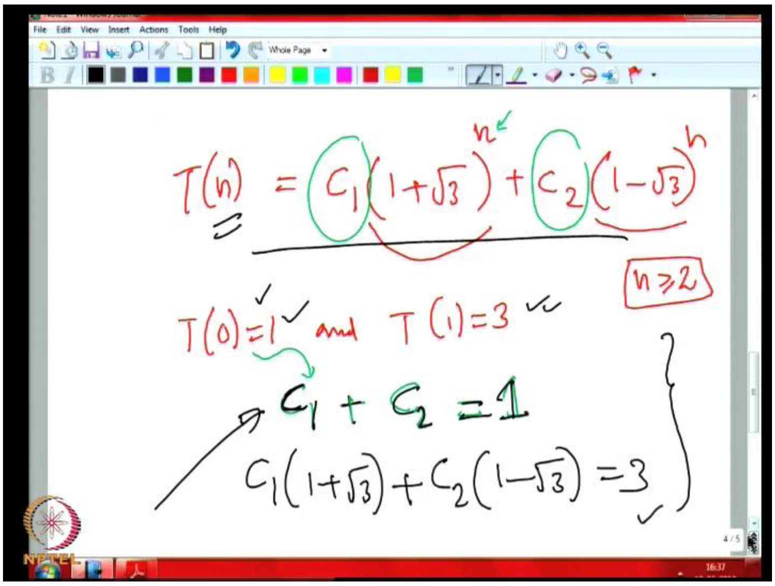

Because now, this and this is satisfied, for n greater than equal to 2 and this equation was satisfied for any c 1 and c 2. Now, we are giving special values for c 1 and c 2, right, which also satisfies this. That is all, right. So now, so everything will be satisfied. So, this is the way we solve it.

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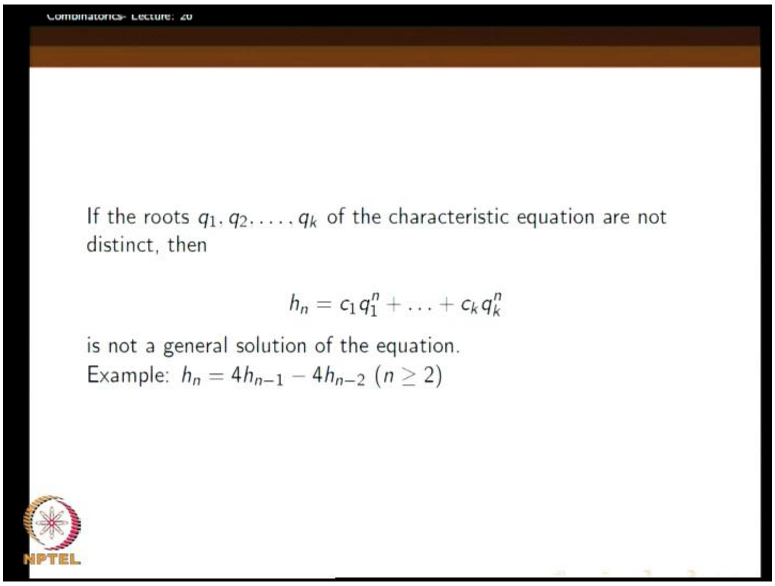

Now, next one, so we will, anyway, so we can take any number of examples of this sort, right. This was this was also the problem, there was a problem and this problem was formulated using the recurrence relation. I mean, the counting, how many words can be formed. We first noted that it will be the solution of recurrence relation t of n equal to 2 times t of n minus 1 plus 2 times t of n minus 2 with some initial conditions and can be solved for it. Of case, so we have to in general, usually there will be problem, which has to be converted into a recurrence relation. Then only we solve the recurrence relation. But again, we can discard that first part for the time being. We can concentrate on just the second part because that is what we have to now study. It is a little dull. Once you learn the technique, it is just the routine, There is nothing much to think. But still, we have to know that. So, therefore we will quickly cover this material. So, my coverage of this material is going to be quite brief.

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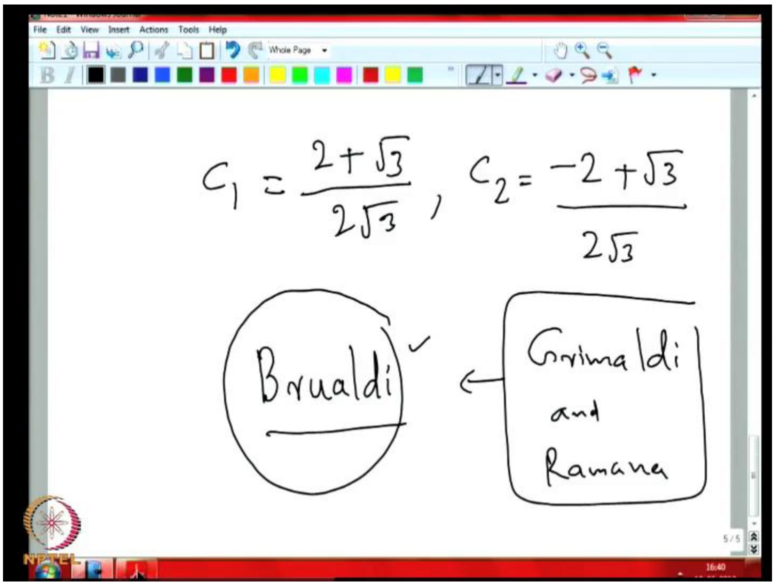

I am following the book of Brualdi, which is mentioned in the first lecture, Introduction to combinatorics. But, in the book Grimaldi and Ramana, we will get more extensive treatment of this topic. Though we are covering all the essential things, and just that in the second book, Grimaldi and Ramana, it is more elaborately treated with more examples and in a more detailed fashion. So, the student is advised to read that presentation once apart from following this presentation because this is brief and still it has all the necessary things.

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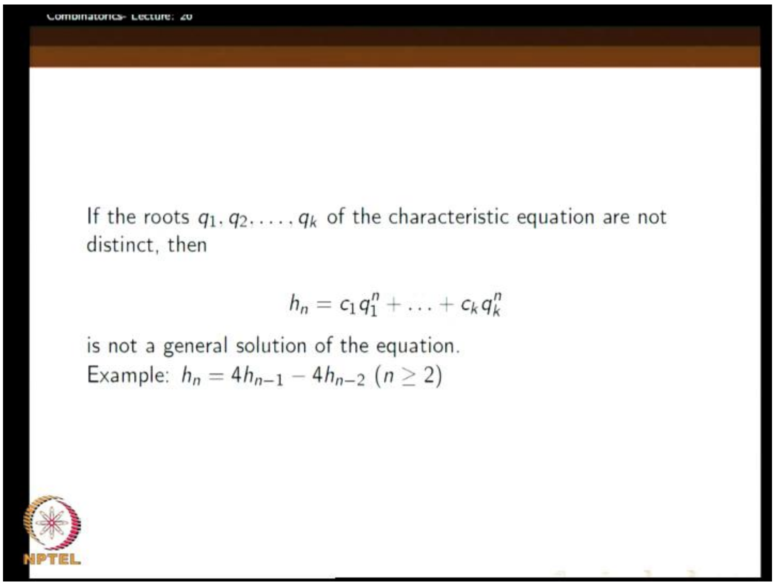

So, that is why because we want to finish of this portion fast and get on to something else, which is where we can discuss more. This is general. Just we have to know how to do this thing because the rest of the things, I believe that it is routine. So, that is why I do not want to spend too much time on this thing. Now, the second thing you want to discuss here after that distinct root k's, the characterization are all root distinct. Now, we will consider the k's, when all the root are not distinct.

So, then you can note that, this h n equal to c 1 times q raise to n plus c k times to q k raise to n. It need not be a general solution for the equation. In the sense, it will be the solution of case by the argument, because q 1 raise to n q 2 raise to n, all those up to q k raise to n, they are all solutions. Then we can also combine it. But, the only thing is, it is not the general solution. In the sense, that for some initial conditions, we may not be able to find the correct coefficients. Because, whichever coefficients you give, it may not work for certain initial conditions.

So, those of few you who are quick can see from this itself. For instance, if q 1 is equal to q 2, and that means q 1 repeats. Then this will, this what h n equal to c 1 times q 1 raise to n plus c 2 times q 2 raise to n up to c k times q k raise to n will simplify to c 1 plus c 2 times q 1 raise to n plus c 3 times q 3 raise to n plus and so on. So, that means, we do not have, we lost a constant. Though, we are writing c 1 plus c 2, it is only one constant there, right. c 1 plus c 2, we do not have much information about that. So, that does not help in which way we split that c 1 plus c 2, right. It is only one constant. c 1 plus c 2 has only one, become one constant.

Then, when we write all these system of equation, we will have, the number of variables will be less compared to the number of equation we write. We know that. So, from linear algebra, if you have done linear algebra, you should know that there is some trouble here. You may not get solution in some cases. Some cases you will get and some cases, you may not get. But, rather than depending on your knowledge of linear algebra, what I will do is, I will give you a small example and make you understand this point. But, those of you who know about all these things need not worry too much. So, it is very clear, right, that it will not work out. You should remember the discussion we did in the last class. We should try to repeat those arguments and see why it is failing. For instance, when you form those equations, so why we have; see you write k equations and we have only less than k variables there, right.

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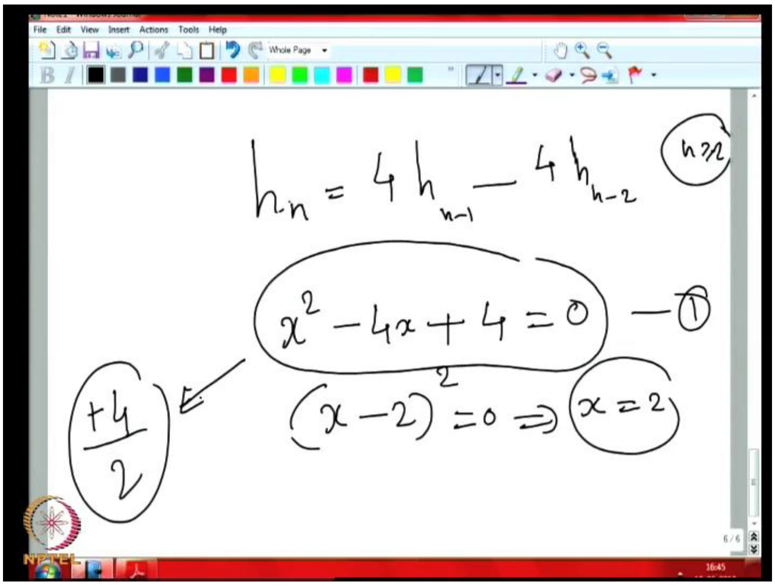

So, we have some trouble there. So, for instance, let us look at this recurrence relation, h n equal to 4 times h n minus 1 minus 4 times h n minus 2 for n greater than equal to 2. h n equal to 4 times h n minus 1 minus 4 times h n minus 2, let us say, for n greater than equal to 2. Now, your characteristic equation is going to be x square minus 4 x minus 4 is equal to, sorry, plus 4 equal to 0. This is the characteristic equation.

Now, this is what? x minus 2 whole square equal to 0, which means x equal to 2 is the only solution. So, if you had used the method to solve the quadratic equations, you would have got x equal to 2 x equal to out of 2 times, right, to the root repeats 2 times. Or for instance, that b square minus 4 a c here, for instance, 4 square minus 4 into 4. That is becoming 0. Therefore, we do not have two roots. It is just that, minus p by 2 will be the only solution we get there, if you use that method. So, otherwise we say that, here the solution x equal to 2 has multiplicity 2 here. Now, what do we mean?

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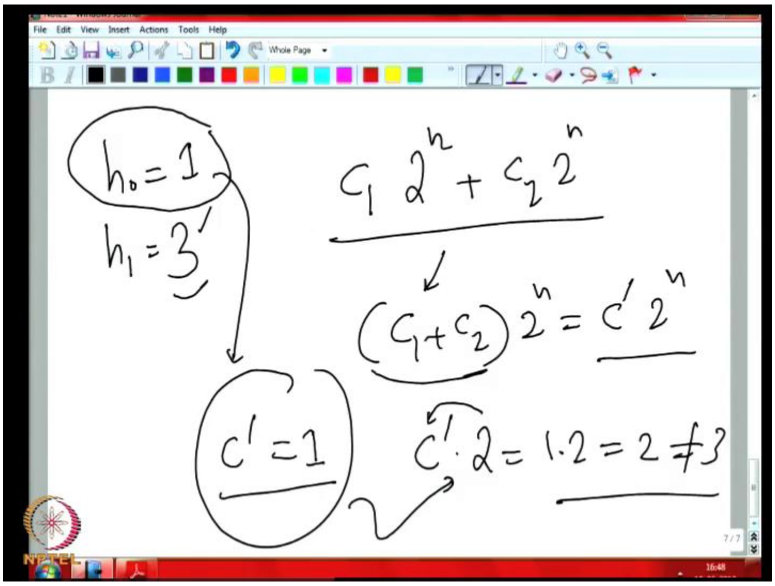

So, we can indeed say that, 2 to the power n is a solution. c 1 times 2 the power n is the solution. If you want c 2 times 2 to the power n is also a solution for any constant. But, that does not help because this is only c 1 plus c 2 times 2 to the power n, which is some c dash times 2 the power n, right. This will not help because this is splitting a constant into two will not help. So, why will it not help? Because, you know, put n equal to 0, we have two initial conditions, so we will have something for h 0 and h 1, right. h 0 is equal to something. h 0 is equal to something. Let us say h 0 equal to 1 and h 1 is equal to something. So, we will decide this something, so that, we will not get answer.

So, put h 0 is equal to 1 n equal to 0. So, this gives us c dash into 2 raise to 0. That t dash is equal to 1 c dash equal to 1, right. Now, when you, this one, right, suppose h 1 is equal to some a. So, put n equal to 1; we will get c dash into 2. That is, 1 into 2 is equal to 2. So, if this a is not equal to, suppose I take 3 here, a equal to 3 and 2 will not be equal to 3, right.

So, in other words, first I selected, to match the value of h 0, we already selected c dash and we have only one constant. That only one constant we have. That constant, if by luck, if this was also 2, it will work out. But, if it was 3, it was something else, 4, it will not work out, right. Because, this is, whichever constant is here, it has to be 2 times c dash, right. So, it will not work out. So, that is, you can see that, when there are repeated roots, it is not working out, right. So, it may not give a solution. So, what we do is, so there is a technique to circum in this.

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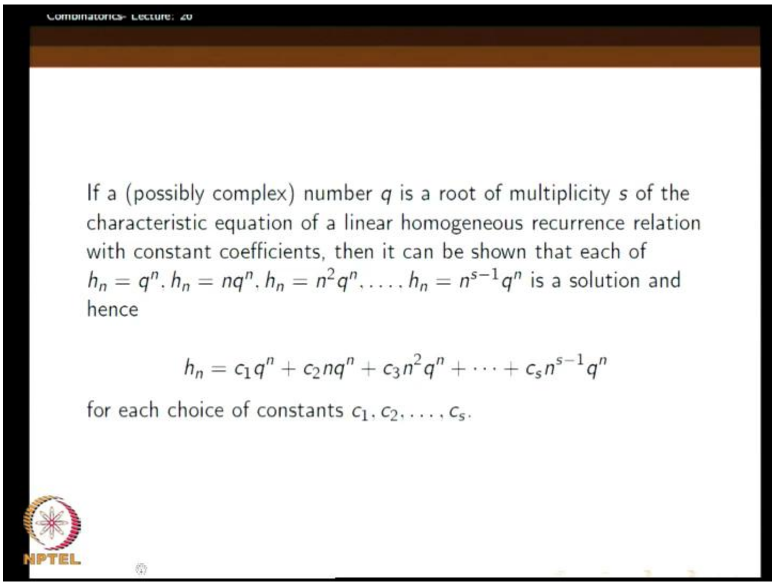

So, if a possibly complex number q is a root of multiplicity s, that means, that q repeats s times of the characteristic equation; of the characteristic equation of a linear homogeneous recurrence relation with constant coefficients and then it can be shown that each of h n equal to q raise to n h n equal to n times q raise to n h n equal to n square times q raise to n h n equal to n times, sorry, n raise to s minus 1 times q raise to n is a solution. Hence, h n equal to c 1 times q raise to n plus c 2 times n q raise to n plus c 3 times n square times q raise to n plus c s times n raise to s minus 1 times q raise to n is a solution for each choice of in the constants c 1 c 2 c s, right. Irrespective of the constant c 1 c 2 c 3 c s, we take, this will be a solution also.

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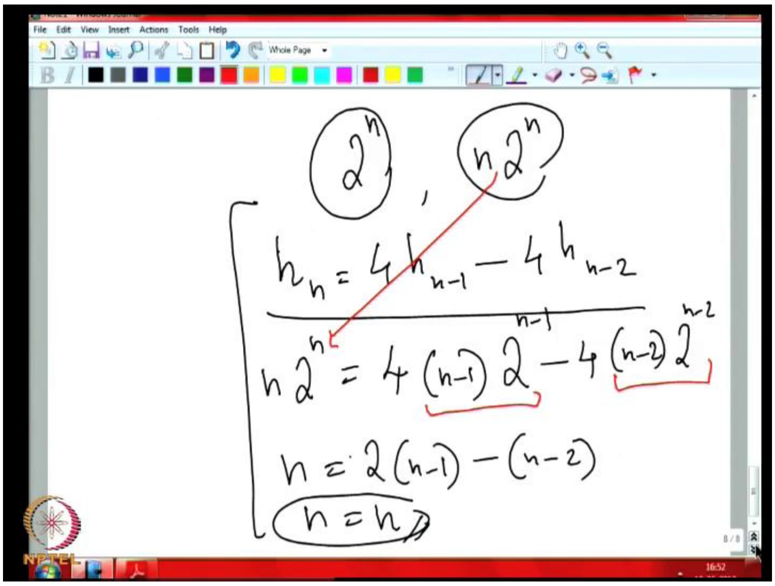

So, in our case, previous case we had 2 coming 2 times. So, we could have taken, we say that 2 raise to n is a solution and also n times 2 raise to n is a solution, right. So, this is h n 4 times h n minus 1 4 times h n minus 2. So, we even raise to 2 raise to n is equal to 4 into n into, right, 2 raise to 4 into, here we say 4 into n minus 1 into 2 raise to n minus 1 minus 4 into n minus 2 into 2 raise to n minus 2. So, we can cancel 2 raise to n equal to 2 into n minus 1 minus n minus 2, so n equal to n. So, minus 2 plus 2. This is true, whatever I have done here.

So, we have already checked that the 2 raise to n is a solution for this thing because characteristic equations has root, 2 as a root of characteristic equation. So, it should be satisfied definitely. Now, we are saying that this will be satisfied. We just substituted it, so and so what is happening here. So, n into 2 raise to n, I put for h n. Here, 4 into n minus 1 times 2 raise to n minus 1 is put. Here, its 4 into 2 raise to n minus 2 is 4. It is verified that, it is indeed satisfied.

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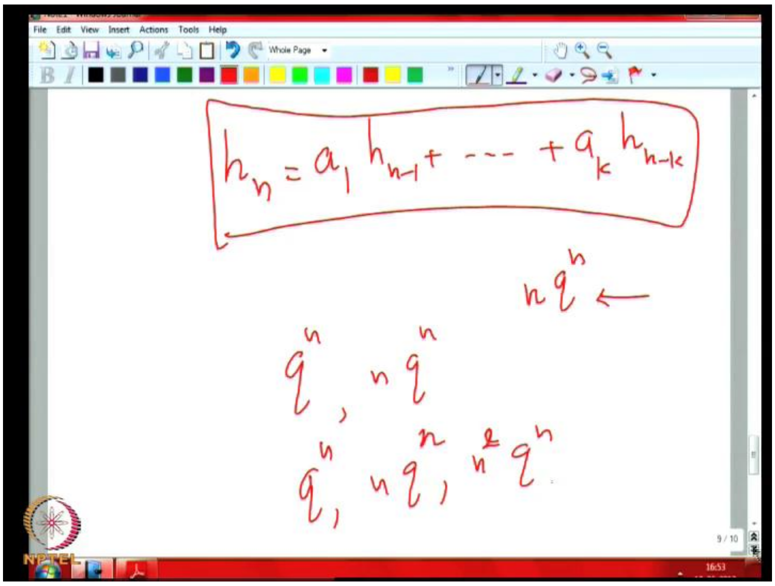

You can check that for the general case of h n equal to a 1 h n minus 1 a k h n minus k. This n times q to the power n. Our k is a solution of the characteristic equation will indeed work for this also, right. So, let us leave it to verify that and now, not only n, because it depends on how many times your solution repeats. If it repeats two times, we just take q to the power n. This is a solution and n times q to the power n is a solution. If it repeats three times, then q to the power n, n times q to the power n and n square times q to the power n is a solution. You can check it.

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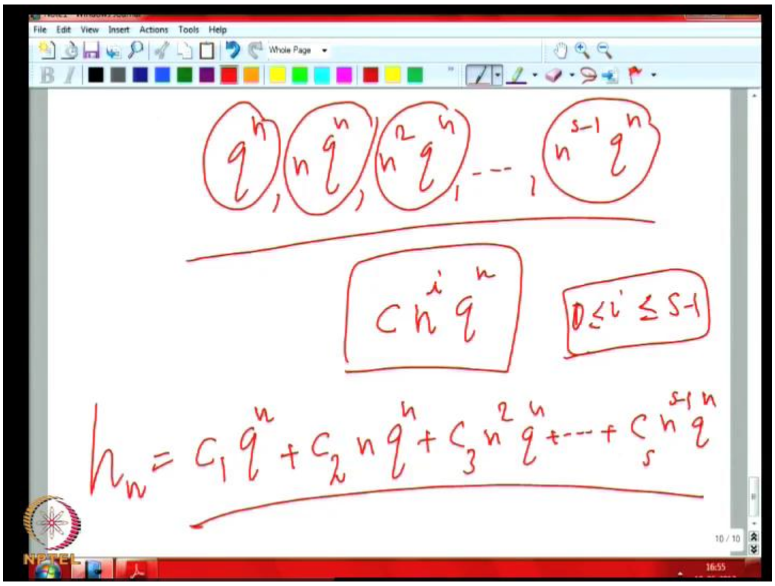

So, if it is in general s times, then q to the power n n times q to the power n and n square times q to the power n, all the way up to n to the power s minus 1 into q (_)). These are all solutions. So, if these are solutions, we know that we can also combine them because this satisfies the recurrence, this satisfy the recurrence, this satisfy the recurrence and this satisfy the recurrence. Any multiple of this, multiplying by constant is not going to disturb that.

For instance, that will be still satisfied. For instance, c times n to the power i into q power n, as far as i in between, 1 and 0 and s minus 1 will again satisfy this. That we know. We can add them together and then their combination, their sum also will satisfy. So, we can infer that c 1 times q raise to n plus c 2 times n times q raise to n plus c 3 times n square times q raise to n plus c s times n to the power s minus 1 into q raise to n, will also be a solution for the recommend solution.

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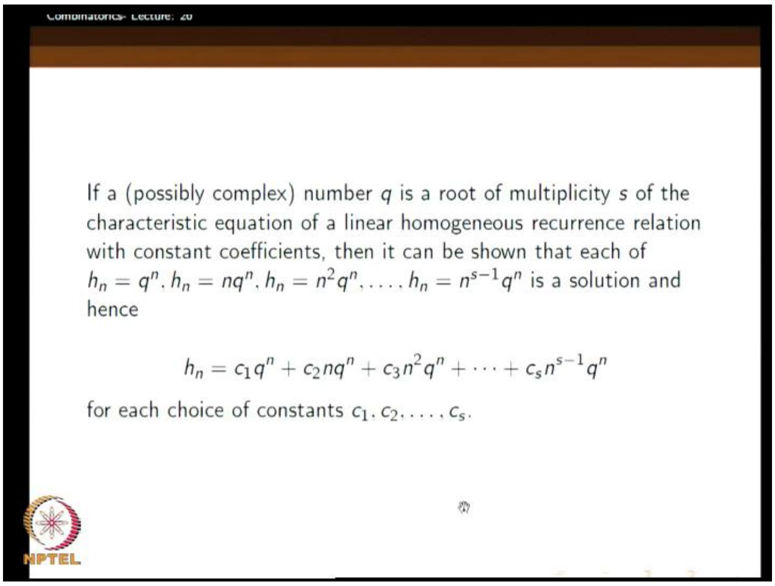

So, this is a solution. This will work for any choice. As obliviously, for any choice of c 1 c 2 c 3 and c s, this will work, right.

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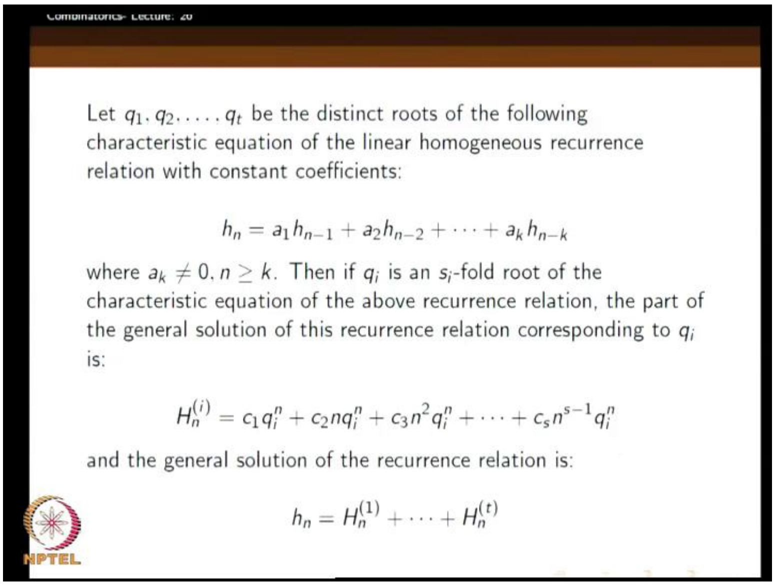

Now, we can, so this is about the same q repeating, right. Now, of case, when I saw the characteristic equation, it may so happen that there are t distinct roots, q 1, q 2, say up to q t. But, it is a degree k and can be bigger than t. So, q 1 may be a repeating, say s 1 times and q 2 will be repeating s 2 times and q 3 may be repeating s 3 times and q t may be repeating s t times, where s 1 plus s 2 plus s t is going to be your k, degree of the polynomial.

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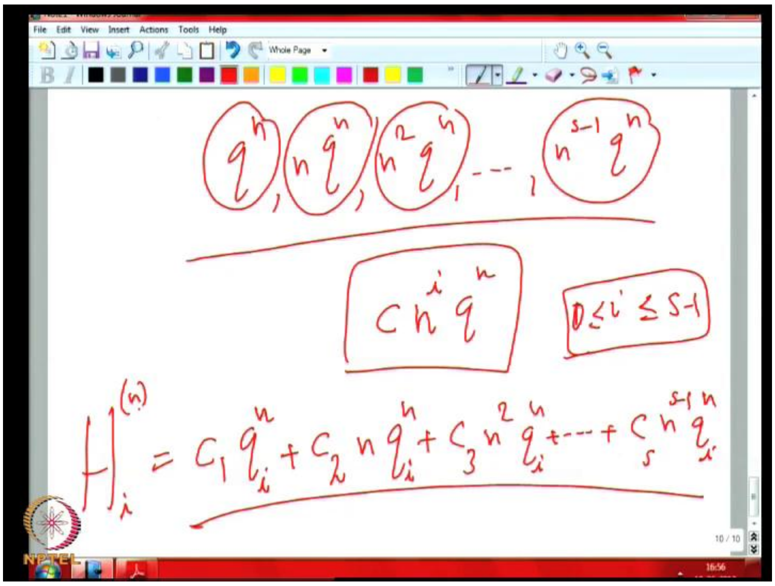

So, here of case, so now, what we do is, for each q i here, we can have a part like this, right. For instance, for any q i, which is satisfying, we can have part like this and let call it as, so that particular part as h i of n, right.

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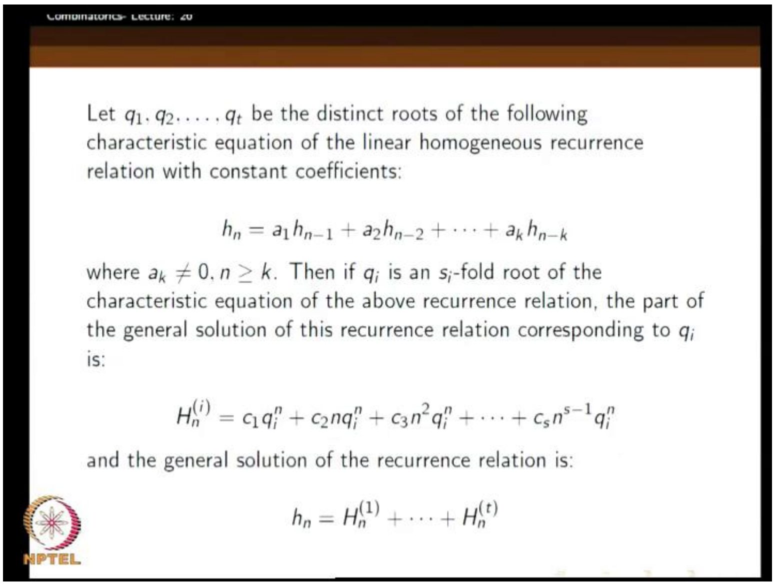

Then, it is clear that this h i of n, this one, c 1 raise to q i raise to n plus c 2 into n times q i raise to n plus c 3 into n square times q i raise to n plus c s into n raise to s minus 1 times q i raise to n will satisfy this. The general solution of the recurrence relation will be the sum of all this parts. There are two parts, because there are distinct roots. Each had s, s i here of case, so here, I have to put s i minus 1, right. Here we will have to say s i minus 1 because now s is not, for q i. it is s i, right, s i minus 1. Now, you sum up h m equal to h n of 1 plus plus h 2 of 2 plus, sorry, h n of 2 plus h n of t. If you add up all these things, you will get it. So, I can, may be, right,

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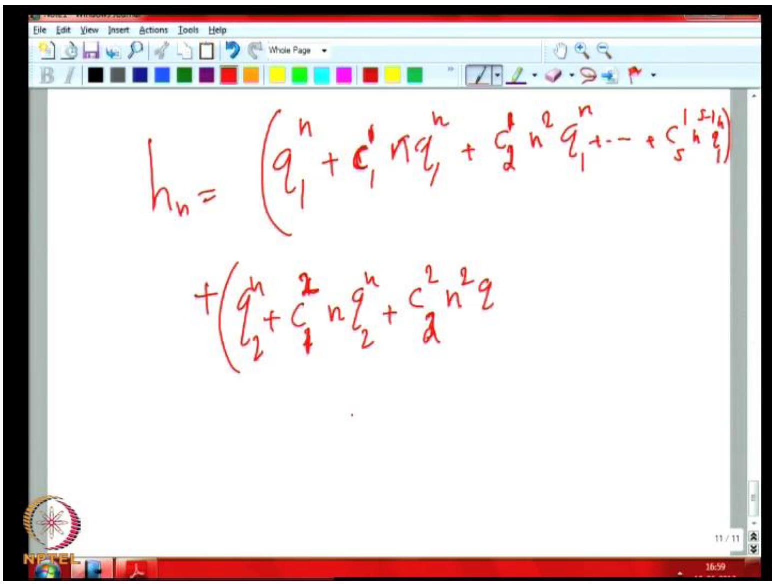

The general solution will be h n is equal to q 1 raise to n plus, so here, it is some c 1 1, right, plus n into q 1 raise to n plus c 1 2. May be, this can be 1, c 1 c 2 1 into n square into q 1 raise to n and so on, plus c s raise 1 n raise to s minus 1 into q 1 raise to n plus, then q 2 raise to n plus c 2 1 n q 2 raise to n plus, sorry, this is c 1, c 1 raise to 2 c 2 2 n square into, sorry, here coefficient have to be, will be same.

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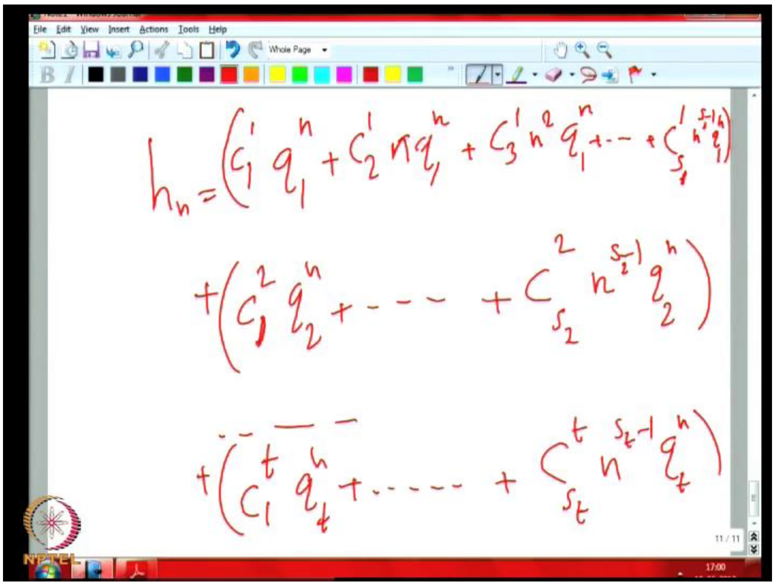

C 1, this is c 2, this is c 3 and this is c s. We have 1 here to indicate that it is the first one, right. c 1 2 q 2 raise to n up to c s 2 q 2 raise to, sorry, this is n raise to s minus 1. This is s

1, this is s 1, and this is 2 minus 1 q 2 raise to n and so on. Lastly, we have c 1 t q t raise to n plus c s t t n raise to s t minus 1 q t raise to n. This will be the general solution. That is what.

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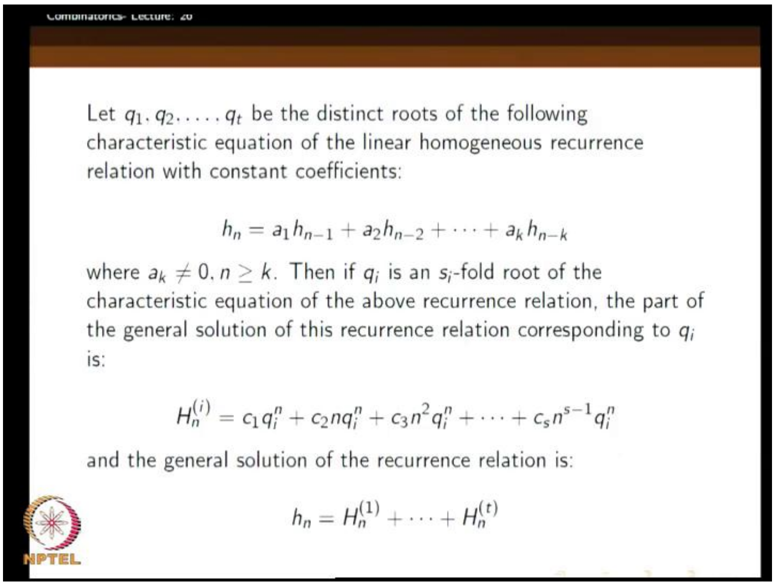

Of case, this is, I have added all of them together and written.

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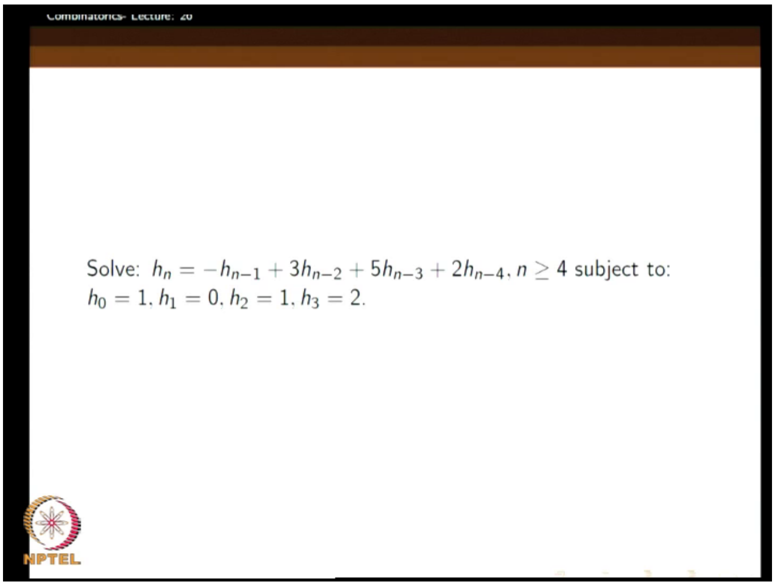

Now, we can, of case, this is the way we have to do. So, we can take an example and see what will happen.

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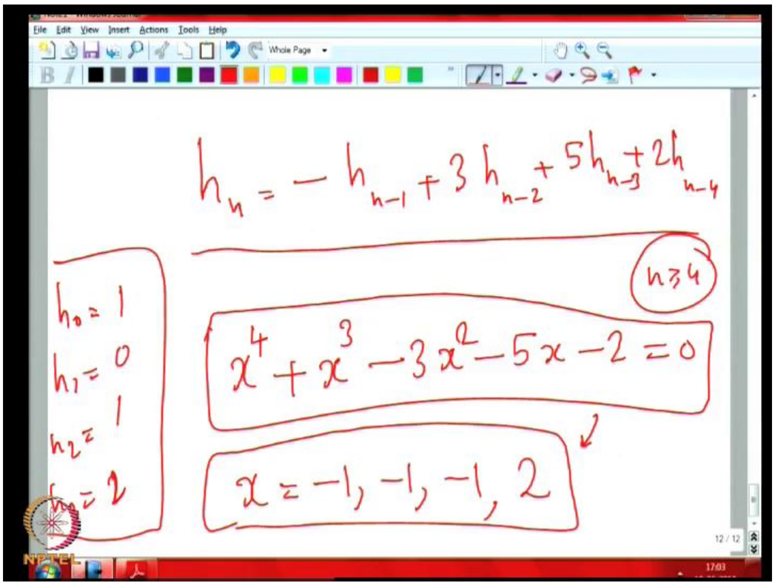

So, solve h n equal to minus of h n minus 1 plus 3 h n minus 2 plus 5 h n minus 3 plus 2 h n minus 4 and this will work for m for n greater than equal to 4 and we have some initial conditions. 1 0 1 2, for h 0, h 1, h 2, h 3, these are the initial condition. The first four values have to be given and this for n greater than equal to 4 onwards, this will work. This recurrence relation will work. The characteristic equation is x raise to 4 minus, sorry, this plus x cube, minus 3 x square minus 5 x plus 2, sorry, this is again minus 2, minus 2 equal to 0. We are taking all these terms to this side and equating to 0 and cancelling the n raise to 4 power x to the power n raise to 4 or q to the power n raise to 4 and writing the characteristic equations, right.

Now, you have to solve this thing. So, if you solve it, characteristic equation, the characteristic equation is, we will have roots; x equal to minus 1 minus 1, 3, minus 1 repeat 3 times and 2. These are the solutions for or the roots of the characteristic, this characteristic equation. x equal to 5 x minus 2 equal to 0. The roots are minus 1 minus 1 minus 1 and 2. Minus 1 has multiplicity 3 and 2 has multiplicity 1.

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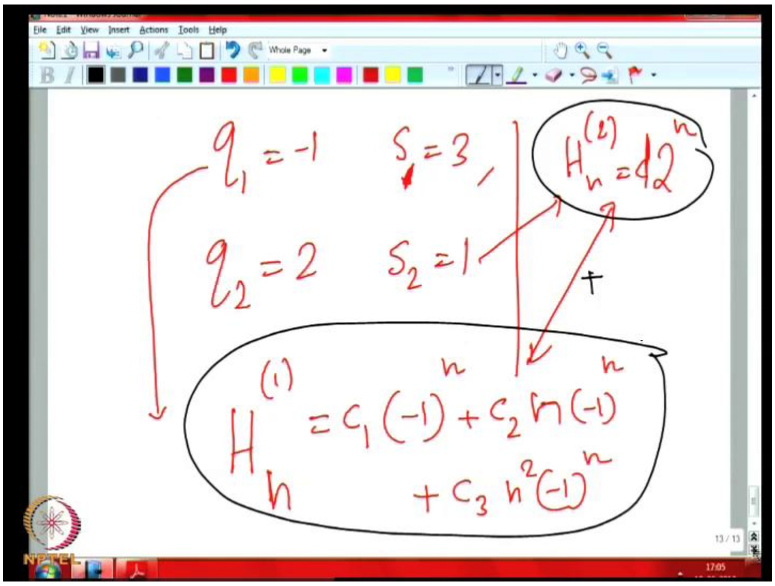

Now, we can write the part h this side. So, we see that q 1 is equal to minus 1. Its multiplicity is s 1 is equal to 3, q 2 is equal to 2 and here, s 2 is equal to 1. Its multiplicity is 5. Now, the path corresponding to this, in the general solution, the power corresponds to this. It will be h 1 1, this is the right, right, first minus 1 raise to n, so will have a constant for it. c 2 times minus, sorry, n into minus n raise to n plus c 3 into 3. There are 3 terms. 3 into n square into minus n and for this portion, q 2, so h 1, h 2 of 2, sorry, this is h n of 1. This is what I wrote. h n of 2 is equal to just 2 to the power n, right, c times 2 to the power n or we can write some other constant, let say d times 2 to the power n. So, we can, the general solution we will add this and this together, right. So, we will sum up this; this part and this part and these are two different parts and add together, right.

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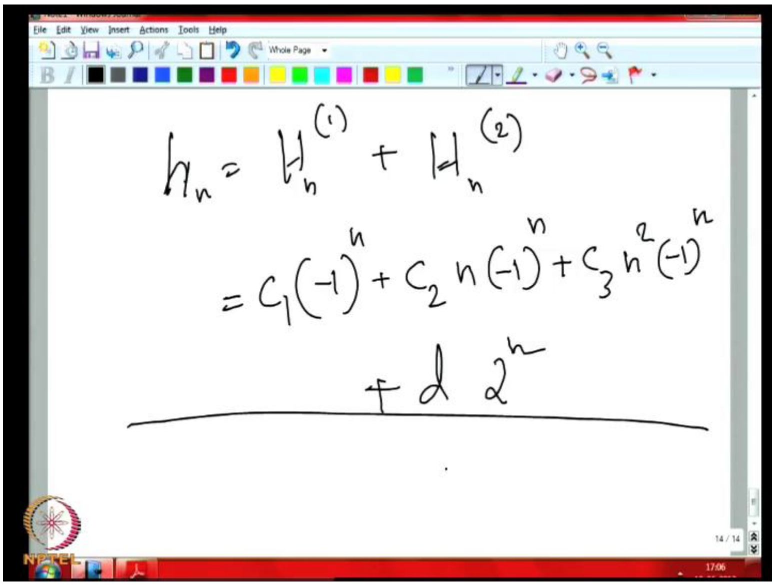

So, we will get h n equal to h n 1 plus h n 2. So, this is minus 1 raise to c1 raise to minus 1 raise to n plus n into c 2 into n into minus 1 raise to n plus c 3 into n square into minus 1 raise to n plus d into 2 raise to n. This is what we get. Now, we have to find the values of c 1 c 2 c 3 and d, so that the initial conditions are met. The initial conditions were this. H 0 is equal to 1 and from that what will you get? Put n equal to 0 here or this will become c 1 plus c 2 and c 3 will go away.

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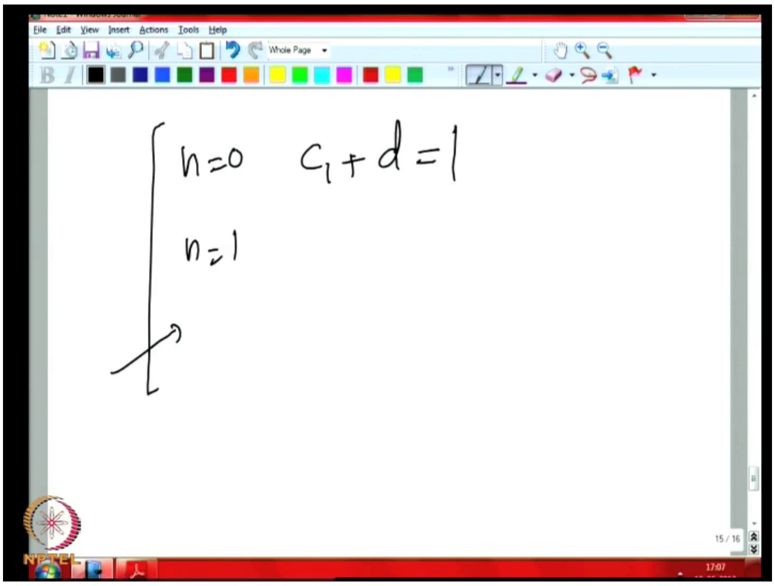

c 1 plus d equal to, right, c 1 plus d is equal to 0, this is what, sorry, it is 1. So, h 1 equal to 1. Once again go back to the initial conditions. h 0 is equal to 1 and h 1 equal to 0. Let us say because we have 12 more minutes to go. So, we have to substitute for n equal to 0. So, then n equal to 1 we have to substitute. We will get something from this thing and then substitute for each other and solve. That is what we want. Of case, it is not worth solving it completely. So, I leave to the student. Anyway, it is a routine task. There are no new things involved in it, right. We just save sometime by just keeping it.

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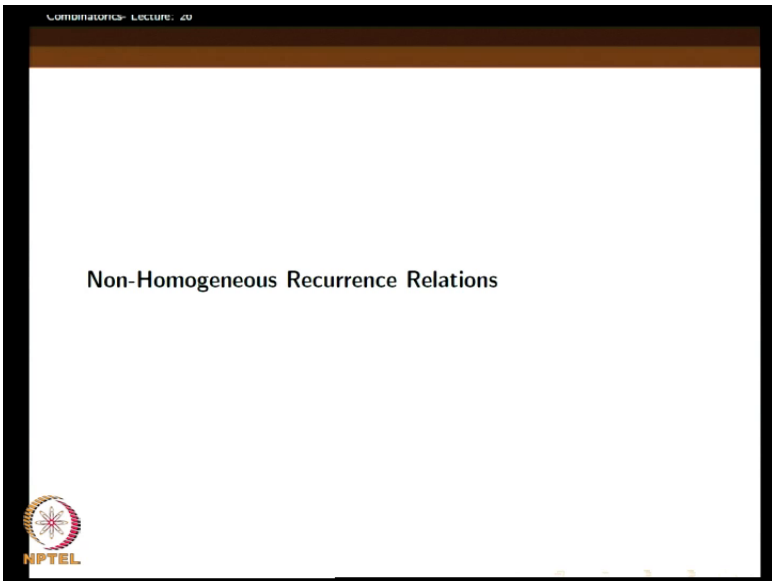

So, we will go on and look at the next section, non homogeneous recurrence relation. What is this non homogeneous recurrence relation? See, in the previous case, we were considering the homogeneous case, where that v m part was 0.

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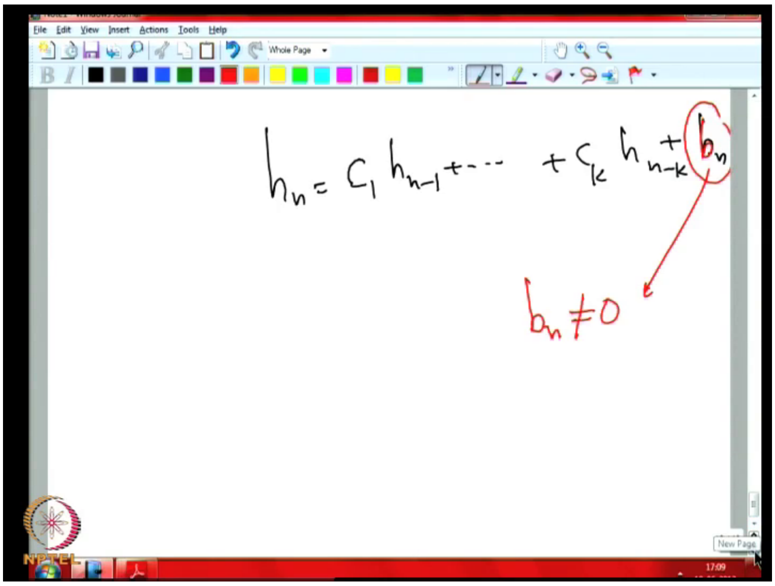

Again, if we go back to our, the way we wrote the recurrence relation, it was h of n equal to c 1 times h n minus 1 plus c k times h n minus k. This is the homogeneous thing. In the non homogeneous case, we will also have a plus b n here, right. This one, plus b n here. This is 0 for homogeneous case. In the non homogeneous case, b n will not be equal to 0, right.

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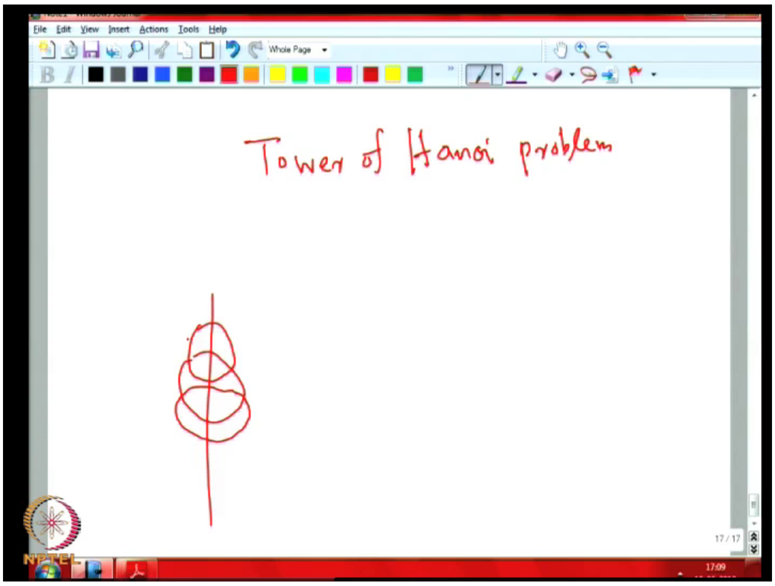

So, we can give an example. The simplest, say k equal to 1 case. The order 1 case, one of the most famous examples, the Tower of Hanoi problem. What is this Tower of Hanoi problem? So, we have these pegs, right. So, you know, so it is a, we have a peg here. In the peg, we have some disc of decreasing size. I am not good at drawing things.

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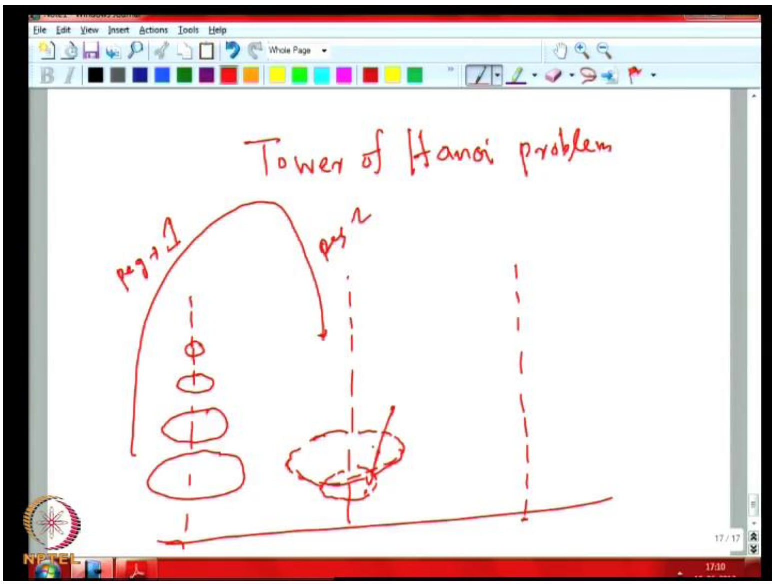

So, let us assume that there is disc 1 here, 2 here, 3 here like this. So, it is a peg here. Then we have another peg here. This is peg number 1 and there is a peg number 2 here. So, we want to move all these discs to this, right. We can use a third peg as a temporary place to keep the disc. We are not supposed to keep the disc on the floor. It can stay only in this peg. Otherwise, it will get destroyed, say. But, there is another problem. You cannot put a peg here like this and on top of that, we cannot put a bigger peg. May be the smaller peg will, sorry, bigger disc, right. This is not an allowed thing. May be this will get damaged because of its weight. So, weight was something or in other words, what we say is, we can only keep the disc on the peg. Hence, in descending order of their sizes, means, the biggest one should be in the lowest position and so on. So, you can never put a bigger disk on top of a smaller disk, right.

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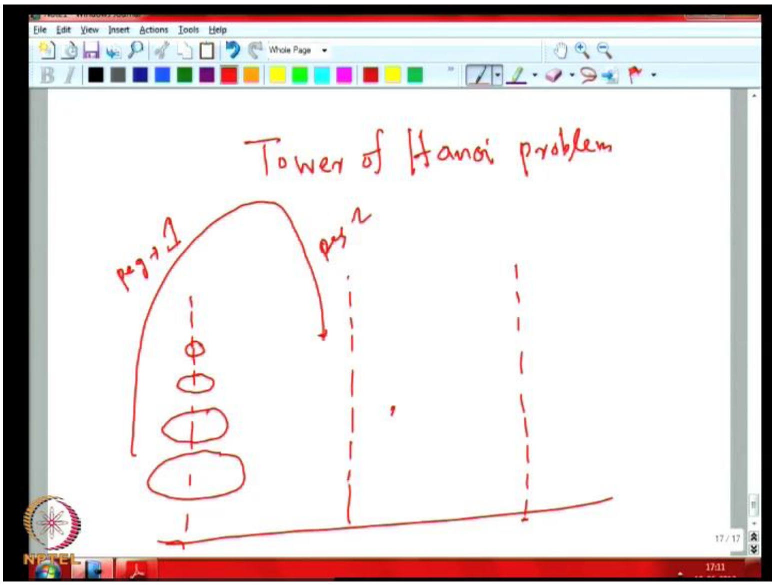

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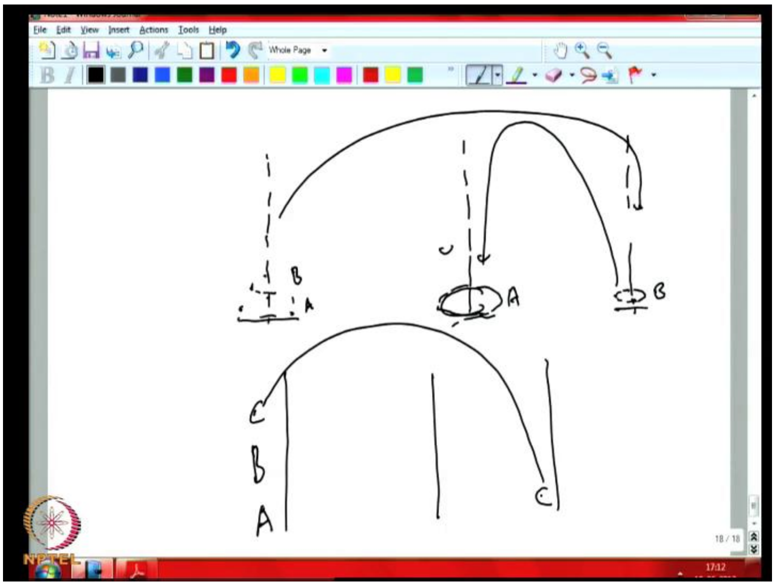

Then, how come you move this disc from one peg to other. So, one may wonder, is it possible at all. So, we can just take some simple examples. We can easily see that, we can do it. How because we have three pegs, right. There is a temporary place to put, right. So, one possibility is, we have only one peg. There is nothing to worry because we just take the peg and put it. There are two, sorry, there is only one disk. It is quite easy, but, if there are two of them, what I do is, let us say, this is a And this is B. What I do is, first I take this B, right from the peg and transfer it to the third one. So, I keep the B here. Now, I can move it here. So, A will be put here and then B can come here, right. So, this is the way one can do, right. If there are more of them, you have to be more careful. If there are three of them, A B C, first I will remove C here, and then I have to remove B here, right. Then it is ok.

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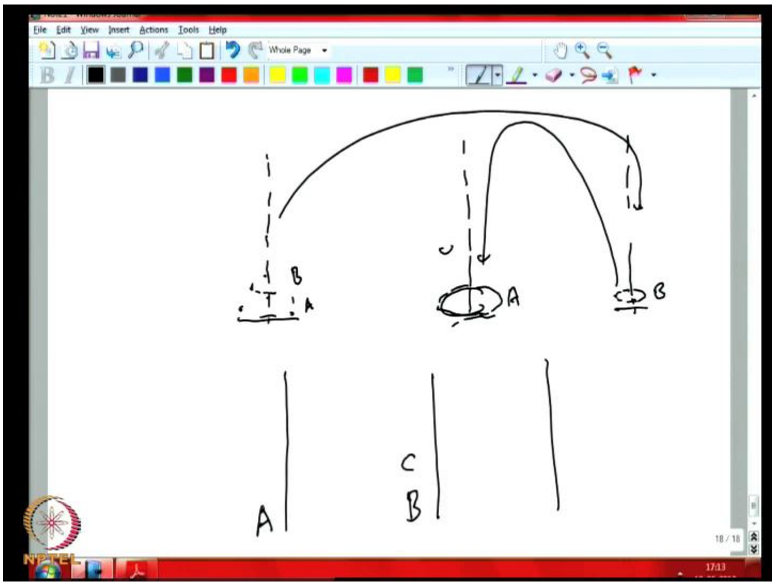

First, I have to move C here and then I move B here. I will see, for instance, A B C and C being the smallest, what I do is, I take C here first. Then I take B and put it here. Then we can have C on top of this. So, it should have been in the other way.

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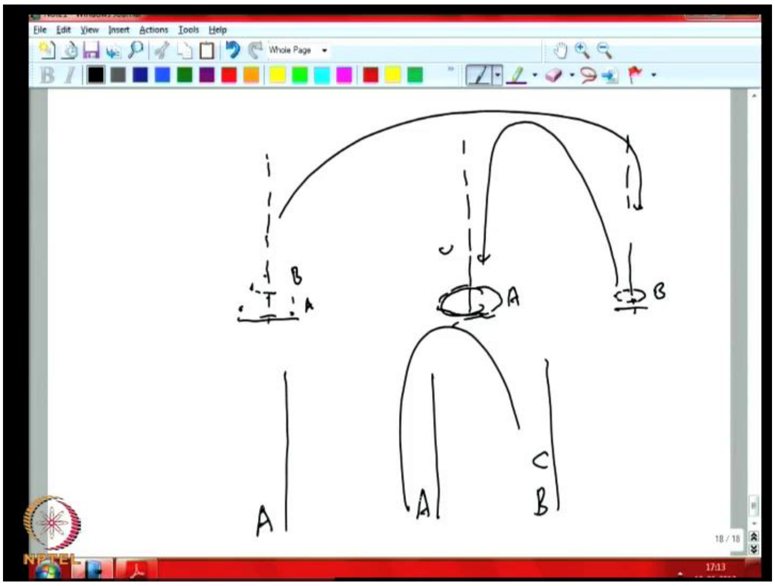

So, first C should go here and then B should go here and then I can put C from here to here and then A can come here. So, this is the way should move, right.

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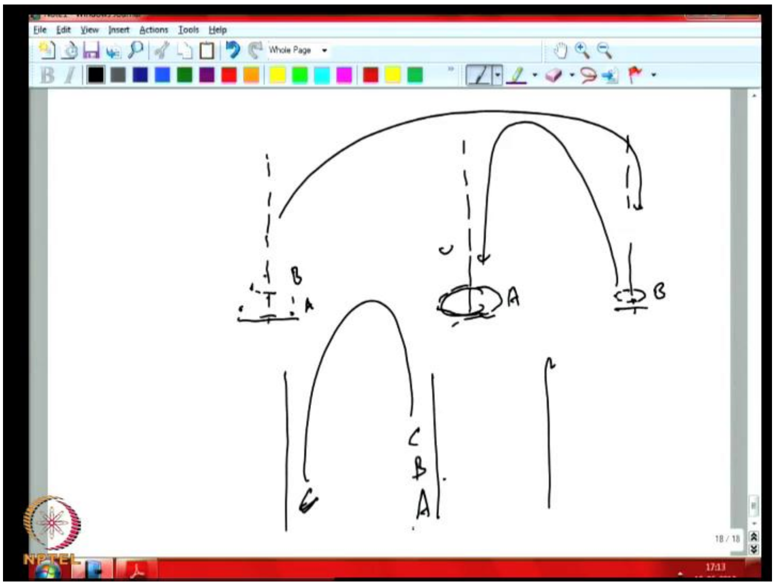

Now, what should I do? I should take that C here, right. C will come here and then B can be moved here, B. Now, C can move here. So, these are the kinds of moments we are looking for. One has to be clever, so that, we will never end up putting a bigger disc on top of a smaller disc. The question is how many moves are required, right. So, the number of moves required can be modeled using a recurrence relation.

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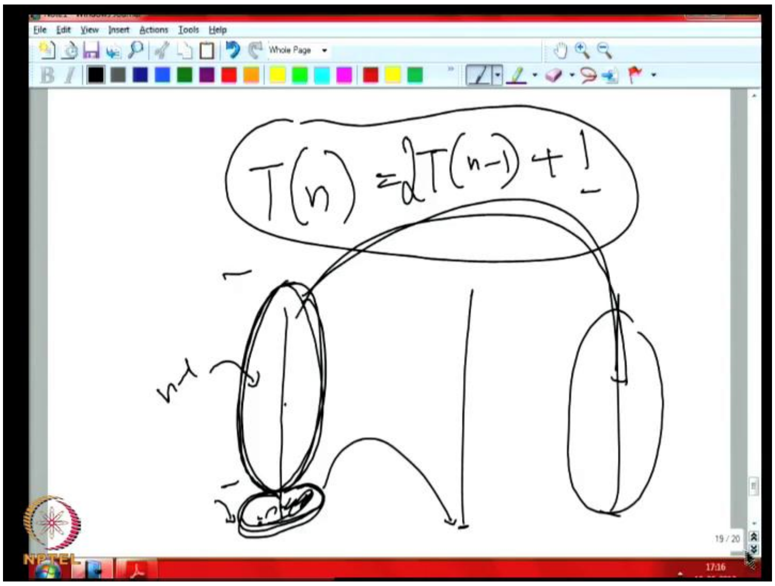

So, this is the way you do it. The t of n will be the number of move moves. So, it is clear that, so from this first peg to the, if you want to move the lowest disc, that means the biggest one from here to here, this should be empty at some point of time because it should go here and all these remaining n minus 1 disc should be moved to here, right. They should be in the proper order, right. This takes t of n minus 1 steps in whatever way you do and then you can move it to here in one step, right. So, these are the recurrence relation.

First you will move all this n minus 1 disc, which we see on top of the biggest disk, above the biggest disk, n minus 1. This should be moved to here in some way. You can use this intermediate peg here. The role of this first, second, third peg will change because now the third peg will take role of the second peg, right and now, we can move this one, that that is one more. It is very clear that you need this much also, right. Because, at some point of time, you should have this situation, that this entire thing has gone here and this is here and this is here. Then only you can move.

So, if is the number of moves required and how will you solve it? Sorry, this is the, of case, then what happens? We should also bring it back. So, therefore that is 2 into t of n minus 1 plus 1 2 into t of n minus 1 plus 1. This is the number of moves required. So first, we transfer into; we have to come back to the initial situation, right. The previous should get back to the first peg. That is why 2 into t minus 1.

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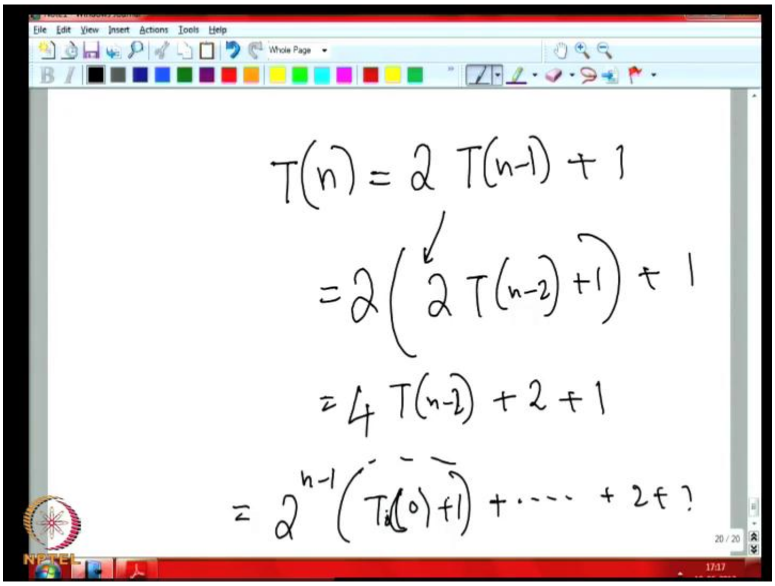

Now, t of n, so we have, because we are familiar with the recurrence relation from your algorithm class and all, so we know that, so this solution is usually, so you write it as t of n, this one and then what you do you is, substitute 2 into 2 of t of n minus 2 plus 1 plus 1. This will give 4 times t of n minus 2 plus 2 plus 1 and so on, right. So, this will essentially lead to finally, this unrolling will lead to 2 to the power n minus 1 into h, sorry, t of 0, sorry, t of 0 plus 1 plus, adding together all these things, 1 plus 1 up to here. This, if there is 0 moves, there is nothing to do.

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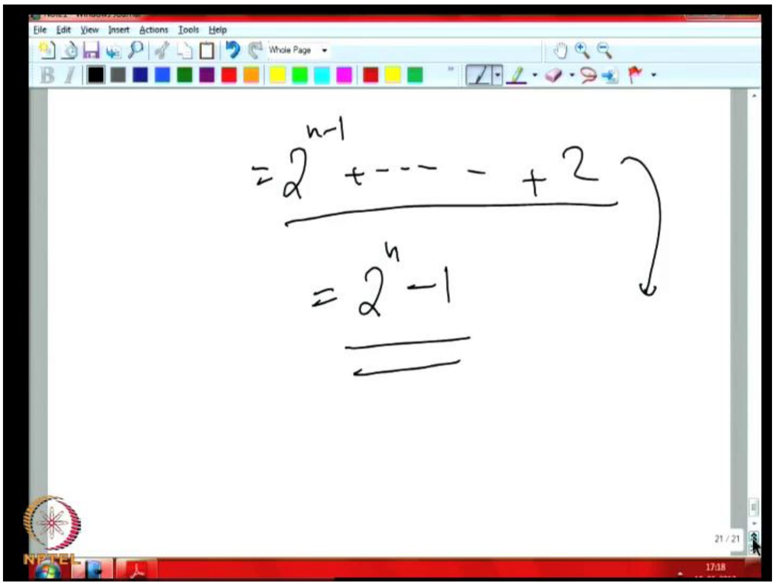

Therefore, it just t of n minus 2 raise to t of n minus 1 up to 2 and this we know is 2 to the power n minus 1, right. This is the answer for this. How did we do this thing? We just unrolled, right. So, we wrote, substituted, substituted and then we finally came to the last. So, n minus 1 steps, starting from n minus 1, we ended up with this, right. So, that is why we could do this thing. But here, the thing work because you know, we finally got this sum and we could sum it up, right. You know that depends on whether we can do that sum or not. We will discuss further on this non homogeneous surfaction in the next class.