Combinatorics
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Lecture - 25
Recurrence Relation - Part (3)

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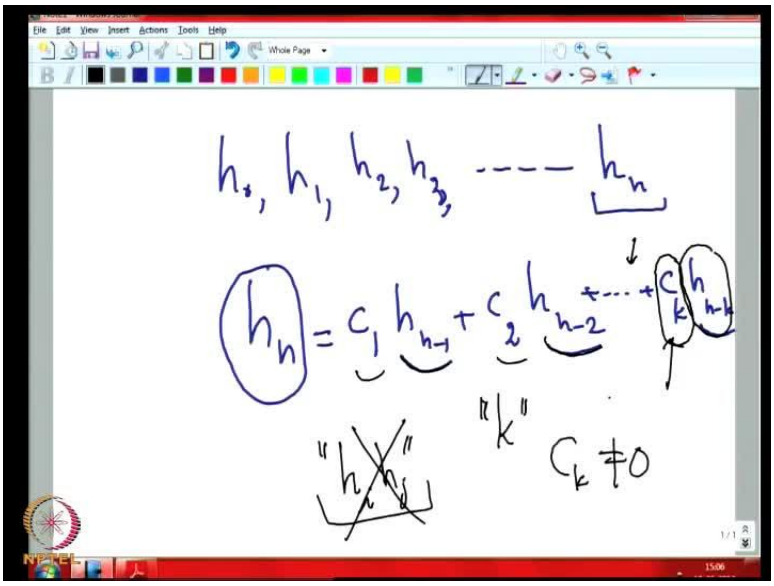

Welcome to the twenty fifth lecture of combinatorics. So, in the last class we were discussing linear homogeneous recurrence relation. I repeat what this thing was. So, we are talking about the sequence like h 0, h 1, h 2, h 3 and so on. So, the… and the recurrence relation is told; it is called as linear homogeneous recurrence relation of order k. I f the n th term can be expressed in terms of at most (( )) k previous terms; an at most k means, you do not have to use all the k previous, but you should know that you should not go below k. right. So, it is something like this. So, C 1 times h n minus 1 plus C 2 times h n minus 2 plus like that C k times h n minus k.

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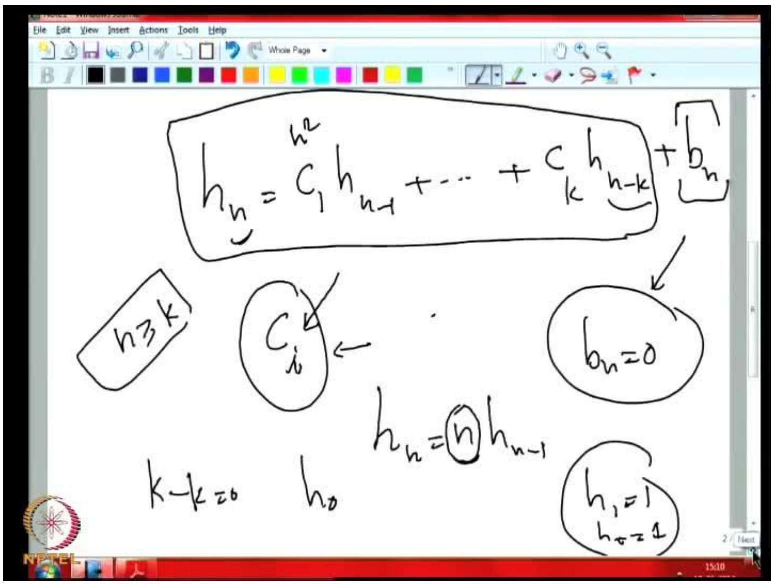

So, we are using this h n minus 1 h n minus 2 up to h n minus k. Combining them in some fashion to get the new term h n. right. So, you should note that when I say this is of order k, this recurrence relation is of the order k, what we mean is, there are… The lowest term we are taking is h n minus k. So, which means that we are assuming C k not equal to 0 here; because if C k is equal to 0, we could have stopped saying at the previous point, C k minus 1 times h n minus k plus 1. Right. Then, we can say that we have used only at most k minus 1 previous term starting from h n minus 1. Right. So, therefore at this; so, to say that it is of order k, we should make sure that C k is not equal to 0.

So, these terms can be 0, cannot be 0; may or may not be 0. Right. And also not that, this is called a linear recurrence relation because of, so in each of this term we are only using h n minus 1, h n minus 2, etcetera. We are not combining them. For instance, no terms of this form h i into h j. right. So, (( )) h i square, such terms are not used.

So, therefore in that sense, it is linear. And, finally the homogeneous recurrence relation; because we do have the constant term; so that means, sorry not constant terms, it is the term which does not involve any previous terms. So, we wrote h n equal to C 1 times h n minus 1, all the way up to C k terms h n minus k. So in general, if it is just linear and the homogeneous word is not there, then we can also have a b n here. Right. So, in the homogeneous case we do not allow this b n. right. So, b n has to be 0 for homogeneous case; in non-homogeneous case, b n can be non-0. Then note that this C i’s, this coefficient for this h i’s. So, they can be constant.

If they are constant, then we say that this recurrence relation is a linear homogeneous recurrence relation with constant coefficients. And, but in general it need not be. So, it can be a function of n. right. It can be a function of n. So, for instance, it can be something like here; n square times h n minus 1. So, if for instance, if I write recurrence relation for factorials, I will write h n equal to n into h n minus 1 with h 1 equal to 1, like h 1 equal to 1 and h 0 equal to 1. Right. So, here the coefficient is not constant. It depends on an… it is a function of n, but when we will be studying mostly the recurrence relations, where this coefficient are constants. So, the… and then there is, yes, right, these are the main points. So, we wanted to tell. And also, not that, so this relation can only be true for n greater than equal to k because we are using a k minus 1 previous term.

So, from kth terms onwards only we will get k minus 1 previous term, yes, sorry, so k; so, we are using k previous terms. So, there should be the n minus kth term available. Right so, from kth terms it is k n equal to k; so, k minus k will be 0. So, we are stating from h 0.

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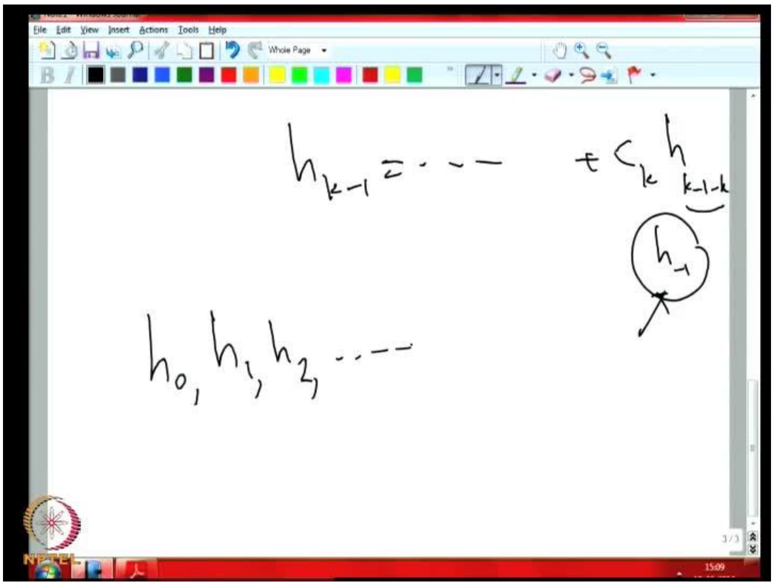

For instance, if we had tried to write this expression for h k minus 1, so that would be a problem; because h k minus 1. So, this C k times h k minus 1 minus k will come here, which would be h minus 1. So, which is not defined because sequence starts from h 0; h

0, h 1, h 2, like that. So, this relation is valid only for n greater than equal to k. These are the main points about this kind of recurrence relation. Again, linear homogeneous recurrence relation of order k and then constant coefficients; this, that is what we are going to study from now onwards.

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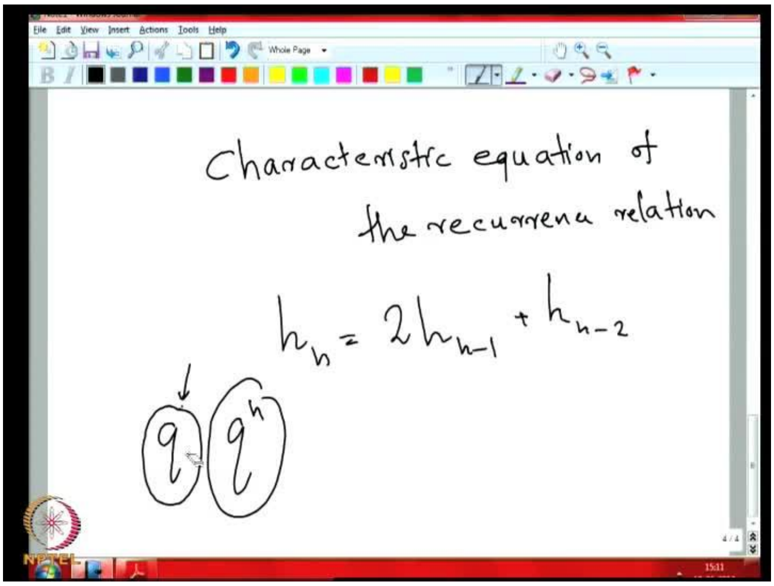

So, now the next thing is to see how a general solution can be obtained for this thing. So, say the key technique we are going to learn as like this. So, we will learn. So, we will introduce a notion called the characteristic equation of the recurrence relation. What is the characteristic equation of a recurrence relation? So, for instance, if the recurrence relation is h n equal to 2 h n minus 1 plus h n minus 2, say, then so we can, suppose so we can guess that q. so, let q be some number and suppose we have guessed that, so the solution for this recurrence relation will be of the form q to the power n.

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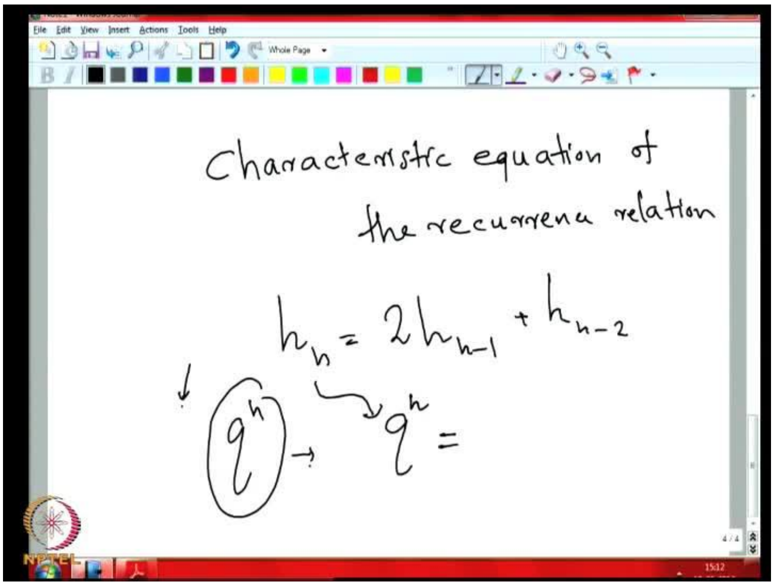

Suppose we guess like that, and then you substitute here q to the (( )), suppose it is like that, I do not know for which q it will be satisfied or whether it will be satisfied or not. So, we substitute hence, try whether it will work out or not. So, h n will become q to the power n.

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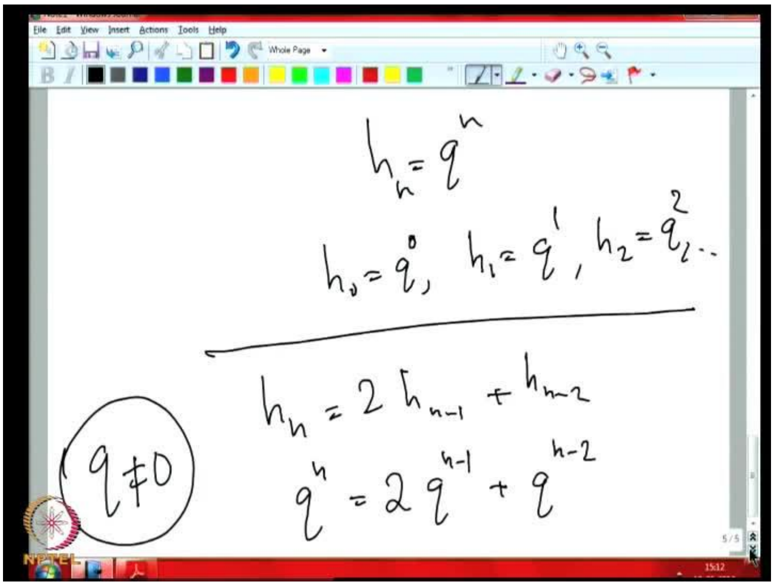

So, when I say h n will be equal to q to the power n, what it mean is, h 0 equal to q to the power 0, h 1 equal to q to the power 1 and h 2 is equal to q to the power 2 and so on. Right so, if that is the case, then for this h n equal to 2 h n minus 1 plus h n minus 2, when I substitute this q to the power n here, and then this will become 2 into q to the power n minus 1 plus, this is q to the power n minus 2. Now, we will assume that q is non-0; q not equal to 0. Then, we can cancel on both sides or in every term q to the power n minus 2.

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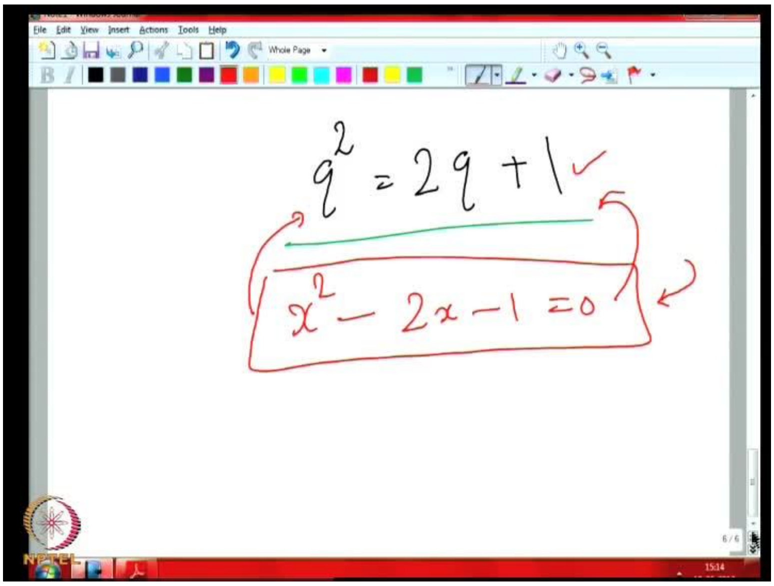

So, that will become q square; because q to the power n minus 2 is g1; 2 into q plus 1. That is what it is coming. Right. This goes away; this goes away. And, here we have 2 q left and here we have q square. So, this is the thing; now we know that. So any q, it satisfies this equation x square minus 2 x plus 1 equal to 0, sorry, say x square minus 2 x minus 1 equal to 0, it will satisfy this thing, this equation. And, we know we can solve for this thing by the usual techniques. And those, so they will be… you can use it that formula for solving the quadratic equation and those values can be given to q and this will be satisfied.

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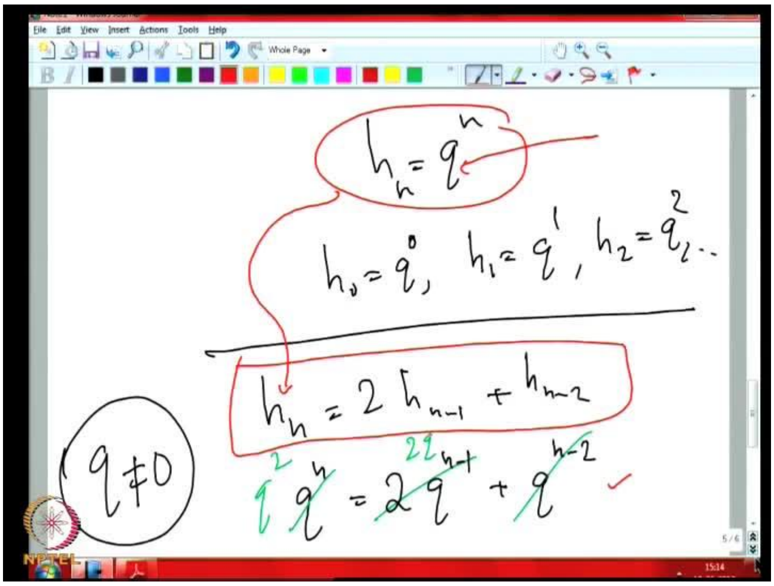

Now you notice that by our argument, if this is satisfied, so again I will head this point and say this is the characteristic equation of the given recurrence relation. And, if a particular value satisfies this characteristic equation, then of case, so this is satisfied; that means this will be q to the power n 2 into q to the power n minus 1 is equal to 2 into the 2 into q to the power n minus 1 plus q to the power n minus 2 will be satisfied. That means for this recurrence relation, this h n equal to q raise to n is indeed a solution.

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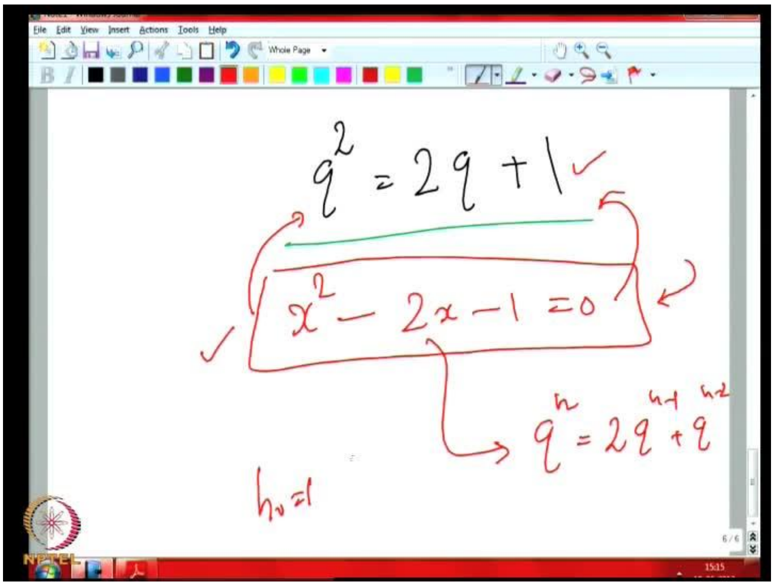

That means, if you substitute h n equal to q raise to the n for that particular value of q, is obtained by solving this characteristic equation. It will be satisfied. So, this recurrence relation will be satisfied. The only worry is that there will be some initial condition, which may not be satisfied. You know that q to the power n. right. With that particular q which is coming from this equation; so, it will satisfy the 2 into q to the power n minus 1 plus q to the power n minus 2 solution.

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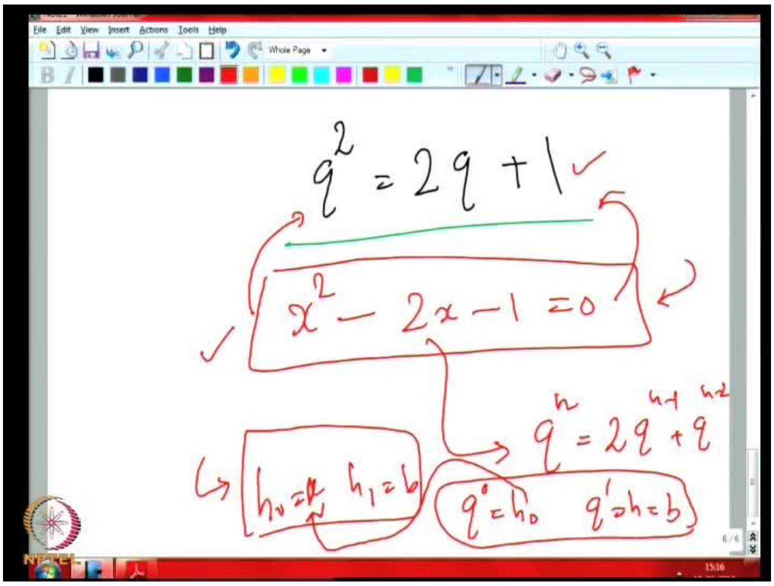

But, we also need that h 0 is equal to certain thing. So in this case, you know, this recurrence relation is written for all n greater than equal to 2; so, h 0 equal to something; h 1 equal to something. We will have 2 initial conditions for this kind of a recurrence relation. The point is we have to make sure that these initial conditions are also satisfied by a solution. So, if you just take that the root of this equation, characteristic equation and say that let that be q, so we take that q to the power n and then put n equal to 0, that will be q to the power 0; this will be h 0. So, this has to match our initial condition that h 0 is equal to a. Similarly if you put n equal to 1, q to the power 1, that has to be equal to b. So, this may not happen all the time. Right. So, we should develop some techniques for that. So, that trick is this.

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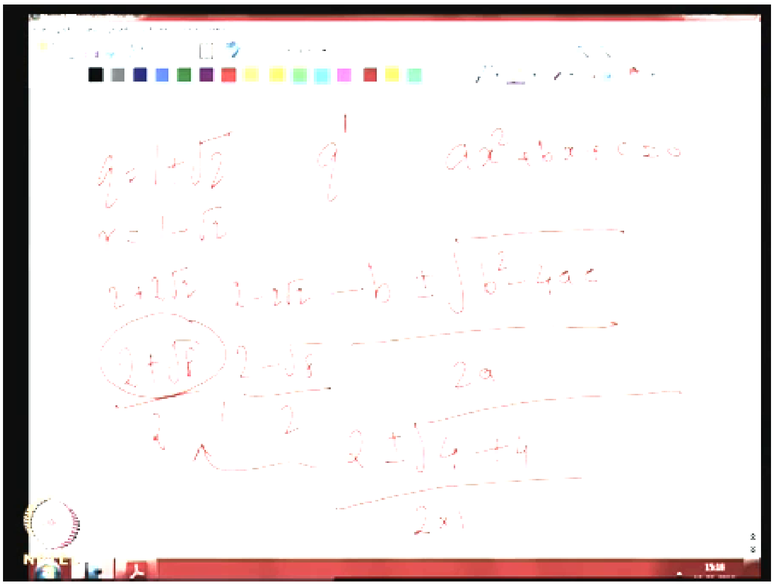

So, we will say that q to the power 1 is a solution. So, suppose we have a… suppose let us look at the characteristic equation here, x square minus 2 x minus 1. right. So, if I had solved it, that will be minus b plus 2 plus or minus this square minus 4, that is 4, sorry, solution is minus b plus whole minus b square minus 4 a c by 2 a; this kind of a solution; a x square plus b x plus c equal to 0.

Now, in this case we have a equal to 1, b equal to minus 2, c equal to minus 1. So, this will be 2 plus or minus, so, four minus… because a is also minus 1 here. So, minus four a c will be just four; plus four by 2 into 1. So, this is what? This will be 2 plus root eight by 2 and 2 minus root 8 by 2. These are the 2 solutions. Right.

Now, if you take q is equal to 2 plus root eight by 2, that means 1 plus you take 4 out from here; root 2, right, 1 plus root 2. Or similarly, see what I do is, I can take, I can write this portion 2 into 2 plus 2 into root 2. Similarly, this is 2 minus 2 into root 2. So, I cancel of 2. So, this is this q and they, let us say other solution is r. so, that is 1 minus root 2. Right. These are the 2 solutions. so, now when I take, put q equal to 1 plus root 2, that will indeed satisfy the recurrence relation, but not necessarily the initial condition.

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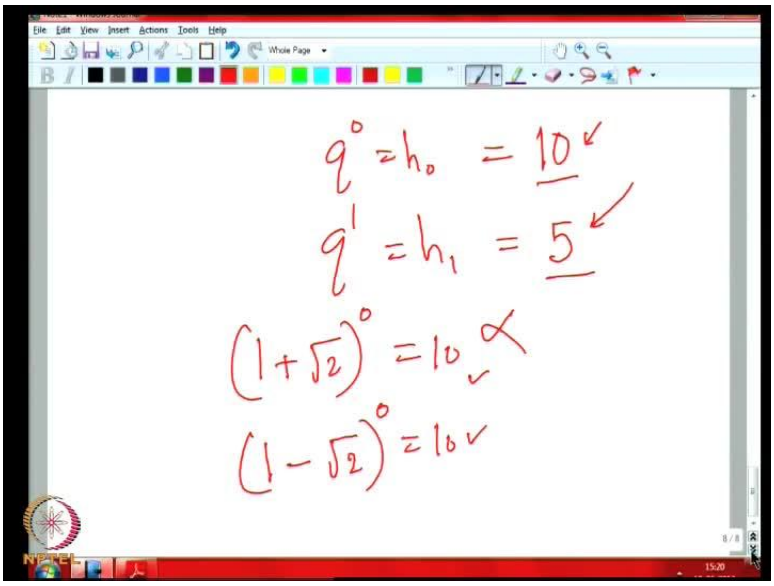

In general, the initial condition can be q raise to 0 equal to h 0 equal to some a. right. Some a; maybe we can say this is equal to ten. Similarly, q equal to q raise to 1 equal to h 1 is equal to something; it can be 5 (( )) or may be. So, this is given as part of the problem. So, we cannot control this thing. Right. So, then we get that. For instance, if you have taken q equal to this 1, there was 1 plus root 2. We will just get 1 plus root 2 to the power 0 is equal to ten, which is not true; because this is definitely wrong. Similarly, 1 minus root 2 to the power 0 is equal to ten is also wrong. So both choice of q, either we take the solution 1 plus root 2 or 1 minus root 2, both are not valid for the initial condition.

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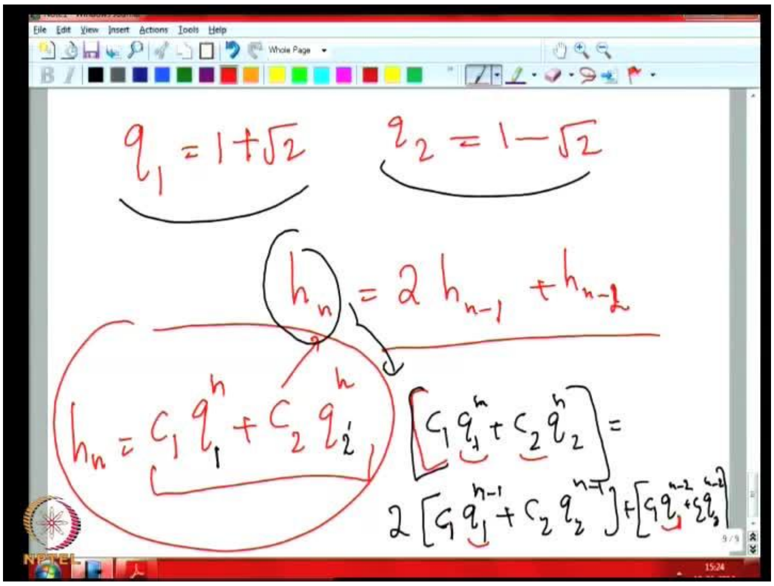

How will you, how will you tackle this situation? Right. so, the answer to this question is that, suppose say let us say q, I will write q 1 equal to 1 plus root 2 and q 2 is equal to 1 plus minus root 2, these are the 2 solutions.

So, we will notice that our recurrence relation h n equal to 2 into h n minus 2 plus h n minus 1 will also satisfy any combination of these solutions. That means, if I had taken C 1 times q to the power n plus C 2 times q to the power n, it is supposable solution for h n; that would also be satisfied. That is also correct because if we, if I substitute h n equal to this and h n minus 2 is equal to C 1 into q to the power n minus 2 plus C 2 into q to the power n minus 2 and h n minus 1 is equal, sorry, h n minus 2 is equal to C 1 into q to the power n minus 2 plus C 2 into q to the power n minus 2, then also this will be satisfied. Why is it so? Because you just have to substitute and see; because on this side, sorry, in this case, in this special case, I will show the substitution.

So, for instance, in the… when I substitute for this h n, so what we get is C 1 into… so here C 1 and C 2 are just some constants. So, C 1 raise to q and plus C 2 into q n. This is what. This will be equal to 2 into C 1 into q raise to; so this q 1, q 2, q 2; q 1 raise to n and q 2 C 2; C 1 q 1 raise to n plus C 2 into q 2 raise to n. See, what I am trying to do here is, I have 2 possible solutions; q 1 raise to n and q 2 raise to n.

So, now I am saying that, if I had taken some constants, 2 constants C 1 and C 2 and put up in new solution h n is equal to C 1 into q q 1 raise to n plus C 2 into q 2 raise to n.

This would also be a valid solution for our recurrence relation. So, this is a very simple fact because see, if we substitute here like this, right, so this will just be like C 2 into q 2 raise to n minus 2. Now, you know combining things; this is q 1 raise to n 1 on this side. So, I will only consider the terms involving q 1 raise to n. So, here we have q 1 raise to n. So, here we have q 1 raise to n minus 1; here we have q 1 raise to n minus 2. So, here this is C 1 into q 1 raise to n; here it is C 1 into 2 times q 1 raise to n; here C 1 into q times, sorry, C 1 into q 1 raise to n minus 2. So, now we know, earlier we know that if C 1 was not there, C 1 is just a multiplier on both sides; q 1 raise to n is indeed equal to 2 times q 1 raise to n minus 1 plus q 1 raise to n minus 2. right.

So, that is still correct. We can substitute with that here on this thing; just C 1 is another multiplier. So, both sides we have C 1. Similarly, q 2 raise to the n is equal to q times q 2 raise to n minus 1 plus q 2 raise to n minus 2. And, we have the extra multiplier; this C 2 on both sides. Therefore, they just cancel off. Right.

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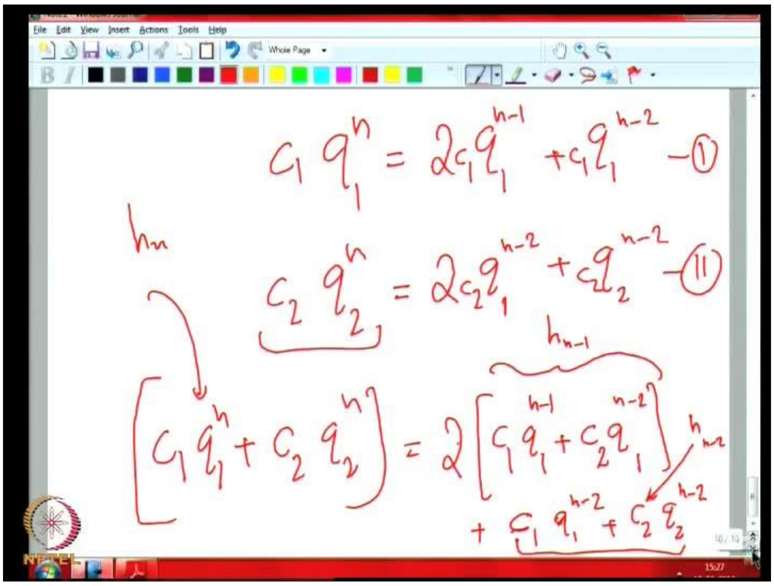

So, if you have any confusion in this discussion, I put it like this, we already know that q 1 raise to the n is equal to C 1 C 1 raise to, sorry, forgot about the C 1. We already know that q 1 raise to n is equal to 2 into q 1 raise to n minus 1 plus q 1 raise to n minus 2. This is equation 1. Now, similarly you know that q 2 raise to n; because this is also, q 2 is also a value which is obtained by solving the characteristic equation; this q 2 raise to n minus 2 plus q 2 raise to n minus 2. Right.

Now, you can simply multiply this equation 1 by C 1. So, here it will be C 1 C1; you can multiply C 2 here, C 2 C 2. Now, add it together; so that will, this will become C 1 raise to q 1 raise to n plus C 2 into q 2 raise to n. right. This was the proposed solution for h n now. right. And, now if you multiply it here, we will get 2 times C 1 raise to q 1 raise to n minus 1 C 1 into q 1 raise to n minus 1 C 2 into q 1 raise to n minus 2. And, this is, this portion is the h n minus 1, according to our new solution; new proposed solution. right. Here if you, when you multiply it, C 1, this is C 1 into q 1 raise to n minus 2 plus C 2 into q 2 raise to n minus 2. Right. This is the proposed solution for h n minus 2.

So this is indeed, h n is equal to 2 into h n minus 1 plus h n minus 2. So, this also works. To summarize what we are saying here is, if q 1 raise to n is a solution and q 2 raise to n is a solution, for any 2 constants C 1 and C 2, C 1 times q 1 raise to n plus C 2 times q 2 raise to n is also a solution. right. But, what is the good about it? So, we hooked up all this things. The good thing is C 1 and C 2 are 2 constants, which we are free to select.

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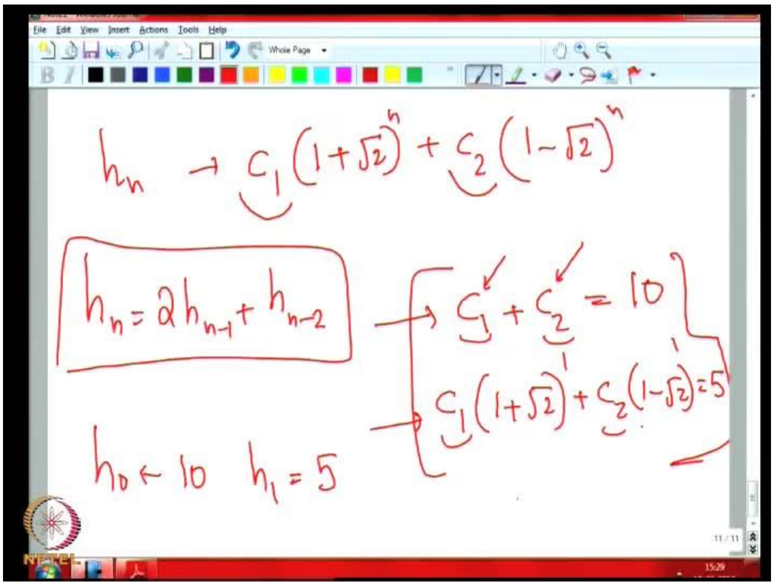

And, we can select those constants in such a way that, the initial conditions are satisfied. So, it is something like this. So, for instance, C 1 into, so in our case it is 1 plus root 2 raise to n plus C 2 into 1 plus root 2 raise… 1 minus root 2 raise to n. right. If this is the equation, sorry, this is the new solution proposed for h n, so we are claiming that for any C 1 and C 2, this is correct. That is what we have seen there. Right. This is recurrence relation only. The recurrence relation will be satisfied; that means, recurrence relation is

2 into… this relation will anyways be satisfied. Right. The only thing we are worried about is the initial conditions. Mainly when n equal to 0, what will happen? When n equal to 0, this will simply become C 1 plus C 2 is equal to… So, if it was 10, we can write it like this.

And then, the next thing; so, when n is equal to 1 what will happen? C 1 into 1 plus root 2 raise to 1 plus C 2 into 1 minus root 2 raise to 1 is equal to… if you put, what was the thing you put here? Five, suppose. Right. So, we will get like this. Right. Now we know that this can be solved (( )) simultaneously, simultaneous equations. And, you know this is definitely not a multiple of this thing. So, you can try to solve it. So, I do not, I cannot waste time solving it here. Right. So, now you see that the advantage is, now you can get to values for C 1 and C 2 here by solving these equations such that, these initial conditions are indeed met. When you put n equal to 0, this proposed solution, that is, h 0 will indeed be equal to the given value ten. Right. Similarly h 1; when equal to 1, they will indeed be the value, say five. right. So because we will be selecting the constants in such a way. Right. this is the technique.

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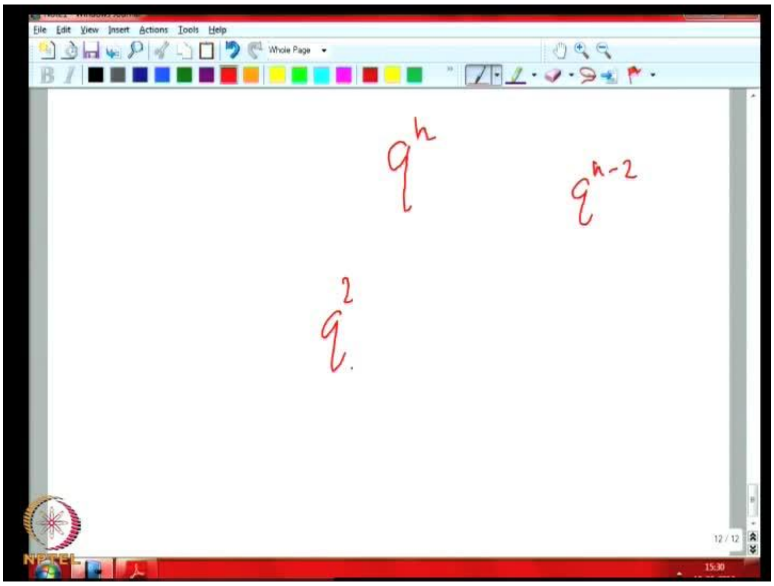

Again, once I summarize what we have d1 as. So, we introduce this concept of characteristic equation. That means, this was by… we wanted to make sure that for some q, let q to the power n will be a solution for that recurrence relation. For that thing, we cancelled that terms say q to the power n minus 2 from both the sides after substituting q raise to n 4 h n or q raise to n minus 1 four h n minus 1 and q raise to n minus 2 four h n minus 2. And then, we got 1 equation. It is a quadratic equation because we only had maximum q to the power 2 there. And, we solved that quadratic equation and got the 2 possible values, which will work for that. That means, if those 2 values are used for q, then indeed that recurrence relation will be satisfied.

Then, we saw that the only trouble was about the initial condition. Now, when you put n equal to 0, we are getting some value for once, say 1 of the solutions of that characteristic equation was taken for q and that q to the power 0 is not necessarily equal to the h 0 value we have given. Right. Similarly, if you try the other root of the characteristic equation, even then it may not work.

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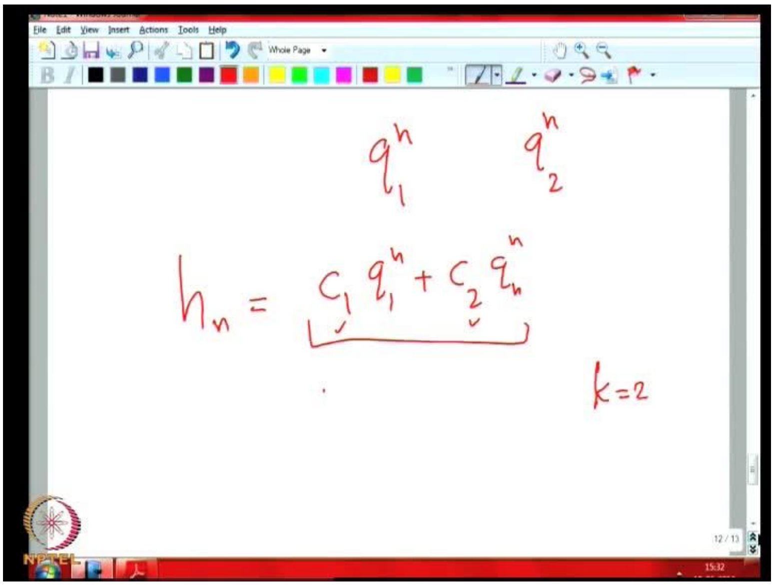

Now to, the solution now was to not that, if q 1 raise to n is a solution for h n and q 2 raise to n is a solution for n, that means, if they were satisfying the recurrence relations, then C 1 times q 1 raise to n plus C 2 times q 2 raise to n also will be satisfied. So, we showed it by explicit writing 2 equations; multiplying by C 1 1 of them and C 2 by the C 2, the other equation and then adding them together. We showed that it is indeed the case. And therefore, so we can think that this is a solution for this thing. Right.

So, for instance, if we substitute this in the recurrence relation for any value of C 1 and C 2, it is working out. Now, we just match; put n equal to 0, and then see what is happening. It was just C 1 plus C 2 equal to something. Right. And, similarly put n equal to 1, we get some equation. So, we get a simultaneous equation in 2 unknowns; name C 1 and C 2. We wanted to figure out for which C 1 and C 2, this initial condition will be met. And, we can get that right. Now, we will generalize these things. See, I use the small recurrence relation which order, whenever the k s, where k equal 2; that means, recurrence relation of order 2 to illustrate these things to, right, it is easier to explain. . That is why it was selected, but we can do it for order k k's without much difficulty. Right.

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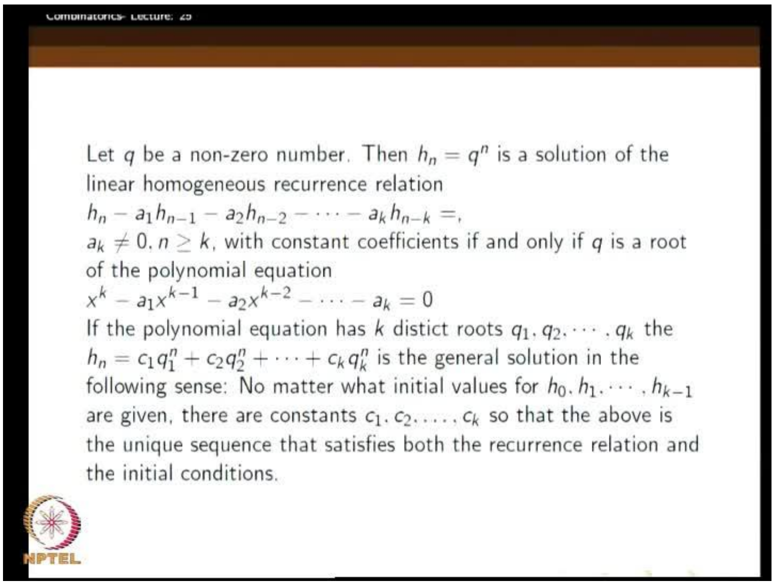

So, that is what is written in this thing. So, here we are considering a recurrence; general recurrence; linear homogenous recurrence relation with constant coefficient. So now, we can rewrite that the recurrence relation is h n minus a 1 times h n minus 1 minus a 2 times h n minus 2 a k times h n minus k equal to 0 with; so, here I should have put 0. So with a k not equal to 0, n greater than equal to 0, and then the corresponding characteristic equation will be x raise to k into a 1 into, a 1 into x raise to k minus 1 minus a 2 into x raise to k minus 2 minus up to a k equal to 0. Right. This is the characteristic equation.

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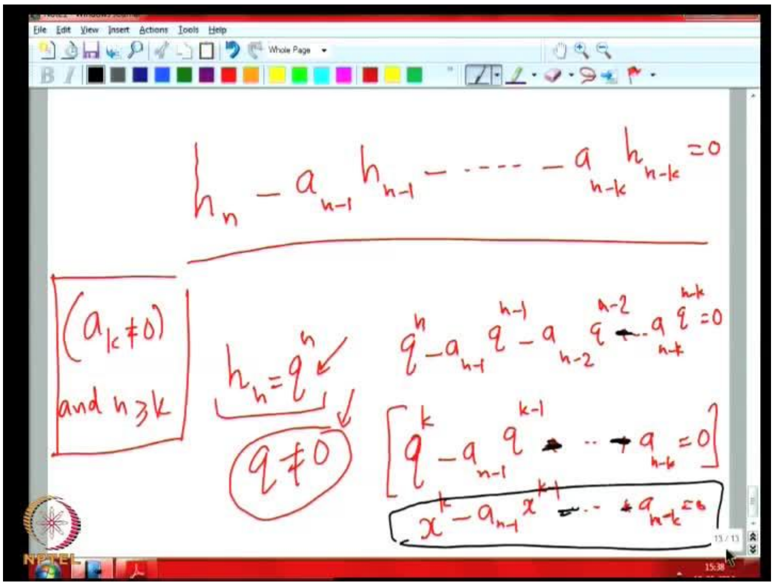

So, I can make it clear here. So, we first rewrite that recurrence relation, general recurrence relation, like this. this is the (( )) because h n equal to this, we wrote all the terms from the right hand side to the left hand side. So, here the conditions were a k not equal to 0. And, what is written here will work only for n greater than equal to k. right.

Now, we just guessed that h n equal to q raise to n. So, for some q, you have to decide. It will be a solution for this thing. You substitute it, you will get something like q raise to n minus a n minus 1 q raise to n minus 1 minus a n minus 2 q n minus 2. All the way, a a n minus k times q raise to n minus k equal to 0. You can cancel off q to the power n minus k. we are assuming that q not equal to 0. We will select some q which is not equal to 0. Right.

Now, so we can cancel q to the power n minus k. We get q to the power k then alright, k minus a n minus 1 q to the power k minus 1 and plus, here a n minus k is equal to 0. And, we can see that, this means q is a solution of this following equation namely x to the power k a n minus 1 into x to the power k minus 1 plus a n minus k equal to 0. And, this equation is the characteristic equation; x to the power k minus a n minus 1 x to the power k minus 1 plus a n, until a n minus k equal to 0. That is easy to get. 1 does not have to memorize it. We just have to follow this procedure and you will easily get; x to the power k minus a n minus 1 x to the power k minus 1 plus into… so, this is minus, 1 minus here also. So minus, minus, this is going to be the… everything is minus is here; because all terms are minus. ok. So, this is going to be the characteristic equation.

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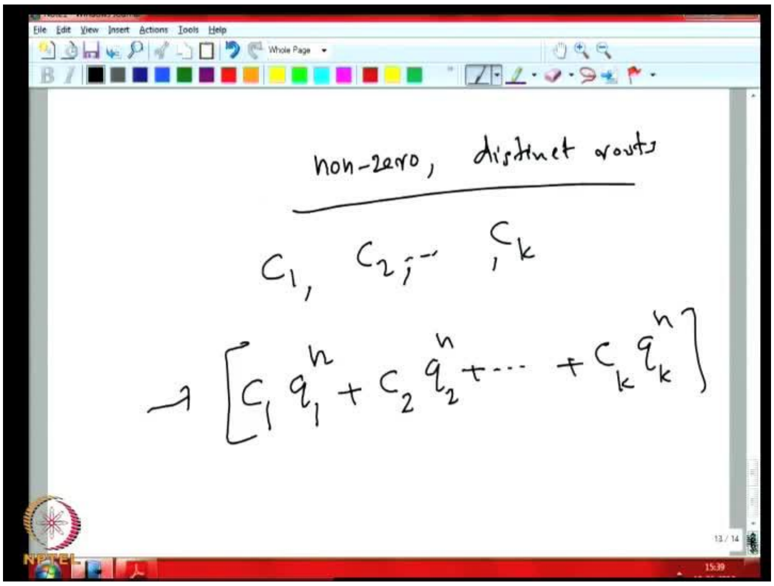

Now, of course we have to find the roots of this characteristic equation. This is a degree k polynomial here, and that is equal to 0 what you are saying. It may have, it will have k roots. Right. So, you can take any root of it; say q is a root. So, it is clear that this equation will be satisfied by q. That means, q raise to k minus a n minus 1 raise to q raise to k minus 1 minus, minus is it, that a n minus k is equal to 0 will be true. Right. You have to take a non-0 k. Now, yes if you get; so, we will consider this situation, where this characteristic equation has all non-0 and distinct roots; distinct roots and non-0. Right. All are non-0s and distinct.

Then, we can also see that, we can see that, by the same procedure we did in the case of order 2, we can select constants; C 1, C 2, C k, sorry not necessarily C 1 C 2 C k, let this constants. So, here we are right. So, this constant, this n minus 1; so, we can use some constant C 1, C 2, C k. actually, any constant will work. Right. Does not matter which constant you select. So, if you combine the solutions C 1 into q 1 raise to n plus C 2 into q 2 raise to n plus C k into q k raise to n, then this will also be a solution. This is nothing surprising.

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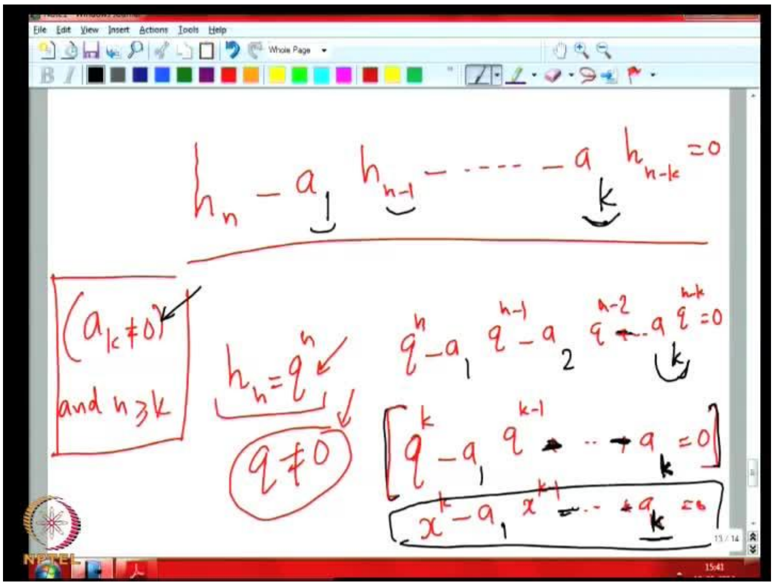

You see when you substitute q 1, this kind of an equation is valid. Right. And then, when you substitute q 2, then also q 2 raise to k minus a n minus 1 into q 2 raise to k minus 1 minus a n minus 2 into q 2 raise to q minus 2 like that, minus a n minus k equal to 0 is valid. Now, you can multiply the first equation by C 1. That is still valid. And then, you can multiply the second equation by C 2 and so on and add them together. So, it will be indeed correct. Right. So, and we should also note that, here when I say non-0 solutions, that is not very important because for this characteristic equation, so, I, the characteristic equations; here it is h n a 1 to a k, sorry I made a mistake here. So, h 1 minus a 1 raise to h n minus 1 minus, so a k; because see, I was just trying to use the same subscript as this thing. But, the notation says the first coefficient a 1 and (( )). That is why we have written here; so, everywhere 1, 2, k, k right.

Now if in this characteristic equation, you remember this a k is not 0, sorry in this equation in this recurrence relation a k is not 0 by our assumption. Now, when you write this thing, so after dividing by q raise to n minus k, we get an equation where this constants term is not 0. So, q equal to, x equal to 0 will not be valid for this thing. So, x is equal to 0 is not a solution of this thing.

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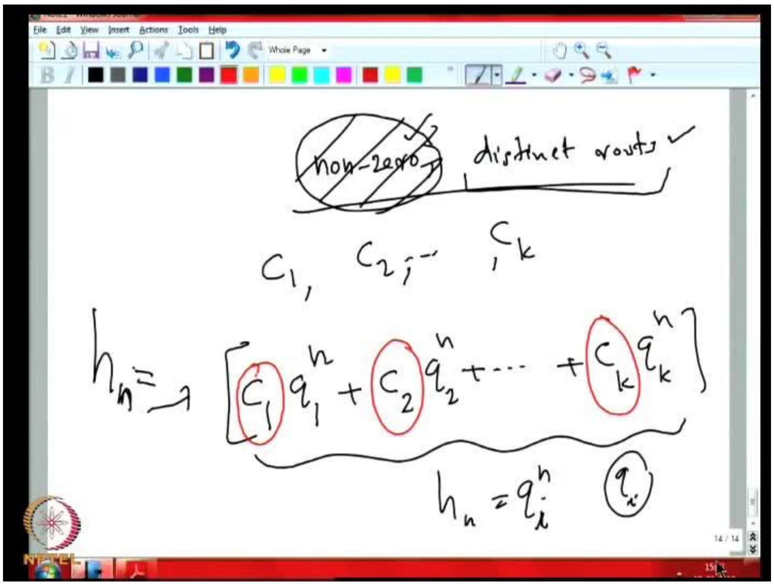

So, the solution will indeed be non-0, right, solutions will be ((_)). So, this is not required. We can just assume that they are all non-0. Right. That is the way we have set up things because our coefficients where selected in such a way that, also we had that assumption that a k equal to non-0. Right. Therefore, there will be the constant term there. It is non-0. Therefore, x equal to 0 will not be a solution for that. Right. Now, we just have to say, suppose there are all the roots are distinct; C 1, C 2, C k.

So, now we can combine the solutions q 1 raise to n. So, h n equal to q i raise to n is a solution for each of this i k roots, where q i is the i th root. Ok. So, that is what it is; because it is a root, it should be a solution. That should, that it should satisfy the recurrence relation. So, it is clear. Right. That is the way we have set up this thing. Now, we also had seen that we can combine it in this way. And, this is also a solution because h n, if I substitute h n equal to this in that recurrence relation, it will be satisfied. The good thing in this thing is that, so here we have this coefficient C 1, C 2, C k. right, C 1, C 2, C k. And these coefficients, see as of now it can be any constants.

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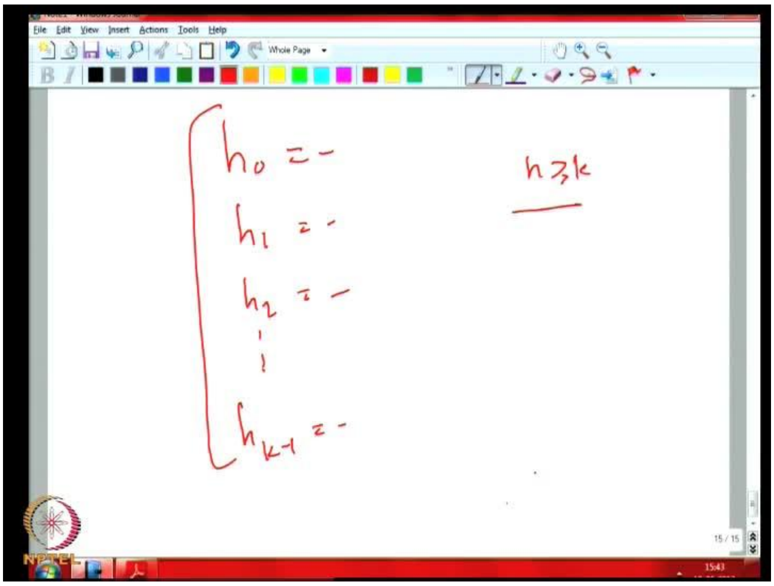

Now, we use that freedom to match the initial conditions. How many initial conditions will be there? We will have h 0 given, h 1 given, h 2 give up to h k minus 1 given; because our recurrence relation is only valid for n greater than equal to k.

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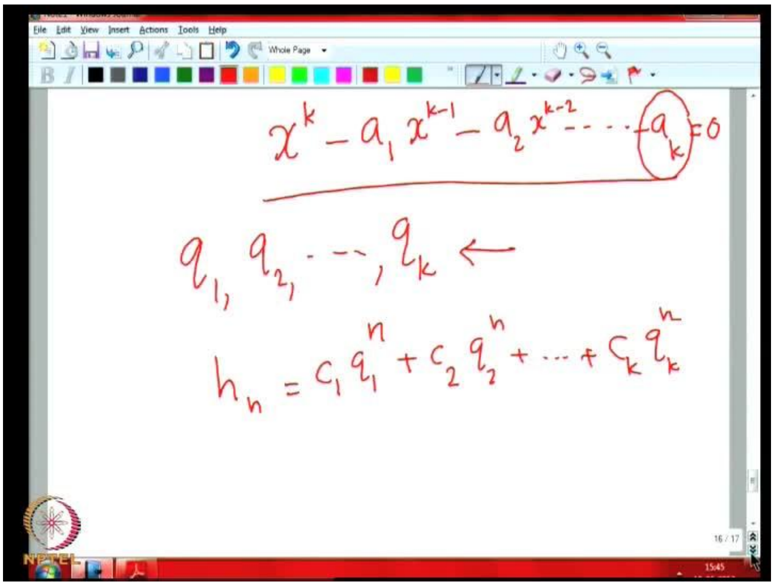

So, these initial conditions should be given. So, what we do is we substitute n equal to 0. So, when you substitute n equal to 0 in that characteristic equation, I write the characteristic equation here once again. So, this is x raise to n, sorry, x raise to k minus a 1 raise to x raise to k minus 1 minus a 2 x raise to k minus 2, finally minus a k equal to 0.

This was the characteristic equation. Now put n equal to 0, so this is the characteristic equation. Now, we are seeing that; so, this characteristic equation will be satisfied by; so, any of the solutions q 1, q 2, q k of this characteristic equation. And, we are assured that n1 of these are 0 because a k is not 0. And, we noted that then the recurrence relation should satisfy; should be satisfied, where solution of this sort C 1 into q 1 raise to n plus C 2 into q 2 raise to n plus C k into q k raise to n.

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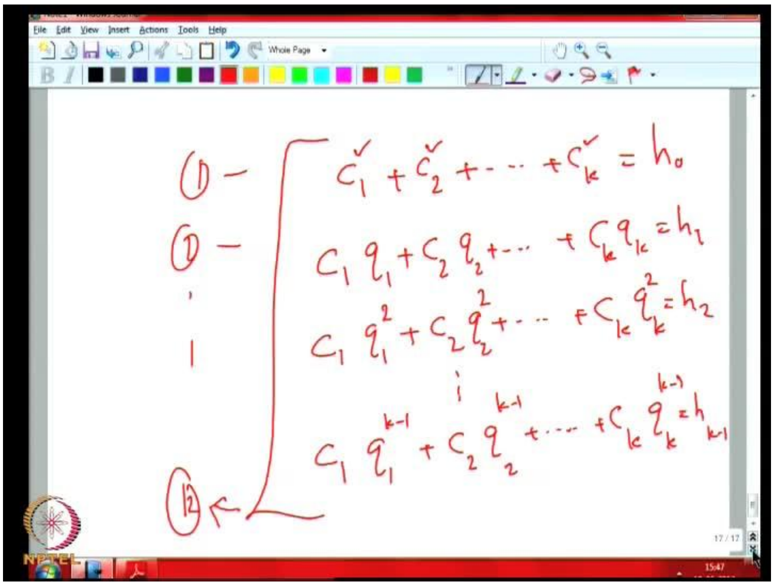

And, we put n equal to 0 in this thing. What do we get? So, we get C 1 raise to q 1 raise to 0, that is this 1, plus C 2 into q 2 raise to 0. That is again 1. So, C k is equal to something. This can be the value. This is h 0. Right now, put n equal to 1. We will get C 1 raise to q 1 plus C 2 q 2 plus C k q k is equal to h 1. Now, put n equal to 2. We get C 1 q 1 square plus q 2 square plus C k q k square is equal to h 2. like that, when you put n equal to k minus 1, we will get C 1 into q 1 raise to, sorry, k minus 1 plus C 2 into q 2 raise to k minus 1 plus C k into q k raise to k minus 1 equal to h k minus 1. So, we want to find values for C 1, C 2, C k such that, all these equations are satisfied. This is the first equation; this is the second equation; so, the k th equation, the k equations. Right. Is it possible always? So, when we say that it is a general solution, those word general is used in the sense that irrespective of the initial conditions, we can find those coefficients C 1, C 2, C k such that the solution C 1 into q 1 raise to n plus C 2 into q 2 raise to n plus up to C k into q k raise to n will indeed be a solution for an recurrence relation. The initial conditions will be met and the recurrence relation will be met.

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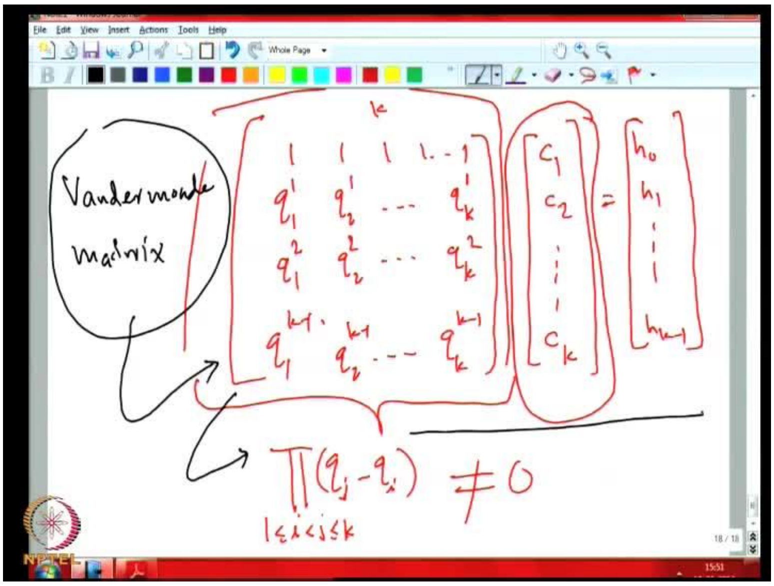

So, can we always find values for C 1, C 2, C k? So, if you remember from linear algebra, this will correspond to solving this matrix equation C k equal to h 0, h 1, h k minus 1. So, this is 1, 1, 1, 1, 1 and this is q 1 raise to 1, q 2 raise to 1, up to q k raise to 1; this is q 2, sorry, q 1 raise to 2, q 2 raise to 2, q k raise to 2 and finally this is q k minus, sorry, q 1 raise to k minus 1 up to q k raise to k minus 1. So, q 2 raise to k minus 1 and so on. So, this is a k by k matrix. And, the question is do we have a solution for this. So, can we find values for C 1, C 2, C k such that, for any value of h 0, h 1, h k minus 1. So, you know from linear algebra that this is possible when this determinant is non-0. And, this is…

So, we do not get into the details of this. So, it is known that this matrix, which is called the Vander Monde matrix. Indeed has the, the determinant is indeed non-0 because the determinant of this matrix is known to be product of for every pair i j, q i minus q j. This is what the determinant of this thing; q j.

So, we write like this. k , right, there are k q values here. Right. So, q j minus q i; this is the term. Not that, here we are resuming that all the roots are distinct. So, that means q i is never equal to q j. So, this q i q j minus q i is never 0. So, this will evaluate to non-0. So this is, this determinant is non-0 that, so this matrix. So, I am not… (( )) this matrix is Vander Monde matrix. And, this is a very famous matrix. And, you can check some linear algebra books to verify that, this determinant is indeed this. So, we want to save some time. That is why we want to get into the calculation of that.

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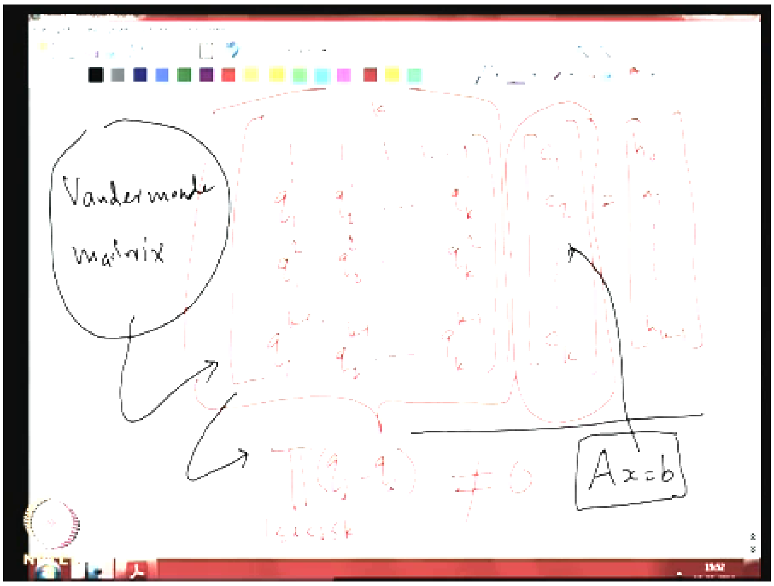

And, also here there is some linear algebraic ideas are used. Namely so the, here this linear equation, system of the linear equation is converted to this matrix equations; so that a x is equal to b kind of equation; which is you should be familiar with this stuff. Right.

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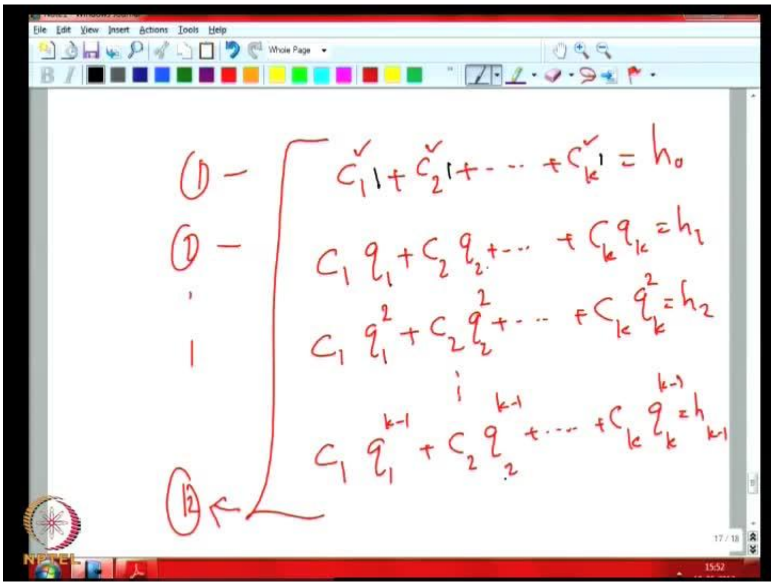

So, here instead of x, I have written this vector C's; C 1 C 2 C k. And then, this A matrix is the Vander monde matrix here. Where, yes, so the matrix comes like this. Why the matrix comes like this is because we are collecting the matrix from here. So 1, here 1, these are the coefficient C 1 into C 2 into 1. That is why that first row is coming like that. The second row q 1, q 2, q k. So, the third row q 1 raise to 2, q 2 raise to 2 and q k raise to 2 because we were substituting n equal to 2 there. And, that is how this matrix is coming from.

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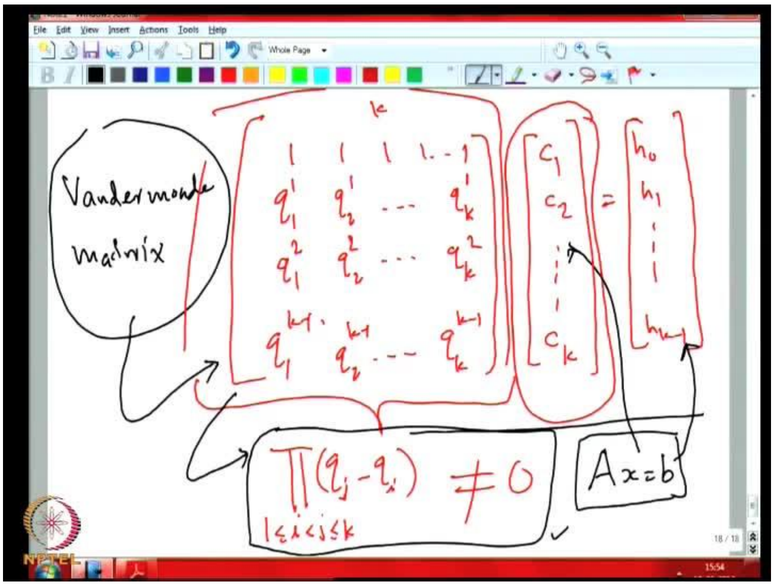

And, this answer has to be h 0, h 1; that b has to be; so, this is the b here. Right. So, this is very familiar thing in linear algebra. And then, it is known that, see this is indeed a k by k matrix and this kind of the system of equations will have a solution always irrespective of what b is. If this matrix determines this non-0, then the matrix is invertible. And, we can; the inverse of this matrix, with the inverse of this matrix we can multiply by this b. So, we will get the solution for this x. right. So that, this is what, this is some ideas from linear algebra. Right.

So, the only nontrivial thing in this part was just to know whether this matrix indeed has a non-0 determinant or not. By luckily, it is so happened that it is a very famous matrix. And, that matrix determinant is already known. Though it is not very difficult to work it out, but to save time we can just avoid it. And, indeed requires lot of knowledge and then we have to explain all the details. Therefore, we would just keep it. Right. Interested reader should read it from some other sources. Right. So, this is non-0. Therefore, indeed we can always solve for C 1, C 2, C k such that the initial conditions are met.

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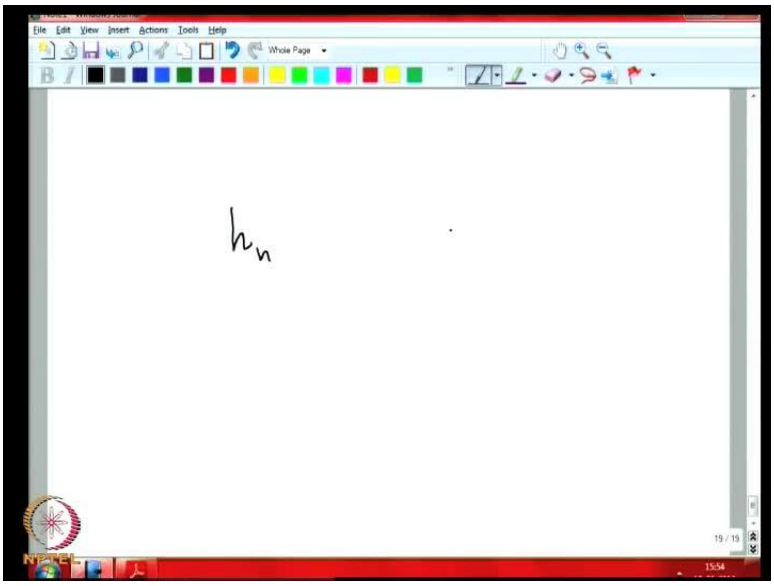

So, now we have already dealt with the general case. Right. So, what are the key points? The key points are that, so we have, we were given this linear homogeneous recurrence relation with constant coefficients. And, yes of case, so this, we found out a way to get some solutions for h n, so that the recurrence relation is valid. So, that is by solving the characteristic equation.

And this, whatever we have discussed works only when all the roots of the characteristic equations are distinct. We observed that the root because of our assumption that C k, sorry, a k, the last coefficient we used; a k times h n minus k that, a k is assumed to be non-0. This characteristic equation cannot have s 0 solution; root, 0 root. It cannot have 0 s root. Therefore, all roots are non-0, just that we are assuming that they are all distinct.

In that case, we also noted that when we combine C 1 times q 1 raise to n plus C 2 times q 2 raise to n; that is also going to be a solution. It is a trivial fact. Just multiplying by the corresponding coefficients and adding together, we show that it is indeed solution. So, the only nontrivial aspect was to somehow make sure that irrespective of the initial conditions; that means h 0, h 1, h 2, h k minus 1. These things are given values. It is not in our hands. That can be given any value by the 1's, who writes down that sequence. Right. Because the root works only starting from n greater than equal to k; n equal to k onwards only it works. The initial k minus, h 0 to k minus 1, h 0 to h k minus 1 can be set up in anyway.

Therefore, we understood that how will be the key point to us to figure out those constants; C 1, C 2, C k. So, in that combination C 1 raise to C 1 into q 1 raise to n plus C 2 into q 2 raise to n such that, all the linear initial conditions are met. That we show that, we write down all the equation n equal to 0, n equal to 1 n equal to up to k minus 1. And then, we showed that always we have solution for that. So, we can uniquely solve for C 1, C 2, C three, C k minus, C k, sorry. Then once you said those after solving that, we fix those values for C i's. Then, indeed the initial conditions will be met and the recurrence relation will be met. This is the technique right.

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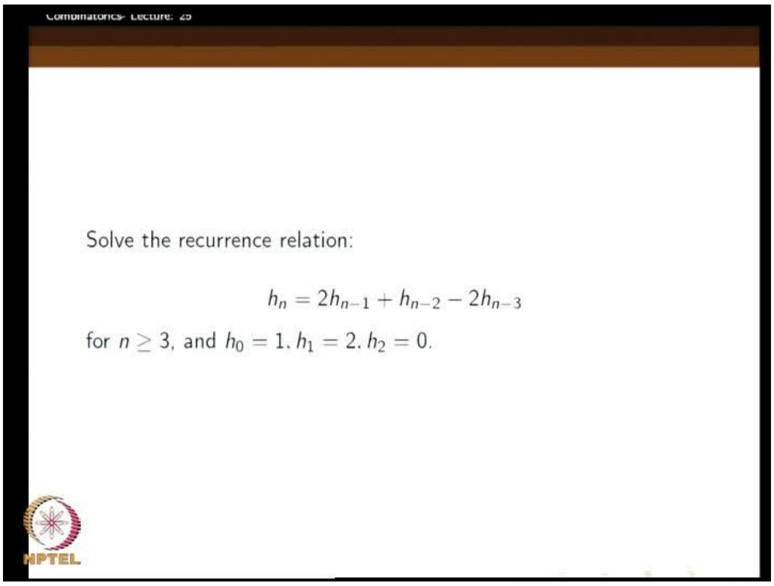

Now, of course we can quickly consider some simple examples if you want. Let us look at. h n equal to 2 h n minus 1; so, little more complicated. h n equal to 2 h n minus 1 plus, sorry, h n minus 2 minus 2 into h n.

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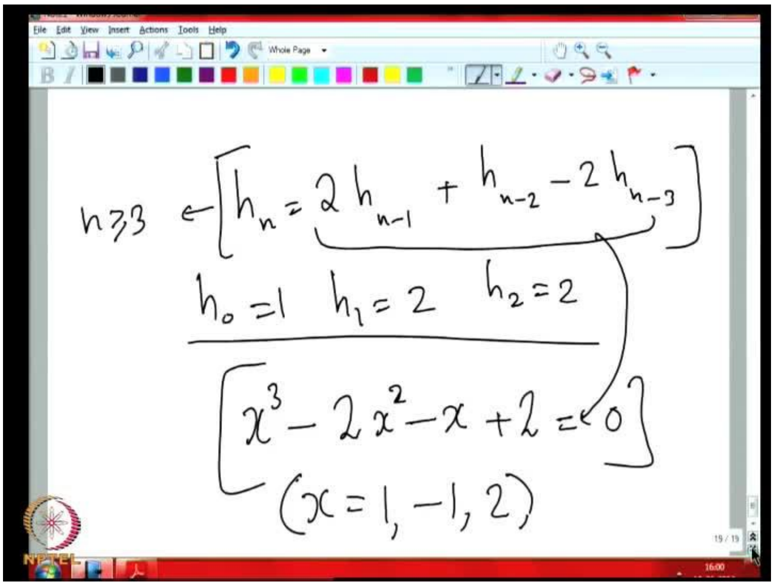

h n equal to 2 h n minus 1 plus h n minus 2 minus 2 into h n minus three. Just look at once again. And, h 0 equal to 1, h 1 equal to 2, h 2 equal to 2. Suppose this is our recurrence relation, not that this first relation will be valid only for n greater than equal to three; because if you try to write it for h 2 n equal to 2, so of course n minus 2 minus three h 1 h negative, and so that it is not defined. Right. How will you do this? First, you write the characteristic equation for that. So, recall how to do that. So, assume that for some non-0 q, write q to the power n is a solution for (( )) substitute it and cancel q to the power n minus three. So, we will get x raise to three minus 2 into x square minus, so, minus x plus 2 into plus 2, sorry, equal to 0 as the characteristic equation. What I did is, I took all the terms to this side and then cancelled off q to the power n minus three terms.

So, that will correspond to q cube minus 2 into q square minus q and then plus 2. Right. So, q is a solution for this term, this equation. This is the characteristic equation. Now, we can solve for this characteristic equation. If you solve this thing, see let us not waste time in trying to see what are the solution are known. This will give the solutions as 1, minus 1 and 2. That is easy to verify this thing. Right. That means, so 1 raise to n, so, put h equal to 1 raise to n; that will be the solution for this thing. Right. So, that means what is this? 1 raise to n, 1, 1, 1, 1, 1, like that. So, that is indeed a solution.

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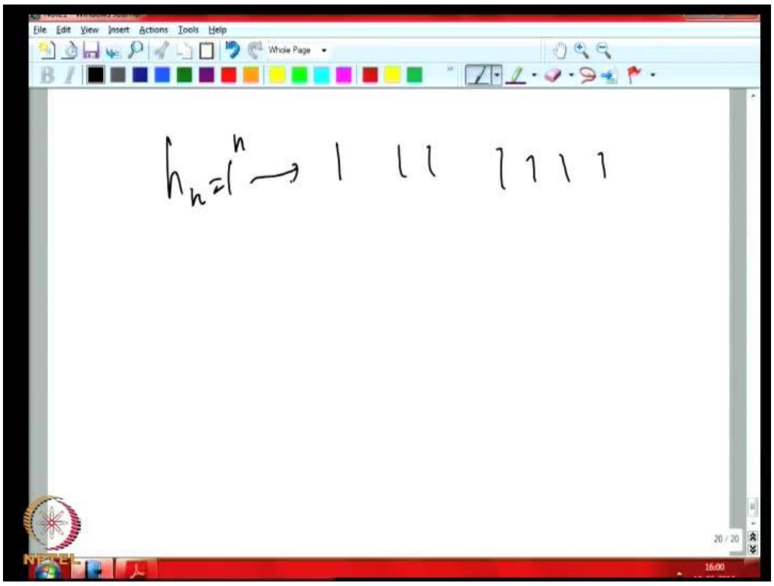

We can see that, yes, you can verify here for… if everything is 1, this sequence, right, 1, 1, 1, 1, 1, 1, 1, so on. h n equal to 1 raise to n is indeed a solution for this thing. Right. Similarly, so you substitute it. So, you can see that it is correct. Minus 1 raise to n is also a solution; 2 raise to n is also a solution.

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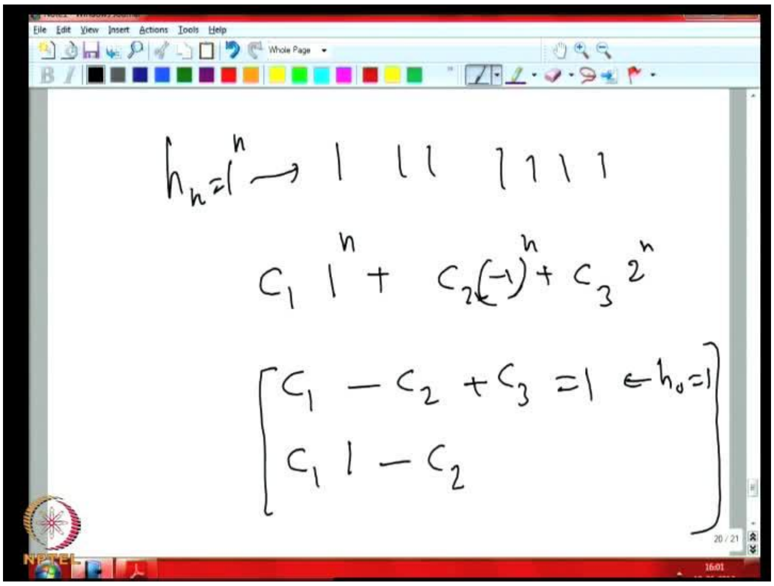

And more than that, you can select any constants C 1, C 2, C three and C 1 raise into 1 raise to n plus C 2 into minus 1 raise to n plus C three into 2 raise to n, also will be a solution for this thing. That is what we told. Right. Now, the work is only to meet the initial conditions; namely h 0 equal to 1, h 1 equal to 2, h 2 is equal to 2. Substitute for n equal to 0; that will be C 1 minus C 2 plus C three equal to 1. This is what h 0 is equal to 1 gives us. Sorry, h 0 is equal to 1 gives us. h 1 equal to 2 will give us C 1 raise to, sorry, h 1, 1 minus C 2. So, I think it is time. Therefore, I will finish off it in the next class or maybe I need not do this thing; because this is just a simple thing. 1 can; you can write the equations for n equal to 1 and equal to 2 and solve for C 1, C 2, C three. And, that will be your solution. So, we will start with something else in the next class.