Combinatorics
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Lecture - 23
Recurrence Relations - Part

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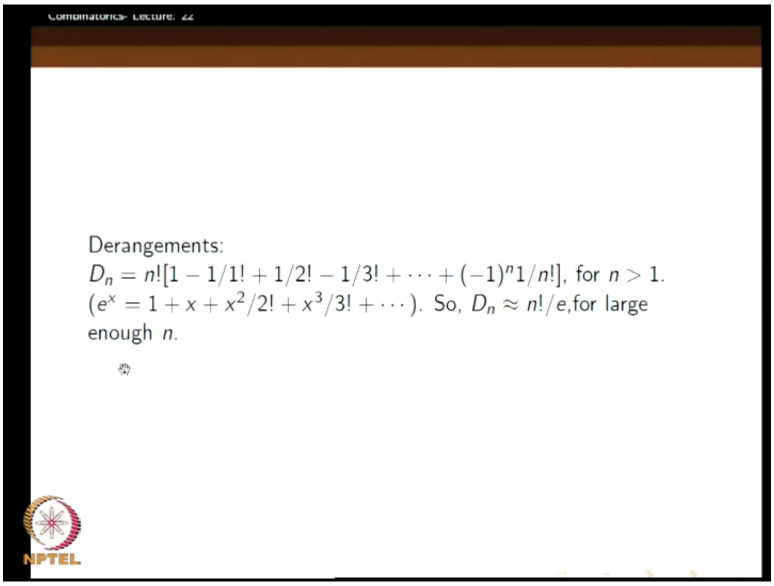

In the last class, we were discussing the derangement problem. So, we discarded the problem in the last class.

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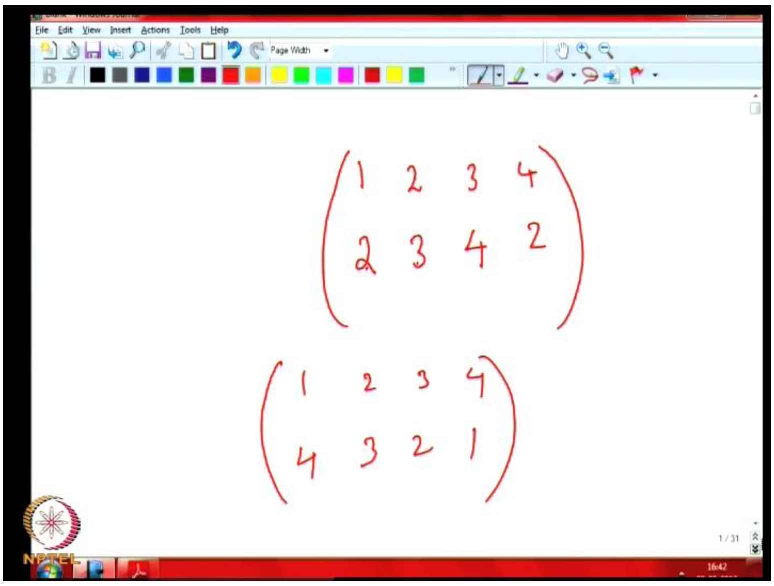

For instance we can consider the derangement, so 1 2 3 4, right? So, some derangements I can write for instance; I will write like this to make sure what we mean. So, each permutation will be written like this. So, I can start with 2, I can give a 3 here, I can give a 4 here, I can give a 2 here. This is a derangement because at i th position we do not see i. So, in one we have 2, then second position we have 3, third position we have 4 and forth position we have 2 or another one I can write 3 4, say, 4 3 2 1. This is also a derangement, right, because in the i th position if you look we are getting a number which is different from i, right? So, the question now we are going to address is how many derangements are there?

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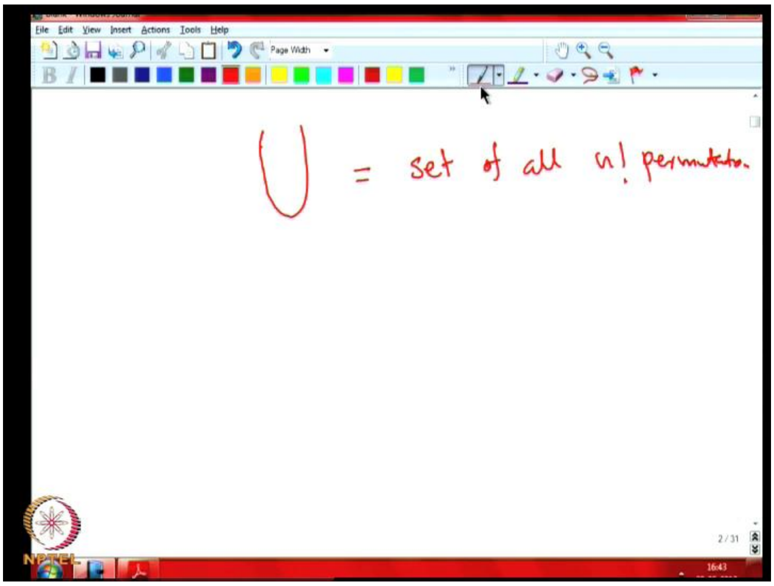

So, as usual we take the universe. Consider a universe namely the set of all permutations set of all n factorial permutations, right, this is the universe.

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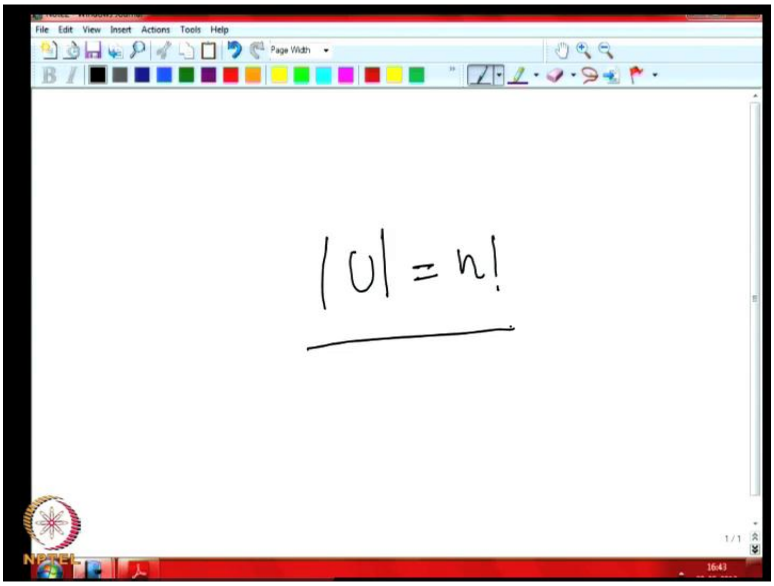

So, cardinality of U then it is obviously, cardinality of U is equal to n factorial, right? Now what we do is, we define the properties which we do not want undesirable properties.

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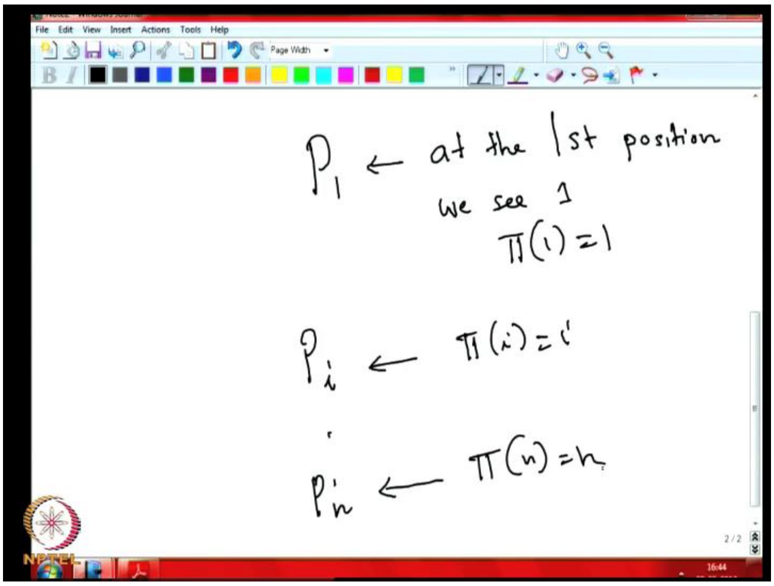

The property p 1 is that at the first position we see 1, right? So, the permutation is such that say the pi is a permutation pie of 1 equal to 1. Similarly p 1 p 2 p 3 pi will be defined as pi of i is equal to i, the i th position we see i, right? So, similarly the p n is the last property, pi of n is equal to n, right?

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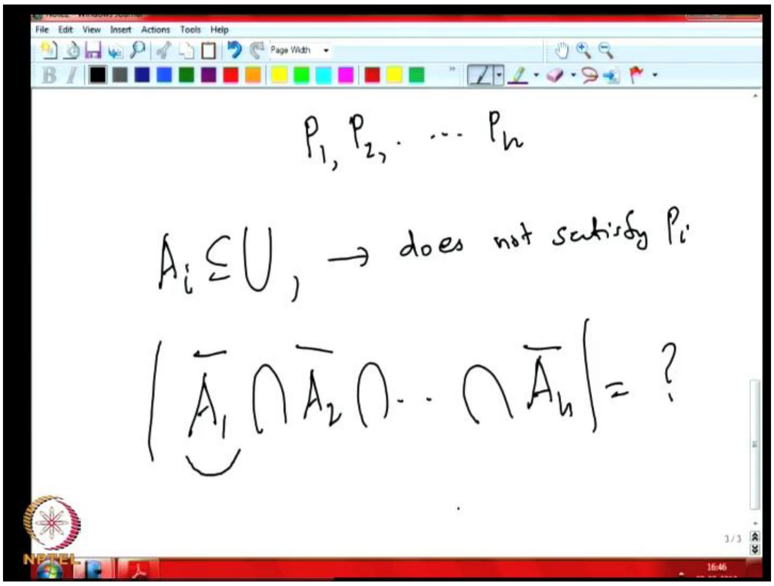

So, what we are interested in is the set of permutations such that those permutations do not satisfy p 1, p 2 or p n, right, easily formulated, right? Let A i be the set of permutations such that the permutations in A i does not satisfy p n. Then now what we are interested in is A 1 complement intersection A 2 complement intersection A n complement its cardinality; clearly that means the permutations where the first position does not have 1, the second position does not have 2 and like that n position does not have n which is exactly the derangements, right? This is what we want to find what is the cardinality of this we can use inclusion exclusion principle.

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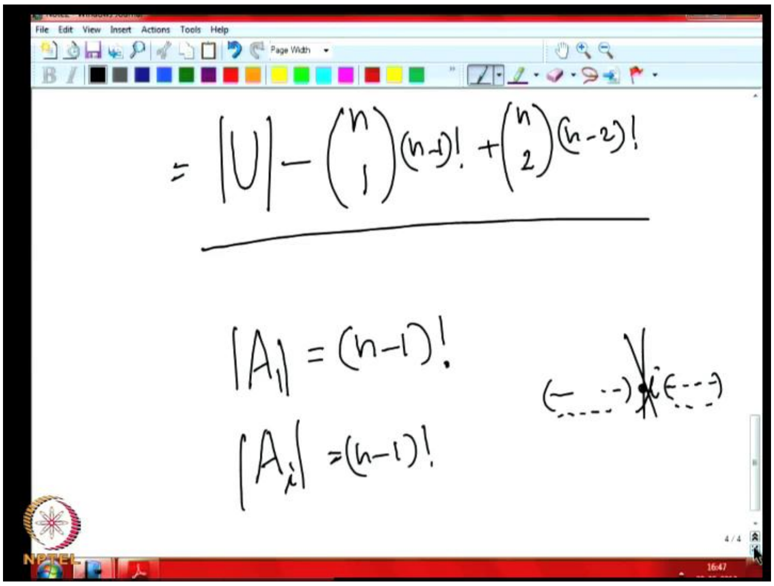

And the answer will be cardinality of U minus n choose 1 into, say, cardinality of a single one see for instance we can use the formula. So, here I will say that this is the special case where when you consider t sets out of k sets their intersection depends only on t not on individual which t sets we say take. So therefore, we can use the formula correspondingly. So, in this A 1 for instance what you get? First position is fixed; that means in the first position we see 1, how many permutations are there like that? Really the first position we have to give 1 there is no other choice. So therefore, n minus 1, but remaining n minus 1 position we can get any permutation of the remaining n minus 1 numbers.

So, that is n minus 1 factorial, right? And this is true for any A i, right, the cardinality of any A i is going to be n minus 1 factorial, why? Because in the i eth position you have to have i; in the remaining positions we can take any permutation. So, if you remove that i in the remaining positions we take any of the permutations of the remaining n minus 1 numbers. So therefore, this is n minus 1 factorial here and then plus n minus 2 into n minus 2 factorial, why is it so?

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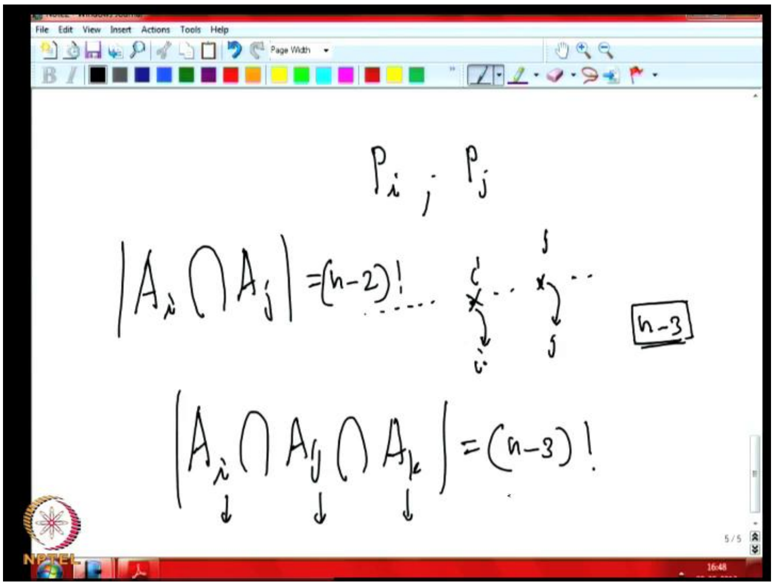

Because once you want to fix any two positions so you are saying that how do you say that property p i and p j together are satisfied. So, how many permutations satisfy? That means in the i th position we have i and in the j th position we have j, the two positions we have only one choice each. So, in the i th position I have to put i, in the j th position I have to put j. So, in the remaining n minus 2 positions I can take any permutation of the remaining n minus 2 numbers. So, in property p i and p j are together satisfied; that means the cardinality of A i intersection A j is going to be that n minus 2 factorial, right?

Similarly the cardinality of A i intersection A j intersection a k, three things together, right, that is n minus 3 factorial, right, because three positions and in i th position and in the j th position and in the k th position, we have to have i and j and k respectively. The remaining n minus 3 things from you can be permuted in any n minus 3 factorial possible ways right? So, these are the possible permutations, right? So, how many? So, now of course this does not depend on which i j position or which i j k position we are talking. It is always n minus 3 factorial for any three positions you fix, right? So therefore, we can go to this formula.

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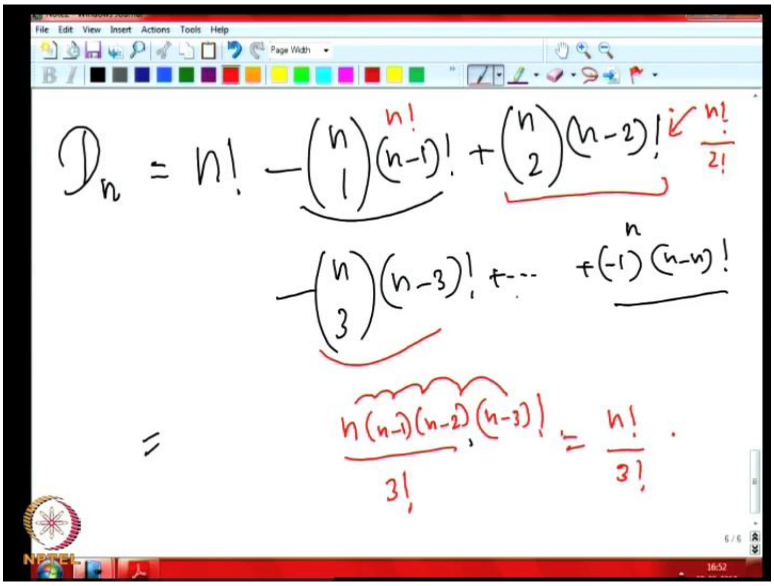

So, we get universe as n factorial. So, when you fix one position that is n minus 1 choose n minus 1 factorial and when you fix two positions that is n choose two ways you can fix that two position and n minus 2 factorial and then we have three positions can three ways, so n minus 3 factorial and so on. So, that is minus 1 raise to n n minus n factorial, because if all the n positions are fixed then there is nothing to be done, because there is only one permutation namely 1, 2, 3 n same write like this. So, this permutation only will be there in the last, this is just one with what about minus or plus sign whichever sign we are ready to give, right?

So, then this is the value we get for the derangements; we will write d n for the number of derangements of 1 to n, right, among the permutations of 1 to n d n things have this property namely when the i th position we do not see i, right? So, this can be simplified by taking n factorial out, right? So, this is actually 1 minus this is what n into n minus 1 factorial by 1 factorial. So, n into n minus 1 factorial will be again n factorial, right? So, this is n into n minus 1 factorial by 1. So, I will do one thing; I will simply it first. Let us take a typical term; this is n into n minus 1 factorial by 1 that is n factorial 1. This turns out to be n factorial itself I will write here n factorial, right, and this is what, this is n into n minus 1 by 2 factorial in fact divided by n minus 2 factorial right?

So, when I combine this and this here. So, this will become n factorial that will be n factorial by 2 factorial. So, this entire term is n factorial by 2 factorial and what is this?

This is you know n choose 3 into n minus 3. So, we have n n minus 1 into n minus 2 in the top in the numerator divided by 3 factorial in the denominator, we have n minus 3 factorial here, right? So, here we multiply all these things we get n factorial divided by 3 factorial and each term will be, so now i th term will correspond to.

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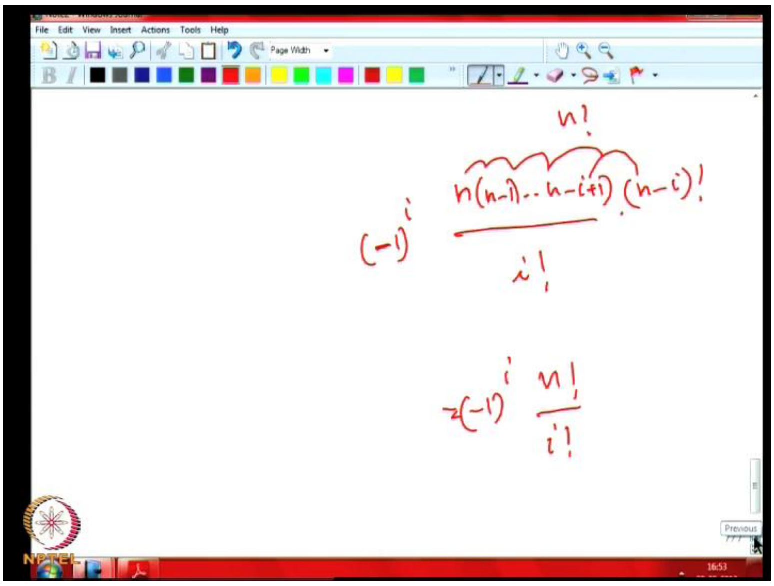

So, n into n minus 1 into up to n minus i plus 1 divided by i factorial, here it will be minus 1 raise to i of s. So, into we have an n minus i factorial here, right? So, when you combine these terms with this we get n factorial number and this will become minus 1 raise to i into n factorial by i factorial, right?

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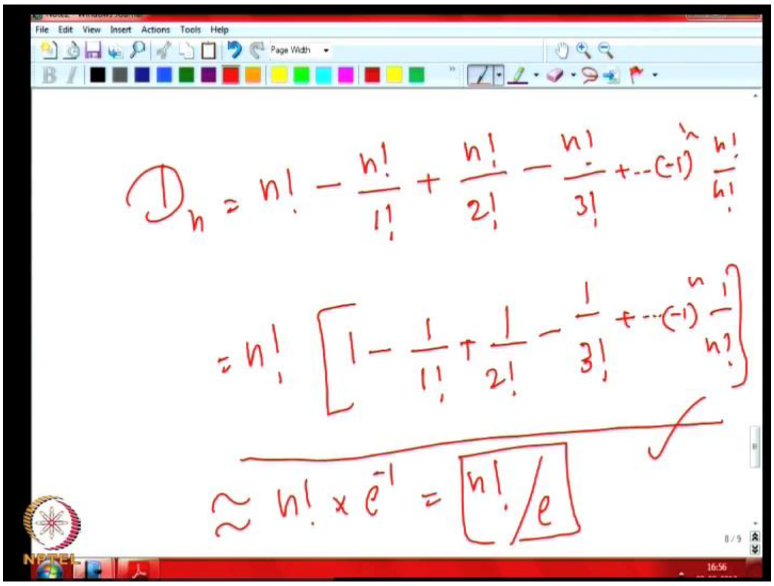

So, what we see is this will simplify to that D n is equal to n factorial minus n factorial by 1 factorial plus n factorial by 2 factorial minus n factorial by 3 factorial and so on till minus 1 raise to n n factorial by n factorial, see here the last term is n minus 1. So, it was just 1 that will become n factorial by n factorial, right? So, now n factorial can be taken out. So, this will be n factorial into 1 minus 1 by 1 factorial plus 1 by 2 factorial and minus 1 by 3 factorial and so on minus 1 raise to n 1 by n factorial; this is what we will get, right, this is the formula. So, then this will remind us of the formula which you have studied in calculus. So, namely e to the power x is equal to 1 plus x plus x square by 2 factorial, so the expansion of e to power x.

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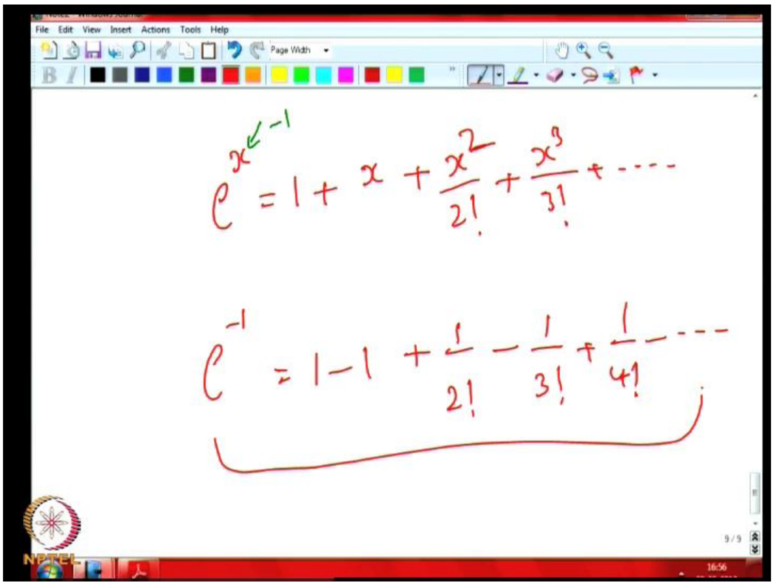

So, expansion of e to the power x, it will be like this e to the power x is equal to 1 plus x plus x square by 2 factorial plus x cube by 3 factorial; this is an infinite thing, right? Now what will happen if I put x equal to minus 1? So, e to the power minus 1 will be equal to 1 minus 1 plus this is x square, right, minus 1 square will be 1, 1 by 2 factorial and another is minus because x cube will become minus 1 cube, that is minus 1 by 3 factorial plus 1 by 4 factorial minus and so on. This will be an infinite alternating signs and exactly this is what we see here but only n terms are seen, right?

So, we can say that as n becomes large this approximately equal to n factorial into e raise to minus 1, that is n factorial divided by approximately equal to. You see that if it is n is greater than equal to 7 or 8, it is very close. You can just do some computation and see; it is already getting very close. After some time when n is reasonably large this difference between n factorial by e and this formula will be very less. So, the infinite series we are just approximately by discarding the remaining terms because as you can see as n becomes large 1 by n factorial kinds of terms will be very very small, right, they are not going to affect the total sum much. So, that is why we say that, right?

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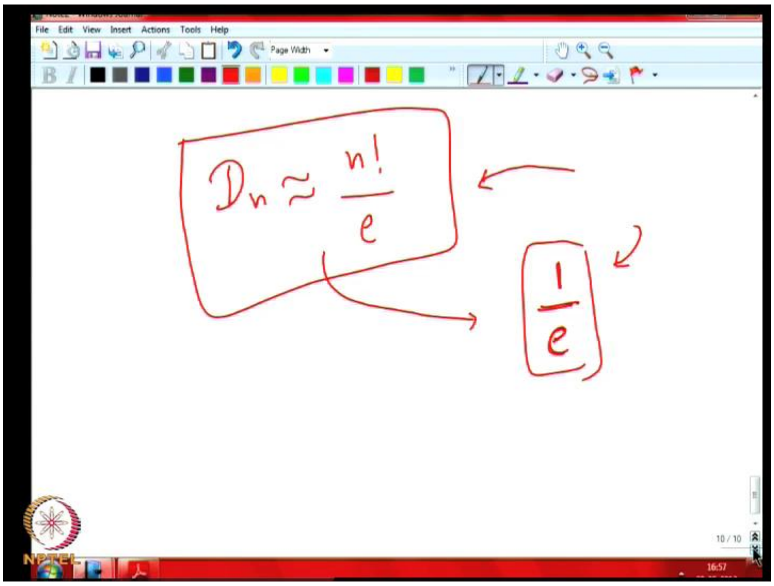

So, the derangements will be D n will be approximately equal to n factorial by e. So, this kind of a formula will be useful when we find the probability of certain things; for instance one can ask what is in the first question we discussed. The gentlemen came to the party, they placed their hats on the table; while going away they took their hats, right, some hat they took and went away. So, what is the probability that nobody will get their own hat, right? This we can say that 1 by e will be the probability if the distribution is uniform; that means given person has the same probability of taking any other hats. So, that means all the permutations are equally possible, right, in that case. So, 1 by e will come.

So, let us not get into the technical details of that, because we are not doing probability now. So, intuitively if you understand the ideas of probability. So, it will be useful to know this formula in such situations, right, so 1 by e. Now we will not discuss more than this about the derangements p; there is interesting material, but then probably this is not the time to discuss about them. So, we will move to the next topic, because we have spent enough time on inclusion exclusion principle and examples, right? We have covered lot of interesting and important examples also.

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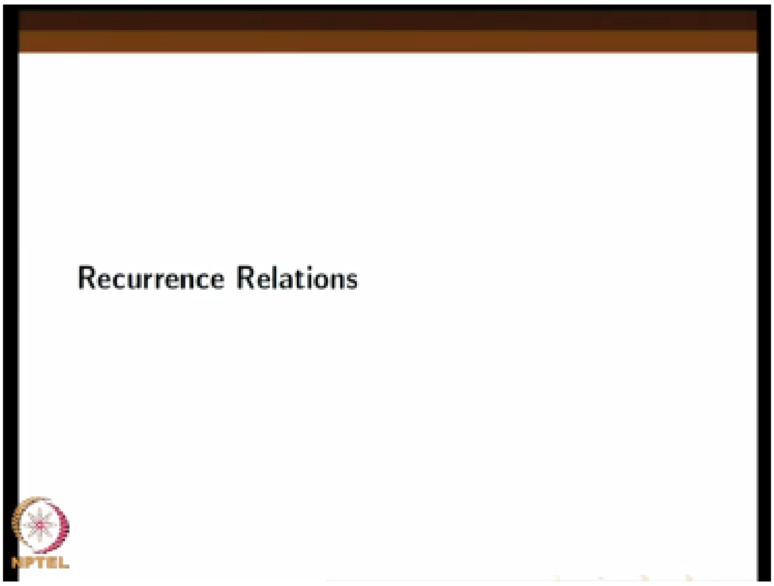

So, now we will get into the next question namely the recurrence relations because counting problems; of course so we have seen several methods now to count and get some information about the combinatorial to answer combinatorial questions counting questions. Recurrence relations are another technique which is very useful because many times in the underlying counting problem we see such relations. Their technique is very simple; we just see that the counting problems as such it are not necessarily that it is just one problem. It is a sequence of problems, so there will be some problem for each n usually most of the time; it is not that, it comes isolated.

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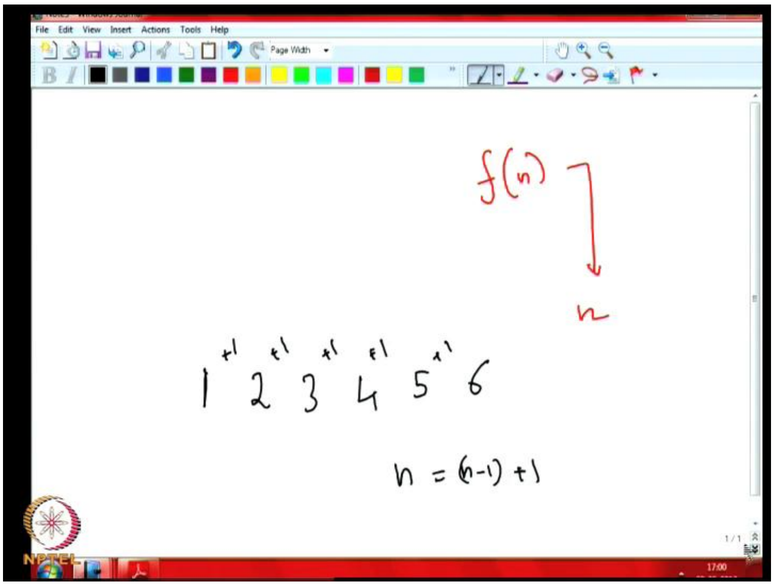

So, it may happen that our count is F of n when the underlying problem has a parameter n, right? So, therefore this solution is corresponding to n can come as a sequence of numbers, right? So, from up to now whatever you have studied like even in your twelfth standard you have studied sequences, some of the more important things we have studied about sequences; I mean one of the more popular sequences you have seen are, say, for instance the ones which have there is a common difference. For instance we start with 1 2 3 4, this is a sequence. So, here the first term we add one term to the previous term then you get the second term, right? So, this is an arithmetic progression we have seen. So, you know the formula, so you know the n eth term is obtained by the n minus 1 eth term plus 1 here, right; of course you need not start with one.

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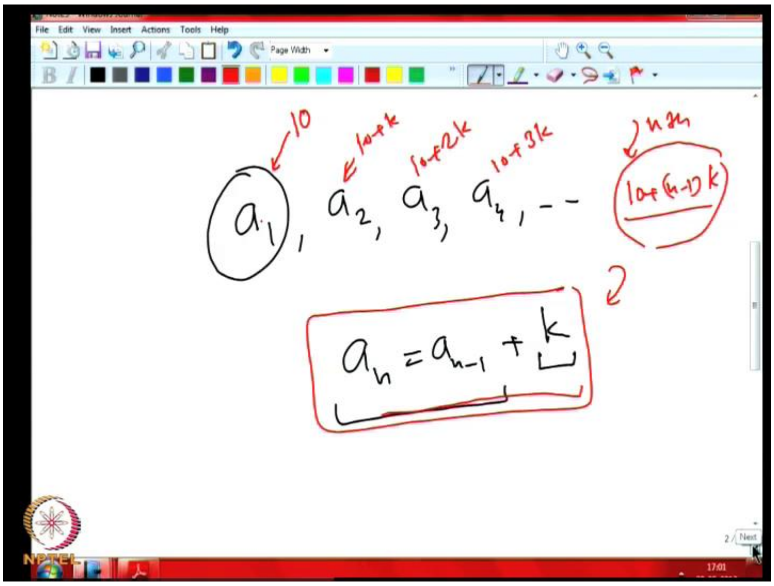

So, you can start the sequence can be like this a 1, a 2, a 3, a 4 like this. So, here you write a n like a n minus 1 plus some constant k, right; this k is the common difference. So, this kind of progressions you are very familiar with and some other kind of progressions, and we will definitely need to know of course what is the first number? This is the very important thing, right; we need to know what is this? Once you know this thing with this formula we will be able to figure out; for instance if you give this as 10 then this will be 10 plus k and this will be 10 plus 2k and this will be 10 plus 3k and so on. So, for instance the n eth term will be 10 plus n minus 1 into k, right? So, that is the good thing about this kind of relations. So, you can infer what is n th term, right somehow. So, therefore another kind of sequence we are very familiar with is the geometric progressions.

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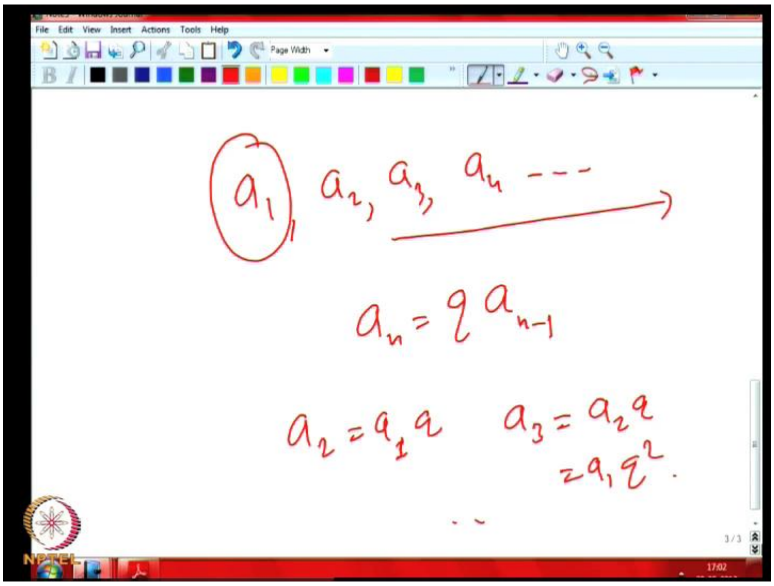

So, there we sequence like a 1, a 2, a 3, a 4, etcetera where a n is some q times a n minus 1, right, q times a n minus 1. We need to know what is a 1; then you know a 2 will be a 1 times q, a 3 will be a 2 times q that is a 1 times q square and so on, right, like that we can figure out all the values, right? So, of course these are sequences given with that property, but many times in combinatorial problems we can identify such a sequence occurring, right? So, we can probably see that like the problem has a parameter n and then we already know the values the answer for the problem when the parameter is n minus 1, right, or n minus 2, then we can say, n minus 2, n minus 3, etcetera.

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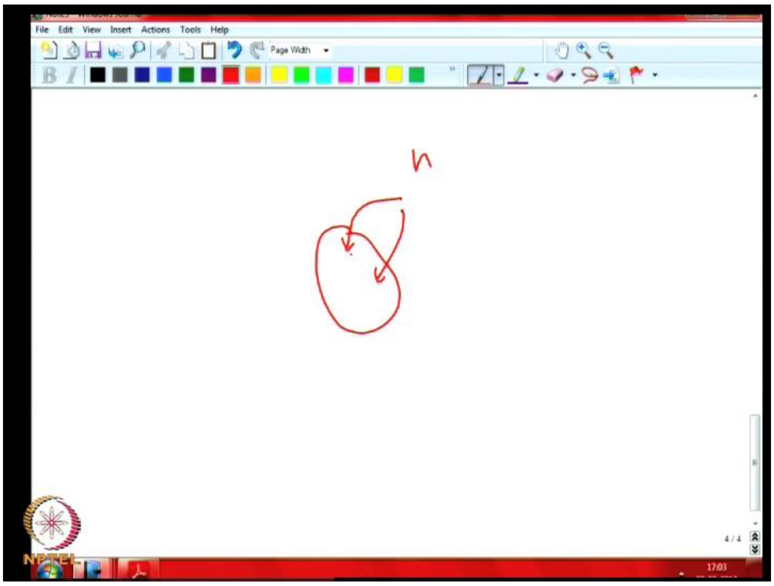

We can find the solution for the problem with parameter n from the solutions for the smaller parameters, right? So, let us consider probably one of the most famous problems of this sort which was proposed by Leonardo Pisano.

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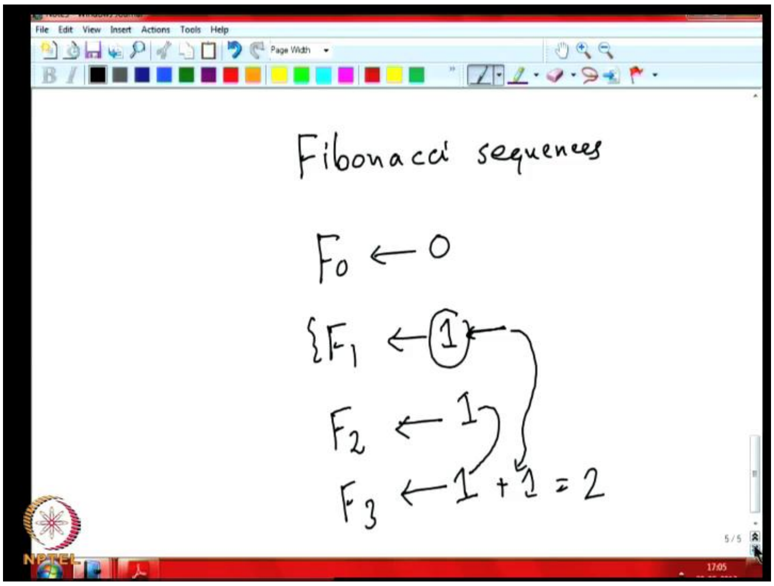

This is about Fibonacci sequences. So, he proposed this problem. So, we have a pair of rabbit's one female, one male. So, what they do is they put two rabbits in a cage, they will have a special cage for the rabbit, and then see the when they are born after every month I mean the first rabbits are there, after one month they will produce a new rabbit, right and then those two rabbits, right, but they produce one female rabbit and a male rabbit. And then they take this female and male rabbit and put in another cage, right, and of course the first month they would not produce new ones, but the second month again second month onwards they will produce a pair of rabbits which is one male and one female, right? And we want to know after n months, n th month how many rabbits will be there? This is the question, right?

So, here we will say that the first month we will write as F 0 the number of rabbits at the 0th months is 0 and F 1 s 1 because we have initially one rabbits and F 2 is such that after. So, this one pair of rabbits produces a new one; that is, sorry they produce at the second month only not first month. So, this will be one only, right, but F 3 what happens is we have this existing rabbits are there, they will not die, right? And then they will also produce, so for instance two months before how many rabbits were there? Those many rabbits those many pairs of rabbits, how many pairs of rabbits were there; those will produce a new pair of rabbits, we will get 2 here.

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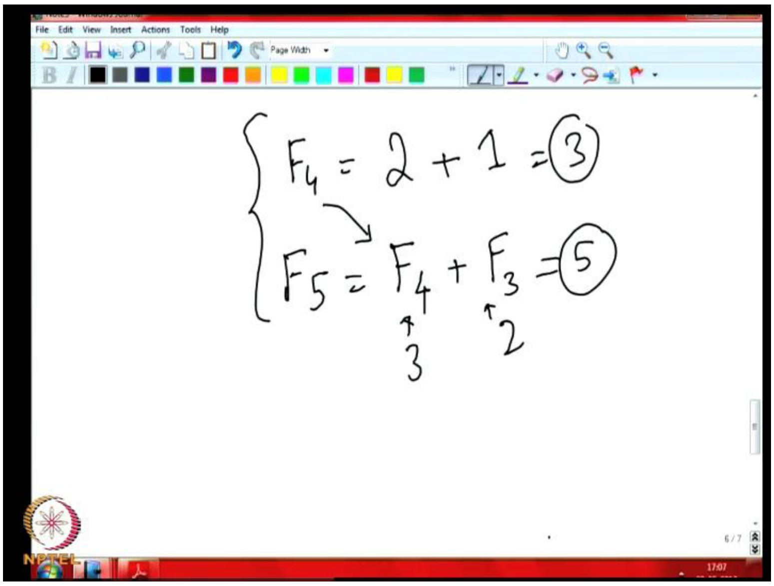

And now what will happen is F 4 in the fourth month. See in the last month we were having 2 rabbits, right, two pairs of rabbits; not 2 rabbits 2 pair of rabbits plus but then the rabbits which were there 2 months before; that means not the last month last last month they will produce young ones now, right? Suppose there was one pair of rabbits in the last last month they will produce one more pair. So, that is 2 plus 1 is 3, right? Now in the fifth month what happens is we have the total number of rabbits will be; in the fourth month which ever number of rabbits are 3, right, those rabbits are still there. But then new rabbits will come which are produced by the pairs which were available 2 months ago; in the fifth month 2 months ago means F 3 is the number of rabbits which were. So, that is 2 F 3 that is 2; this is 3 plus 2 is 5, right? So, this is what. So, then he was asking in the n th month how many rabbits will be there? So, this is a typical example of recurrence relation.

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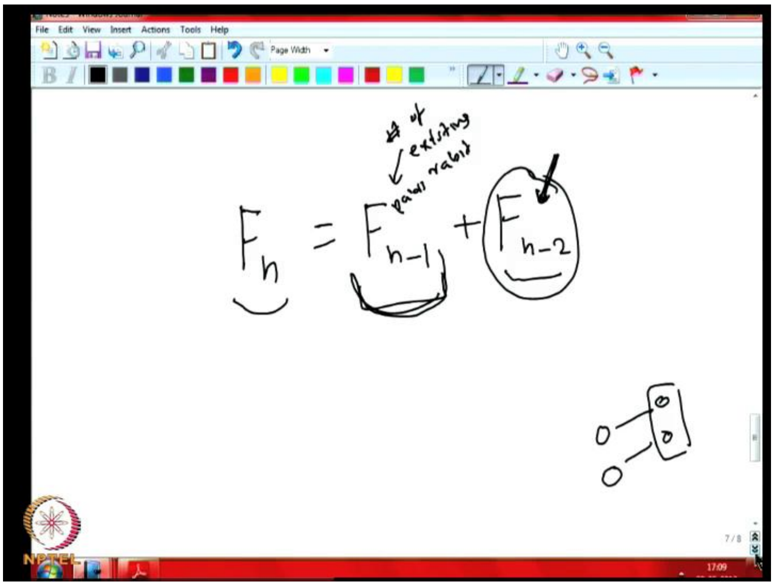

So, now n th month how you will find the number of rabbits is you will notice that the number of rabbits will be equal to the sum of rabbits which were; see when I say rabbits I mean pairs of rabbits. So, the number of pairs of rabbits which were existing in the n minus 1 th month; that means F n minus 1 plus the number of rabbits which are newly produced, but because those rabbits which were newly born last month will not produce new rabbits now.

So, only the rabbits which were existing last but one month, I mean one month before, I mean they need one month to grow up. So, the next next month only after they are born next next month only they will produce the young ones. So therefore, we have to go to F n minus 2; that means in the last last month there were F n minus 2 rabbit’s pairs of rabbits and each pair will produce a new pair. So therefore, F n minus 1 plus F n minus 2 will be the new one. This is existing pairs of rabbits number of existing pairs of rabbits at the n minus 1 month. This were actually number of pairs of rabbits which existed at the n minus 2th month, but then what we counting here is that pairs they are producing, right, because those rabbits which are existing at the n minus 2 are also existing at the n minus 1 month, right

So, they produce new ones. So, those extra count is this. So, that is how this is. Of course you can argue that this is a cooped up problem because of course how do you make sure that one male and female rabbit together will always produce one male and female rabbit, right, and how do we say that there are only two rabbits produced by them. So, these are all there, but this is a problem which is proposed to illustrate this sequence.

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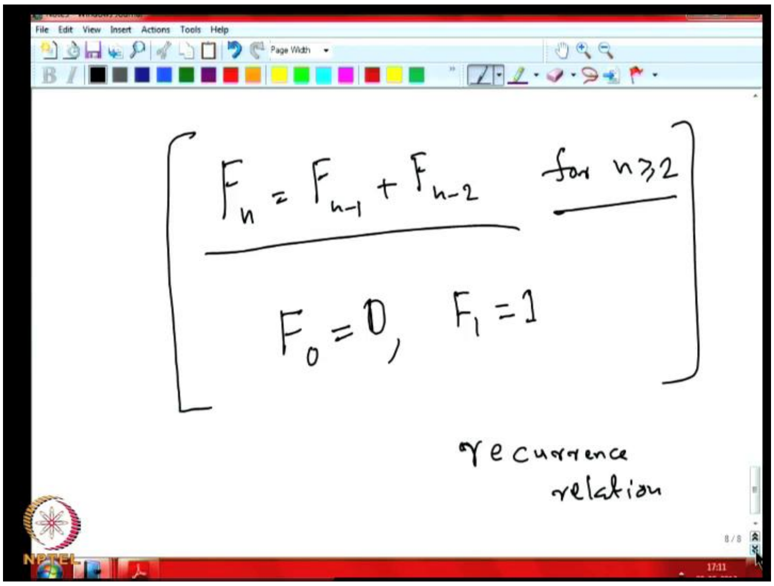

So, it is clear that we get the sequence F n equal to F n minus 1 plus F n minus 2 as the answer for this. This works for all n greater than or equal to 2, right, because the first 2, F 0 we are just defining as 0. F 1 we have one pair of rabbits and F 2 onwards we can add 0 plus 1, because this pair of rabbits is not going to produce new ones now; new ones are not going to come and the next three. So, this is producing new one and this is existing, right?

So, this plus this, right, 1 plus 1 is 2. So therefore, for n greater than equal to 2 this is true and the initial conditions you call this initial condition F 0 is equal to 0 and F 1 equal to 1; these are the initial conditions, right? This is called the Fibonacci sequence and you see these kinds of relations are called recurrence relations. I assume that students are familiar with recurrence relation already. So, for instance they have studied it in their algorithm course and many courses. As of now we will not formally define it, but if you understand that this is a sequence of numbers.

So, if for instance is a function of natural numbers and this F n means the n th number in the sequence, right; that is obtained by combining certain numbers which had appeared before in the sequence. So, Fibonacci number takes the just previous number and one before, right, just two previous numbers are taken and added together and then that is how the new number is produced, right? And now you will see because this Fibonacci numbers are very important and they appear in many problems. So therefore, we will study them in little detail. So, this maybe it is a good starting point to study the recurrence relations, right, being one of the most important recurrence relations which we see in combinatorics. So, let us see. So, now, we will take some other properties of it.

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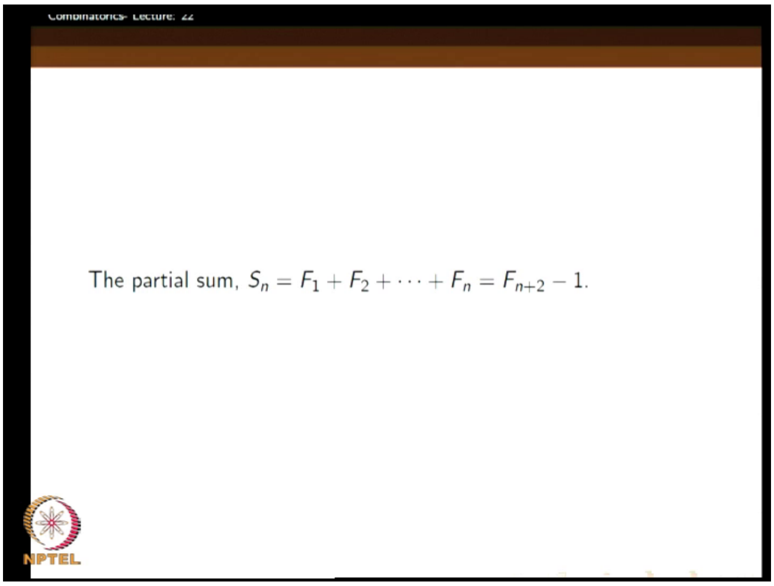

So, one first property we will study is what is the partial sum? Suppose we sum from the first 10 Fibonacci numbers F 1 plus F 2 plus up to F n. So, we have F 0, but F 0 is 0; therefore, whether we sum it or not is not important. So, we sum from F 1 F 2 F n, the first 10 terms what will you get? So, here we see that we always get F n plus 2 minus 1, right?

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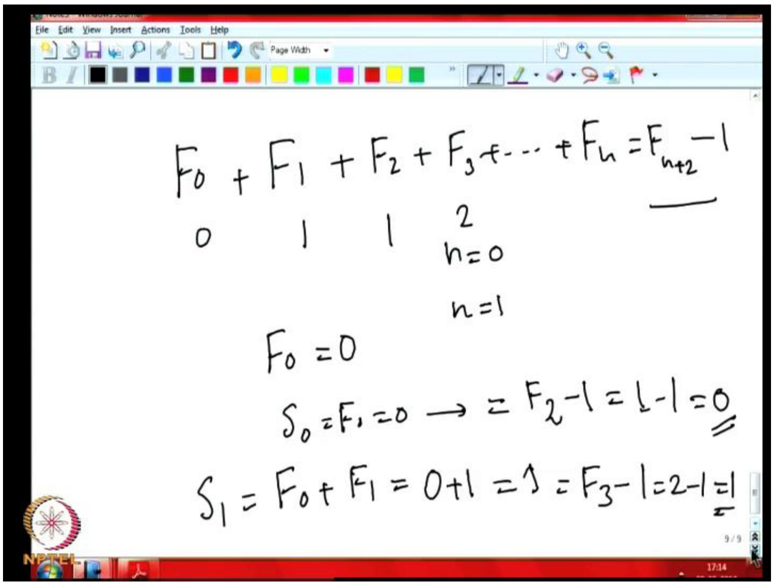

This is the first formula we want F 1 plus F 2 plus, you can even write F 0 does not matter, 0 plus up to F 3 plus up to F n, then we will get F n plus 2 minus 1,right, that is all. So, we can verify it for small values; put n is equal to 0, n equal to 0 means we are just taking F 0, F 0 is just 0. So, we are saying that. So, the S 0 is just F 0 that is 0, then we are saying that this is equal to F n plus 2; that means F 0 plus 2 is F 2 minus 1, F 2 is what? F 2 is 1 only, 1 minus 1 is 0, right? And for n equal to 1 we get F 0 plus F 1 is the S 1, alright; that is 0 plus 1 is equal to 1. This is equal to F 3 minus 2, F 3 is how much? F 3 is 2, this is 0, this is 1, this is 1, this is 2, right? So, we get 3 minus 1, this is 2 minus 1 this equal to 1. So, that is also correct. So, we have verified this formula for small values of n and which seems to be working out. So, we want to give the proof for the general case, right?

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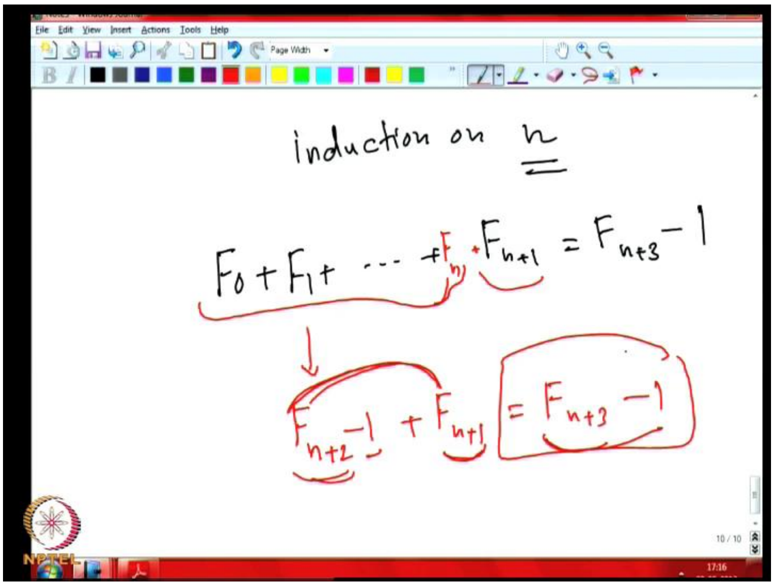

So, what we do is we do induction on n. This is the proof technique we use here induction on n. So, how do we use induction on n? So, we have verified for n equal to 0 case, n equal to 1 case, even n equal to 2 case. So, we will assume that it is true up to n. Now we will try to prove it for n plus 1, right? So, what we want to show is F 0 plus F 1 plus F n plus 1 is equal to F n plus 3 minus 2 is what minus 1 is what we want to show, right, this is what we want to show.

But when this one we can split it up to here; that is this is F n, right, up to F n up to here and then this. This is what; by induction hypothesis this is F n plus 2 minus 1 and this is F n plus 1. So, this plus this minus 1, but this plus this is what by Fibonacci recurrence relation that is F n plus 3 minus 1, right? So, I am just combining these two terms F n plus 2 n F n plus 1, that is the next term in the sequence namely F n plus 3 and this minus 1 is there. So, this is so simple. So, we can prove it so easily by just simple induction. So, now let us look at the next formula.

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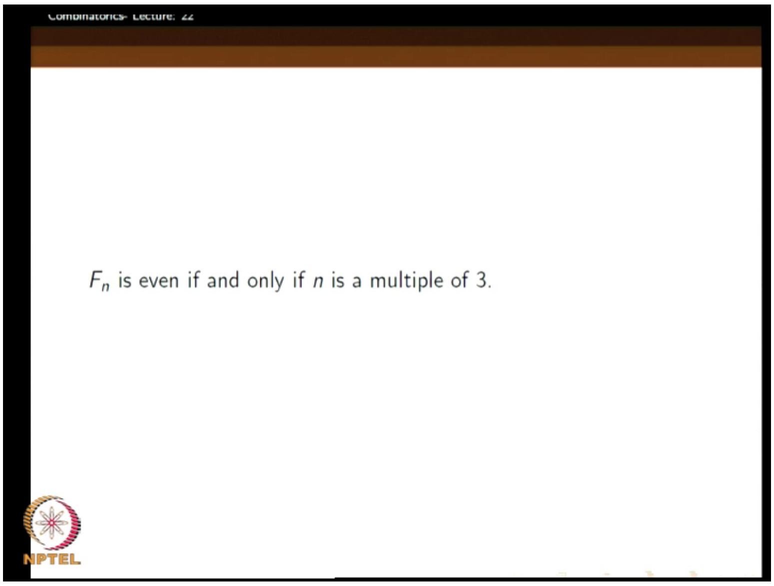

So, the next term property of Fibonacci numbers we want to study. F n is even if and only if n is multiple of 3; this also very easy to prove, so that we can verify first.

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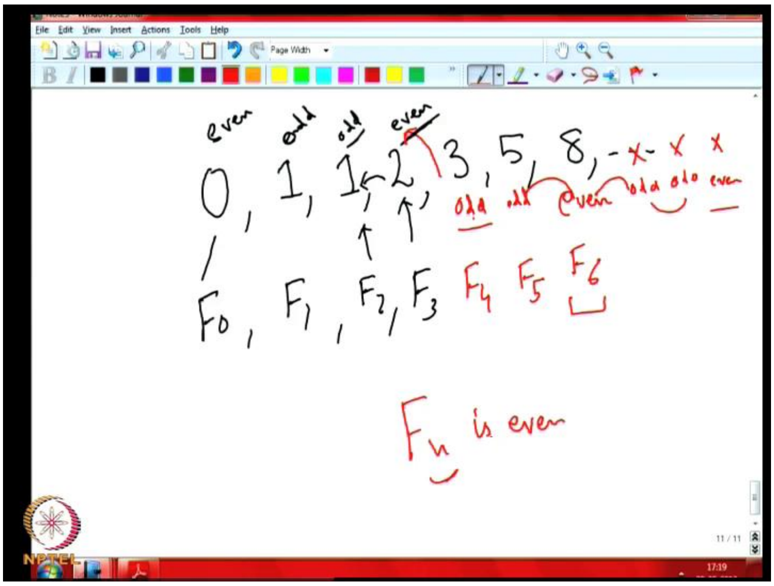

For instance Fibonacci numbers if I write. So, we start with 0, right, then the first number is F 1 is 1, F 2 is 1. So, F 3 is 2, then next is 3, next is 5, next is eight and so on, right? So, this is F 0, this is F 1, this is F 2 and so on, right? So, if here is an even number, here is an odd number, here is an odd number, here is an even number, right? Now first what happens are you have an even number; second is an odd number, right? These are by initial conditions are like that. The third will definitely be an odd number, sorry this is F 2; F 2 will definitely be an odd number because we are adding an odd and even here, we will get an odd number only, but the next one is going to be an even number again. So, we got F 3 to be even as wanted.

Now we want to show that next two numbers are always going to be odd and after that we will get one even number. So, here this is an even number; the previous one, we have already seen there is an odd number. So, even and odd will give us an odd. So, give us an odd number, this is an odd number, right? Now next also has to be odd, why because now is an odd and the just previous number is an even number. So, even and odd is going to be an odd number for us, right? So, now the next one we say that is again going to be multiple of this.

So, this was F 4, this is F 5, this is F 6, right? Now odd and odd will give an even, right? So, this is the reason. So, now it is very clear from here that now if you consider the next two numbers they are going to be odd because here even and even this is odd and here odd and even this is going to be odd, now this odd and odd will give an even, right, two numbers odd then even, right, this is the pattern. So, F n is even if and only if n is a multiple of 3; that is 0, 3, 6, like that.

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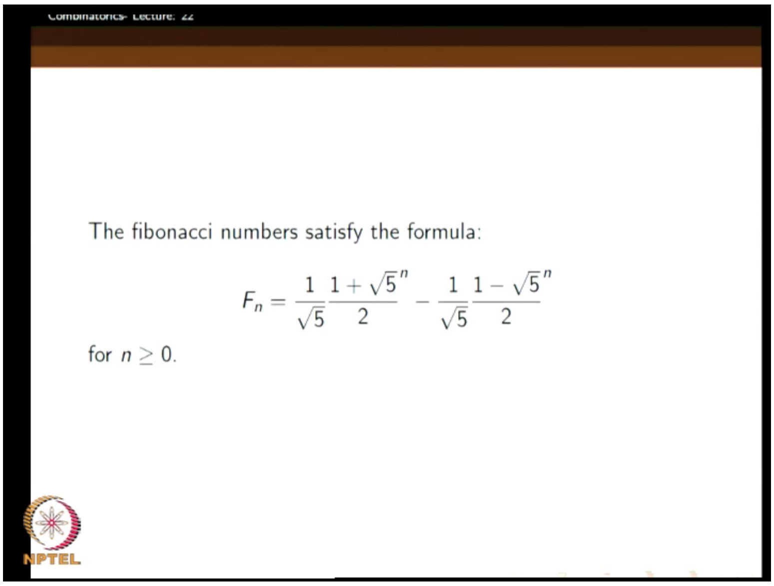

Next we will consider the general formula. So, formula to compute F n we get formula; we do not have to write the entire sequence, because as of now if you want to find F 10 for instance what will you do, how will you find F 10 for F 100, right?

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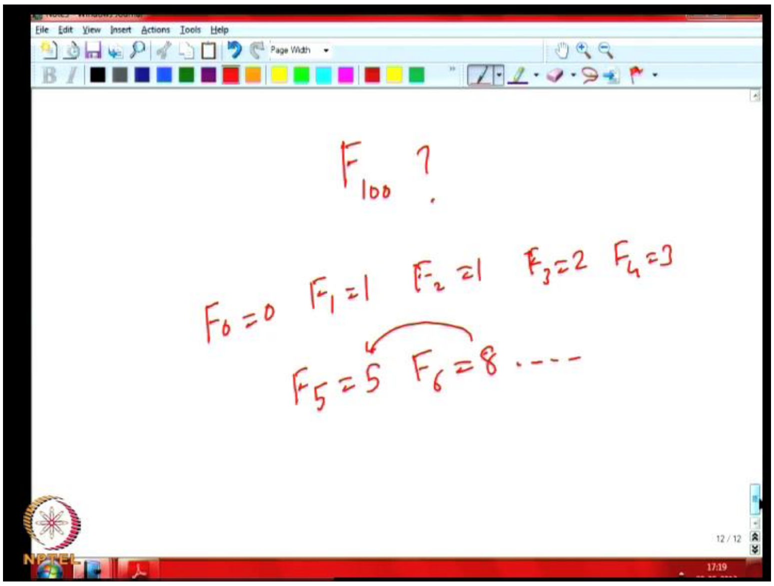

So, we will say that we start with F 0 is equal to 0, F 1 equal to 1. So, then now F 2 is equal to 1. So, F 3 is equal to 2, F 4 equal to 3 and F 5 is equal to 5 and F 6 equal to 8 and like that we can keep on writing till 100 by adding always we add the previous two numbers, right, the next number and till 100 we can go but it will take some long time. So, rather we would like a close form expression for computing F n.

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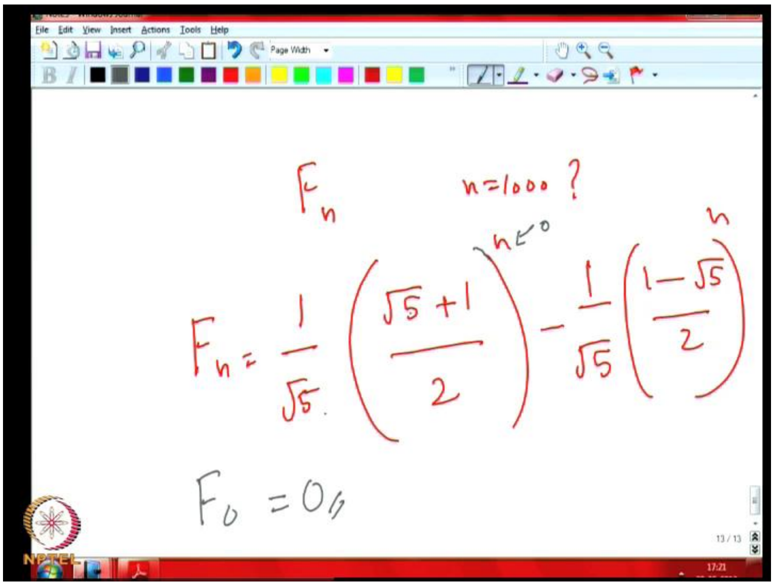

For instance if you want to find F n for n equal to 1000, right, so we can use that formula, right, that is what the next this thing. So, we claim that there is a formula and the formula is given by this thing, 1 by root 5 into 1 plus, this is 1 plus root 5 whole raise to n, sorry 1 plus root 5 by 2 whole raise to n. That n we should have put a bracket here, so I will just put it once again and write; the formula is this F n equal to 1 by root 5 into, so this bracket is missing here, 1 by root 5 plus 1 by 2 whole raise to n minus 1 by root 5 into 1 minus root 5 by 2 raise to n, right? So, this will work for all n. So, here we need a bracket, here also we need a bracket; that is what is missing. This will work for all n.

We can just try for small values of n. So, for instance if we put n equal to 0 what will happen is. So, this will become one. So, if you put n equal to 0 here this quantity will turn out to be one, this quantity will turn out to be this, we will 1 by root 5 minus 1 by root 5 that will be equal to 0. So, F 0 will be equal to 0 according to this. So, if you put n equal to 1, right, then what will happen? So, then you get root 5 plus 1 by 2 here minus 1 by root 5 into 1 minus root 5, then put n equal to 1. So, we can take root 1 by root 5 out.

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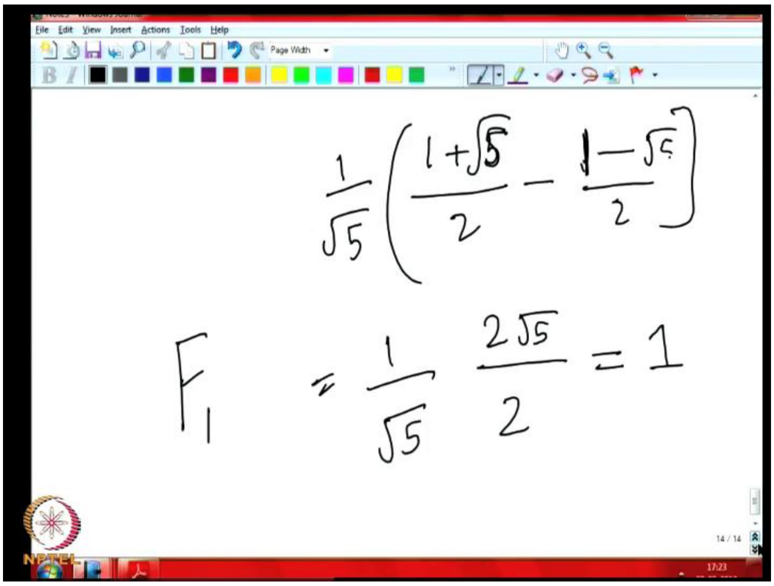

So, it will look like 1 by root 5 into 1 plus root 5 by 2 minus 1 minus root 5 by 2, right, This 2 is common, so this root 5 cancels off; sorry 1 cancels off and then we get this 1 by root 5 into 2 root 5 by 2, this is 1. So, F 1 we are getting as 1, right? So, it is working even for n equal to 0, 1, etcetera. So, when we put bigger values it is more difficult to compute. What you do is for instance put n equal to 10, you expand it using binomial expansion binomial theorem and then we see that all these irrational numbers root 5 and all will cancel off, and finally we will get a integer number positive integer which is the value of F n right? But how does this magic formula work and why is it true; that is all we want to find out now. So, the technique is simple. So, this is what we should do. So, we will write.

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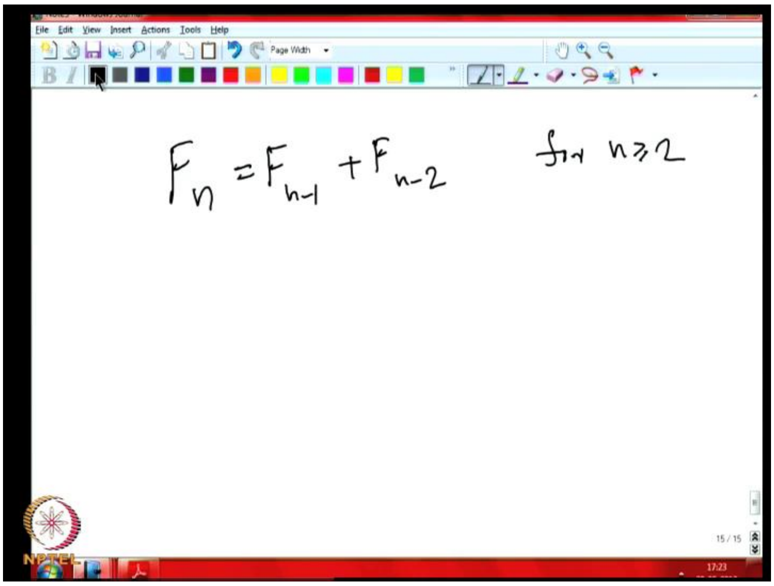

So, we know that for n greater than or equal to 2 we have F n is equal to F n minus 1 plus F n minus 2; for the time being we will assume that n is greater than or equal to. All the manipulation we are going to do now is kind of true only for n greater than equal to 2, right; we are assuming n greater than or equal to 2 as of now, right? So, what we do?

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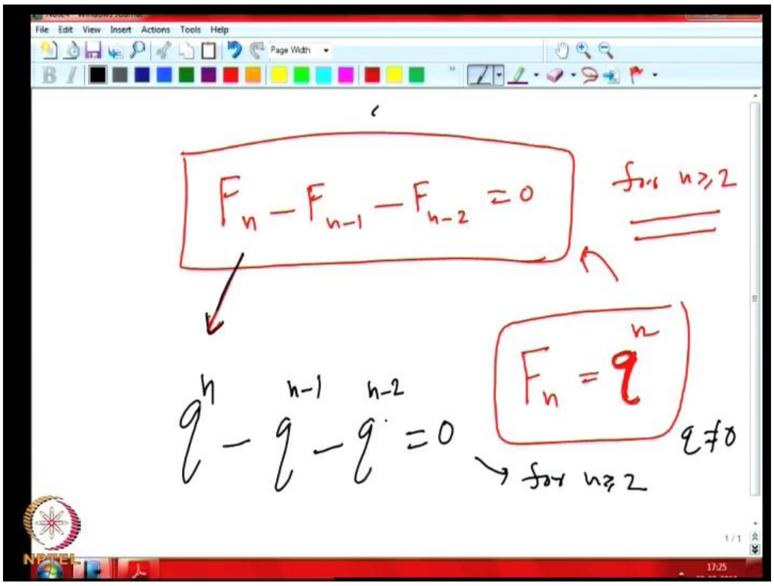

So, we can rewrite it as F n minus F n minus 1 minus F n minus 2 is equal to 0. This is true for n greater than or equal to 2, right, because if you had taken n is 1 then we should have to worry about what is F n minus 2; that will be F minus 1 and that is not defined, right? So, this is only true for this is written only for n greater than 2, right? Now what we do is to guess the formula for F n. So, how do we guess? So, we just think that this is some q to the power n, but q is some quantity which we will figure out later, right? So, this is from experience and then we try it out and see for what values of q this will work or will it work at all, right? So, that means we will get something like.

So, by substituting we can get; this is q to the power n minus q to the power n minus 1, this is minus q to the power n minus 2 is equal to 0 and this is also for n greater than or equal to 2. Now assume that q is not 0; we will assume q not equal to 0. If q was equal to 0 the sequence will be 0 0 0 0 right? So, that q is not 0, right, because if you put q equal to 0 every number for every n we will get 0 F of n equal to 0. If we know that that is not what we want because we are trying to solve find a close form expression for the Fibonacci sequence which we know is not all 0's, right? So therefore, q is not 0, q to the power n minus 2 is also not 0 in the worst case in any case because n is at least2, q raise to 0 can be one, but whatsoever.

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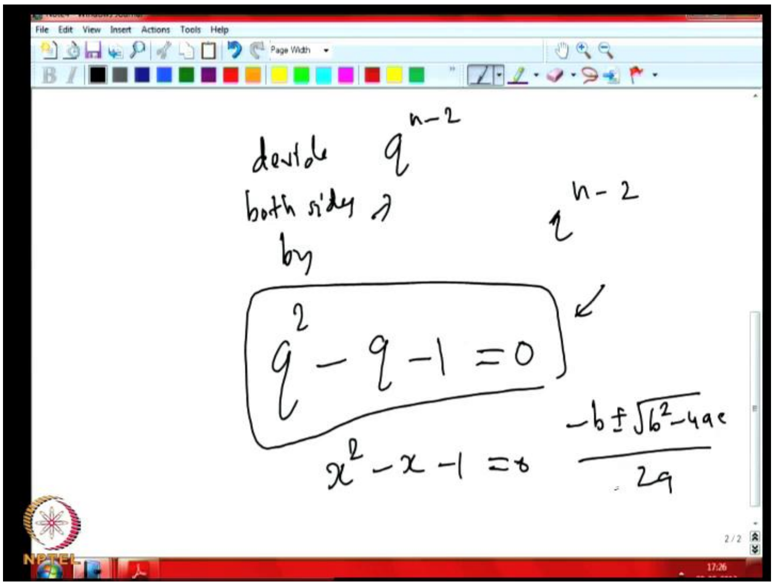

So therefore, for any n we can divide the q to the power n minus 2, divide by both sides right, divide both sides by q to the power n minus 2. So, what do we get? q square minus q minus 1 equal to 0 is what we get. So, that q to the power n minus 2 goes common to everything. So, we divide it off on the 0 side from the other side also. Now what is possible value of q? Is there a possible value of q? You know there is a quadratic equation, you know x square minus x minus 1 equal to 0. If you can find the solution for this thing and that can be taken as the value of q and you know this by using the minus b plus or minus root b square minus 4 a c by 2 a formula, right, for instance you know that.

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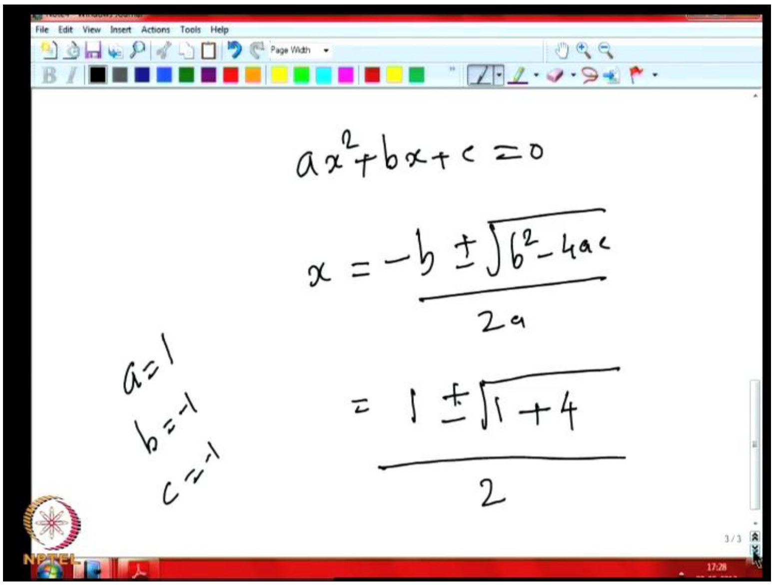

So, you remember that a x square plus b x plus c equal to 0. The roots are x equal to minus b plus or minus b square minus 4 a c by 2 a. So, here you know a equal to 1, b equal to minus 1 is it not? b equal to minus 1 and c equal to minus 1, try putting it. So, we get 1 plus or minus 1 minus 4 into, a is 1, c is again minus 1, this will be plus and divided by 2 into a is 2. So, this is root 5 here.

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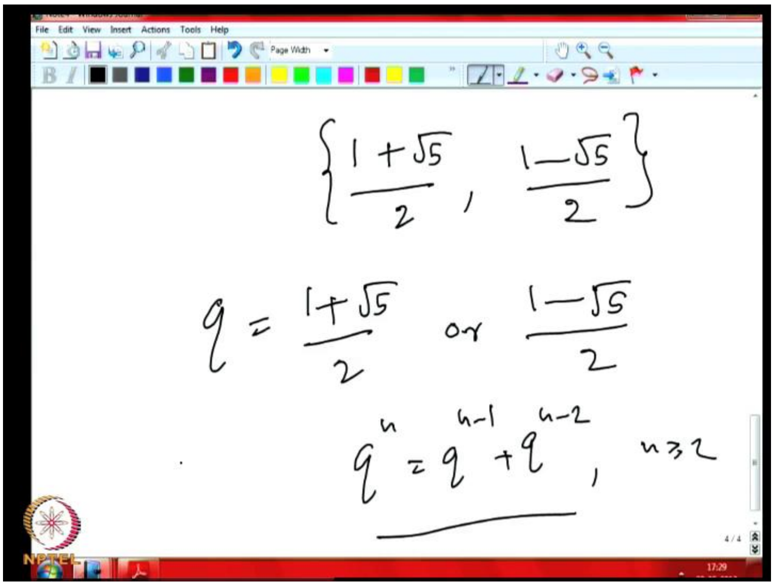

So, root 5 we have 1 plus root 5 or 1 minus root 5 there are two solutions. 1 plus root 5 by 2 is the solution and 1 minus root 5 by 2 is the solution for this thing. So, what it means is. So, you can take the q is equal to 1 plus root 5 by 2 or you can take 1 minus root 5 by 2. In both cases this recurrence for instance q to the power n equal to q to the power n minus 1 plus q to the power n minus 2 will work, right, this is true. So, it will work. So, this would be a general solution, but the point here is that we have only considered the recurrence relation.

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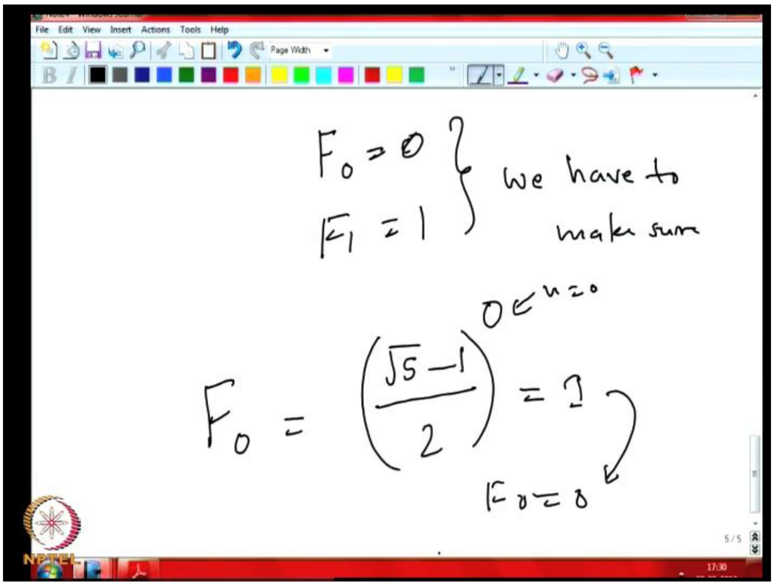

We have not considered for instance the two values initial values namely which is equal to 0 and F 1 equal to 1 we have not considered, right? We have to make sure that this also works make sure that this is consistent. So, now for instance if you take root 5 minus 1 by 2 as q and you just put q to the power 0, right? So, if this was the value actually; F 0 is equal to this, I am just putting n equal to 0 here. So, then this will be one right, but we know F 0 has to be 0. So, there is something wrong about it. So, then what can we do? So, we have to somehow adjust for that.

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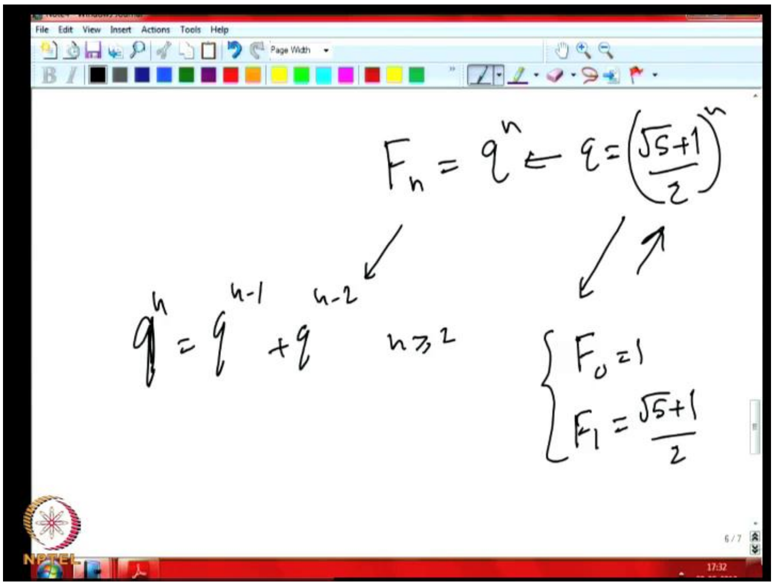

We have to make sure that the initials values are also satisfying this thing; otherwise what happens is. So, it is true that if I take F n equal to q to the power n that will satisfy the recurrence relation, but it would not give the Fibonacci sequence because initial values of q to the power n with q is equal to, say, root 5 plus 1 by 2 whole to the power n, right? So, when I put n equal to because recurrence relation when we start with the correct initial value only we will get that same sequence; we start with another initial value we will get definitely a wrong thing, right?

So, for instance here if we put n equal to 0 we are only getting only F 0 is equal to 1 and when we put F 1 n equal to 1, we are getting root 5 plus 1 by 2. So, if you had started with these two initial values definitely this formula will give the correct solutions, right, because that is what it says. So, q n is actually equal to q n minus 1 plus q n minus 2, right, for n greater than equal to 2. But how will you make sure that initial value is met, right, because otherwise we would not get; we will get some strange sequence here. If you keep on substituting this thing we will get strange sequence; we would not get the Fibonacci sequence that we are looking for.

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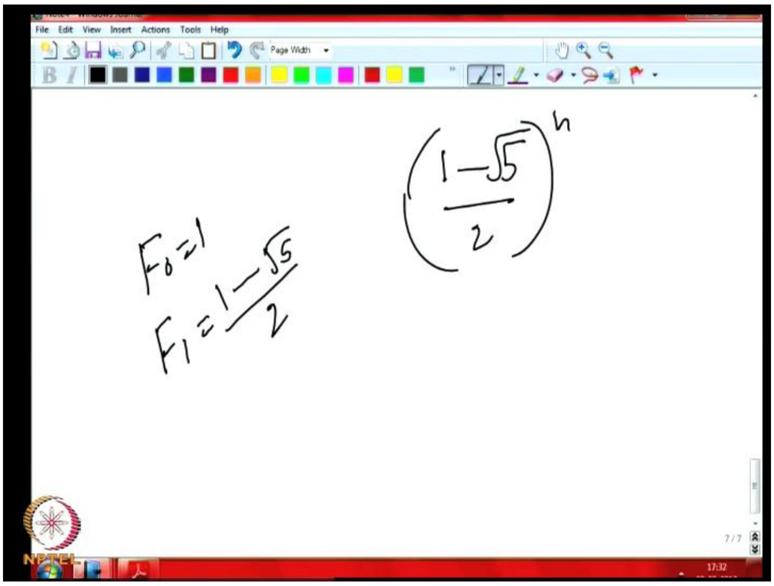

The same is true if I try 1 minus root 5 by 2 whole power n, right? So, when n equal to 0 we are again getting 1. So, F 0 is equal to 1 is what we are getting. Similarly when I put up n equal to 1 we are getting only 1 minus root 5 by 2, it is not even an integer. So, definitely the next one will be the square of this thing. It is not the Fibonacci sequence, then how will you get it? So, what we observe is that while the recurrence relation is satisfied.

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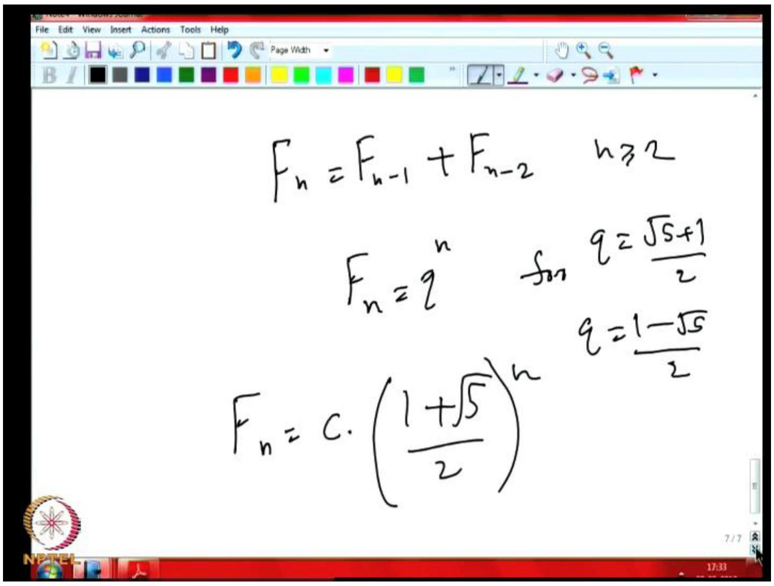

For instance this F n is equal to F n minus 1 plus F n minus 2 will be satisfied if you take F n equal to q to the power n for q equal to root 5 plus 1 by 2 or q is equal to 1 minus root 5 by 2. We have also other expressions which will satisfy this thing. For instance if I had taken F n equal to c into some constant into root 1 plus root 5 by 2 raise to n, then also this will be satisfied, why is it so?

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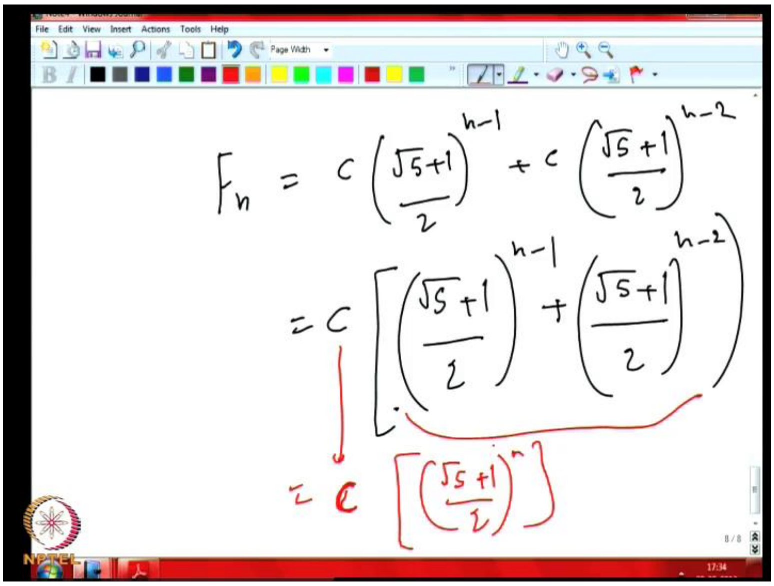

You can just verify it; for instance what we get is F n equal to c into root 5 plus 1 by 2 to the power n minus 1 plus c into root 5 plus 1 by 2 to the power n minus 2, right? We can take c as common c into. So, this is root 5 plus 1 by 2 whole power n minus 1 plus root 5 plus 1 by 2 whole power n minus 2. But we have already seen that this two add up to c into because that q to the power n is equal to q to the power n minus 1 plus q to the power n minus 1 when you put q is equal to this value. So, we will get this as c into root 5 plus 1 by 2 whole power n, right, that is true. So, the c also comes. So, if we had defined F n is c times some constant times this root 5 plus 1 by 2 whole power n also, that recurrence relation is indeed satisfied.

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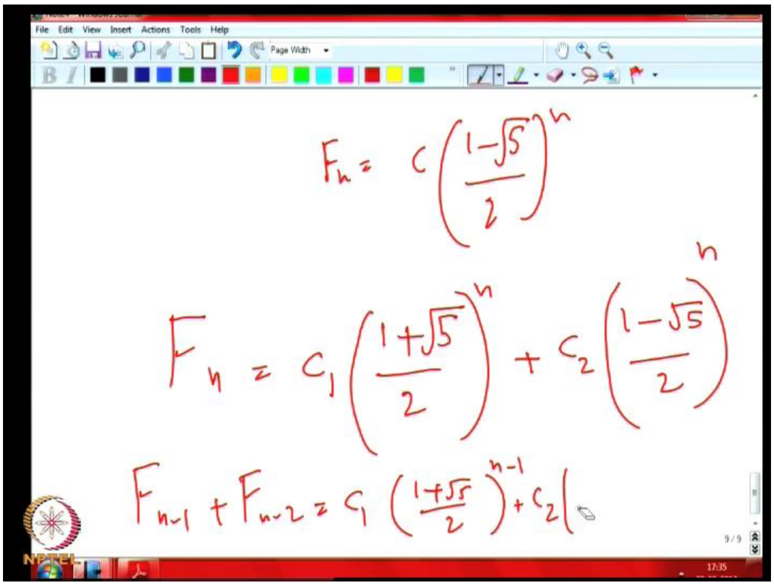

Same is true that if you had defined F n c into some 1 minus root 5 by 2 whole power n, then also recurrence relation will be satisfied, right? And more generally we could have defined F n as some constant c 1 times 1 minus root 5 by 2. So, 1 plus root 5 by 2 whole power n plus some constant some other constant times 1 minus root 5 by 2 whole power n. You substitute it for instance you take what is F n minus 1 plus F n minus 2 according to this definition. Then what happens is this will be c 1 into this quantity same quantity 1 plus root 5 by 2 n minus 1 plus c 2. So, what happens is you should do this calculation and see that, I need a different page here; anyway I will write like this.

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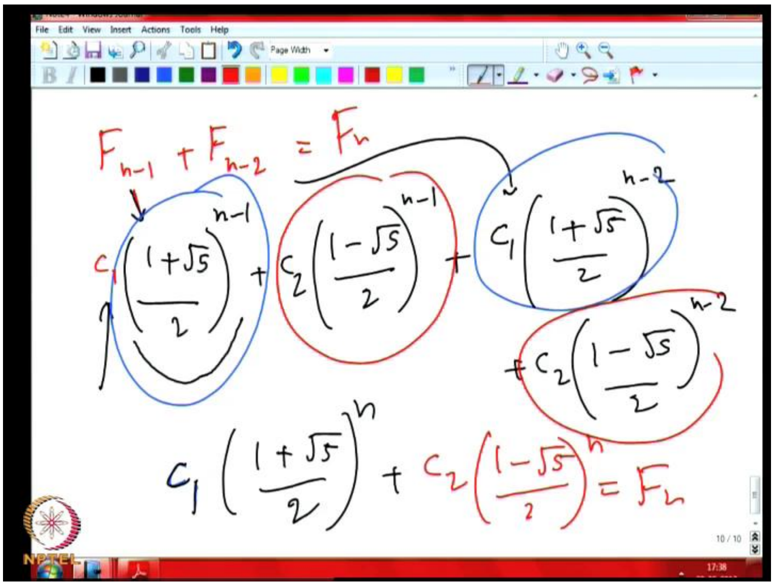

So, we may try calculating F n minus 1 plus F n minus 2. This will be equal to c 1 times, this I am expanding first, right, c 1 times 1 plus root 5 by 2 whole power n minus 1 plus c 2 times 1 plus 1 minus root 5 by 2 whole power n minus 1 plus, then again I expand this. So, c 1 times 1 plus root 5 by 2 whole power n minus 2 plus c 2 times 1 minus root 5 by 2 whole power n minus 2.

Now we can combine the terms together; for instance I can take c 1 into 1 plus root 5 times n plus 1 and 1 plus root 5, this two terms I can combine. So, this term and this term I can combine that is c 1 times this plus this, which is essentially, we know that is going to be c 1 times 1 plus root 5 by 2 whole power n, because we know that q to the power n minus 1 plus q to the power n minus 2 is q to the power n when q is taken as 1 plus root 5 by 2.

Similarly, these two terms I can combine this term and this term right? This is c 2 times 1 minus root 5 by 2 raise to n minus 1 plus 1 minus root 5 by 2 to the power n minus 2; that we know is 1 minus root 5 by 2 to the power n. So, this is essentially our definition of F n also, right, this is the way we have to. So, they are adding up to F n; this two are adding up to F n F n minus plus 1 F n minus 1 plus F n minus 2 is giving F n.

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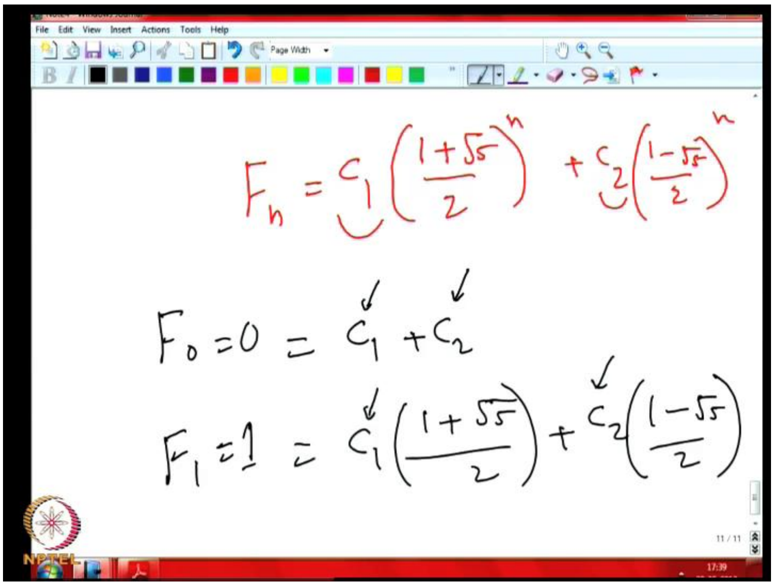

So, indeed we can define because we have two solutions. So, we can take the general solution as c 1 times that two solutions, first solution and c 2 times the second solution, right? The first solution is 1 plus root 5 by 2 whole power n, the second is 1 minus root 5 by 2 whole power n. The next thing is okay, if this is what is the use? The use is only that we have not defined c 1 and c 2 here. I can select it; any value I can select such that the initial conditions are met.

What are the initial conditions? One is F 0 is equal to 0 which means when I put n equal to 0 I will get c 1 plus c 2 is 0; this is what I get. The second is F 1 is equal to 1. So, when I put n equal to 1 I get c 1 into 1 plus root 5 by 2 plus c 2 into 1 minus root 5 by 2 is equal to 1. So, this is simultaneous equations; c 1 and c 2 we want to find out such that these two conditions are met, c 1 plus c 2 is equal to 0 and c 1 times 1 plus root 5 by 2 plus c 2 times 1 minus root 5 by 2. You know that this can be solved, right?

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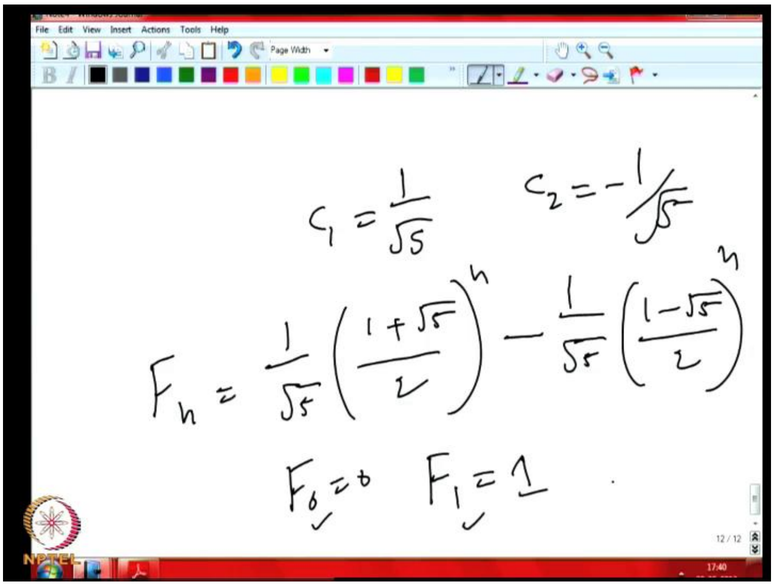

So, you can try to solve it, and then we get c 1 equal to 1 by root 5 and c 2 is equal to minus 1 by root 5. So, I am not wasting time on solving it as such. So, finally our formula will be F n equal to 1 by root 5 into 1 plus root 5 by 2 whole power n minus 1 by root 5 into 1 minus root 5 by 2 whole power n. So, the good thing here is when I put n equal to 0, I get the correct value; F 0 will be equal to 0; that is the way I have adjusted c 1 and c 2. Similarly when I put n equal to 1 I will get F 1 equal to 1; that is the way I have decided my constancy. When I could have selected c 1 and c 2 as anything, but because there are two initials values to be met. I carefully selected c 1 and c 2 such that those things are met, but once this F 0 and F 1 are met the rest of the things are taken care of by the recurrence relation namely.

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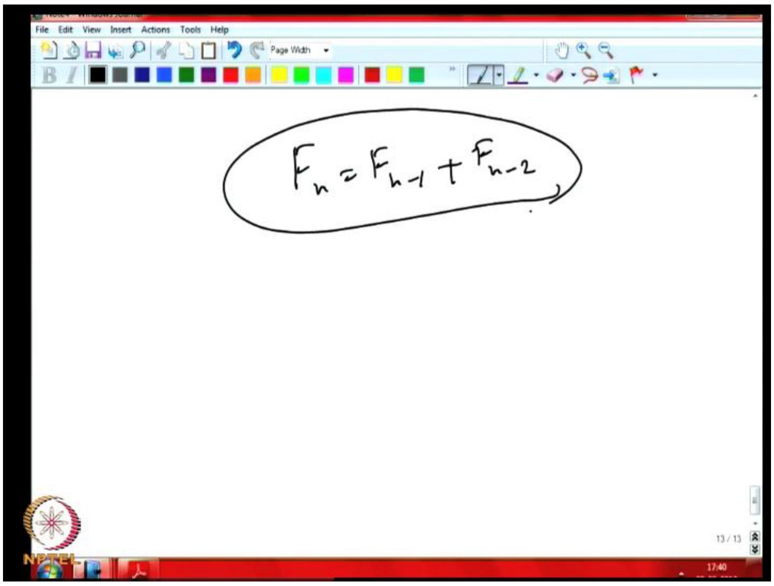

F n is equal to F n minus 1 plus F n minus 2, because for the expression we select it; this is valid, right? So, Fibonacci numbers will come. We will discuss in the next class.