Combinatorics
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Lecture - 22
Inclusion Exclusion Principle - Part (5)

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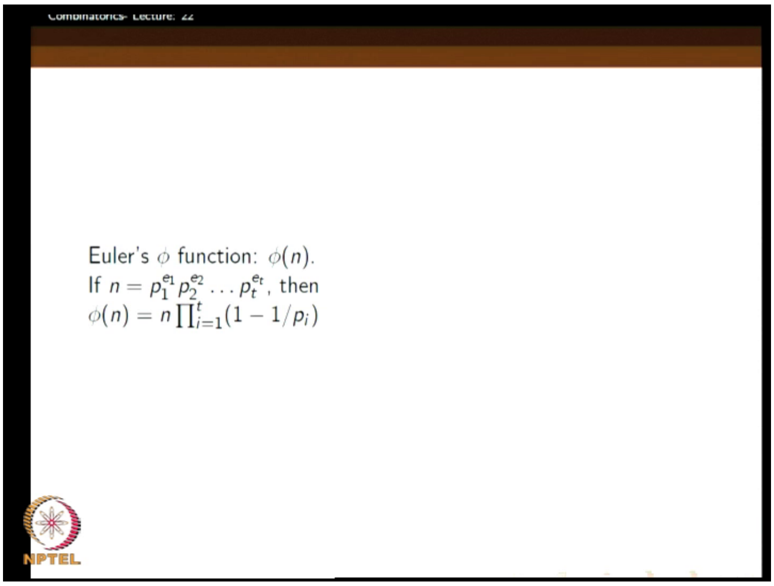

Welcome to the twenty second lecture of combinatorics. In the last class, we were discussing one of the interesting applications of inclusion exclusion principle, namely finding formula for the Euler phi function in terms of the prime factors of n the given number n.

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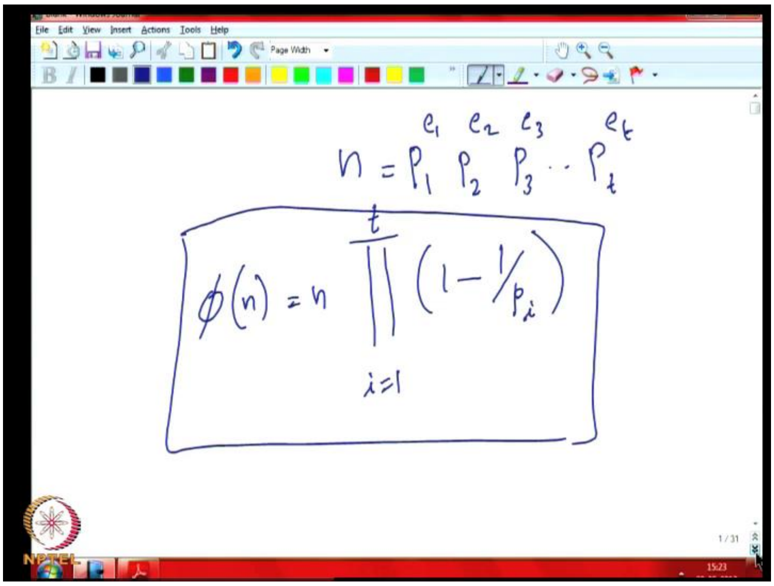

Suppose n equal to P 1 to the power e 1 P 2 to the power e 2, this is the prime composition of n, suppose P 3 e 3, so P t to the power e t. So, we want to prove the following formula phi of n this equal to n into product of 1 minus 1 by P i, i ranges from 1 to t; this is what we want. We have verified it in the case of prime numbers and like some other smaller numbers. So, now how do you prove this thing, how do you derive this formula? So, we want to do it using the inclusion exclusion principle.

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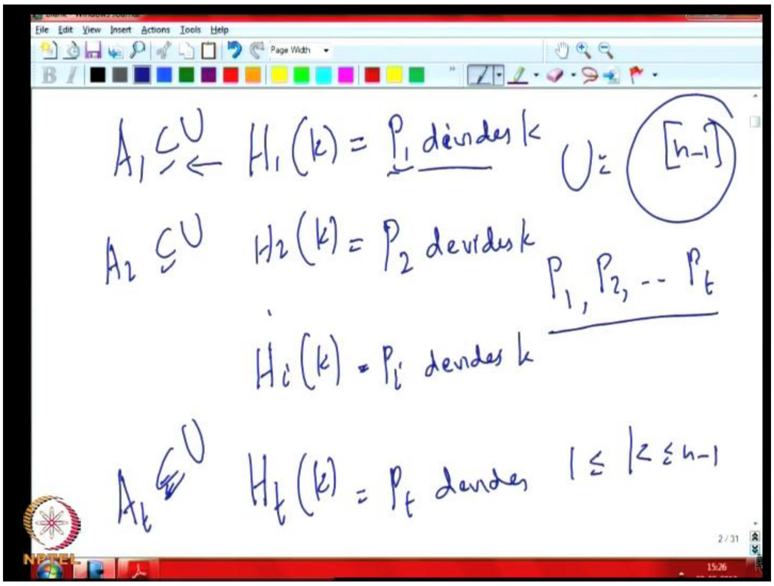

So, for that purpose we define certain properties H 1 H 2 up to H t; see this H 1 H 2 H t are the undesirable properties, right, which we do not want because we are looking for numbers. When we count phi n phi of n when we want to figure out the value of phi of n which all numbers from 1 to n in the universes see this n minus 1, right, 1 to n minus 1 which all numbers are such that they do not have any common factor with n other than 1. I mean relatively prime k n this is what we are looking for. So, when there is a common factor that is not desirable, right. So, therefore suppose there is a common factor then there should be a prime common factor; if there is a common factor there should be prime common factor also either P 1 or P 2 or P t should be common to k and, say, n, right. So, if we are analyzing k.

So, let us say we define H 1 of k H 2 of k H t of k for k in the range 1 to n minus 1. So, that H 1 of k says P 1 divides k of course P 1 divides n also, P 1 will be a common factor between n and k, right. So therefore, this k would not be a good number because if this property is satisfied by k, then P 1 divides k; of course n being P 1 to the power A 1 into P 2 to the power A 2 into P t to the power A t. P 1 will be a factor of n also; therefore, P 1 is a common factor; therefore, k is not relatively prime to n. So, this is not the kind of things we want, right; we want to avoid k which satisfies H 1. So, this is the first undesirable property. The second property we want to avoid is P 2 divides k. Similarly then H i of k would be P i divides k. Similarly the last one P t divides k. So, correspondingly we can also define if the set A 1 A 2 and A t as subsets of the universe, universe being this one n, right, universe being 1 to n minus 1, right.

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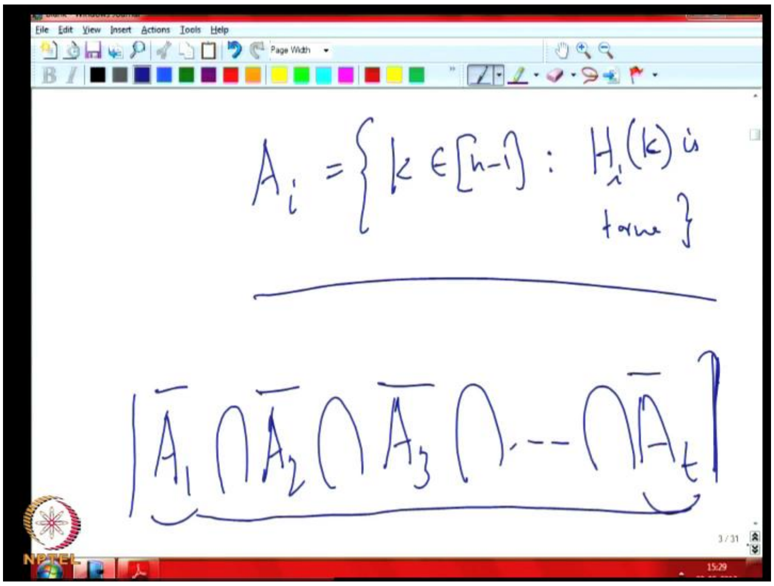

So, this A i would mean A i is this set of numbers from the universe such that those numbers satisfy property, right. So, H i of k is true is satisfied as usual. Now it is very clear that what we are looking for is the cardinality of A 1 complement intersection A 2 complement intersection A 3 complement intersection A t complement, why? Because you know this A 1 means of course P 1 is common factor between the numbers of A 1 and n. So, A 1 bar will not have P 1 as common factor, A 2 bar will not have P 2 as common factor; I mean the numbers in A 2 bar will not have P 2 as common factor with n. So, actually the numbers in A 2 bar will not have P 2 as a factor unit. Similarly numbers in A 1 bar will not have P 1 as a factor unit, numbers in A 3 bar will not have P 3 as a factor unit and then number in A t bar will not have A t compliment will not have P t as a factor unit.

So, when you take the intersection of all these things we get the numbers which do not have P 1 P 2 P 0033 or P t any of them as the factor in them. So, such numbers cannot have a common factor with n because if at all you have a common factor with n, there should be a prime common factor and these are the only primes which are in n. And therefore, if these primes are missing in k then this k has to be relatively prime to n, and this is the only way k can be relatively prime to n because even any of these P i's is a deviser of k, then there is that P i is a common factor and then we can apply the inclusion exclusion principle here, we know how to use it.

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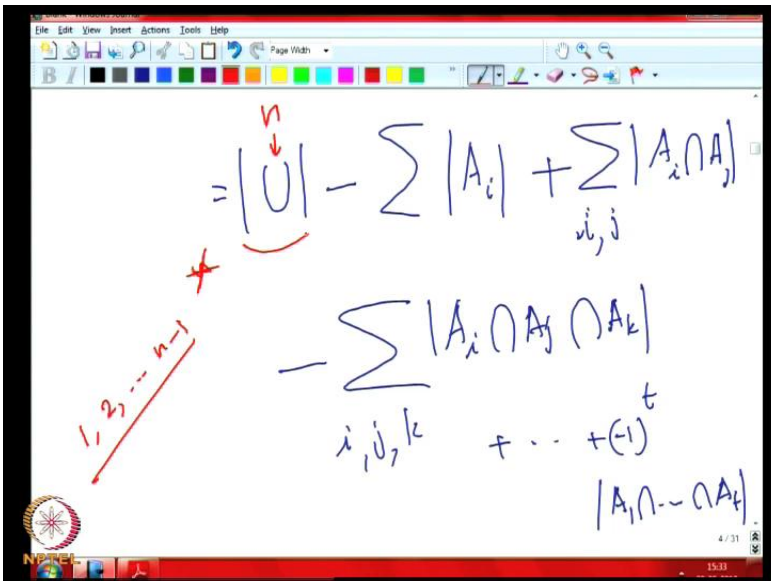

The value will be cardinality of U, right, minus sigma cardinality of A i, right, plus sigma for every i and j A i intersection A j, right; A i intersection A j if you take n minus sigma i j, k, I take A i intersection A j intersection A k, right, and so on, right. Finally, this is minus 1 raise to t, right, the last one A 1 intersection to A t, this is that we should, alright, estimate. This will be the answer.

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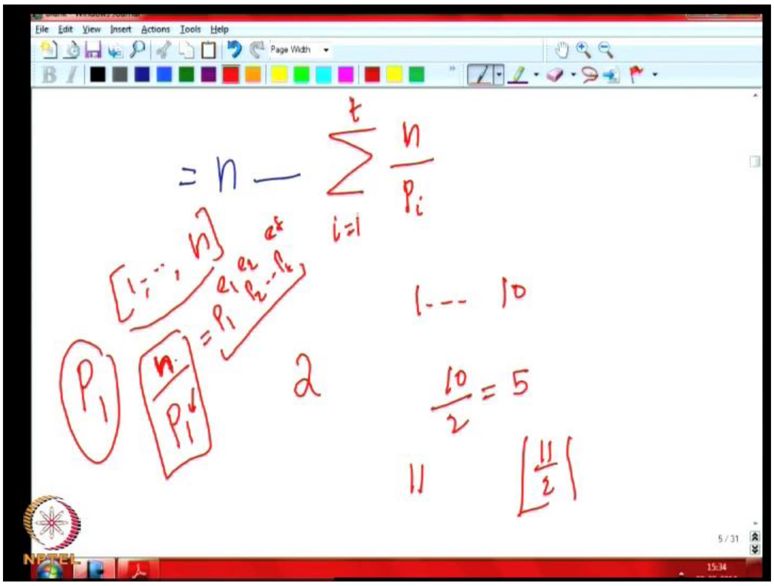

And this is what, the cardinality of u is n and the second term is you know for instance when P 1 is not a factor how will I find? Sorry, P 1 is a factor. The numbers I have to consider numbers from 1 to n minus 1 and then how many numbers have P 1 as a factor. We know that that is n by P 1 because we are minusing some numbers from here because I think, I was telling n minus 1.

So, this has to be n because we consider the universe now as n here. We can do it in n minus 1 also, but it is more convenient to consider the universe as n, so I will change it. So, here the universe let us take as n. As I told you as far as the numbers, which do not have a factor or I mean common factor with n; that means relatively prime numbers whether you take up to n minus 1 or n it does not matter because you know n will not be relatively prime to n.

But on the other hand when we define these properties the problem is that we are saying P 1 divides k. Now it depends on whether if you take the universes as n, n will be definitely there in this set, and if we do not take n then it will not be there, but it is more convenient to take n as the universe so that later the manipulations will be easier. So, we will consider n; that means 1 to n as the universe. So, that now when I wrote this thing A i is equal to k element of n H of k is satisfied. So, this is n, correct not n minus 1.

So, we take the universe cardinality as n not n minus 1, because we are including n also in the universe. So, I repeat; if you are interested in the numbers from 1 to n which are relatively prime to n. So, in a sense we only have to consider numbers from 1, 2, 3 up to n minus 1, because n is anyway not going to be part of this collection, but the difficulty here is that we are actually defining this undesirable properties, right. So, then P 1 divides k; I am trying to see how many numbers are satisfying that properties. So, there it makes a little difference because if you have define the universes up to 1 to, n minus 1 only. So, then n cannot be in this set.

So, that will change the numbers; it will make the manipulation more complicated. So, therefore, we define u as n. So, that n also will be in each of this a i's. So, that is what this will be taken as n now, this will be taken as n and then the next one as you see up to n. So, we have a how many numbers are divisible by P 1 here. So, as we have seen before it is n by P 1, right; for instance how many numbers from 1 to 10 are divisible by 2, we just have to take 10 by 5, right, not that n by P 1 is an integer. It is not a fraction because n if you remember is P 1 raise to e 1 into P 2 raise to e 2 into P t raise to e t. So therefore, when you divide by P 1 you will get an integer P 1 divides n, right.

So, this P 1 we will get a correct number like here 10 by 2 is 5; for instance, if you have taken 11 and asked the question you should have taken 11 by 2 floor, but in this case we do not have to do that because P 1 divides n. So, we say that for each P i n by P i is the number from 1 to n which is divisible by P i. So, this is from i equal to 1 to t for each of them, right. So, we have written this sum for A i the value is n by P i. So, we just wrote it like this. So, now what we do is look at the second term; second term is this one. So, if we take any 2 i and j. So, what is the intersection cardinality? A i intersection A j.

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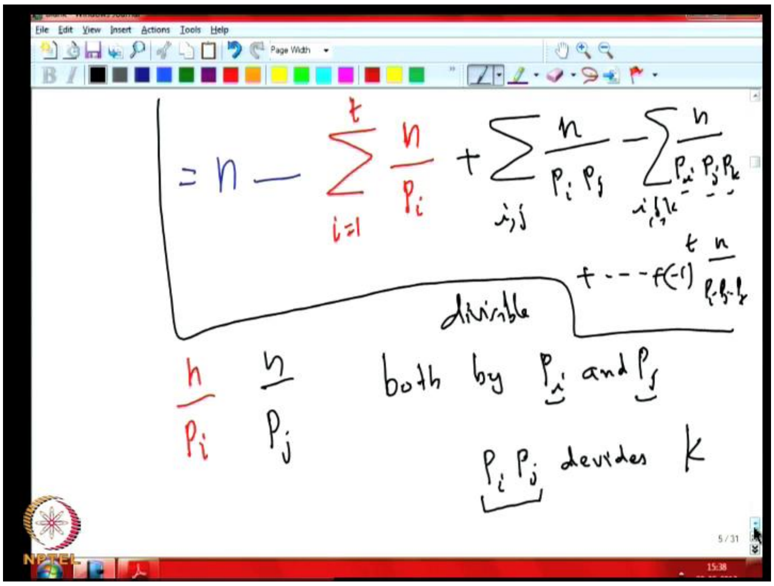

To find this thing what we can do is so that means the numbers which satisfy both the property P i and the property P j that is what we are talking about, but property h i and h j. What does it mean to say that the property P i is satisfied means it is the number is divisible by prime P i, right. So, there will be n by P i of them and similarly there will be n by P i numbers which are n by P j numbers which are divisible by P j, but when you say the numbers which are divisible both by P i and P j, how will you find out?

Because P i and P j both divides means P i, P j divides it, P j divides it divides k. If a number k is divisible by both P i and P j because this P i and P j both are prime numbers, if both of them divides k P i into P j has to divide and that is the only way it can divide it right. So, it is clear that n number is divisible by P i into P j then both P i and P j divides and also if P i and P j both divides and P i P j has to divide. So, instead of asking for the numbers which are divisible both by P i and P j, we can ask for the numbers which are divisible by P i into P j.

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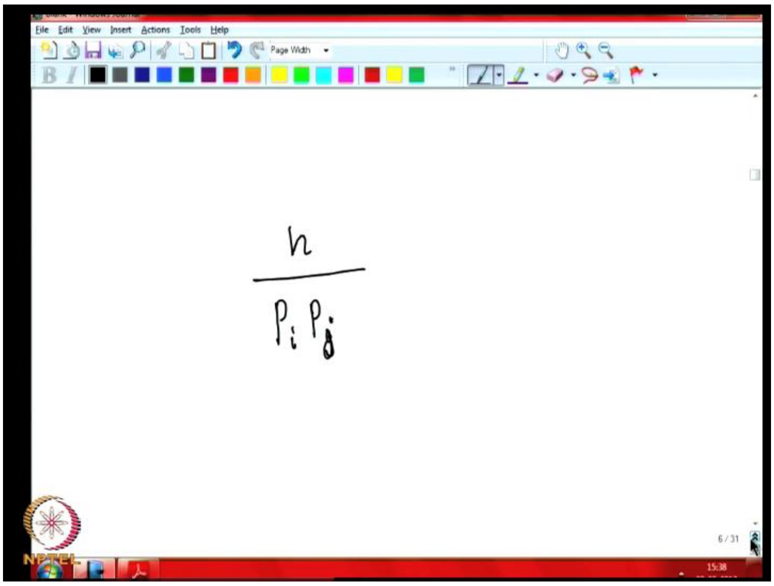

So, this number is n by P i into P j. So, before this is an integer because n has both P i and P j as factors. So, when you divide n by P i into P j we will definitely get an integer. So, the second next term here will come sigma for every i, j we will get every pair i, j from 1 to t. So, we can right n by P i into P j here, right. And next term would be similarly n by P i into P j into P k for i, j, k, right, because we now want all the three properties - h i h j and h k should be satisfied for selected i, j, k. We are selecting any three properties h i h j and h k and then we want to count the numbers which satisfy from the universe, 1 to n, the numbers from 1 to n, which satisfy all the three properties simultaneously right. That means that those numbers should be divisible by P i, those numbers should be divisible by P j and P k.

So, which is equivalent to asking for the numbers which are divisible by P i into P j into k; that is definitely n by P i into P j into k, this is what we will get. So, like this we can get a formula minus 1 raise to t. So, n by P i into P j into P t, this is what we will get, right. So, this is the formula we will get when we substitute the values for all these terms here, here, here and all of these terms, right, this is what we will get. Now we can try to do a little bit of simplification here.

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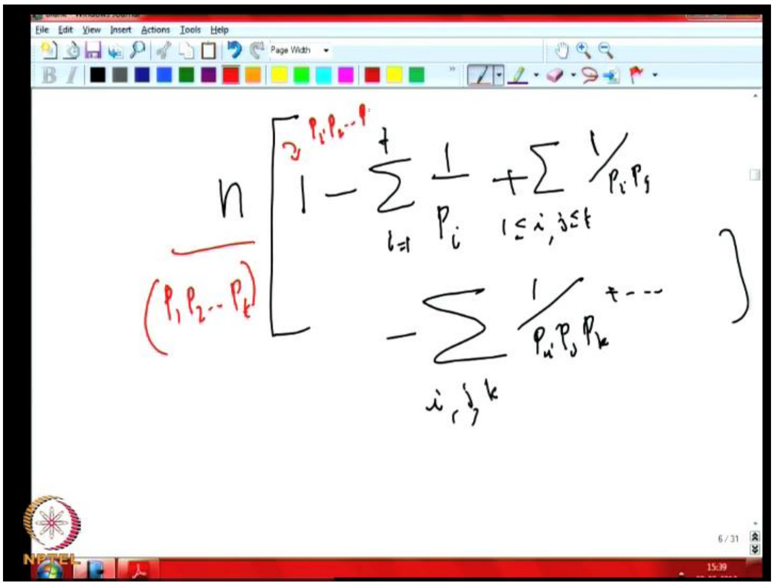

What we do is, we take n out and then we get 1 minus sigma 1 by P i, right; for every i equal to 1 to t and then this is sigma for every i and j on t, in the range 1 and t we are taking 1 by i into P j and minus sigma for every i, j, k. So, we are taking 1 by P i into P j into P k like this, right, this final term being minus 1 raise to t 1 by P 1 into P 2 into up to P t, right. So, what we can do now is we will divide it by P 1 into P 2 into P t and then of course you had to multiply if you divide it by this then here in the numerator you have to multiply by P 1 into P 2 into P t, right, every term should be multiplied.

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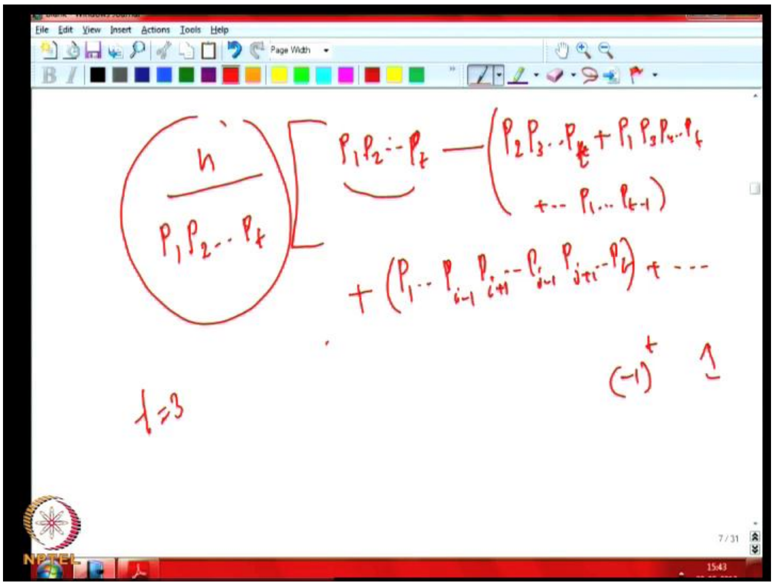

Then what we get is n by P 1 into P 2 into P t and then first term will be P 1 P 2 P t, right; second term will be some terms of this form P 1, P 1 will be omitted first, P 2 P 3 maybe I can write it fully. So, what we can do is first P 2 P 3 up to P k. So, what happens is here we have 1 by P 1 then you multiply by P 1 into P 2 into P t that one P 1 will go away, all the P 2 to P t will remain and then next. So, these are all inside one sum, right. Then P 2 will go, but P 1 P 3 P 4 up to p, sorry this is t P t will remain and so on.

Similarly P 1 to P t minus 1 will remain here and so on and plus. Now the next term would be of the form because here there are terms P i and P j's; then other than say, P i and P j for every P i P j selected all the other terms will remain here; say P 1 up to P i minus 1 then P i will be missing P i plus 1, and then up to P j minus 1. So, P j plus 1 ends in P t, this kind of terms will come in that.

So, like that we can enumerate all the terms, right, the last term will be just 1 because that is one by P 1 minus 1 raise to, sorry we have a negative or positive term minus 1 raise to t into 1, right. So, this is what. Now if you carefully look at it, right. So, see it would be for instance if the student is finding it difficult to understand this thing what is advisable is to take the case of, say, t equal to 3 and try writing all the terms then it will be clear, because I already spend a lot of time in this inclusion exclusion principle I am not spending time to take an example and write it, but it is very easy for the student to try it out but it is not difficult.

And the next important step is this thing. So, see what we have achieved now is that this kind of here everything see one by form fractions were available here. Now by multiplying by P 1 into P 2 into P t all over and what we have made is in the numerator we have only integer values. Now fractions are all removed, but then this is familiar because if you carefully look at it. So, of course you have to notice that if you just discard it for the time being just look at what is inside here, right.

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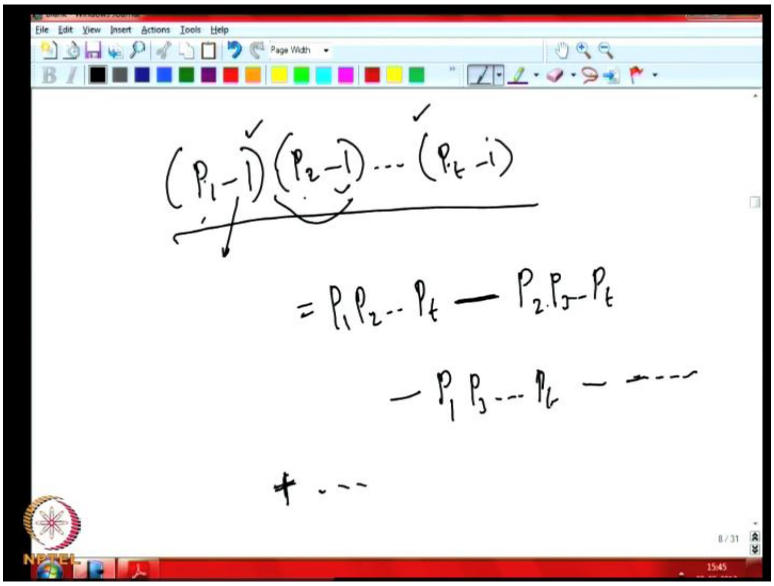

That is this stuff; this is P 1 minus 1 into P 2 minus 1 into P t minus 1. If you expand this thing what we get is the first term will be P 1 into P 2 into P t, right. Here also we have P 1 into P t; what is happening is we are taking P 1 from here, P 2 from here, P t from here and then we will leave out, what we will do we will take minus 1 from any of you remember how we proved the binomial theorem and all. So, we will take P i’s from each of the terms except one from that one we will take a minus 1.

So therefore, the sign will be minus 1, but all others of this form, right, P 2; for instance when I take minus 1 from this thing P 2 P 3 P t; from here I will get minus P 2 P 3 up to P t or if I take P 1 from here and only minus 1 from here and P 1 P 2 P 3 to P t then that is a minus P 1 P 3 P t, right; like that we can formulate all the forms, right. So, now finally the next type of things is when I take 2 minus 1’s and rest all P i’s; for instance I select from any of the two positions 2 minus 1’s and then collect all the other P i’s. So, we will leave out 2 P i so that P i and P j we divided off, right, that will correspond to minus 1 into minus 1 which is a plus, right. So, like that. So, this essentially the upper whatever we see in the upper part is nothing but this P 1 minus 1 into P 2 minus 1 into all the way to P t minus 1.

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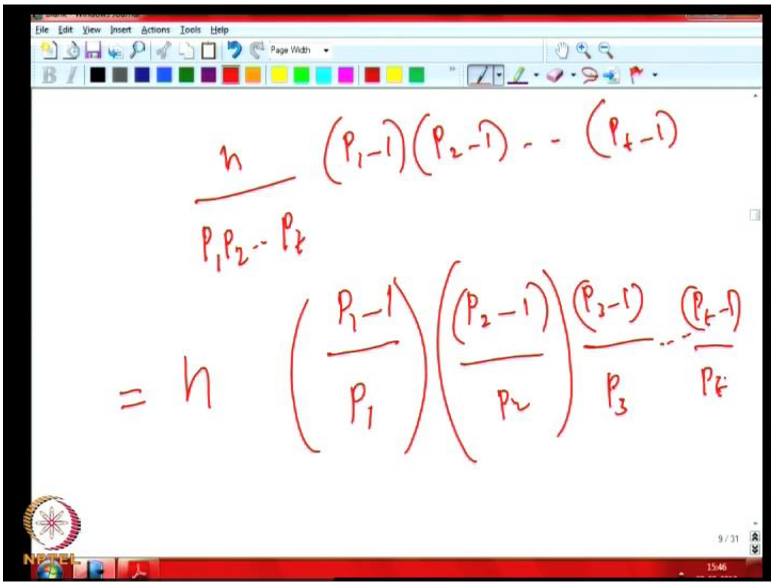

So, we can substitute it there like this n by P 1 P 2 up to P t P 1 minus 1 into P 2 minus 1 all the way to P t minus 1. Now you divide this by this, right. So, what we can do now is we take n out this is equal to n into. So, we will write it is a product n into this is P 1 minus 1 divided by P 1, right. So, maybe instead of writing the product we will write like this and this is P 2 minus 1 divided by P 2 and then P 3 minus 1 divided by P 3 and so on up to P t minus 1 divided by P t.

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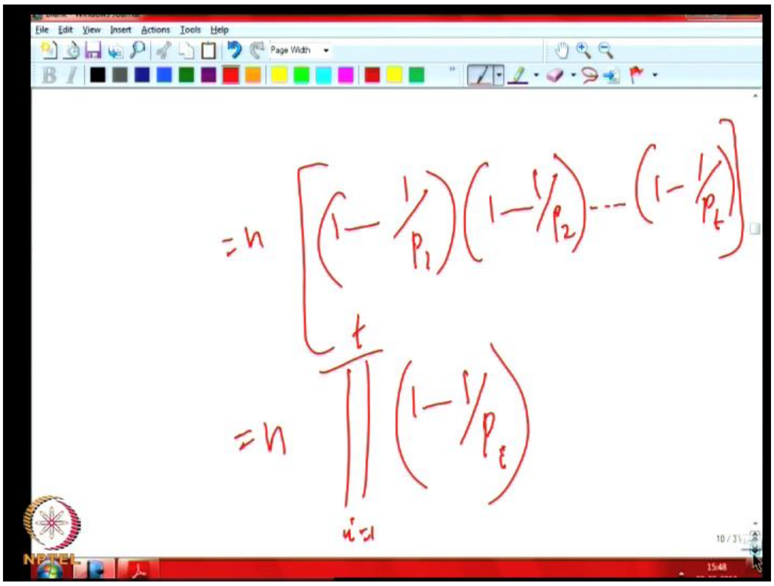

This is n into 1 minus 1 by P 1 into 1 minus 1 by P 2 into 1 minus 1 by P t. This is what we get. So, this is actually product of i equal to 1 to t 1 minus 1 by P i as promised; the last two steps are only manipulation. So, we have just rearranged this terms that for below P i minus 1 we wrote P i that is what. So, because below also there is a product above also there is a product, we can arrange them properly. So, I am just rewriting it and then P i minus 1 by P i is rewritten as 1 minus 1 by P i, that is all right. The crucial steps are here noticing that what we see in the upper part in the numerator other than this, right, can actually be expressed like this; that was the expansion of this form. This people may miss, but you have to observe that, right.

And then the earlier one this one by multiplying these are all usual steps to get rid of the fractions you multiply it by the all the things which have appeared in the denominator, right, the common LCM of what we see in the denominator the numbers which we see in the denominator and got rid of all the fractions, right. So, that is what we did, right; before that it was the application of the just only thing is estimating the sizes of those intersection of several sets. So, this is the way we proved prove this thing this inclusion exclusion principle gave us this formula finally, we will move out to the next one.

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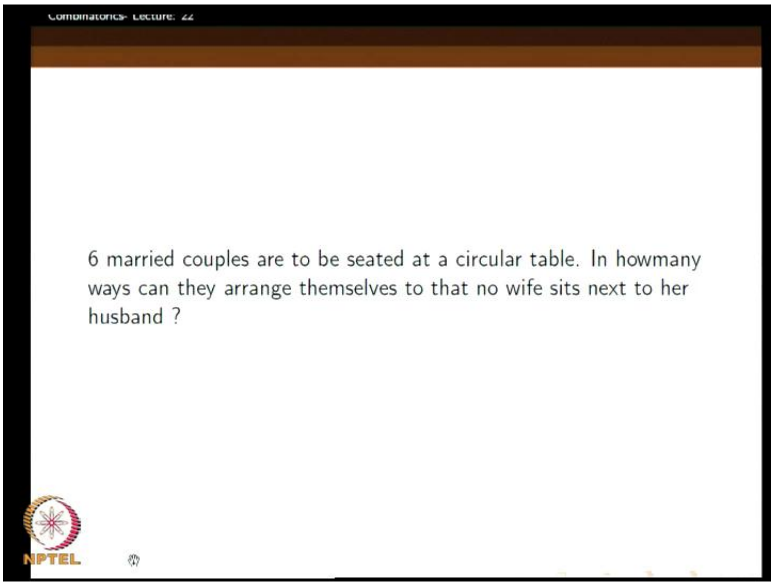

And here is another question. A 6 married couple are to seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband? So, 6 married couples are to be seated at a circular table; it is a typical system we were looking at and there are 12 people then, right. So, how many ways they can they arrange themselves so that no wife sits next to her husband.

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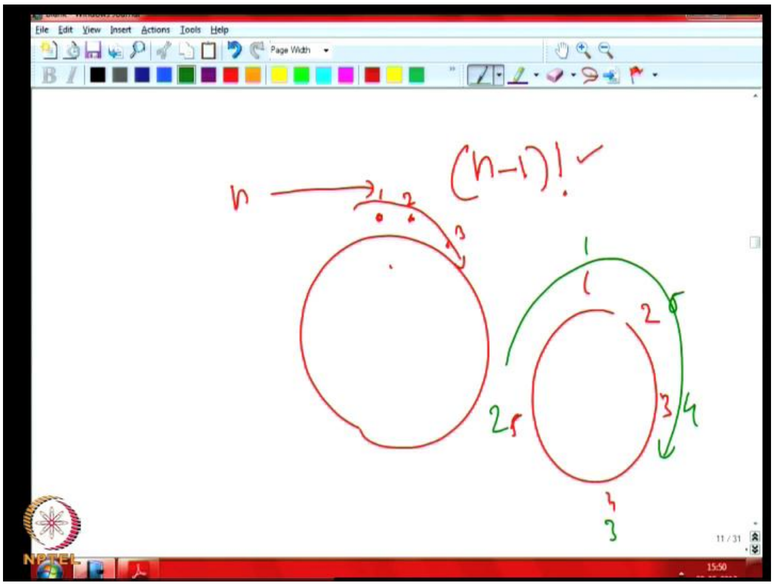

So, we had discussed the circular arrangement problem very elaborately when we learned the division principle. So, we know that n people can be arranged around a circular table in n minus 1 factorial; here what we say is that 1 2 3 this kind. If you rotate them, say, in one direction either clockwise or counterclockwise it is still the same, right, it is considered the same.

So, not that if you write 1 2 3 4 5 this is not the same as writing, say, 1 2 3 4 5, but on the other hand you can rotate either in the clockwise direction or in the counterclockwise direction that is, say, we had defined it as always we see the same left neighbor and the right neighbor, right. So, neighbors would not change. So, it is not neighbors would not change it is that if you have a notion of left neighbor because the clockwise neighbor counterclockwise neighbor they are not going to change, that is what it means, right. So, of course you can go back to that discussion once again if you have any confusion about the problem. So, that is we are referring to the kind of arrangements.

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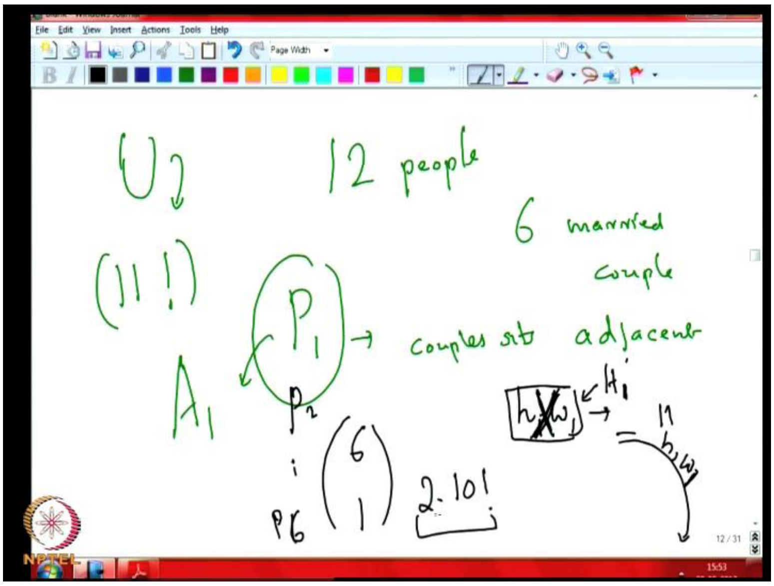

So, here we have 12 people just that they are 6 married couple 6 into 2, right. And the universes all possible arrangements as we say 11 factorial arrangements, right, this is the universe U. Now the properties we want to avoid. The first property P 1 will be that the first couple we can say couple one, couple two, couple three; couple one sits nearby, I mean consecutive positions are adjacent sits adjacent, right. But in how many ways we can I mean how many of this members elements from the universe, right, the arrangements from the universe will belong to this category namely.

So, we can call A 1 as the subset of the universe the arrangements which satisfy property one namely the first couple sits adjacent. We know the technique to count that namely, say, the couple one. So, the husband one and wife one say we will just tie them together, say; that means we will introduce some h dash notation for that. So, some combined they will consider them as one person and then we will arrange the resulting 11 persons around the circular table; that will result in 10 factorial arrangements right because 11 minus 1.

But now wherever we see this special symbol, right. So, let us say we call H 1, right. So, then we will replace this couple there, but then this couple can be placed there either as a h 1 w 1 or as w 1 h 1, right. There are two possible ways we can. So, as if we are reading from in the clockwise direction. So, around the table we are going in the clockwise direction. We can first place h 1 and w after that or first the wife and then husband, right;

so therefore, 2 into 10 factorial possible ways to arrange them, right. So, now we see that there are 6 couples, we can select 6 choose 1 couple we can select, right. So, there are how many properties? So, P 2 P 6, right there are 6 properties, any properties can course this thing.

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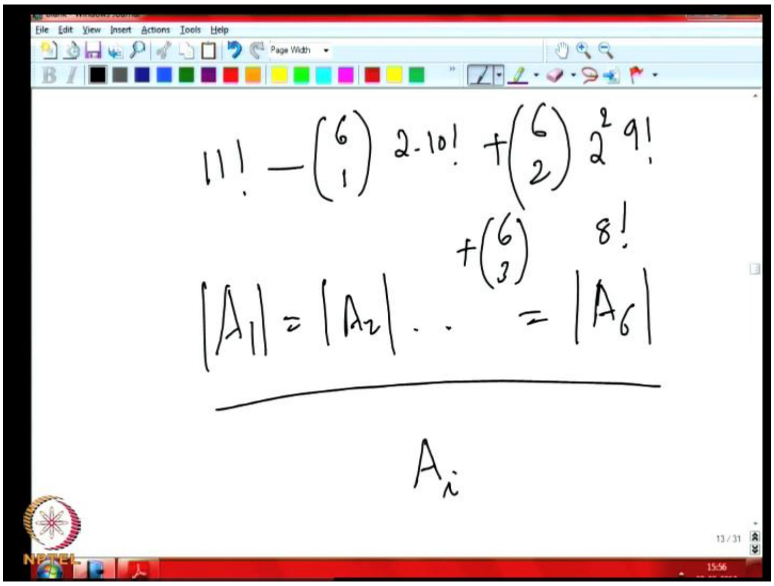

So, we say that this is 11 factorial is the universe size minus that sigma cardinality of A i's will correspond to 6 choose 1 into 2 into 10 factorial because here this is a special case that if you consider a set A i its cardinality does not depend on which set it is it only depend on. So, we are talking about. So, the case where A 1 is equal to A 2 is equal to A 6 because you know which couple is sitting next to next is unimportant, because that count will be still 2 into 10 factorial, right, this is that case where we simplified, right, we gave a formula saying that. So, in the situation that the cardinality of some collection of set, say, P set selected from the k sets, right. If their intersection cardinality is always the same then we can use a special formula namely first in the universe size U is written as alpha 0 minus k choose 1 into alpha 1 plus k choose 2 into alpha 2 that formula we remember, this will come in that category.

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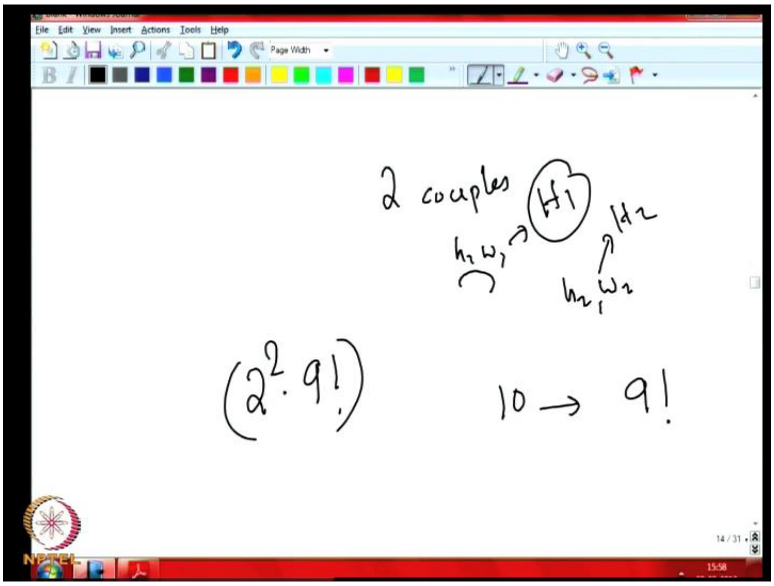

So, in the next case we are talking about two couples two married couples are sitting next to next. So, whichever is the couple two of them they are sitting. So, what we do is we first replace say h 1 w 1 by H 1 by one symbol, so h 2 w 2 by another symbol H 2. Now we have you know initially 12 people were there. Now the two are reduced; only 10 people can be seated around 9 factorial ways and wherever we see this H 1 we will replace either by h 1 w 1 or w 1 h 1 in two possible ways, here also h 1 w 1 or h 2. So, 2 into 2 possible ways of replacing, so 2 square into 9 factorial ways of replacing is there, right.

So therefore, this will correspond to 6 choose 2; any two couple can be selected for that. So, 2 square into 9 factorial. Then next term we can easily see is 6 choose 3 because any three couple are selected this will become 8 factorial why because you know for each couple each couple is replaced by 1 symbol now. So from 12, 3 will reduce. So, there will be 9 people, the circular arrangement will be one more less. So, 9 people are arranged around a circular table in 8 factorial ways, right. So, now this 8 factorial after getting this for each of the special symbols; we have two ways of placing the husband and wife, either husband earlier wives later or wife earlier husband later, right.

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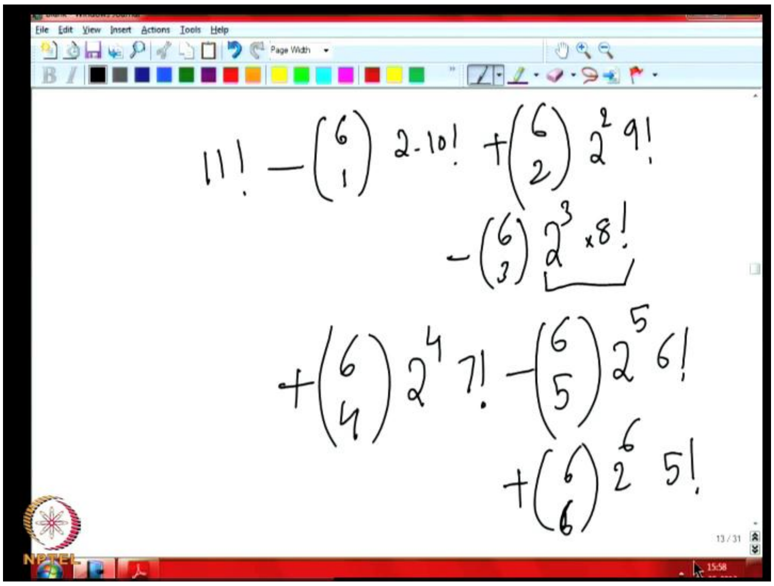

So therefore, 2 to the power 3 into 8 factorial possible ways of doing it and 6 choose 3 ways of selecting the three couples right. So, there are three sets that i j k from 1 to k can be selected in. So, of course k choose three ways. Sorry this was minus, and then next is plus. So, that would be 6 choose 4 into 2 to the power 4 into 7 factorial. So, the student can easily figure out what is happening here 6 choose 5 into 2 to the power 5 into 6 factorial and next one, the last one will be 6 chose 6 into 2 to the power 6 into; all the couple are sitting next to next and then here it is 5 factorial, right. So, 5 factorial 2 to the power 6 ways; so, this would be the answer right.

So, of course this is just direct application of thing; what you have to remember is only the number of ways of arranging n people around a circular table; that is n minus 1 factorial and then here once the properties that we want to avoid is written out, then finding this cardinalities are easy because you know a couple is now replaced by one symbol one representative symbol and then now the number of people reduces because of that and then we know the circular arrangements accordingly and then only thing is really this couple can sit in two different ways any couple. Now it is only one symbol; when you replace the couple back there, there are two ways of replacing back, right.

So therefore, if there is some t couples we have converted to like combined or maybe represented by symbols then each symbol has two ways of being, right, replaced back by the couple, right. Therefore 2 to the power t will come there; that is only thing, right. So, this is simple. So therefore, I do not think there are many new ideas in that, right. So, now the next question we can consider.

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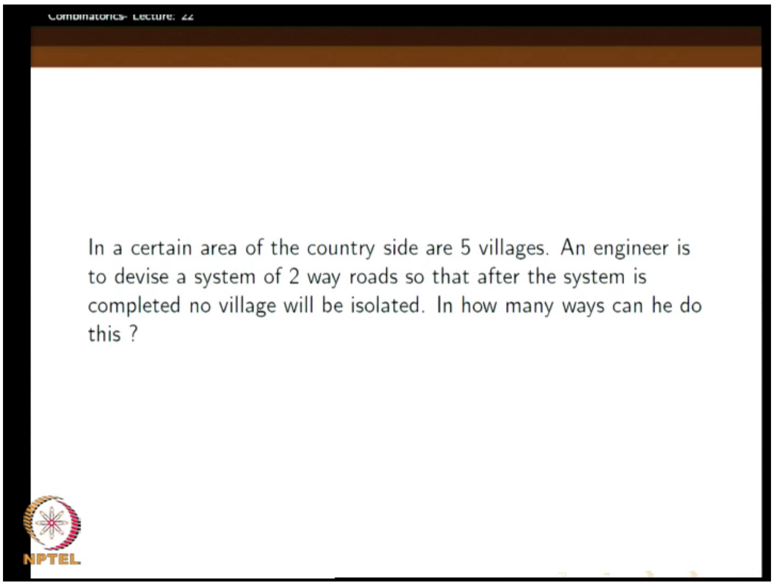

So, here is a graph problem. In a certain area of the country side are 5 villages. An engineer is to devise a system of 2 way roads so that after the system is completed no village will be isolated. In how many ways can we do this thing?

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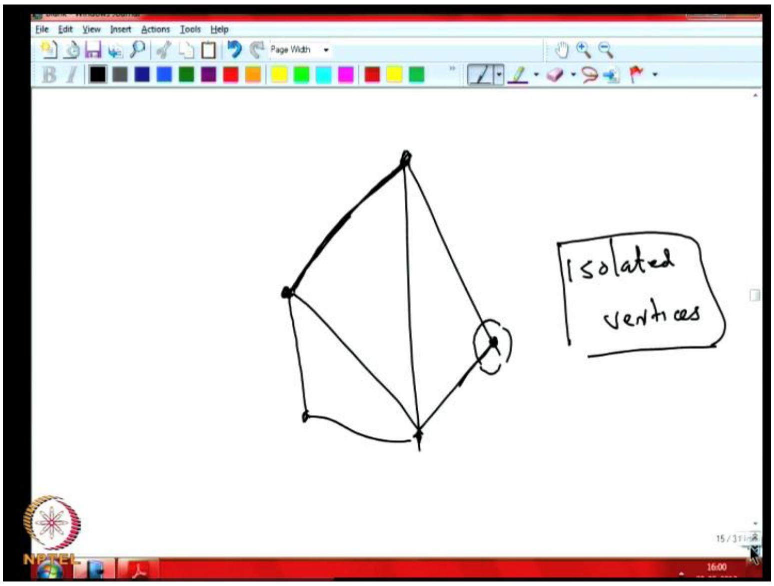

So, here of course this is a question about graphs because 5 villages I will place here. So, some 5 villages, now 2-way road means just an edge. So, as if we are drawing an undirected graph on undirected edge means you can go this way or this way; there is none like, direction is not important here, and then we want to draw these edges. These are the roads, right, but the only condition we are saying that it should not happen that no road goes to this village. So, we are creating a graph here but making sure that there are no isolated vertices; isolated vertices means the degree of the vertex in the graph is zero, right, not that we are not talking about connected graphs here while of course this is indeed a valid way to assign the roads, rights.

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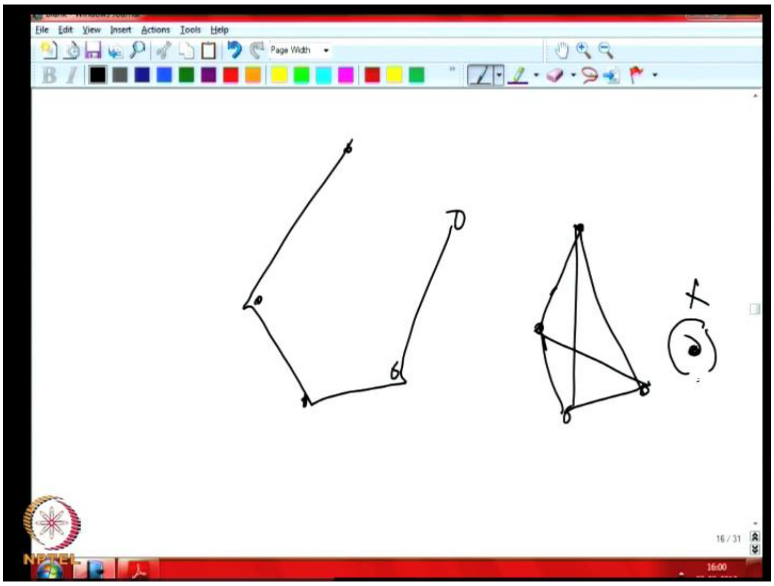

This is a valid way to assign the roads, but this is not a valid way. So, how much ever roads you put here this is not a valid one, right.

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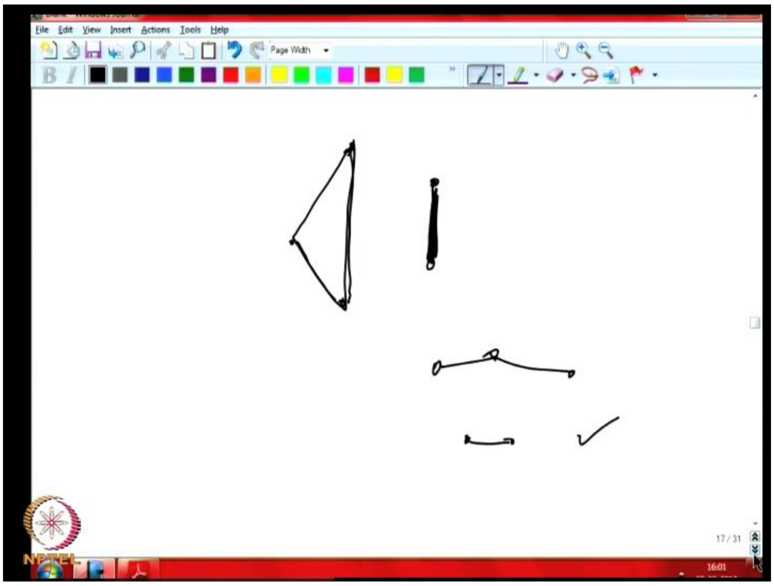

Now because there was an isolated vertex here, this we do not want but not that this is valid; see we can always have this kind of a system. So, though there is no connectivity here for instance from here we cannot reach here, but still as far as each village is concern they are not isolated; there is a road between two villages nearby. So therefore, they are not isolated, right. So, this is valid. Similarly yeah so correct, this is also valid. So, these kinds of things are okay, right. So, we just want to avoid isolated vertices. Now how many ways we can do it? It is a matter of counting, say, the first.

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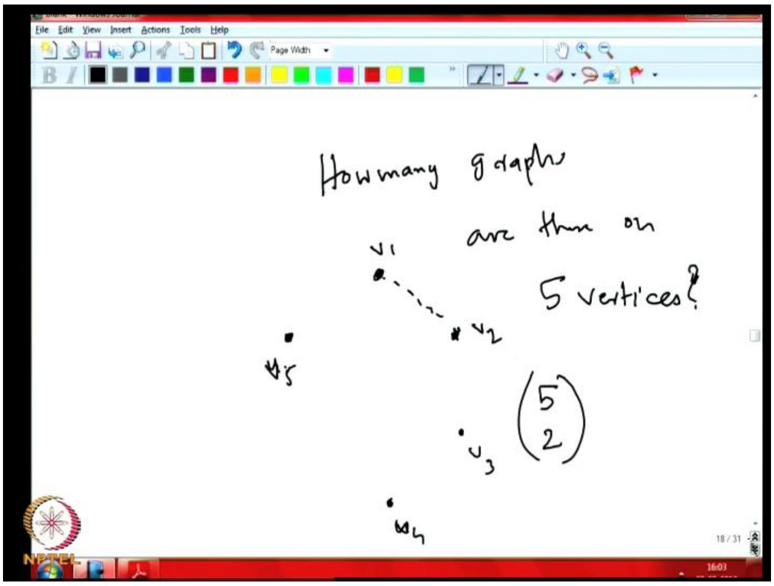

Suppose if we do not have any restriction, it is just a question of how many graphs can be drawn? Are there graphs on 5 vertices are what we are asking if the condition no restriction was there? See for instance you note that we are talking about simple graphs in the sense that there are no loops, nothing like this. So, similarly we do not put double roads here, right. So, it is only between any two villages we have one road and then there is nothing like a village the same village road kind of thing; that does not make sense in this problem at least. When we talk about graphs we should worry about it; there are no self loops, there are no multiple edges multi edges, right.

Now, how many graphs can be there? So, it is just a question, it is a simple combinatorial problem. So, there are 5 vertices, how many edges are possible as what you should ask. So, 5 vertices how many edges are possible? Definitely there are 5 choose 2 edges possible because any pair can actually define an edge. There can be an edge or may not be edge, there can be or may not be, how will you express this thing. So, we can say that for instance if there is an edge. So, I will list down all these possible edges, right. So, for instance this is, say, v 1 v 2 v 3 v 4 v 5, right.

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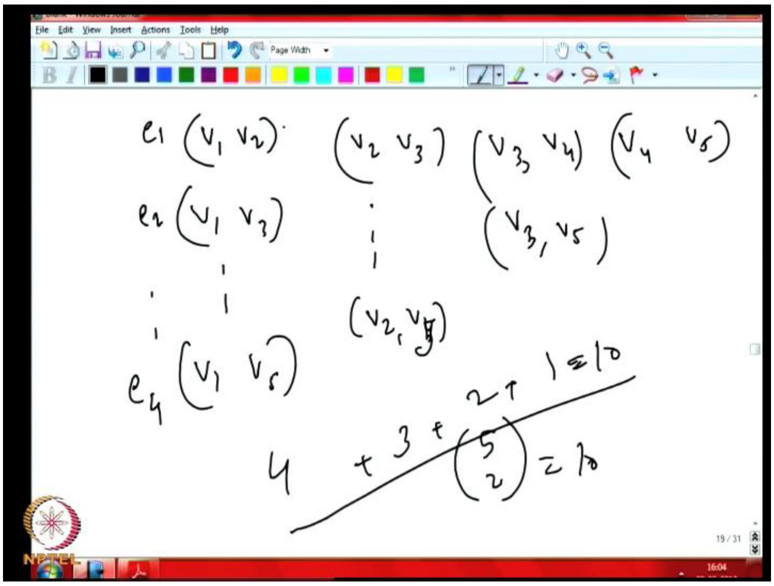

So, now the possible edges are v 1 v 2, v 1 v 3, v 1 v 5 then v 2 v 3. So, up to v 2 v 4 and then we have v 3 v 4, sorry this is v 5, and v 3 v 5 and then finally we have v 4 v 5, right, total this many. This is actually 5 choose 2 is equal to 10, this 4 plus 3 here 2 here and then 1 in the end, right. This is total 10, 10 possible edges are there. Now let us say this is first edge, second edge, say, fourth edge, this is the fifth edge and like that we number edges also, right.

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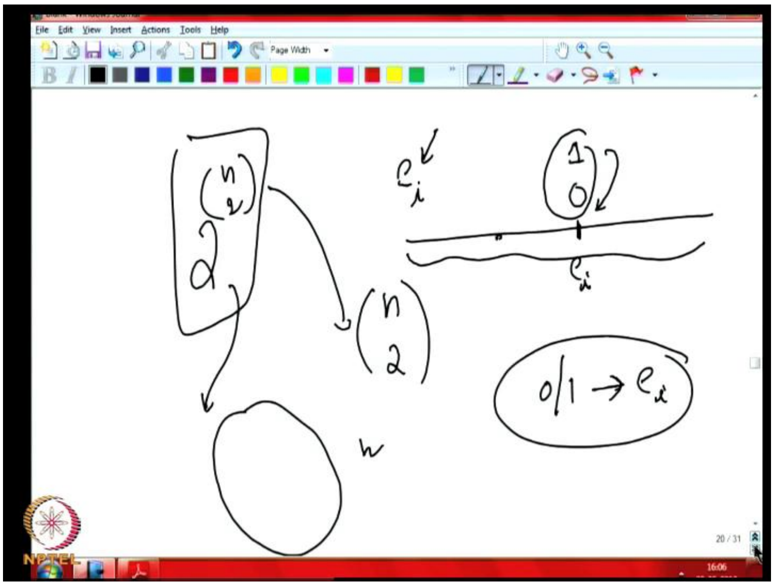

So, now, we can decide whether the edge e i is present in the graph or not by saying whether it is 1 or 0 by associating with 1. So, if you put 1 for e i then it is present; otherwise, it is a 0. So, we have a binary string of length n choose 2 being defined, because we have a long string and the position i th position correspond to the i th edge the way we have defined i th edge. So, if we decide to have that edge in the graph then we put 1 in that position the string position the i th position of the string; otherwise, we will put a 0 there, right; 0 or 1 can be placed at the e i th position.

Now how many strings are possible? There are 2 to the power n choose 2 strings possible, because there are n choose 2 positions available. And you can see that any graph will correspond to a string this way, and the converse is that if you have any string we can define the graph based on the edges. You just look at the i th position, if it is 1 put an edge there, otherwise, do not put, right. So, what I mean is if I get a binary string of length n choose 2 any binary string of length n choose 2, I can get a unique graph from that one n vertices, right, in this way procedure. The converse is that if you get any graph on n vertices, we can write down the corresponding binary string. We just look at the i th edge and if that edge is present put one in the i th position, otherwise, put 0 in the i th position, you get. So therefore, it is same as counting how many binary strings are there of length n choose 2. So, that is 2 to the power n choose 2, right.

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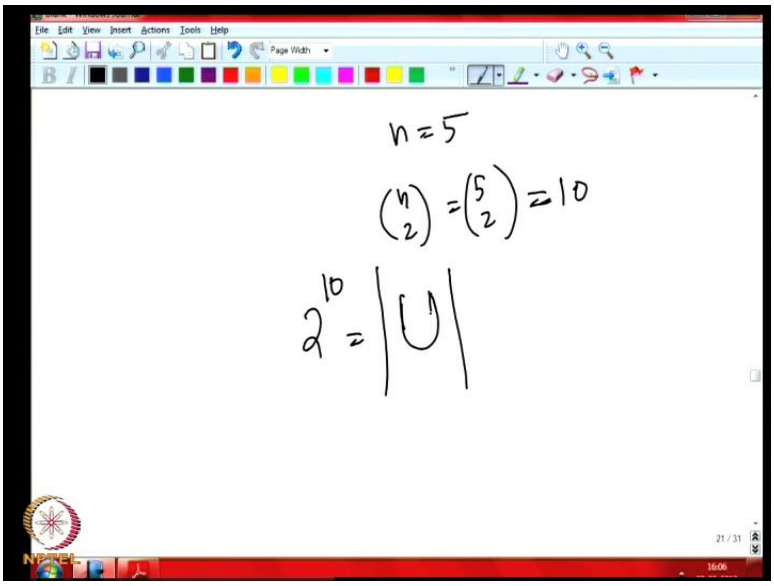

So, in this case if we do not have any restriction n equal to 5. So, n choose 2 is equal to 5 choose 2 is 10, right. There are 2 to the power 10 possible graphs. This will be the cardinality of our universe all possible graphs without any restriction.

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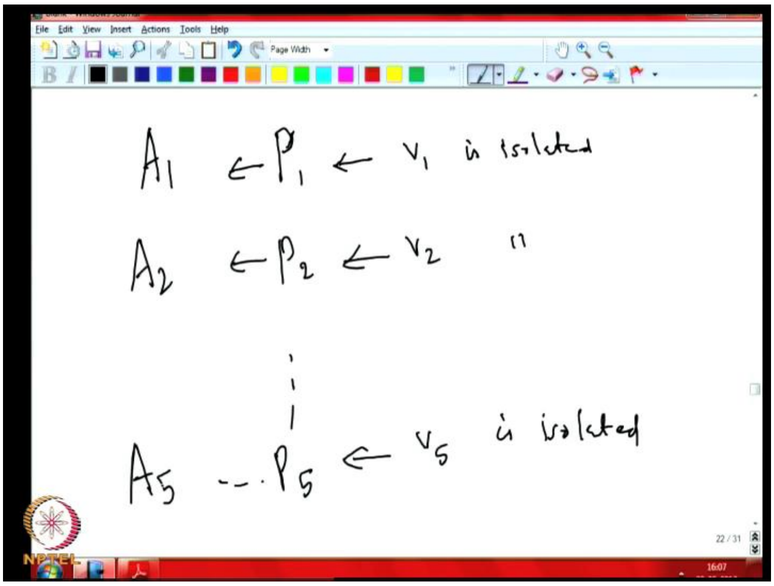

Now the properties that are to be avoided, P 1 is that vertex v 1 is isolated. Similarly, P 2 will be the property that vertex v 2 is isolated; similarly sp 5 is the property that the vertex v 5 is isolated. All these properties are undesirable, because we do not want any vertex to be isolated. So, now we know what we are looking for. So, suppose A 1 is the set of graphs around 5 vertices which satisfy property P 1; that means vertex v 1 is isolated in that. Similarly let A 2 be the set of graphs which satisfy property P 2; that means the vertex v 2 is isolated in the graph and similarly A 5.

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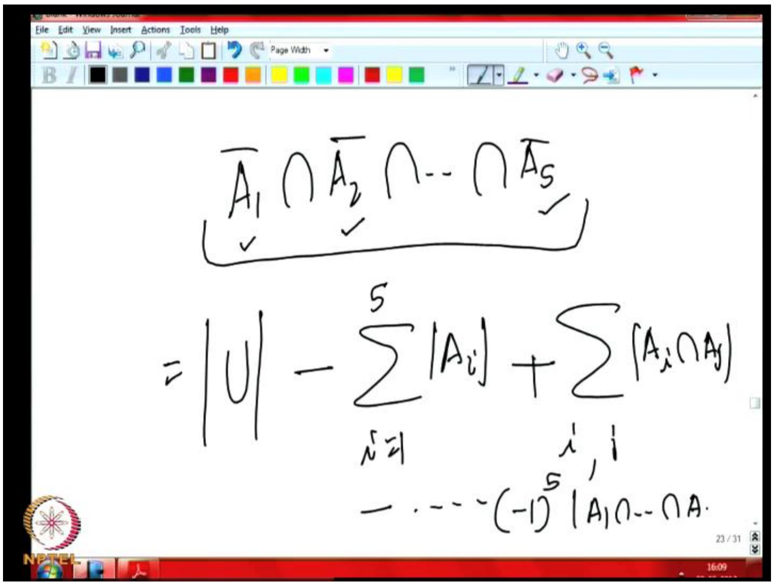

Now we know we are looking for this set A 1 complement intersection A 2 complement intersection A 5 complement, why? Because we are looking for graphs on 5 vertices where the vertex v 1 is not isolated and the vertex v 2 is not isolated and the vertex v 3 is not isolated and the vertex v 4 is not isolated and the vertex v 5 is not; this is just this intersection, right. And this we know, this is cardinality of U minus sigma A i cardinalities for i equal to 1 to 5 plus sigma for every i and j pair i j we have we have to find A i intersection A j and so on and so forth, right.

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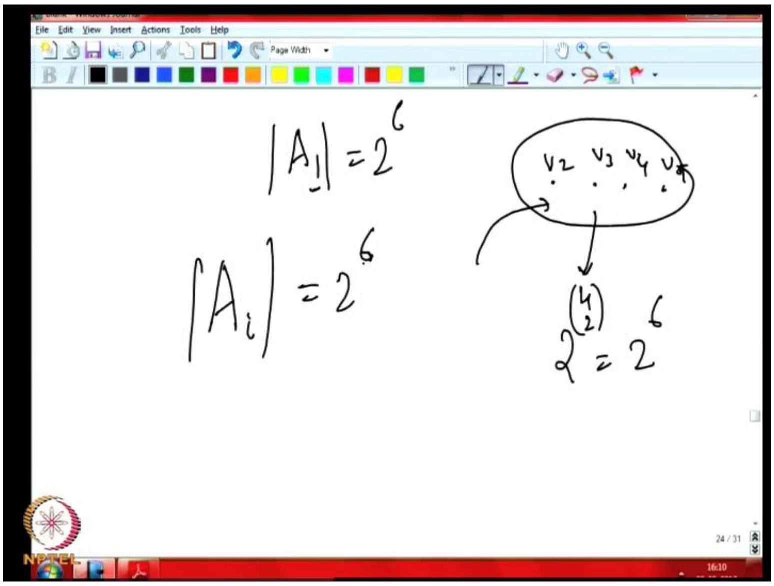

Now how do we find the cardinality of A 1? That is easy because you know that the vertex v 1 is not i is isolated. It is as good as asking how many graphs are possible on the vertex at v 2 v 3 v 4 and v 5. So, there are only four vertices and then how many graphs are possible, because we are never going to put an edge incident on v 1. So, it is same as asking how many graphs are there on 4 vertices, right. Discard one vertex; that is definitely 2 to the power 4 choose 2 namely 2 to the power 6 that is 2 to the power 6, right. And there is nothing special about vertex v 1; that same counting is true for vertex v 2 v 3 v 4 v 5 anything. So therefore, for each i, i equal to 1 to 5 we get this 2 to the power 6, right, 2 to the power 4 choose 2.

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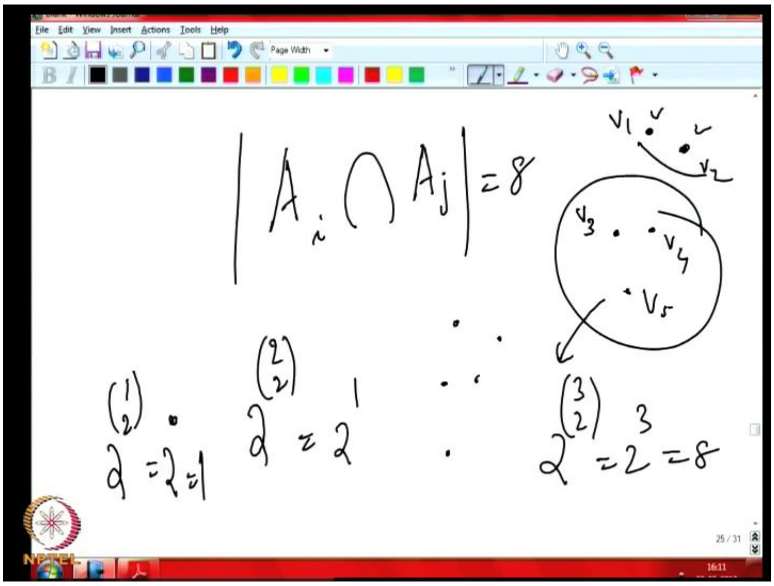

Now what about A i intersection A j cardinality? This is also easy to find, because we want two vertices to be isolated simultaneously. So, for instance v 1 and v 2 are to be isolated, and it is as good as asking how many graphs are possible on the remaining vertices v 3 v 4 v 5, right; that is what we are asking, and we know this is 2 to the power 3 choose 2 namely 2 to the power 3 which is 8. So, this is 8 and for every i j it does not matter which two vertices we are leaving out, right. So therefore, these are same and finally, if you want, say, A i same way we can find; for instance when you take three vertices together simultaneously has to be isolated that will end up as 2 to the power 2 choose two namely 2 to the power 1, right.

Because three vertices has gone and then how many graphs can be drawn on this two vertices; that is only two ways, either there is an edge here or no, right. And then finally, four vertices are gone; there is only one way of doing it, that is 2 to the power 1 choose 2 is 2 to the power 0 is 1, right. So, now we know how to find the cardinalities of all the intersection of all the required sets. So, we just use that simplified formula; I mean the formula which we derived for the simplified case where the particular I mean suppose t sets are taken and the intersection is made and that cardinality only depends on t does not depend on which t sets, right, that is this case belongs to that.

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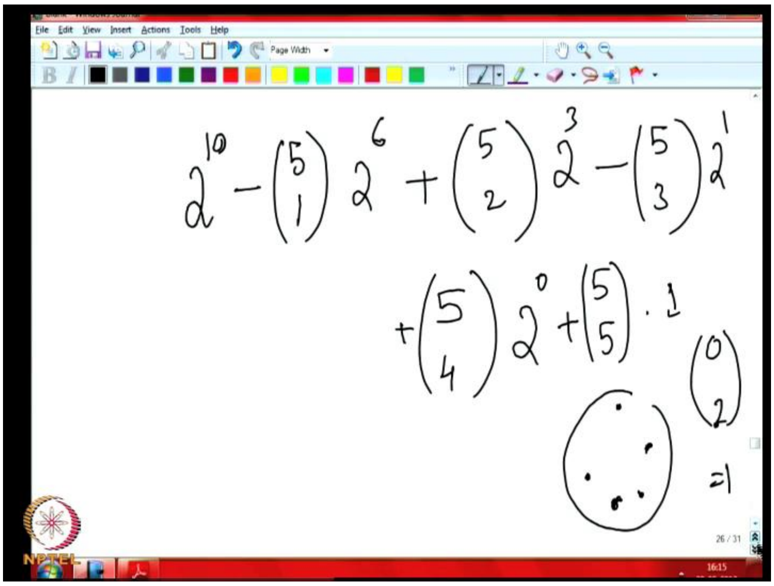

Therefore, that would be this cardinality of universe namely 2 to the power 10 minus 5 choose 1 into 2 to the power 6 plus 5 choose 2 into 2 to the power 3 and then minus 5 choose 3 into 2 to the power 1 plus then 5 choose 4 into 2 to the power 0 and then we do not have anything, right. Yeah, but the last one for instance when we say how many graphs are there with all 5 vertices isolated; that is like there is nothing 0 choose 2 is what we are asking that will be just one because you know there is a graph here with everything is isolated. So, therefore, that also will come, right, 2 choose 0. So, finally that 5 choose 5 and that is again this one the; last one is just one because all the properties together are satisfied, all the 5 vertices are isolated; that means there is only one graph like that isolated graph, right. So, now what we do is the next one.

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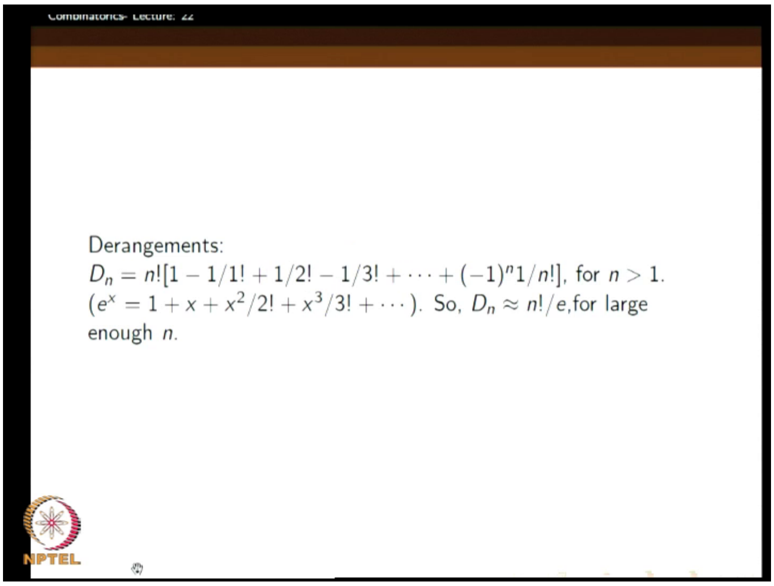

So, the next problem is about derangements; this is also an important problem. So, we will quickly do this thing, derangement means what?

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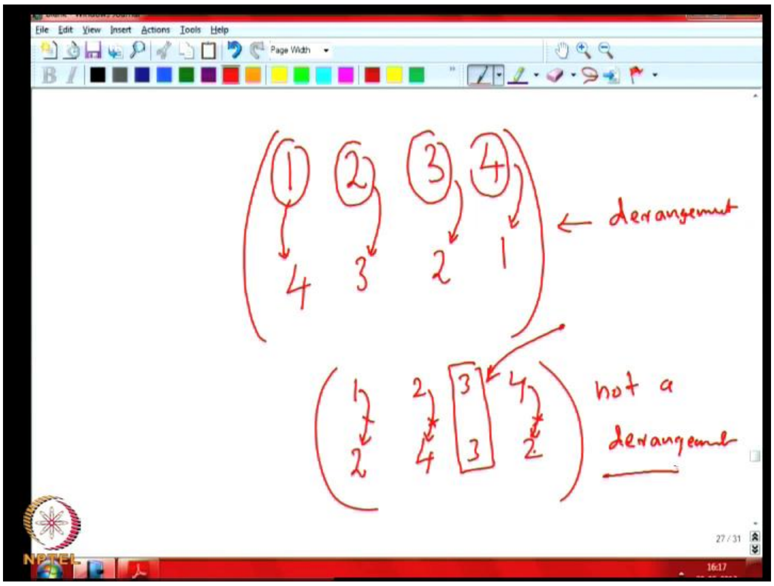

So, we are talking about permutations now, suppose 1 2 3 4. So, we consider the permutations of 1 2 3 4. So, we can always represent the permutation like this. So, we first write the positions. So, 1 2 3 4 here and then maybe the permutation can be written below this 1 2 3 4, say, 3 2 1. Now we ask at this first position do we get 1, no; the second position do we get 2, no. So, in the third position do we get 3, no; in the fourth position do we get 4, no. Here every position we are getting a number different from the position numbers, right.

So, such a permutation is called a derangement. So, an example of derangement would be like this; for instance 1 2 3 4, see suppose if I take 2 4 3 2 here the first position we have no problem because it is not 1, the second position we have no problem because in the second position we have seen 4 not 2, the fourth position also we do not have a problem because the fourth position we have seen it 2 not 4, but in third position we are seeing 3 itself. So, this is something we want to avoid, right. So, this is not a derangement, this problem can be right. There are usually we say that one situation we use to explain this derangement as.

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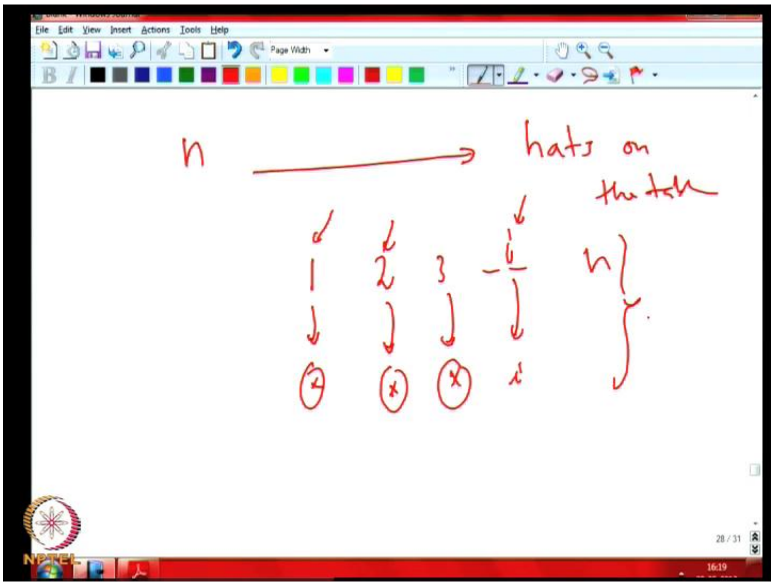

So, say n gentlemen come to a party and they keep their hats on the table and then while going back each of them takes their hats back; he takes the hats back, he takes one hat from the table. Now the question is in how many ways they can take back their hats such that nobody gets his own hat, right. He is taking a hat from the table, but there is no one among this n people who gets the same hat. So, that number of ways is the number of derangements because their original hats are numbered by their, say, the person number 1, person number 2 up to person number n, the hats also has the number say 1 2 3 up to n.

Now the person number 1 comes and takes a hat which is never 1 something else; so, when person number 2 comes and takes a hat which is something else, right. So, these numbers will be whatever the person number 1 2 3 n takes the those hat numbers are written below them then in the i th position you will never see i because if that is like that, then that means i th person has taken his own hat, right. So, in this situation we are asking for how many derangements are there, right, for the permutation 1 to n.

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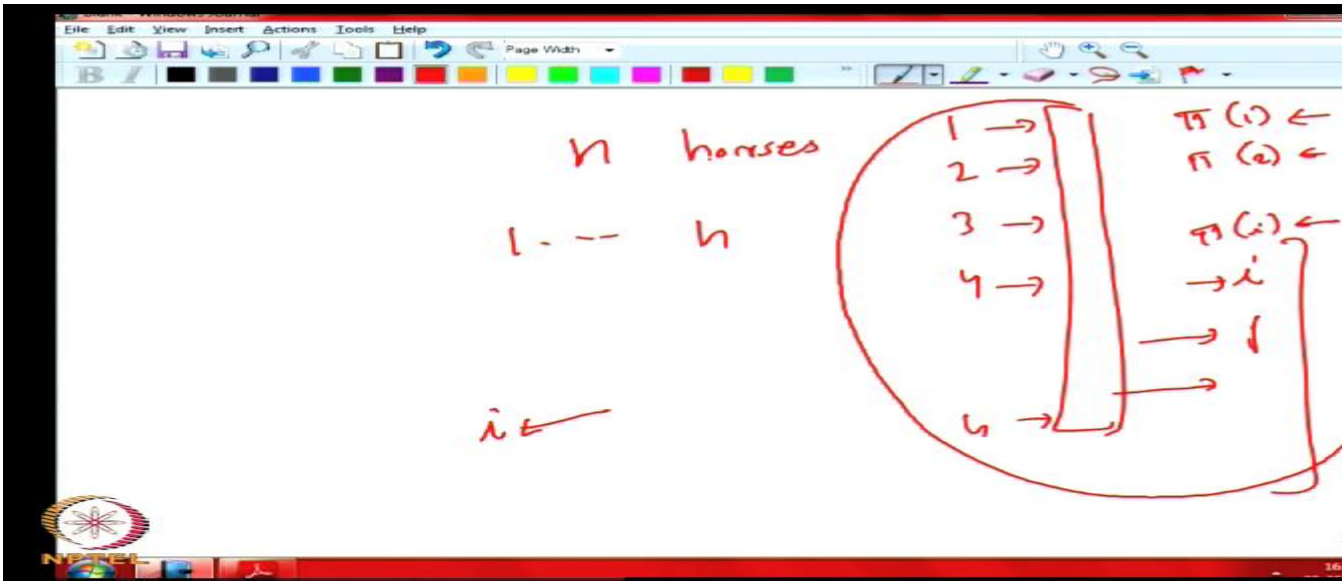

In other words another way to ask this question is suppose there is a horse race and there are n horses, right, you can make a bet saying that, so the horses are numbered from 1 to n. So, you can say that the first prize to the winner, the first place will be for horse number i, I can bet on that, right, and then second position will be horse number j like that.

So, if I can bet for each position the first will be this, second will be this horse, third will be the one who finishes the race first, right, second, third, fourth, all the n horses. Suppose somebody puts a bet like that essentially he is taking selecting a permutation of the horses, right. So, from the first his permutation pi pi of 1 correspond to according to him that horse will come first, right, and pi of 2 means according to him that particular horse will come second, i of pi means the i th position will be obtained by that horse. Now how many permutations are there? How many ways you can lose all the bets? I mean none of his bets are correct means there is no i such that in the i th position the horse he predicted comes. So, in that case we are again counting the derangements. This problem we will discuss in the next class and this will be the last one inclusion exclusion thing.